

# The Pitman-Koopman theorem and informationally relevant quantities

## research notes

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23 June 2018; updated 24 June 2018

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I believe that the greatest unsolved problem in statistics  
is communicating the subject to others.

N. Davies (Copas et al. 1995 p. 445)

## 1 From qualitative to quantitative statements

We often have the need of quantifying a qualitative property or phenomenon, for example to ascertain whether it does or doesn't occur in some situations. This quantification is usually difficult because the property itself is only vaguely defined in our minds.

At times we sweep this difficulty under the carpet by choosing a quantitative measure that 'feels right' and proceeding with the calculations. It's obvious that the quantitative results we obtain this way are mostly smoke: despite the deceiving presence of numbers, our results are only qualitative, since our starting point was vague and our quantitative measure wasn't unique. Which results would we have obtained if we had started from a slightly different qualitative intuition or used a slightly different quantitative measure?

Other times we choose several quantitative measures that 'feel right', proceed with the calculations, and then compare the quantitative results obtained from all these measures. Results obtained this way are more robust – for example if all the chosen measures lead to the conclusion that the property is there – but unfortunately still not unequivocally quantitative.

In science there are successful examples of qualitative definitions that, formulated as a set of desiderata, lead to unique quantitative measures or theories. The most impressive example is probably that of the probability

calculus: simple intuitive desiderata about reasoning in situations of uncertainty, such as those examined by Pólya (1971; 1949; 1954; 1968), turn out to have a unique mathematical translation: the laws of the probability calculus (Cox 1946; Paris 2006; Jaynes 2003 ch. 2; Snow 1998; 2001; Aczél 2004; Dupré et al. 2009; Terenin et al. 2017). Another example is Shannon’s entropy (Shannon 1948 § 6) as a measure of uncertainty, although Shannon’s desiderata explicitly include the use of addition as the mathematical representation of ‘combination’ (cf. Jaynes 2003 § 11.3, p. 347), a requirement which is not universally agreed upon. In physics, semi-quantitative invariance requirements can often be translated into group-invariance statements leading to fully fledged theories; classical examples being Einstein’s requirements (Einstein 1905 § 2) leading to the Lorentz transformations and special relativity, or Noether’s theorem (Noether 1918; partially anticipated by Cosserat et al. 1909).

✚ to be continued

## Thanks

PGLPM thanks Mari & Miri for continuous encouragement and affection; Buster Keaton and Saitama for filling life with awe and inspiration; the developers and maintainers of L<sup>A</sup>T<sub>E</sub>X, Emacs, AUCT<sub>E</sub>X, Open Science Framework, Python, Inkscape, Sci-Hub for making a free and unfiltered scientific exchange possible.

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