Memos on factorial moments

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Note: Dear Reader & Peer, this manuscript is being peer-reviewed by you. Thank you.

Consider a quantity x assuming integer values $\{0, 1, \dots, N\}$, and a distribution of degrees of belief $\mathrm{p}(x \mid I)$ for it. The raw moments of this distribution are

$$\mathrm{E}(x^m \mid I) \coloneqq \sum_x x^m \, \mathrm{p}(x \mid I). \tag{1}$$

Now think of x as the number of units, among a total population of N, that have some property. We can ask how many distinct *ordered* pairs, triplets, m-tuples have that property. Their number is the falling factorial

$$x^{\underline{m}} \coloneqq m! \binom{x}{m}. \tag{2}$$

The mth factorial moment of our distribution is

$$\mathrm{E}(x^{\underline{m}} \mid I) \equiv m! \, \mathrm{E}\left[\binom{x}{m} \,\middle|\, I\right] \coloneqq m! \sum_x \binom{x}{m} \, \mathrm{p}(x \mid I). \tag{3}$$

The expectation of the number of distinct *unordered* m-tuples, $\binom{x}{m}$, is therefore $1/m!$ times the mth factorial moment.

The total number of possible unordered m-tuples is $\binom{N}{m}$. The fraction of unordered m-tuples having the property under study is therefore $\binom{x}{m}/\binom{N}{m}$, and its expectation is the normalized factorial moment

$$\mathrm{E}\left(\frac{x^{\underline{m}}}{N^{\underline{m}}} \,\middle|\, I\right) \equiv \mathrm{E}\left[\frac{\binom{x}{m}}{\binom{N}{m}} \,\middle|\, I\right]. \tag{4}$$

for $m = 1$ we have the population mean $\mathrm{E}(x/N \mid I)$ as a special case.

The use of factorial moments is useful because of their relation with raw moments:

$$\mathrm{E}(x^m \mid I) = \sum_{n=0}^{m} \left\{ {m \atop n} \right\} \mathrm{E}(x^{\underline{n}} \mid I), \tag{5}$$

where $\left\{ {m \atop n} \right\}$ is a Stirling number of the second kind (Knuth 1992).

Bibliography

Knuth, D. E. (1992): *Two notes on notation*. Am. Math. Monthly **99**^{5}, 403–422.