

# The Pitman-Koopman theorem and informationally relevant quantities

## research notes

Y. Roudi

[<yasser.roudi@ntnu.no>](mailto:yasser.roudi@ntnu.no)

P.G.L. Porta Mana

[<piero.mana@ntnu.no>](mailto:piero.mana@ntnu.no)

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I believe that the greatest unsolved problem in statistics  
is communicating the subject to others.

N. Davies (Copas et al. 1995 p. 445)

## 1 From qualitative to quantitative statements

We often have the need of quantifying a qualitative property or phenomenon, for example to ascertain whether it does or doesn't occur in some situations. This quantification is usually difficult because the property itself is only vaguely defined in our minds.

At times we sweep this difficulty under the carpet by choosing a quantitative measure that 'feels right' and proceeding with the calculations. It's obvious that the quantitative results we obtain this way are mostly smoke: despite the deceiving presence of numbers, our results are only qualitative, since our starting point was vague and our quantitative measure wasn't unique. Which results would we have obtained if we had started from a slightly different qualitative intuition or used a slightly different quantitative measure?

Other times we choose several quantitative measures that 'feel right', proceed with the calculations, and then compare the quantitative results obtained from all these measures. Results obtained this way are more robust – for example if all the chosen measures lead to the conclusion that the property is there – but unfortunately still not unequivocally quantitative.

In science there are successful examples of qualitative definitions that, formulated as a set of desiderata, lead to unique quantitative measures or theories. The most impressive example is probably that of

the probability calculus: simple intuitive desiderata about reasoning in situations of uncertainty, such as those examined by Pólya (1971; 1949; 1954; 1968), turn out to have a unique mathematical translation: the laws of the probability calculus (Cox 1946; Paris 2006; Jaynes 2003 ch. 2; Snow 1998; 2001; Dupré et al. 2009; Terenin et al. 2017; see also Aczél 2004). Another example is Shannon's entropy (Shannon 1948 § 6) as a measure of uncertainty, although Shannon's desiderata explicitly include the use of addition as the mathematical representation of 'combination' (cf. Jaynes 2003 § 11.3, p. 347), a requirement which is not universally agreed upon. In physics, semi-quantitative invariance requirements can often be mathematically translated into group-invariance statements leading to fully fledged theories; classical examples being Einstein's requirements (Einstein 1905 § 2) leading to the Lorentz transformations and special relativity, or Noether's theorem (Noether 1918; anticipated for the Euclidean group by Cosserat et al. 1909).

✂ to be continued

'A bewildering variety of statistical measures has been reported in the literature on spike-train analysis. Unfortunately, there has been little theoretical justification for the use and interpretation of many of these techniques' (Moore et al. 1966 p. 494)

## Bibliography

- (‘de  $X$ ’ is listed under  $D$ , ‘van  $X$ ’ under  $V$ , and so on, regardless of national conventions.)
- Aczél, J. (2004): *The associativity equation re-revisited*. Am. Inst. Phys. Conf. Proc. **707**, 195–203.
- Chatfield, C. (1995): *Model uncertainty, data mining and statistical inference*. J. Roy. Stat. Soc. A **158**<sup>3</sup>, 419–444. See also discussion (Copas, Davies, Hand, Lunneborg, Ehrenberg, Gilmour, Draper, Green, et al. 1995).
- Copas, J. B., Davies, N., Hand, D. J., Lunneborg, C. E., Ehrenberg, A. S., Gilmour, S. G., Draper, D., Green, P. J., et al. (1995): *Discussion of the paper by Chatfield [Model uncertainty, data mining and statistical inference]*. J. Roy. Stat. Soc. A **158**<sup>3</sup>, 444–466. See (Chatfield 1995).
- Cosserat, E., Cosserat, F. (1909): *Théorie des corps déformables*. (A. Hermann et fils, Paris). <http://ebooks.library.cornell.edu/cgi/t/text/text-idx?c=math;idno=06420001>.
- Cox, R. T. (1946): *Probability, frequency, and reasonable expectation*. Am. J. Phys. **14**<sup>1</sup>, 1–13. <http://algomagic.org/ProbabilityFrequencyReasonableExpectation.pdf>.
- Dupré, M. J., Tipler, F. J. (2009): *New axioms for rigorous Bayesian probability*. Bayesian Anal. **4**<sup>3</sup>, 599–606.
- Einstein, A. (1905): *Zur Elektrodynamik bewegter Körper*. Ann. der Phys. **17**, 891–921. Transl. in (Einstein 1989a), Doc. 23, pp. 140–171. Repr. in (; Einstein 1989b).

