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Note on conditional exchangeability

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1 Exchangeability of pairs

Our domain of discourse consists of a countable set of atomic statements

$$\begin{aligned} &X_{ni}\,, \quad n \in \mathbf{N},\ i \in \mathfrak{X}, && X_{na}, X_{nb} \text{ mutually contradictory for } a \ne b, \\ &Y_{nj}\,, \quad n \in \mathbf{N},\ j \in \mathfrak{Y}, && Y_{na}, Y_{nb} \text{ mutually contradictory for } a \ne b. \end{aligned} \tag{1}$$

Typically these statements express observations or measurement outcomes of some quantities and have the form "$Z_n = z_i$".

A probability distribution is called *jointly exchangeable* in X and Y if

$$\begin{aligned} &\mathrm{P}(X_{1i_1}, Y_{1j_1}, \cdots, X_{ni_n}, Y_{nj_n}, \cdots, X_{mi_m}, Y_{mj_m}, \cdots \mid H) = \\ &\qquad \mathrm{P}(X_{1i_1}, Y_{1j_1}, \cdots, X_{ni_m}, Y_{nj_m}, \cdots, X_{mi_n}, Y_{mj_n}, \cdots \mid H) \\ &\qquad\qquad \text{for all } m, n\,. \end{aligned} \tag{2}$$

A probability distribution is called *conditionally exchangeable* in Y given Y if

$$\begin{aligned} &\mathrm{P}(Y_{1j_1}, \cdots, Y_{nj_n}, \cdots, Y_{mj_m}, \cdots \mid X_{1i_1}, \cdots, X_{ni_n}, \cdots, X_{mi_m}, \cdots, H) = \\ &\qquad \mathrm{P}(Y_{1j_1}, \cdots, Y_{nj_m}, \cdots, Y_{mj_n}, \cdots \mid X_{1i_1}, \cdots, X_{ni_m}, \cdots, X_{mi_n}, \cdots, H) \\ &\qquad\qquad \text{for all } m, n\,. \end{aligned} \tag{3}$$

This definition correspond to the alternative definition given in Appendix 2 of Lindley & Novick (1981).

If the probabilities for X, Y are jointly exchangeable then the conditional probabilities for Y given X are conditionally exchangeable. If the conditional probabilities for Y given X are conditionally exchangeable and X is exchangeable, then X, Y are jointly exchangeable.

If the conditional probability for Y given X is conditionally exchangeable, then we can write

$$\mathrm{P}(Y_{1j_1}, Y_{2j_2}, Y_{3j_3}, \dotsb \mid X_{1i_1}, X_{2i_2}, X_{3i_3}, \dotsb, H) = \int \Bigl(\prod_n F_{j_n|i_n}\Bigr)\, \mathrm{p}\Bigl[(F_{b|a}) \Bigm| \bigwedge_n X_{n i_n}, H\Bigr]\, \mathrm{d}(F_{b|a}) \quad (4)$$

Note in particular that the probability $\mathrm{p}\bigl[(F_{b|a}) \mid \dots, H\bigr]\, \mathrm{d}(F_{b|a})$ can depend on the long-run frequencies of X in the conditional.

Bibliography

("de X" is listed under D, "van X" under V, and so on, regardless of national conventions.)

Lindley, D. V., Novick, M. R. (1981): *The role of exchangeability in inference*. Ann. Stat. **9**^{1}, 45–58. doi:10.1214/aos/1176345331.