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# Logic and the Jeffreys-Lindley "paradox"

**[draft]**

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## 1 An apology

So much has been written on the Jeffreys-Lindley "paradox", that maybe an apology is in order for writing even more about it. My purpose here is not to tell you to avoid or use point-hypotheses, or to avoid or use improper priors, nor to draw comparisons between Bayesian probability theory and frequency statistics (it's difficult to compare a theory and a trade). Our discussion is fully within the Bayesian probability-calculus. I only want to point out some logical inconsistencies that recur in the literature about the paradox.

The "paradox" is that some derivations with the probability-calculus lead to particular results which are different from what some people expect to find with those derivations. But the very set-up of such derivations, as presented in the literature, is often vague and logically inconsistent, even before unexpected results are obtained. For example, usually two hypotheses are presented: say '$\mu = 0$' versus '$\mu \neq 0$'. But an unhurried analysis shows that '$\mu \neq 0$' is not a hypothesis, or at most it is an ill-defined one. Some expressions are used, such as "negligible effect"; but what does *negligible* mean?

I shall now discuss some of these inconsistencies. To understand some of them better, it is useful to discuss a simple problem in formal propositional logic first.

## 2 Formal logic and well-defined hypotheses

Consider an atomic statement $A$. We want to find its truth-value. This is possible only if we have a sufficient set of sentences whose truth-values are given: the premisses or axioms. This set must be sufficient in the sense that the truth-values of its sentences determine the truth-value of $A$ according to rules of the truth calculus.

Take $H \Rightarrow A$ as a premiss; that is, this composite statement has value true. We shall keep this premiss fixed; for this reason I call it our 'context'. The context $H \Rightarrow A$ is not sufficient to determine the truth-value of $A$. We consider other possible additional premisses besides our context. I shall call an additional premiss a 'hypothesis' only if together with the context it determines the truth-value of $A$.

The statement $H$ is a valid hypothesis within our context: its truth determines the truth-value of $A$ as true, as can be seen by a simple application of the truth calculus. On the other hand the premiss $\neg H$, the negation of $H$, does *not* determine the truth-value of $A$. It is therefore not a hypothesis, according to our terminology.

We thus see that even if a statement is a hypothesis, its negation may not be a hypothesis – with respect to a specific context and a definite set of statements whose truth-values we want to determine.

The premiss $\neg H$ must be conjoined with some additional premiss, for instance $\neg H \Rightarrow \neg A$, in order to form a hypothesis. So the sentence $H$ and the sentence $\neg H \land (\neg H \Rightarrow \neg A)$ are two alternative and mutually exclusive hypotheses within our context, but they are not each other's negation.

## 3 What are the hypotheses in the paradox?

The Jeffreys-Lindley paradox is usually presented with this set-up:

- a quantity $x$ with unknown value in $\mathbf{R}$;
- a statement $I$ to the effect that the probability of $x \in S$, for every (Borel) set $S$ in $\mathbf{R}$, has the functional form

$$\int_S \frac{1}{\sqrt{2\pi}} \exp\left[-\tfrac{1}{2}(x-\mu)^2\right] \mathrm{d}x \tag{1}$$

for some specific $\mu \in \mathbf{R}$;

- a statement $H$, equivalent to $\mu = 0$;
- the negation $\neg H$, equivalent to $\mu \neq 0$;
- a statement $L$ to the effect that the probability of $H$ is, say, $1/2$ (and such a statement does not alter the effect of statement $I$).

It is clear that the statement $I$, which is analogous to the context in the discussion of the previous section, does not determine the probability-value of any set $S$, except the whole real line.

The statements $I$ and $H$ together do determine the probabilities of every set $S$; for example

$$\mathrm{P}\bigl(x \in [1,2] \bigm| I \land H\bigr) = \int_1^2 \frac{1}{\sqrt{2\pi}} \exp\bigl(-\tfrac{1}{2}x^2\bigr)\, \mathrm{d}x \approx 0.14\ . \tag{2}$$

We can thus call $H$, that is, $\mu = 0$, a hypothesis in the context $I$, since it allows us to determine probability-values for the ranges of the $x$.

The statement $\neg H$ or $\mu \neq 0$, however, does not allow us to determine any definite probability-values for $x$, as is clear by the assumptions listed above. Therefore, according to our terminology, *in the context $I$ or $I \land L$ the statement $\mu = 0$ is a hypothesis, but its negation $\mu \neq 0$ is not a hypothesis*. This is analogous to the situation in sentential logic examined in the previous section; it is not surprising nor a sign of inconsistency.

Please note that my point is not about the word 'hypothesis'. If you want to call both statements 'hypotheses', fine. What matters here is that *we cannot "test" the statement $\mu = 0$ versus the statement $\mu \neq 0$*, or more precisely we cannot determine their posterior probabilities in the context $I \land L$, because the likelihood of the second statement is undefined.

Similarly to the sentential-logic example, ***

This point is neglected or glossed over by many authors discussing the paradox. The hypotheses are usually presented as '$\mu = 0$' vs '$\mu \neq 0$', and for the second the author later adds "it is necessary to introduce a prior distribution for $\mu$". But it is an important point, because it completely changes our inference.