

Bayesian Plinko

Study notes

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How does a Bayesian robot do at Plinko, using an infinitely exchangeable model and borrowing a human participant's prior?

We are human, after all
Much in common, after all
(Daft Punk 2005a)

1 Introductory remarks on the methods

The experimental setup is open to a huge number of analyses and interpretations on the participants' part, inspired by past experience. As a participant, we can surmise that there's a connection between different trials, some sort of 'constant mechanism' at some level. Or we can surmise that there's no such connection, hence observation of past trials doesn't say anything about the next one. Or we can surmise that the next trial is influenced by the participant's own bar-height assignments. And many other hypotheses. We can also entertain all these hypotheses at the same time, and shift from one to another during the experiment. For example, if we suddenly wondered whether the computer program is actually using our bar distribution, we could suddenly move all probability to a slot at the edge and check if this seems to influence the next outcome (cf. participant 31).

The same goes for the choice of initial distribution. As a participant, we can say 'alright, there are 40 slots', and just give a uniform distributions to the 40 possibilities. Or we can consider the pyramidal mechanism of the game, which leads to a binomial distribution. Or we can consider that this is a computerized version of the game. The computer could simulate the physics of the actual game; but the image of the mechanism could also be just for show, the computer being programmed to distribute the outcomes according to a predetermined, completely arbitrary distribution. From this point of view we could again decide to assign a uniform distribution.

✠ *L's opinion* Should a participant's 'performance' be assessed against a probability distribution that supposedly lies behind a computer program? This is a tricky question. Let's assume an operational point of view. We have a computer program in which we specify two parameters: 'mean' and 'standard deviation'. The program then generates a sequence of, say, 100 numbers between 1 and N . Finally we ask the participant to guess a distribution that supposedly lies behind these 100 outcomes; suppose we also ask him to choose a normal distribution. This is equivalent to ask him to guess the two parameters we gave to the computer program. But here's the catch: there are two parameter values under which the data are more likely than under the values we chose: the *empirical* mean and standard deviation. From this point of view it's unfair to assess the participant's performance against the values we chose, given that there are other values under which the data are more likely. It's as if we made a carpenter – who turns out to be a good cook – prepare a delicious dish, then asked a person to guess who prepared it, and then assessed their answer – say, 'a chef' or 'a baker' – as bad if they didn't guess that it was a carpenter. ✠ *L: alright, I must find a better example*

The assessment problem just described is worse if the data are few compared with the number of possible outcomes, because the empirical mean and standard deviation will then usually be very different from our parameter choices. It is even worse if we ask the participant to choose not among normal distributions but among all possible distributions. There will always be a distribution under which the data are more likely than under our normal. On top of this, the likelihood of the data is not the only criterion to make a sound judgement. If the empirical mean of the data were 17.0362, for example, it'd be a good bet that the one chosen by the human operator was 17.

And, given that the computer program is generating the data deterministically, shouldn't we assess the participant according to the prediction of the *exact* outcome sequence then?

This kind of paradoxes always appear when we try to judge a probability assignment based on some fictitious distribution 'out there'. It'd be best to judge it by checking its consistency with the assumptions that went into it. We can thus find other assessment measures. For example, we can set up several inference robots, constructed according to different assumptions, and compare a participant's response to the robots'.

De Finetti introduced the notion of exchangeability, to be presented in the next section, in order to avoid thinking in terms of ‘mechanisms’ or of ‘true unknown distributions’ (cf. Good 1965 ch. 3). He characterized a statistical model in terms of *features of the predictive distribution* – for example, that it be invariant under exchanges of the outcomes. This point of view was taken up by many other statisticians and has been used to characterize many other statistical models, like Markov ones for example (Freedman 1962; Diaconis et al. 1980; Zaman 1984; Fortini et al. 2002; 2014).

