Memos on measures and metrics on simplices

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Draft of 14 January 2019 (first drafted 11 January 2019)

Some memos on measures and metrics on simplices.

Consider $N+1$ mutually exclusive and exhaustive propositions. The belief distributions and the relative-frequency distributions for them form an N-dimensional simplex. Label the propositions $\{0,\dots,N\}$, denote a distribution by $(q_0,\dots,q_N) =: \boldsymbol{q}$. In the rest of this memo it's always implicitly assumed that $q_i \geqslant 0$, also in the integration domains. Denote

$$\Delta_N \coloneqq \{(x_1,\dots,x_N) \mid x_i \geqslant 0, \textstyle\sum_i x_i \leqslant 1\}, \tag{1}$$

which is the $(N+1)$-simplex asymmetrically embedded in $\mathbf{R}^{N+1}$. As basic volume element we can take either

$$\mathrm{d}q_1 \cdots \mathrm{d}q_N, \qquad (q_1,\dots,q_N) \in \Delta_N, \tag{2}$$

or

$$\mathrm{d}q_0 \cdots \mathrm{d}q_N \; \delta(1-\textstyle\sum \boldsymbol{q}), \qquad (q_0,\dots,q_N) \in [0,+\infty[^{N+1}. \tag{3}$$

The latter leads to more symmetric formulae. The two volume elements are equivalent, and their integration gives $1/N!$, as can be proven inductively (Δ_k is the base of Δ_{k+1}: multiply its k-volume by a unit height and divide by $k+1$) or as shown in Jaynes (2003 § 18.10). Let's denote either measure by $\mathrm{d}\boldsymbol{q}$. In integrations, when (2) is intended, any q_0 that appears in the integral must be understood as $1-\sum_{i=1}^{N} q_i$.

The N-simplex has a natural convex structure. Thus the ratio of two N-volumes is well-defined. There's only one measure that gives (a) the same degree of belief to any two N-volumes having unit ratio, (b) unit degree of belief to the full simplex:

$$N!\,\mathrm{d}\boldsymbol{q}. \tag{4}$$

This document is designed for screen reading and two-up printing on A4 or Letter paper

Bibliography

('de *X*' is listed under D, 'van *X*' under V, and so on, regardless of national conventions.)

Jaynes, E. T. (2003): *Probability Theory: The Logic of Science*. (Cambridge University Press, Cambridge). Ed. by G. Larry Bretthorst. First publ. 1994. https://archive.org/details/XQUHIUXHIQUHIQXUIHX2, http://www-biba.inrialpes.fr/Jaynes/prob.html.