- (1989a): *The Collected Papers of Albert Einstein. Vol. 2: The Swiss Years: Writings, 1900–1909. (English translation).* (Princeton University Press, Princeton). Transl. of (Einstein 1989b) by Anna Beck and Peter Havas.
- (1989b): *The Collected Papers of Albert Einstein. Vol. 2: The Swiss Years: Writings, 1900–1909.* (Princeton University Press, Princeton). Ed. by John Stachel. Transl. in (Einstein 1989a).
- Jaynes, E. T. (2003): *Probability Theory: The Logic of Science.* (Cambridge University Press, Cambridge). Edited by G. Larry Bretthorst. First publ. 1994. <https://archive.org/details/XQUHIUXHIQUHIQUXUIHX2>, <http://www-biba.inrialpes.fr/Jaynes/prob.html>, <http://omega.albany.edu:8008/JaynesBook.html>.
- Moore, G. P., Perkel, D. H., Segundo, J. P. (1966): *Statistical analysis and functional interpretation of neuronal spike data.* Annu. Rev. Physiol. **28**, 493–522.
- Noether, E. (1918): *Invariante Variationsprobleme.* Ges. d. Wiss. Nachrichten (Gött.) Math.-Phys. Kl. **1918**, 235–257. <http://cwp.library.ucla.edu/articles/noether.trans/german/emmy235.html>. Transl. in (Noether 1971).
- (1971): *Invariant variation problems.* Transport Theory Stat. Phys. **13**, 186–207. <http://cwp.library.ucla.edu/articles/noether.trans/english/mort186.html>. Transl. by M. A. Tavel of (Noether 1918).
- Paris, J. B. (2006): *The Uncertain Reasoner's Companion: A Mathematical Perspective*, reprint. (Cambridge University Press, Cambridge). See also (Snow 1998).
- Pólya, G. (1949): *Preliminary remarks on a logic of plausible inference.* Dialectica **3**<sup>1-2</sup>, 28–35.
- (1954): *Mathematics and Plausible Reasoning: Vol. I: Induction and Analogy in Mathematics.* (Princeton University Press, Princeton).
- (1968): *Mathematics and Plausible Reasoning: Vol. II: Patterns of Plausible Inference*, 2nd ed. (Princeton University Press, Princeton). First publ. 1954.
- (1971): *How To Solve It: A New Aspect of Mathematical Method*, 2nd ed. (Princeton University Press, Princeton). First publ. 1945.
- Shannon, C. E. (1948): *A mathematical theory of communication.* Bell Syst. Tech. J. **27**<sup>3, 4</sup>, 379–423, 623–656. <https://archive.org/details/bstj27-3-379>, <https://archive.org/details/bstj27-4-623>, <http://math.harvard.edu/~ctm/home/text/others/shannon/entropy/entropy.pdf>.
- Snow, P. (1998): *On the correctness and reasonableness of Cox's theorem for finite domains.* Comput. Intell. **14**<sup>3</sup>, 452–459.
- (2001): *The reasonableness of possibility from the perspective of Cox.* Comput. Intell. **17**<sup>1</sup>, 178–192.
- Terenin, A., Draper, D. (2017): *Cox's theorem and the Jaynesian interpretation of probability.* arXiv:1507.06597. First publ. 2015.