2 The Bayesian robot

In the context of these notes and of the Plinko experiments (Filipowicz et al. 2014; 2016) we call ‘model’ any set of assumptions that allows us to assign a probability to a new observation, given a number of observations of a similar kind. Denote such assumptions by a proposition M – a proposition surely very difficult to express in writing. Denote the proposition ‘The outcome of the i th observation is d ’ by D_d^i , with $d \in \{1, \dots, N\}$. Then M allows us to give a numeric value to

$$P(D_{d_{m+1}}^{m+1} | D_{d_1}^1 \wedge D_{d_2}^2 \wedge \dots \wedge D_{d_m}^m \wedge M), \quad (1)$$

We will abbreviate logical conjunction ‘ \wedge ’ with a comma, for simplicity. Our statistical terminology and notation follow ISO standards (iso 2009; 2006) otherwise.

We shall consider a robot who uses either of these two equivalent assumptions:

- the joint distribution for any number of observations is symmetric with respect to their order; that is, the order of the observations is irrelevant for inferential purposes;
- for inferential purposes, only the relative frequencies of past observations are relevant. Any additional data about past observation is irrelevant and can be discarded.

Distributions for different number of observations must of course be consistent with one another through marginalization.

The two equivalent assumptions are technically called *infinite exchangeability*. This notion was introduced by de Finetti (1930; 1937; Heath et al. 1976); it is described in detail in Bernardo et al. (2000 § 4.2).

Infinite exchangeability determines this form of the probability above:

$$P(D_{d_1}^1, D_{d_2}^2, \dots, D_{d_m}^m | M) = \int_{\Delta} \left(\prod_{i=1}^m q_{d_i} \right) p(q | M) dq, \quad (2)$$

where q is a normalized N -tuple of positive numbers: $\Delta := \{q \in \mathbb{R}^N \mid q_i \geq 0, \sum_{i=1}^N q_i = 1\}$. This N -tuple can be thought of the relative, long-run frequencies of the possible outcomes¹, and $p(q | M) dq$ as their probability density. From this point of view it is as if the robot first assumes to know the long-run frequencies of the different outcomes and, not knowing their particular order in the observation, assigns to the occurrence of each a probability proportional to its frequency: this is the term $\prod_{i=1}^m q_{d_i}$ in the integral. Then, not being sure about the long-run frequencies, the robot assigns to them the density $p(q | M) dq$ – which is determined by additional assumptions besides exchangeability.

As an explicit example, say with $N = 40$,

$$P(D_{37}^1, D_6^2, D_{25}^3, D_{37}^4 | M) = \int_{\Delta} q_6 q_{25} q_{37}^2 p(q | M) dq. \quad (3)$$

In the following we omit the integration domain Δ .

From Bayes's theorem we obtain the expression for the predictive probability (1) of an infinite exchangeable model:

$$P(D_{d_{m+1}}^{m+1} | D_{d_1}^1, \dots, D_{d_m}^m, M) = \int q_{d_{m+1}} p(q | D_{d_1}^1, \dots, D_{d_m}^m, M) dq, \quad (4a)$$

$$p(q | D_{d_1}^1, \dots, D_{d_m}^m, M) = \frac{(\prod_{i=1}^m q_{d_i}) p(q | M)}{\int (\prod_{i=1}^m q'_{d_i}) p(q' | M) dq'}. \quad (4b)$$

Continuing our numeric example (3) this could be

$$P(D_6^5 | D_{37}^1, D_6^2, D_{25}^3, D_3^4, M) = \int q_6 p(q | D_{37}^1, D_6^2, D_{25}^3, D_3^4, M) dq, \quad (5a)$$

$$p(q | D_{37}^1, D_6^2, D_{25}^3, D_3^4, M) = \frac{q_6 q_{25} q_{37}^2 p(q | M)}{\int q_6 q_{25} q_{37}^2 p(q' | M) dq'}. \quad (5b)$$

¹'But this *long run* is a misleading guide to current affairs. *In the long run* we are all dead.' (Keynes 2013 § 3.I, p. 65)

Formula (4) tell us how our robot would update its predictive probabilities at each new observation of a Plinko outcome.

