

# The relation between cross-validation log-scores, weight of evidence, and Bayes factors

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The probability calculus unequivocally tells us how our degree of belief  $P(H_h | D K)$  in a hypothesis  $H_h$  given data  $D$  and background knowledge  $K$  is related to our degree of belief  $P(D | H_h K)$  in observing those data when we entertain that hypothesis as true:

$$P(H_h | D K) = \frac{P(D | H_h K) P(H_h | K)}{P(D | K)} \quad (1a)$$

$$= \frac{P(D | H_h K) P(H_h | K)}{\sum_{h'} P(D | H_{h'} K) P(H_{h'} | K)}. \quad (1b)$$

$D, H_h, K$  denote propositions, which usually are about numeric quantities. I use the terms ‘degree of belief’, ‘belief’, and ‘probability’ as synonyms. By ‘hypothesis’ I mean a scientific (physical, biological, etc.) hypothesis, a state or development of things capable of experimental verification, at least in a thought experiment.

Don’t we too often abuse of the fact that the probability calculus, just like the truth calculus, proceeds purely syntactically rather than semantically? That is, if I tell you that  $H$  and  $H \Rightarrow D$  are true, you can conclude that  $D$  is true; similarly, if I tell you that  $P(H | K) = p$  and  $P(H \Rightarrow D | K) = q$ , you can conclude (try it as an exercise) that  $P(H D | K) = p + q - 1$ . In either case you don’t need to know what  $H$  and  $D$  are about – they could be about Donald Duck or parallel universes. And so we often say ‘under model  $\theta$  our belief about the value of quantity  $x$  is expressed by such and such distribution  $p(x | \theta) = f(x)$ ’, without explaining what  $\theta$  really is and why it leads to  $f$ . Aren’t terms such as ‘model’ and ‘hypothesis’, as often used in probability and statistics, convenient and respectable-looking carpets under which we can sweep the fact that we barely know what we’re speaking about? The need to look under the carpet arises, though, the moment we have to specify our pre-data belief, the prior, about the mysterious  $\theta$ .

But semantics can very well be a by-product of syntax, or the distinction between the two be a chimera (Wittgenstein 1999; Girard 2001; 2003). Such important matters are unfortunately rarely discussed in probability and statistics.

Expression (1b) assumes that we have a set  $\{H_h\}$  of mutually exclusive and exhaustive hypotheses under consideration, which is implicit in our knowledge  $K$  – in fact, the right side is only valid if

$$P(\bigvee_h H_h \mid K) = 1, \quad P(H_h \wedge H_{h'} \mid K) = 0 \quad \text{if } h \neq h'. \quad (2)$$

Only in extremely rare cases does the set of hypotheses  $\{H_h\}$  encompass and reflect the extremely complex and fuzzy hypotheses lying in the backs of our minds. The background knowledge  $K$  is therefore only a simplified picture of our actual knowledge. That's why  $K$  or the hypotheses  $\{H_h\}$  are often called *models*. 'A theory cannot duplicate nature, for if it did so in all respects, it would be isomorphic to nature itself and hence useless, a mere repetition of all the complexity which nature presents to us, that very complexity we frame theories to penetrate and set aside. If a theory were not simpler than the phenomena it was designed to model, it would serve no purpose. Like a portrait, it can represent only a part of the subject it pictures. This part it exaggerates, if only because it leaves out the rest. Its simplicity is its virtue, provided the aspect it portrays be that which we wish to study' (Truesdell et al. 1980 Prologue p. xvi).

Expression (1a) is universally valid instead, but it's rarely possible to quantify its denominator  $P(D \mid K)$  unless we simplify our inferential problem by introducing a possibly unrealistic exhaustive set of hypotheses, thus falling back to (1b). We can bypass this problem if we are content with comparing our beliefs about any two hypotheses through their ratio, so that the term  $P(D \mid K)$  cancels out. See Jaynes's (2003 §§ 4.3–4.4) insightful remarks about such binary comparisons, and also Good's (1950 § 6.3–6.6).

If our problem is to finally choose a hypothesis, discarding its competitors for future calculations, or more generally to make a decision (for example, choice of medical treatment) based on the observed data, the post-data belief (1) is necessary but not sufficient. We also need to specify a utility or cost function to calculate the expected gains of choosing one or another hypothesis or making one or another decision (Kadane et al. 1980; DeGroot 2004; Bernardo et al. 2000 ch. 2).

If our problem has an exploratory nature instead – for example, evaluating which hypotheses to include in our simplified set, or examining whether a hypothesis leads to peculiar beliefs for peculiar kinds of data –

then all terms appearing in expression (1) are usually freely examined. In particular the term  $P(D | H_h K)$ , called the *likelihood* of the hypothesis given the data (Good 1950 § 6.1 p. 62), or its logarithm

$$\log P(D | H_h K). \quad (3)$$

The likelihoods of several hypotheses are often compared, through their ratios for example, called *relative Bayes factor*, or its logarithm, called *relative weight of evidence* (Good 1950 ch. 6; 1975; 1981; 1985; and many other works in Good 1983; Osteyee et al. 1974 § 1.4; MacKay 1992; Kass et al. 1995; see also Jeffreys 1983 chs V, VI, A). ‘It is historically interesting that the expression “weight of evidence”, in its technical sense, anticipated the term “likelihood” by over forty years’ (Osteyee et al. 1974 § 1.4.2 p. 12).

