Inferences about an urn with changing content

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Draft of 17 July 2019 (first drafted 22 June 2019)

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Note: Dear Reader & Peer, this manuscript is being peer-reviewed by you. Thank you.

1 A magic urn

Quantifying our degree of belief about the contents of an urn that contains, say, white and black balls in an unknown proportion is a standard textbook topic. Our belief depends on our drawing data about the urn and on our state of knowledge about the urn before we knew those data. For example, we may know the colours of five balls drawn from the urn, and how or why the urn was prepared. Jaynes's book (2003 chs 3, 6) gives a superb analysis and discussion of this problem and of the probability formulae that it typically involves; and even of more realistic variants, such as an increased frequency of drawing and replacing from the top layer of the urn (ibid. § 3.9).

In this note I derive some probability formulae for a more complex version of the urn problem, and some approximations of these formulae. They can be useful in several scientific questions that, sufficiently simplified, can be mapped to the present problem. examples

The problem we consider is the following. There's an urn with N balls, each of which can be white or black. At regular intervals of time every ball can change its colour: a white can turn black, or vice versa, or it can keep the colour it has. We observe n out of the N balls for T time intervals. The n observed balls are initially chosen in a way unknown to us. Note that we observe the *same* n balls at all times. Some variations of this set-up will be discussed later on. We ask several questions about this magic urn:

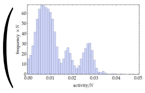
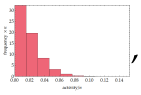
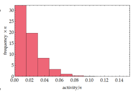
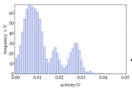
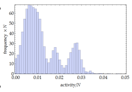
Q1 What was the proportion of white and black balls in the urn at some specific time, among the times we observed? For example, did the urn contain 383 white and 617 black balls during the 3rd time step?

- Q2 How frequently did each possible proportion of white and black balls appear, during the times we observed? For example: did the urn have a white/black proportion of 887/113 during 37 out of 10 000 time steps?
- Q3 What was (or will be) the proportion of white and black balls in the urn at some specific time, among the times we did not observe?
- Q4 How frequently will every possible proportion of white and black balls appear, during all times the urn exists?

Our goal is to quantify how our belief is distributed among the possible answers to these questions. This belief depends on our observation of the n balls for T time steps, and on our knowledge about the mechanism behind the magic changes of the urn, expressed as a prior belief.

We consider this specific prior belief: *we are equally uncertain about all sequences proportions of urn colours that differ only by their ordering*. For example, we have equal beliefs that the sequences of proportions $(239/761, 647/353, 17/983, 239/761)$, $(17/983, 647/353, 239/761, 239/761)$, and $(647/353, 17/983, 239/761, 239/761)$ may occur.

2 Formulae (draft)

p( | , I) $\mathrm{d}\boldsymbol{F} \propto$ p( | , I) $\times$ p( | I) $\mathrm{d}\boldsymbol{F}$

$$\mathrm{p}(\boldsymbol{F} \mid \boldsymbol{f}, I)\, \mathrm{d}\boldsymbol{F} \propto \mathrm{p}(\boldsymbol{f} \mid \boldsymbol{F}, I) \times \mathrm{p}(\boldsymbol{F} \mid I)\, \mathrm{d}\boldsymbol{F}$$

$$\mathrm{p}(\boldsymbol{F} \mid I)\, \mathrm{d}\boldsymbol{F} \propto \exp\left(-\frac{|\boldsymbol{S}\boldsymbol{F}|^2}{2\sigma^2}\right) \exp[-L\, \mathrm{H}(\boldsymbol{F}; \boldsymbol{R})]\, \mathrm{d}\boldsymbol{F}$$

$$\exp[-L\, \mathrm{H}(\boldsymbol{F}; \boldsymbol{R})] \approx \binom{L}{L\boldsymbol{F}} \prod \boldsymbol{R}^{L\boldsymbol{F}}$$

$$\boldsymbol{S} := \begin{pmatrix} 1 & -4 & 6 & -4 & 1 & 0 & 0 & \dots \\ 0 & 1 & -4 & 6 & -4 & 1 & 0 & \dots \\ 0 & 0 & 1 & -4 & 6 & -4 & 1 & \dots \\ 0 & 0 & 0 & 1 & -4 & 6 & -4 & \dots \\ 0 & 0 & 0 & 0 & 1 & -4 & 6 & \dots \\ 0 & 0 & 0 & 0 & 0 & 1 & -4 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

$$\mathrm{p}(\boldsymbol{f} \mid \boldsymbol{F}, I) \propto \exp\left(-\frac{|\boldsymbol{M}\boldsymbol{F} - \boldsymbol{f}|^2}{2\delta^2}\right)$$

$$\mathrm{p}(\boldsymbol{F} \mid \boldsymbol{f}, I)\, \mathrm{d}\boldsymbol{F} \propto \exp\left(-\frac{|\boldsymbol{M}\boldsymbol{F} - \boldsymbol{f}|^2}{2\delta^2}\right) \times \exp\left(-\frac{|\boldsymbol{S}\boldsymbol{F}|^2}{2\sigma^2}\right) \exp[-L\, \mathrm{H}(\boldsymbol{F}; \boldsymbol{R})]\, \mathrm{d}\boldsymbol{F}$$

$$\mathrm{p}(\boldsymbol{F} \mid \boldsymbol{f}, I) \propto \exp\left[-\frac{|\boldsymbol{M}\boldsymbol{F} - \boldsymbol{f}|^2}{2\delta^2} - \frac{|\boldsymbol{S}\boldsymbol{F}|^2}{2\sigma^2} - L\, \mathrm{H}(\boldsymbol{F}; \boldsymbol{R})\right]$$

δ small, L large, L/δ small $\implies$

“ mode of $\mathrm{p}(\boldsymbol{F} \mid \boldsymbol{f}, I)$ ” $\approx$ “ $\boldsymbol{F}$ that minimizes $\mathrm{H}(\boldsymbol{F}; \boldsymbol{R})$ subject to $\boldsymbol{M}\boldsymbol{F} = \boldsymbol{f}$ ”

Bibliography

('de *X*' is listed under D, 'van *X*' under V, and so on, regardless of national conventions.)

Jaynes, E. T. (2003): *Probability Theory: The Logic of Science*. (Cambridge University Press, Cambridge). Ed. by G. Larry Bretthorst. First publ. 1994. https://archive.org/details/XQUHIUXHIQUHIQXUIHX2, http://www-biba.inrialpes.fr/Jaynes/prob.html.