3 General remarks on the robot's behaviour

The exchangeable-model formula (4) leads to some characteristic features of the robot's beliefs and of their evolution:

- The robot's predictions can be interpreted in several ways. One is this: the robot believes that there's 'something' constant in all trials; loosely speaking, a 'constant mechanism'. Another, maybe preferable, is this: the robot has memory of past observations, but not of their order; any trends are therefore invisible to it.
- As data accumulate, the robot's probabilities for the next outcome approach the observed frequencies. Such approach happens independently of the form of the prior $p(q|M) dq$ – unless the latter is zero in peculiar regions of the integration domain – but the prior determines the celerity of the approach. A prior heavily peaked on a frequency q' will require a lot of data to move the predictions to a very different frequency q .
- As data D accumulate, the updated density $p(q|D, M) dq$ will become more and more peaked at the N -tuple of observed frequencies.
- Suppose that we first have a long sequence of observations concentrating around frequencies q – say, a very long sequence of 1s in a row – and then a shift to other frequencies q' – say, suddenly 2s only appear. After the shift, the predictive probabilities will eventually become peaked around the new frequencies, but the shift in the peaks will take a larger number of observations around the new frequencies than the number around the old frequencies.

4 Initial prior

The shape of the initial prior heavily determines the predictions in the first observations, so it must be chosen with care. The Plinko data tell us the initial predictive probabilities of the participants,

$$p(D_k^1|M) \equiv \int q_k p(q|M) dq, \quad (6)$$

but not their prior $p(q|M)dq$.

As a first exploration we consider a *Johnson-Dirichlet* prior, proportional to a monomial $\prod_i q_i^{x_i}$ for some values of x_i :

$$p(q|M_J) = \frac{\Gamma(\Lambda)}{\prod_i \Gamma(\Lambda v_i)} \prod_{i=1}^N q_i^{\Lambda v_i - 1}, \quad \Lambda > 0, v \in \Delta. \quad (7)$$

This prior is determined by the additional assumption – call it M_J – that the frequencies of other outcomes are irrelevant for predicting a particular one:

$$P(D_k^{m+1} | Nf, M_J) = P(D_k^{m+1} | Nf_k, M_J) \quad k \in \{1, \dots, N\}, \quad (8)$$

where f is the N -tuple of observed relative frequencies. This assumption is called ‘sufficientness’ (Johnson 1924; 1932; Good 1965 ch. 4; Zabell 1982; Jaynes 1996). This is a conjugate prior (DeGroot 2004 ch. 9; Diaconis et al. 1979) and it has two convenient properties: it updates to a density of the same mathematical form, and its corresponding predictive distribution can be calculated analytically using the formula

$$\int_{\Delta} \prod_{i=1}^N q_i^{x_i - 1} dq = \frac{\prod_i \Gamma(x_i)}{\Gamma(\sum_i x_i)}. \quad (9)$$

We obtain for the initial distribution:

$$P(D_k^1 | M_J) = \int q_k p(q | M_J) dq = v_k, \quad (10)$$

and for the updated density:

$$p(q | D_{d_1}^1, \dots, D_{d_m}^m, M_J) = \frac{\Gamma(\Lambda')}{\prod_i \Gamma(\Lambda' v'_i)} \prod_{i=1}^N q_i^{\Lambda' v'_i - 1} \\ \text{with } \Lambda' = \Lambda + N, \quad v' = \frac{\Lambda v + Nf}{\Lambda + N}. \quad (11)$$

Formula (10) says that the Johnson-Dirichlet prior can produce any initial probabilities assigned by the participants, just by equalling the parameters v to them. The parameter Λ is left arbitrary. We can call it the *stubbornness* of the robot, Here’s the reason.

Suppose that after some observations the predictive distribution is v , and that the next outcome is k . Then the probability for slot k is updated

to $\frac{\Lambda v_k + 1}{\Lambda + 1}$, and that for all other slots j to $\frac{\Lambda v_j}{\Lambda + 1}$. This update corresponds to a participant's raising the bar assignment under slot k , leaving the others untouched, and/or lowering the bar assignments for *all* other slots by the same proportion. The parameter Λ is increased by 1. The larger Λ , the more reluctant the robot is in revising its guesses in the light of new observations. The update formula (11) says that the robot behaves as if it had already made Λ observations with outcome frequencies \mathbf{v} .

5 Examples: participant vs robot

Let's choose a participant, and use formula (10) to choose the \mathbf{v} parameters of the robot's prior, equating it to the initial predictive distribution of the participant. Let's set a value for the robot's stubbornness Λ , and check how the robot updates its predictive distribution, using formula (11), while observing the same outcomes as the participant.

