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Logic and the Jeffreys-Lindley "paradox"

[draft]

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1 An apology

So much has been written on the Jeffreys-Lindley "paradox", that maybe an apology is in order for writing even more about it. My purpose here is not to tell you to avoid or use point-hypotheses, or to avoid or use improper priors, nor to draw comparisons between Bayesian probability theory and frequency statistics (it's difficult to compare a theory and a trade). Our discussion is fully within the Bayesian probability-calculus. I only want to point out some logical inconsistencies that recur in the literature about the paradox.

The "paradox" is that some derivations with the probability-calculus lead to particular results which are different from what some people expect to find with those derivations. But the very set-up of such derivations, as presented in the literature, is often vague and logically inconsistent, even before unexpected results are obtained. For example, usually two hypotheses are presented: say '$\mu = 0$' versus '$\mu \neq 0$'. But an unhurried analysis shows that '$\mu \neq 0$' is not a hypothesis, or at most it is an ill-defined one. Some expressions are used, such as "negligible effect"; but what does *negligible* mean?

I shall now discuss some of these inconsistencies. To understand some of them better, it is useful to discuss a simple problem in formal propositional logic first.

2 Formal logic and well-defined hypotheses

Consider an atomic statement A. We want to find its truth-value. This is possible only if we have a sufficient set of sentences whose truth-values are given: the premisses or axioms. This set must be sufficient in the sense that the truth-values of its sentences determine the truth-value of A according to rules of the truth calculus.

Take $H \Rightarrow A$ as a premiss; that is, this composite statement has value true. We shall keep this premiss fixed; for this reason I call it our 'context'. The context $H \Rightarrow A$ is not sufficient to determine the truth-value of A. We consider other possible additional premisses besides our context. I shall call an additional premiss a 'hypothesis' only if together with the context it determines the truth-value of A.

The statement H is a valid hypothesis within our context: its truth determines the truth-value of A as true, as can be seen by a simple application of the truth calculus. On the other hand the premiss $\neg H$, the negation of H, does *not* determine the truth-value of A. It is therefore not a hypothesis, according to our terminology.

We thus see that even if a statement is a hypothesis, its negation may not be a hypothesis – with respect to a specific context and a definite set of statements whose truth-values we want to determine.

The premiss $\neg H$ must be conjoined with some additional premiss, for instance $\neg H \Rightarrow \neg A$, in order to form a hypothesis. So the sentence H and the sentence $\neg H \wedge (\neg H \Rightarrow \neg A)$ are two alternative and mutually exclusive hypotheses within our context, but they are not each other's negation.

3 What are the hypotheses in the paradox?

The Jeffreys-Lindley paradox is usually presented with this set-up:

- a quantity x with unknown value in $\mathbf{R}$;
- a statement I such that the probability of $x \in S$, for every (Borel) set S in $\mathbf{R}$, has the functional form

$$\int_S \frac{1}{\sqrt{2\pi}} \exp\left[-\tfrac{1}{2}(x-\mu)^2\right] \mathrm{d}x \tag{1}$$

for some specific μ;

- a statement H, equivalent to $\mu = 0$;
- the negation $\neg H$, equivalent to $\mu \neq 0$.

It is clear that the statement I, which is analogous to the context in the discussion of the previous section, does not determine the probability-value of any set S, except the whole real line.

The statements I and H together do determine the probabilities of every set S; for example

$$\mathrm{P}\big(x \in [1,2] \,\big|\, I \land H\big) = \int_1^2 \frac{1}{\sqrt{2\pi}} \exp\big(-\tfrac{1}{2}x^2\big)\,\mathrm{d}x \approx 0.136\,. \tag{2}$$

We can thus call H, that is, $\mu = 0$, a hypothesis in the context I, since it allows us to determine probability-values for the quantity x.

The statement $\neg H$ or $\mu \neq 0$, however, does not allow us to determine a definite probability-value for x.