Bayesian Plinko

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How does a Bayesian robot do at Plinko, using an infinitely exchangeable model and borrowing a human participant's prior?

1 The Bayesian robot

In the context of these notes we call 'model' any set of assumptions that allows us to assign a probability to a new observation, given a number of observations of a similar kind. Denote such assumptions by a proposition M – surely a proposition very difficult to express in writing – and the ith observation by $D_i \coloneqq$ 'The ith observation is d_i', with $d_i \in \{1, \dots N\}$. Then M allows us to give a numeric value to

$$\mathrm{P}(D_{m+1} | D_1 \land D_2 \land \dots \land D_m \land M), \tag{1}$$

with

The proposition D_i In the following we abbreviate logical conjunction '$\land$' with a comma, for simplicity. Our statistical terminology and notation follow ISO standards (ISO 2009; 2006) otherwise.

We shall consider a robot who uses an *infinitely exchangeable* model. This model, introduced by de Finetti (1930; 1937; Heath et al. 1976) and described in detail in Bernardo et al. (2000 § 4.2), is determined by the following assumption of *infinite exchangeability*: the joint distribution for any number of observations is symmetric with respect to their order; that is, the order of the observations is irrelevant for inferential purposes. Distributions for different number of observations must of course be consistent with one another through marginalization. Infinite exchangeability may in turn be motivated by other specific assumptions, but the details of these are irrelevant for the mathematical form of this model.

Infinite exchangeability determines this form of the probability above:

$$\mathrm{P}(D_1, D_2, \dots, D_m | M) = \int_{\Delta} \left(\prod_{i=1}^{m} q_{d_i} \right) \mathrm{p}(\boldsymbol{q} | M) \, \mathrm{d}\boldsymbol{q}, \tag{2}$$

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where $\boldsymbol{q}$ is a normalized N-tuple of positive numbers: $\Delta \coloneqq \{\boldsymbol{q} \in \mathbf{R}^N \mid q_i \geqslant 0, \sum_{i=1}^N q_i = 1\}$. This N-tuple can be thought of the relative, long-run frequencies of the possible outcomes[1], and $\mathrm{p}(\boldsymbol{q} \mid M)\,\mathrm{d}\boldsymbol{q}$ as their probability density. From this point of view it is as if the robot first assumes to know the long-run frequencies of the different outcomes and, not knowing their particular order in the observation, assigns to the occurrence of each a probability proportional to its frequency: this is the term $\prod_{i=1}^m q_{d_i}$ in the integral. Then, not being sure about the long-run frequencies, the robot assign to them the density $\mathrm{p}(\boldsymbol{q} \mid M)\,\mathrm{d}\boldsymbol{q}$ – which is determined by additional assumptions besides exchangeability.

As an explicit example, say with $N = 40$,

$$\mathrm{P}(d_1 = 37, d_2 = 6, d_3 = 25, d_4 = 37 \mid M) = \int_\Delta q_6\, q_{25}\, {q_{37}}^2\ \mathrm{p}(\boldsymbol{q} \mid M)\,\mathrm{d}\boldsymbol{q}. \quad (3)$$

In the following we omit the integration domain Δ.

From Bayes's theorem we obtain the expression for the predictive probability (1) of an infinite exchangeable model:

$$\mathrm{P}(D_{m+1} \mid D_1, \dots, D_m, M) = \int q_{d_{m+1}} \mathrm{p}(\boldsymbol{q} \mid D_1, \dots, D_m, M)\,\mathrm{d}\boldsymbol{q}, \quad (4a)$$

$$\mathrm{p}(\boldsymbol{q} \mid D_1, \dots, D_m, M) = \frac{\bigl(\prod_{i=1}^m q_{d_i}\bigr)\,\mathrm{p}(\boldsymbol{q} \mid M)}{\int \bigl(\prod_{i=1}^m q'_{d_i}\bigr)\,\mathrm{p}(\boldsymbol{q}' \mid M)\,\mathrm{d}\boldsymbol{q}'}. \quad (4b)$$

Continuing our numeric example (3) this could be

$$\mathrm{P}(d_5 = 6 \mid d_1 = 37, d_2 = 6, d_3 = 25, d_4 = 3, M) = \int q_6\ \mathrm{p}(\boldsymbol{q} \mid d_1 = 37, d_2 = 6, d_3 = 25, d_4 = 3, M)\,\mathrm{d}\boldsymbol{q}, \quad (5a)$$

$$\mathrm{p}(\boldsymbol{q} \mid d_1 = 37, d_2 = 6, d_3 = 25, d_4 = 3, M) = \frac{q_6\, q_{25}\, {q_{37}}^2\ \mathrm{p}(\boldsymbol{q} \mid M)}{\int q_6\, q_{25}\, {q_{37}}^2\ \mathrm{p}(\boldsymbol{q}' \mid M)\,\mathrm{d}\boldsymbol{q}'}. \quad (5b)$$

Formula (4) tell us how our robot would update its predictive probabilities at each new observation of a Plinko outcome.

[1] 'But this *long run* is a misleading guide to current affairs. *In the long run* we are all dead.' (Keynes 2013 § 3.I, p. 65)

2 Initial prior

Bibliography

('de *X*' is listed under D, 'van *X*' under V, and so on, regardless of national conventions.)

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