A formula for partial and conditional infinite exchangeability

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[draft] A formula is given for conditionally, infinitely exchangeable probability distributions.

1 Full, partial, and conditional exchangeability

De Finetti's theorem for infinitely exchangeable probability distributions is one of the formulae derived from the probability calculus having the richest practical and philosophical consequences. Leaving for the moment the definition of symbols to intuition, the theorem rewrites a joint probability distribution as a law of total probability:

$$\mathrm{P}(X_1 = x_1,\ X_2 = x_2,\ \dots \mid J) = \int \prod_i f_{x_i}\ \mathrm{p}(\boldsymbol{f} \mid J)\,\mathrm{d}\boldsymbol{f}\ , \tag{1}$$

where $\boldsymbol{f} \coloneqq (f_x)$ is a distribution over the values that can be assumed by each X_i, and the integral is over the simplex of such distributions.

The condition for the theorem to hold is that the joint distribution be infinitely *fully exchangeable*, that is, symmetric with respect to permutations of the x_i, for any set of indices $\{i\}$. This formula is the infinite limit of the sampling formula from an urn of unknown content. From now on 'exchangeable' shall be understood as 'infinitely exchangeable'.

A more general version of the theorem holds if the joint distribution is *partially exchangeable*, that is, if the set $\{X_i\}$ is divided into two or more categories represented by subsets $\{Y_j\}, \{Z_k\}, \dots,$

and permutations are allowed within each subset but not necessarily across subsets. The formula then becomes, with a suitable re-indexing $\{1, 2, \dots\} \mapsto \{1', 2', \dots, 1'', 2'', \dots\}$,

$$
\mathrm{P}(Y_{1'} = y_{1'},\ Y_{2'} = y_{2'},\ \dots,\ Z_{1''} = z_{1''},\ Z_{2''} = z_{2''},\ \dots \mid J) = \iint \prod_j g_{y_j} \prod_k h_{z_k}\, \mathrm{p}(\boldsymbol{g}, \boldsymbol{h} \mid J)\, \mathrm{d}\boldsymbol{g}\, \mathrm{d}\boldsymbol{h}\ , \tag{2}
$$

with distinct distributions $\boldsymbol{g}$, $\boldsymbol{h}$ for each category. If the density $\mathrm{p}(\boldsymbol{g}, \boldsymbol{h} \mid J)\, \mathrm{d}\boldsymbol{g}\, \mathrm{d}\boldsymbol{h}$ is diagonal, that is, if it contains a term $\delta(\boldsymbol{g} - \boldsymbol{h})$, the fully exchangeable form (1) is recovered.

With a little reflection we see that if we know that the quantities X belong to category Y in instances $1', 2', \dots$, and to category Z in instances $1'', 2'', \dots$, then (a) there is some other quantity C that allows us to distinguish the two categories, and (b) the values $C_i = c_i$ of this quantity *are known* for all instances.

Let us say, for example, that the quantities X_i are the results of animal treatments, with values 'S'uccess and 'F'ailure. Y refers to the results for treatments on Yaks, and Z on Zebras. If we write

$$
\mathrm{P}(Y_3 = \mathsf{S},\ Z_5 = \mathsf{F} \mid J) = 0.2\ ,
$$

then we must already know that animal number 3 is a yak, $C_3 = \mathsf{Y}$, and animal number 5 is a zebra, $C_5 = \mathsf{Z}$. This is clear from our very notation, otherwise we would not have known whether to use the symbol Y or Z for those instances. This information is evidently implicit in our background information J.

Let us make the information about the C_i explicit. Their possible values are 'Y' and 'Z'. We rewrite the probability in eq. (2) as

$$
\begin{aligned}
&\mathrm{P}(Y_{1'} = y_{1'},\ \dots,\ Z_{1''} = z_{1''},\ \dots \mid J) \equiv \\
&\mathrm{P}(X_{1'} = x_{1'},\ \dots,\ X_{1''} = x_{1''},\ \dots \mid C_{1'} = \mathsf{Y},\ \dots,\ C_{1''} = \mathsf{Z},\ \dots,\ I)\ .
\end{aligned} \tag{3}
$$

Then it is clear that the partially exchangeable probability distribution (2) or (3) can also be called *conditionally exchangeable*.

The present work gives a representation formula for pairs of quantities (X_i, C_i) such that

1. the distribution for the $\{X_i\}$ is conditionally exchangeable, given $\{C_i\}$,
2. the distribution for the $\{C_i\}$ is fully exchangeable.

The formula is:

$$\mathrm{P}\bigl[(X_1 = x_1, C_1 = c_1),\ (X_2 = x_2, C_2 = c_2),\ \dots \mid I\bigr] = \int \prod_i f_{x_i, c_i}\ \mathrm{p}(\boldsymbol{f} \mid I)\,\mathrm{d}\boldsymbol{f} \tag{4a}$$

$$\text{with} \qquad \boxed{\mathrm{p}(\boldsymbol{f} \mid I)\,\mathrm{d}\boldsymbol{f} = \mathrm{p}[(f_{x|c}) \mid I]\,\mathrm{d}(f_{x|c}) \times \mathrm{p}[(f_{,c}) \mid I]\,\mathrm{d}(f_{,c})} \tag{4b}$$

where

- $\boldsymbol{f} \coloneqq (f_{x,c})$ is a joint distribution over the set of values that can be assumed by each (X_i, C_i) pair,
- $\bigl(f_{,c} \coloneqq \sum_x f_{x,c}\bigr)$ is the related *marginal* frequency distribution for the c values,
- $\bigl(f_{x|c} \coloneqq f_{x,c}/f_{,c}\bigr)$ are the related *conditional* frequency distributions of x given c.

The noteworthy feature of the formula above is that the conditional exchangeability for $\{X_i\}$ given $\{C_i\}$ is expressed by *the factorizability of the density* $\mathrm{p}(\boldsymbol{f} \mid I)$ into the product, eq. (4b), of a density for the conditional frequencies $(f_{x|c})$ and a density for the marginal frequencies $(f_{,c})$.

2 Setup

We have a countably infinite set of statements $X_t = x_t$ with $t \in \{1, 2, \dots\}$, where each X_t is some quantity and each $x_t \in S$, a common finite set of values (the theorem also works with statements that are not about variates). De Finetti's theorem states that if a probability distribution about any subset of such statements is exchangeable, that is,

$$\begin{gathered}\mathrm{P}(X_1 = x_1, X_2 = x_2, \dots, X_T = x_T \mid I) = \\ \mathrm{P}(X_1 = x_{\pi(1)}, X_2 = x_{\pi(2)}, \dots, X_T = x_{\pi(T)} \mid I) \\ \text{for any } T \text{, any set of indices } \{1, \dots, T\} \text{, and any permutation } \pi \text{ thereof,}\end{gathered} \tag{5}$$

all such joint probabilities appropriately related by marginalization, then we can write

$$\int \mathrm{d}\ldots \prod_{ij} \bigl(\xi_{i|j}\, \nu_j\bigr)^{q_{i|j}\, f_j}\ \mathrm{p}(\xi)\, \mathrm{p}(\nu) \tag{6}$$

Bibliography

('de *X*' is listed under D, 'van *X*' under V, and so on, regardless of national conventions.)

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