Bayesian Plinko

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How does a Bayesian robot do at Plinko, using an infinitely exchangeable model and borrowing a human participant's prior?

1 The Bayesian robot

In the context of these notes and of the Plinko experiments (Filipowicz et al. 2014; 2016) we call 'model' any set of assumptions that allows us to assign a probability to a new observation, given a number of observations of a similar kind. Denote such assumptions by a proposition M – surely a proposition very difficult to express in writing – and the ith observation by $D_i \coloneqq$ 'The ith observation is d_i', with $d_i \in \{1, \dots N\}$. Then M allows us to give a numeric value to

$$P(D_{m+1} | D_1 \wedge D_2 \wedge \cdots \wedge D_m \wedge M), \tag{1}$$

with

The proposition D_i In the following we abbreviate logical conjunction '$\wedge$' with a comma, for simplicity. Our statistical terminology and notation follow ISO standards (ISO 2009; 2006) otherwise.

We shall consider a robot who uses an *infinitely exchangeable* model. This model, introduced by de Finetti (1930; 1937; Heath et al. 1976) and described in detail in Bernardo et al. (2000 § 4.2), is determined by the following assumption of *infinite exchangeability*: the joint distribution for any number of observations is symmetric with respect to their order; that is, the order of the observations is irrelevant for inferential purposes. Distributions for different number of observations must of course be consistent with one another through marginalization. Infinite exchangeability may in turn be motivated by other specific assumptions, but the details of these are irrelevant for the mathematical form of this model.

Infinite exchangeability determines this form of the probability above:

$$P(D_1, D_2, \dots, D_m | M) = \int_\Delta \left(\prod_{i=1}^m q_{d_i} \right) p(\boldsymbol{q} | M) \, d\boldsymbol{q}, \tag{2}$$

where q is a normalized N-tuple of positive numbers: $\Delta := \{q \in \mathbf{R}^N \mid q_i \geqslant 0, \sum_{i=1}^{N} q_i = 1\}$. This N-tuple can be thought of the relative, long-run frequencies of the possible outcomes[1], and $\mathrm{p}(q \mid M)\,\mathrm{d}q$ as their probability density. From this point of view it is as if the robot first assumes to know the long-run frequencies of the different outcomes and, not knowing their particular order in the observation, assigns to the occurrence of each a probability proportional to its frequency: this is the term $\prod_{i=1}^{m} q_{d_i}$ in the integral. Then, not being sure about the long-run frequencies, the robot assign to them the density $\mathrm{p}(q \mid M)\,\mathrm{d}q$ – which is determined by additional assumptions besides exchangeability.

As an explicit example, say with $N = 40$,

$$\mathrm{P}(d_1 = 37, d_2 = 6, d_3 = 25, d_4 = 37 \mid M) = \int_{\Delta} q_6\, q_{25}\, q_{37}^{\,2}\; \mathrm{p}(q \mid M)\,\mathrm{d}q. \quad (3)$$

In the following we omit the integration domain Δ.

From Bayes's theorem we obtain the expression for the predictive probability (1) of an infinite exchangeable model:

$$\mathrm{P}(D_{m+1} \mid D_1, \ldots, D_m, M) = \int q_{d_{m+1}} \mathrm{p}(q \mid D_1, \ldots, D_m, M)\,\mathrm{d}q, \quad (4a)$$

$$\mathrm{p}(q \mid D_1, \ldots, D_m, M) = \frac{\left(\prod_{i=1}^{m} q_{d_i}\right) \mathrm{p}(q \mid M)}{\int \left(\prod_{i=1}^{m} q'_{d_i}\right) \mathrm{p}(q' \mid M)\,\mathrm{d}q'}. \quad (4b)$$

Continuing our numeric example (3) this could be

$$\mathrm{P}(d_5 = 6 \mid d_1 = 37, d_2 = 6, d_3 = 25, d_4 = 3, M) = $$
$$\int q_6 \; \mathrm{p}(q \mid d_1 = 37, d_2 = 6, d_3 = 25, d_4 = 3, M)\,\mathrm{d}q, $$
$$(5a)$$

$$\mathrm{p}(q \mid d_1 = 37, d_2 = 6, d_3 = 25, d_4 = 3, M) = \frac{q_6\, q_{25}\, q_{37}^{\,2}\; \mathrm{p}(q \mid M)}{\int q_6\, q_{25}\, q_{37}^{\,2}\; \mathrm{p}(q' \mid M)\,\mathrm{d}q'}. $$
$$(5b)$$

Formula (4) tell us how our robot would update its predictive probabilities at each new observation of a Plinko outcome.

