The Pitman-Koopman theorem and informationally relevant quantities

research notes

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> I believe that the greatest unsolved problem in statistics is communicating the subject to others.
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> N. Davies (Copas et al. 1995 p. 445)

1 From qualitative to quantitative statements

We often have the need of quantifying a qualitative property or phenomenon, for example to ascertain whether it does or doesn't occur in some situations. This quantification is usually difficult because the property itself is only vaguely defined in our minds.

At times we sweep this difficulty under the carpet by choosing a quantitative measure that 'feels right' and proceeding with the calculations. It's obvious that the quantitative results we obtain this way are mostly smoke: despite the deceiving presence of numbers, our results are only qualitative, since our starting point was vague and our quantitative measure wasn't unique. Which results would we have obtained if we had started from a slightly different qualitative intuition or used a slightly different quantitative measure?

Other times we choose several quantitative measures that 'feel right', proceed with the calculations, and then compare the quantitative results obtained from all these measures. Results obtained this way are more robust – for example if all the chosen measures lead to the conclusion that the property is there – but unfortunately still not unequivocally quantitative.

In science there are successful examples of qualitative definitions that lead to unique quantitative measures or theories. The most impressive example is probably that of the probability calculus: simple intuitive desiderata about reasoning in situations of uncertainty, such as those examined by Pólya (1971; 1949; 1954; 1968), turn out to have a unique mathematical translation: the laws of the probability calculus (Cox 1946; Paris 2006; Jaynes 2003 ch. 2; Snow 1998; 2001; Aczél 2004; Dupré et al. 2009; Terenin et al. 2017). Another example is Shannon's entropy (1948 § 6)

✠ to be continued

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('de *X*' is listed under D, 'van *X*' under V, and so on, regardless of national conventions.)

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