Inferences about an urn with changing content

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Note: Dear Reader & Peer, this manuscript is being peer-reviewed by you. Thank you.

1 A magic urn

Quantifying our degree of belief about the contents of an urn that contains, say, white and black balls (identical in all other respects) in an unknown proportion is a standard textbook topic. Our belief depends on our drawing data about the urn and on our state of knowledge about the urn before we knew those data. For example, we may know the colours of five balls drawn from the urn, and how or why the urn was prepared. Jaynes's book (2003 chs 3, 6) gives a superb analysis and discussion of this problem and of the probability formulae that it typically involves; and even of more realistic variants, such as an increased frequency of drawing and replacing from the top layer of the urn (ibid. § 3.9).

In this note I derive some probability formulae for a more complex version of the urn problem, in which the balls can change colour with time. These formulae can be useful in scientific questions that, sufficiently simplified, can be mapped to the present problem. examples And their derivation can be a useful exercise in the probability calculus.

The problem we consider is the following. There's an urn with N identical balls except for the colour: each can be white or black. At regular intervals of time every ball can change its colour: a white can turn black, or vice versa; or it can keep the colour it has. We observe n out of the N balls for T time intervals; note that we observe the *same* n balls at all times. These n balls are initially chosen in a way unknown to us. Some variations of this set-up will be discussed later on. We ask several questions about this magic urn:

Q1 What was the proportion of white and black balls in the urn at some specific time, among the times we observed? For example, did the urn contain 383 white and 617 black balls during the 3rd time step?

Q2 How frequently did each possible proportion of white and black balls appear, during the times we observed? For example: did the urn have a white/black proportion of 887/113 during 37 out of 10 000 time steps?

Q3 What was (or will be) the proportion of white and black balls in the urn at some specific time, among the times we did *not* observe?

Q4 How frequently will every possible proportion of white and black balls appear, during all times the urn exists?

Our goal is to quantify how our belief is distributed among the possible answers to these questions. This belief depends on our observation of the n balls for T time steps, and on our beliefs about the mechanism behind the magic changes of the urn before making the observations.

We consider this pre-observation belief: *we have the same belief about sequences of proportions of urn colours that differ only by their ordering, no matter how long the sequences are*. For example, we have equal beliefs that the sequences of proportions

$$
\begin{aligned}
&(239/761,\ 647/353,\ 17/983,\ 239/761),\\
\text{or}\quad &(17/983,\ 647/353,\ 239/761,\ 239/761),\\
\text{or}\quad &(647/353,\ 17/983,\ 239/761,\ 239/761)
\end{aligned}
$$

may occur, and similarly for other permutations and arbitrarily long sequences. This *exchangeable* belief is clearly inappropriate in many situations, for example if we think that the changes of colour depend on the previous colours of the balls, or if we hold similar hypotheses involving memory. Most important, this belief is *irrevocable*: no data, such as peculiar patterns in the sequences, can ever change it. Exchangeable beliefs can therefore be used as approximations of our real belief only if our inferences involve sequences that are not too long, so that the presence of patterns isn't very important.

2 Set-up and notation

Bibliography

('de *X*' is listed under D, 'van *X*' under V, and so on, regardless of national conventions.)

Jaynes, E. T. (2003): *Probability Theory: The Logic of Science*. (Cambridge University Press, Cambridge). Ed. by G. Larry Bretthorst. First publ. 1994. https://archive.org/details/XQUHIUXHIQUHIQXUIHX2, http://www-biba.inrialpes.fr/Jaynes/prob.html.