Parameter priors for Ising models

research notes

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Study of uniform priors in parameter space and in constraint space for Ising models

'Flat priors do not exist'
(anonymous)

1 A two-unit model with sufficient statistics

Consider a population of two binary units $s := (s_1, s_2)$ with values in $\{0, 1\}$. One observation of this population can thus give four results: $s \in \{00, 01, 10, 11\}$.

Assume that we have m observations $(s^{(1)}, \dots, s^{(m)})$ of this or other populations prepared in similar conditions, so that knowledge of these observations is relevant for our forecast of a new observation s, again in similar conditions. Also assume that only the number, the mean, and the second moments of these past observations are relevant to forecast the new one; that is,

$$m, \qquad \tfrac{1}{m}\big(s^{(1)} + \dots + s^{(m)}\big), \qquad \tfrac{1}{m}\big(s^{(1)} s^{(1)\intercal} + \dots + s^{(m)} s^{(m)\intercal}\big) \tag{1}$$

are sufficient statistics; note that the second sum contains the first as its diagonal. These assumptions are collectively denoted I. Then the Koopman-Pitman theorem says that our probabilistic forecasts must assume this general form:

$$\mathrm{p}(s \mid I) = \int g(s_1, s_2) \frac{\exp(\mu_1 s_1 + \mu_2 s_2 + \lambda s_1 s_2)}{Z(\mu_1, \mu_2, \lambda)} \, \mathrm{p}(\mu_1, \mu_2, \lambda \mid I) \, \mathrm{d}\mu_1 \, \mathrm{d}\mu_2 \, \mathrm{d}\lambda,$$

$$\text{with} \quad Z(\mu_1, \mu_2, \lambda) := 1 + \exp(\mu_1) + \exp(\mu_2) + \exp(\mu_1 + \mu_2 + \lambda), \tag{2}$$

where neither g nor $\mathrm{p}(\mu_1, \mu_2, \lambda \mid I)$ are determined by the theorem: they need to be determined by additional assumptions. For later convenience, denote $\theta := (\mu_1, \mu_2, \lambda)$.