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A formula for partial and conditional infinite exchangeability

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[draft] A formula is given for conditionally, infinitely exchangeable probability distributions.

1 Full, partial, and conditional exchangeability

De Finetti's theorem for infinitely exchangeable probability distributions is one of the formulae derived from the probability calculus with the richest practical and philosophical consequences. Leaving for the moment the definition of symbols to intuition, the theorem rewrites a joint probability distribution as a law of total probability:

$$\mathrm{P}(X_1 = x_1,\ X_2 = x_2,\ \ldots \mid I) = \int \prod_i f_{x_i}\ \mathrm{p}(\boldsymbol{f} \mid I)\, \mathrm{d}\boldsymbol{f}\ , \tag{1}$$

where $\boldsymbol{f} \coloneqq (f_x)$ is a distribution over the set of values that can be assumed by each x_i, and the integral is over the simplex of such distributions.

The condition for the theorem to hold is that the joint distribution be infinitely *fully exchangeable*, that is, symmetric with respect to permutations of the x_i. This formula is the infinite limit for a sampling formula from an urn with unknown distribution. From now 'exchangeable' shall be understood as 'infinitely exchangeable'.

A more general version of the theorem holds if the joint distribution is *partially exchangeable*, that is, the X_i can be divided into two or more groups $Y_j, Z_k, \dots$, and permutations are allowed within each group but not necessarily across groups. The formula then becomes

$$\mathrm{P}(Y_{1'} = y_{1'},\ Y_{2'} = y_{2'},\ \dots,\ Z_{1''} = z_{1''},\ Z_{2''} = z_{2''},\ \dots \mid I) = \iint \prod_j g_{y_j} \prod_k h_{z_k}\ \mathrm{p}(\boldsymbol{g}, \boldsymbol{h} \mid I)\, \mathrm{d}\boldsymbol{g}\, \mathrm{d}\boldsymbol{h}\ , \tag{2}$$

with distinct distributions $\boldsymbol{g}$, $\boldsymbol{h}$ for each group. If the density $\mathrm{p}(\boldsymbol{g}, \boldsymbol{h} \mid I)$ is diagonal, that is, if it contains a term $\delta(\boldsymbol{g} - \boldsymbol{h})$, the fully exchangeable form (1) is recovered.

A little reflection shows that if we know that the quantities X belong to group Y in instances $1', 2', \dots$, and to group Z in instances $1'', 2'', \dots$, then (a) there is some other quantity C that allows us to distinguish the two groups, and (b) the value of this quantity *is known* in each instance, $C_i = c_i$.

Let us say, for example, that the quantity X is the result of a patient's treatment, with values Success and Failure. Y refers to the result for a Juvenile patient, and Z for an Adult patient. If we write

$$\mathrm{P}(Y_3 = \mathsf{S},\ Z_5 = \mathsf{F} \mid I) = 0.2\ ,$$

then we must already know that patient number 3 is juvenile, $C_3 = \mathsf{J}$ and patient number 5 is adult, $C_5 = \mathsf{A}$. This is clear from our very notation, otherwise we would not have known whether to use Y or Z for these patients. This information is evidently implicit in our background information I.

Let us render this information more explicit in our notation. The probability in eq. (2) is rewritten as

$$\begin{aligned} &\mathrm{P}(Y_{1'} = y_{1'},\ \dots,\ Z_{1''} = z_{1''},\ \dots \mid I) \equiv \\ &\mathrm{P}(X_{1'} = x_{1'},\ \dots,\ X_{1''} = x_{1''},\ \dots \mid C_{1'} = c_{1'},\ \dots,\ C_{1''} = c_{1''},\ \dots,\ I')\ . \end{aligned} \tag{3}$$

Then it is clear that the partially exchangeable probability distribution (2) or (3) can also be called *conditionally exchangeable*.

The present work presents a representation formula for pairs of quantities X_i, C_i such that

1. they have, jointly, a fully exchangeable probability distribution,

2. the distribution for Xi is only conditionally exchangeable, given C_i. The formula is:

$$\mathrm{P}(X_1 = x_1, C_1 = c_1,\ X_2 = x_2, C_2 = c_2,\ \dots \mid I) = \int \prod_i f_{x_i,c_i}\ \mathrm{p}(\boldsymbol{f} \mid I)\,\mathrm{d}\boldsymbol{f} \tag{4a}$$

$$\boxed{\text{with}\quad \mathrm{p}(\boldsymbol{f} \mid I) \equiv \mathrm{p}(f_{x_i|c_i} \mid I)\ \mathrm{p}(f_{,c_i} \mid I)\,\mathrm{d}\boldsymbol{f}} \tag{4b}$$

where

- $\boldsymbol{f} \coloneqq (f_{x,c})$ is a joint distribution over the set of values that can be assumed by each (x_i, c_i) pair,
- $\bigl(f_{,c} \coloneqq \sum_x f_{x,c}\bigr)$ is the related *marginal* frequency distribution for over the c values,
- $\bigl(f_{x|c} \coloneqq f_{x,c}/f_{,c}\bigr)$ are the related *conditional* frequency distributions obtained from $\boldsymbol{f}$.

The noteworthy feature of the formula above is that the conditional exchangeability of X given C is expressed by the factorizability of the density

and the integral is over the simplex of such distributions.

2 Setup

We have a countably infinite set of statements $X_t = x_t$ with $t \in \{1, 2, \dots\}$, where each X_t is some quantity and each $x_t \in S$, a common finite set of values (the theorem also works with statements that are not about variates). De Finetti's theorem states that if a probability distribution about any subset of such statements is exchangeable, that is,

$$\begin{gathered}\mathrm{P}(X_1 = x_1, X_2 = x_2, \dots, X_T = x_T \mid I) = \\ \mathrm{P}(X_1 = x_{\pi(1)}, X_2 = x_{\pi(2)}, \dots, X_T = x_{\pi(T)} \mid I) \\ \text{for any } T \text{, any set of indices } \{1, \dots, T\}\text{, and any permutation } \pi \text{ thereof,}\end{gathered} \tag{5}$$

all such joint probabilities appropriately related by marginalization, then we can write

$$\int \mathrm{d}\dots \prod_{ij} \bigl(\xi_{i|j}\ \nu_j\bigr)^{q_{i|j}\, f_j}\ \mathrm{p}(\xi)\,\mathrm{p}(\nu) \tag{6}$$

Bibliography

('de *X*' is listed under D, 'van *X*' under V, and so on, regardless of national conventions.)

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