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[1] 'But this *long run* is a misleading guide to current affairs. *In the long run* we are all dead.' (Keynes 2013 § 3.I, p. 65)

2 General remarks on the robot's behaviour

The exchangeable-model formula (4) leads to some characteristic features of the robot's beliefs and of their evolution:

- As data accumulate, the robot's probabilities for the next outcome approach the observed frequencies. Such approach happens independently of the form of the prior $p(q\,|\,M)\,dq$ – unless the latter is zero in peculiar regions of the integration domain – but the prior determines the celerity of the approach. A prior heavily peaked on a frequency q' will require a lot of data to move the predictions to a very different frequency q.

- As data D accumulate, the updated density $p(q\,|\,D, M)\,dq$ will become more and more peaked at the N-tuple of observed frequencies.

- Suppose that we first have a long sequence of observations concentrating around frequencies q – say, a very long sequence of 1s in a row – and then a shift to other frequencies q' – say, suddenly 2s only appear. After the shift, the predictive probabilities will eventually become peaked around the new frequencies, but the shift in the peaks will take a larger number of observations around the new frequencies than the number around the old frequencies.

3 Initial prior

The shape of the initial prior heavily determines the predictions in the first observations, so it must be chosen with care. The Plinko data tell us the initial predictive probabilities of the participants,

$$p(d_1 = k\,|\,M) \equiv \int q_k\, p(q\,|\,M)\,dq, \tag{6}$$

but not their prior $p(q\,|\,M)\,dq$.

As a first exploration we consider a *Johnson-Dirichlet* prior, proportional to $\prod_i q_i{}^{L_i}$ for some values of L_i:

$$p(q\,|\,M_J) = \frac{\Gamma(\sum_i L_i)}{\prod_i \Gamma(L_i)} \prod_{i=1}^{N} q_i{}^{L_i-1}, \qquad L_i > 0 \tag{7}$$

This prior is determined by the additional assumption – call it M_J – that that the frequencies of other outcomes are irrelevant for predicting

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a particular one:

$$P(d_{m+1} = k | f, M_J) = P(d_{m+1} = k | f_k, M_J) \qquad k \in \{1, \dots, N\}, \quad (8)$$

where f is the N-tuple of observed frequencies. This assumption is called 'sufficientness' (Johnson 1924; 1932; Good 1965 ch. 4; Zabell 1982; Jaynes 1996). This is a conjugate prior (DeGroot 2004 ch. 9; Diaconis et al. 1979) and it has two convenient properties: it updates to a density of the same mathematical form, and its corresponding predictive distribution can be calculated analytically using the formula

$$\int_\Delta \prod_{i=1}^N q_i^{L_i - 1} \, dq = \frac{\prod_i \Gamma(L_i)}{\Gamma(\sum_i L_i)}. \quad (9)$$

We obtain for the initial distribution:

$$P(d_1 = k | M_J) = \int q_k \, p(q | M_J) \, dq = \frac{L_k}{\sum_i L_i}, \quad (10)$$

and for the updated density:

$$p(q | D_1, \dots, D_m, M_J) = \frac{\Gamma(\sum_i L_i')}{\prod_i \Gamma(L_i')} \prod_{i=1}^N q_i^{L_i' - 1} \quad \text{with } L_i' = L_i + f_i. \quad (11)$$

Formula (10) says that the Johnson-Dirichlet prior can produce any initial probabilities assigned by the participants, just by making the parameters L_i proportional to such probabilities. The proportionality factor is left arbitrary. The larger the proportionality factor, the more peaked the prior, and therefore the more reluctant the robot will be in revising its guesses in the light of new observations.

Bibliography

('de *X*' is listed under D, 'van *X*' under V, and so on, regardless of national conventions.)

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