Since recent literature (for example Kass et al. 1995) seems to exclusively deal with *relative* Bayes factors, I’d like to point out that the non-relative Bayes factor for a hypothesis  $H_h$  provided by data  $D$  is actually defined as (Good 1981 § 2)

$$\frac{P(D | H_h K)}{P(D | \neg H_h K)} \equiv \frac{O(H_h | D K)}{O(H_h | K)} = \frac{P(D | H_h K) [1 - P(H_h | K)]}{\sum_{h' \neq h} P(D | H_{h'} K) P(H_{h'} | K)}, \quad (4)$$

where the *odds*  $O$  is defined as  $O := P/(1 - P)$ . Looking at the expression on the right, which can be derived from the probability rules, it’s clear that the Bayes factor for a hypothesis involves the likelihood of *all* other hypotheses as well as their pre-data probabilities. This quantity and its logarithm, the (non-relative) weight of evidence, has important properties which *relative* Bayes factors don’t enjoy. For example, the expected weight of evidence for a correct hypothesis is always positive, and for a wrong hypotheses always negative (Good 1950 § 6.7). See Jaynes (2003 §§ 4.3–4.4) for further discussion and a numeric example.

The literature in probability and statistics has also employed various other ad-hoc measures to make exploratory analyses. Here I consider one in particular: the *leave-one-out cross-validation log-score*, which I’ll just call ‘log-score’ for brevity:

$$\frac{1}{d} \sum_{i=1}^d \log P(D_i | D_{-i} H_h K) \quad (5)$$

where every  $D_i$  is one datum in the data  $D \equiv \bigwedge_i D_i$ , and  $D_{-i}$  denotes the data with datum  $D_i$  excluded. The intuition behind this score is more or less this: ‘let’s see what my belief in one datum should be, on average, once I’ve observed the other data, if I consider  $H_h$  as true’. ‘On average’ means considering such belief for every single datum in turn, and then taking the geometric mean, which is the arithmetic mean on a log scale.

There is an ample literature discussing the properties and use of the log-score (for example Geisser et al. 1979; Vehtari et al. 2002; 2012; Krnjajić et al. 2011; 2014; Gelman et al. 2014; Piironen et al. 2017; Gronau et al. 2019; Chandramouli et al. 2019). Gelman et al. (2014), for example, show among other things that it's approximately equal to expected value of the post-data log-probability of a new datum:

$$E[\log P(D' | D H K) | D H K], \quad (6)$$

where  $D'$  represent the new datum. Krnjajić et al. (2011; 2014) numerically compare it with the deviance information criterion. Regarding its uses I recommend the recent, beautiful debate in Gronau et al. (2019 & Chandramouli et al. 2019), where all authors give very insightful remarks.

I now show an exact relation between the log-score (5) and the log-likelihood (3) which doesn't seem to appear in the literature. I find this relation very intriguing because it says that *the log-likelihood is the sum of all log-scores that can be formed from all data subsets*.

We can obviously write the likelihood as

$$P(D | H K) \equiv \left[ \underbrace{P(D | H K) \times \cdots \times P(D | H K)}_{d \text{ times}} \right]^{1/d} \quad (7)$$

where we have dropped the subscript  $_h$  for simplicity. By the rules of probability we have

$$P(D | H K) = P(D_i | D_{-i} H_h K) \times P(D_{-i} | H_h K) \quad (8)$$

and this holds no matter what specific  $i \in \{1, \dots, d\}$  we choose (temporal ordering and similar matters are completely irrelevant in the formula above: it's a logical relation between propositions). So let's expand each of the  $d$  factors in the identity (7) using the product rule (8), but using a different datum in each of them. The result can be displayed thus:

$$\begin{aligned} P(D | H K) \equiv & \left[ P(D_1 | D_{-1} H K) \times P(D_{-1} | H K) \times \right. \\ & P(D_2 | D_{-2} H K) \times P(D_{-2} | H K) \times \\ & \quad \quad \quad \dots \times \\ & \left. P(D_d | D_{-d} H K) \times P(D_{-d} | H K) \right]^{1/d}. \end{aligned} \quad (9)$$

$\uparrow$   
 log-score

Note that upon taking the logarithm of this expression, the  $d$  factors vertically aligned on the left add up to the log-score (5), as indicated.

Now note that the mathematical reshaping of  $P(D \mid H K)$  – that is, the root-product identity (7) and the expansion (9) – can be done for each of the remaining factors  $P(D_{-i} \mid H K)$ , and so on recursively. Here is an explicit example for  $d = 3$ :

$$\begin{aligned}
 P(D \mid H K) \equiv & \\
 & \left\{ P(D_1 \mid D_2 D_3 H K) \times [P(D_2 \mid D_3 H K) \times P(D_3 \mid H K) \times \right. \\
 & \quad \left. P(D_3 \mid D_2 H K) \times P(D_2 \mid H K)]^{1/2} \times \right. \\
 & P(D_2 \mid D_1 D_3 H K) \times [P(D_1 \mid D_3 H K) \times P(D_3 \mid H K) \times \\
 & \quad \left. P(D_3 \mid D_1 H K) \times P(D_1 \mid H K)]^{1/2} \times \right. \\
 & P(D_3 \mid D_1 D_2 H K) \times [P(D_1 \mid D_2 H K) \times P(D_2 \mid H K) \times \\
 & \quad \left. P(D_2 \mid D_1 H K) \times P(D_1 \mid H K)]^{1/2} \right\}^{1/3}.
 \end{aligned} \tag{10}$$

In this example, the logarithm of the three vertically aligned factors in the left column is, as already noted, the log-score (5). The logarithm of the six vertically aligned factors in the central column is an average of the log-scores calculated for the three distinct subsets of pairs of data  $\{D_1 D_2\}$ ,  $\{D_1 D_3\}$ ,  $\{D_2 D_3\}$ . Likewise, the logarithm of the six factors vertically aligned on the right is the average of the log-scores for the three subsets of data singletons  $\{D_1\}$ ,  $\{D_2\}$ ,  $\{D_3\}$ .

In the general case with  $d$  data there are  $\binom{d}{k}$  subsets with  $k$  data points. We therefore obtain

$$\begin{aligned}
 \log P(D \mid H K) &\equiv \frac{1}{d} \sum_{i=1}^d \log P(D_i \mid D_{-i} H K) + \\
 &\quad \frac{1}{d} \sum_{i \in \{1, \dots, d\}} \frac{1}{d-1} \sum_{j \in \{1, \dots, d\}}^{j \neq i} \log P(D_{-i,j} \mid D_{-i,-j} H K) + \dots + \\
 &\quad \binom{d}{k}^{-1} \frac{1}{k} \sum_{k\text{-tuples}} \log P(D_{i_1} \mid \underbrace{D_{i_2} \dots D_{i_k}}_{\text{order doesn't matter}} H K) + \dots + \\
 &\quad \frac{1}{d} \sum_{i=1}^d \log P(D_i \mid H K), \\
 &\equiv \sum_{k=1}^d \left[ k \binom{d}{k} \right]^{-1} \sum_{k\text{-tuples}} \log P(D_{i_1} \mid \underbrace{D_{i_2} \dots D_{i_k}}_{\text{order doesn't matter}} H K).
 \end{aligned} \tag{11}$$

The post-data log-probability for  $H$  will be equal to this expression, plus the pre-data log-probability for  $H$ , plus a term which is the same for all hypotheses.

I'd like to offer three ways of looking at the relation (11) between the log-likelihood and the log-score.

First, we can see the log-likelihood as a refinement and improvement of the log-score. The log-likelihood takes into account not only the log-score for the whole data, but also the log-scores for all possible subsets of data. Figuratively speaking it examines the relationship between hypothesis and data locally, globally, and on all intermediate scales.

The second point of view only holds for hypotheses  $\hat{H}$  which make any observed data irrelevant:

$$P(D \mid D' \hat{H} K) = P(D \mid \hat{H} K) \quad \text{if } D' \not\Rightarrow D, \tag{12}$$

or super-hypotheses  $H$  about such hypotheses, leading to exchangeable joint beliefs:

$$P(DD' \mid H K) = \sum_h P(D \mid \hat{H}_h H K) P(D' \mid \hat{H}_h H K) P(\hat{H}_h \mid H K) \quad \text{if } D' \not\Rightarrow D. \tag{13}$$

In either case the log-score can be seen as an approximation of the log-likelihood; more precisely of the log-likelihood per datum:

$$\frac{1}{d} \sum_{i=1}^d \log P(D_i | D_{-i} H K) \approx \frac{1}{d} \log P(D | H K). \quad (14)$$

This is in fact an exact equality if property (12) holds for  $H$ .

which lead to exchangeable beliefs about the data

Second, we can see This approximation is only valid

This approximation is reasonable if the amount of data is large with respect to the dimension of the space of a single datum, because \*\*\* (ref to geisser, stone, gelfandetal)

\*\*\* problems calculation with time-relevant hypotheses

‘we cannot give a universal rule for them beyond the common-sense one, that if anybody does not know what his suggested value is, or whether there is one, he does not know what question he is asking and consequently does not know what his answer means’ (Jeffreys 1983 § 3.1 p. 124 ).

\*\*\* with similar procedure we can included all k-fold scores.

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