Figure 1 shows the means and standard deviations of the sequence such predictive distributions, for participant 12 and a robot with stubbornness $\Lambda = 0.1$. This low value makes the robot give great consideration to the first outcomes, as the initial variability in the figure shows. The program generating the outcomes had a change in standard deviation, shifting to a narrower distribution at trial 101. The robot adapted to this change very slowly.

Figure 2 is analogous to fig. 1 but for a robot with stubbornness $\Lambda = 50$. This robot is even more slow to adapt to the narrowing in the standard deviation of the generated outcomes.

Figures 3 and 4 show the same for participant 30. The change in standard deviation was from narrow to large in this case.

The robot with low stubbornness seems to adapt to the widening of the outcome outputs faster than it had for the narrowing of the previous case: the change in the slope of the robot's standard-deviation curve seems steeper in fig. 3 than in 1.

If we look at the sequence of outcomes of figs 1 or 3, we perceive that something changed around trial 100. If we could plot these outcomes while they are generated, we would likely notice the change by around trial 25. Our robot, however, can't detect this change for the reasons explained in §3; any outcomes from narrow or wide generating processes are mingled in the robot's memory.



Figure 1 Comparison of the means and standard deviations of the predictive distributions of participant 12 and of a robot with stubbornness $\lambda = 0.1$

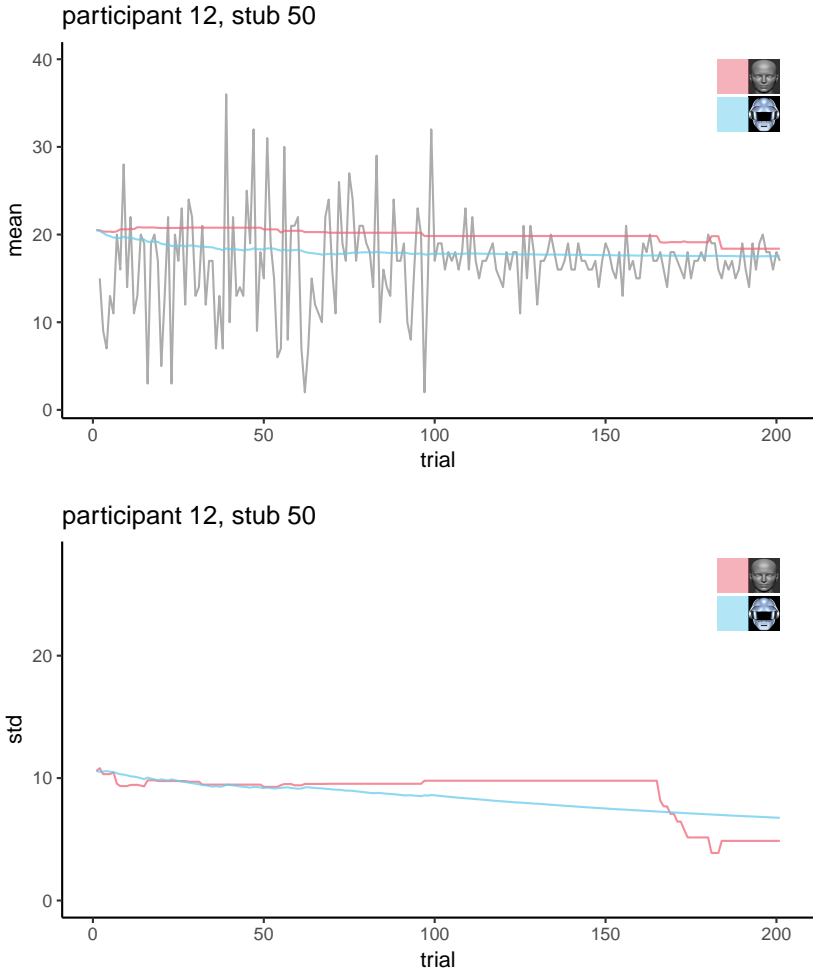


Figure 2 Comparison of the means and standard deviations of the predictive distributions of participant 12 and of a robot with stubbornness $\lambda = 50$



Figure 3 Comparison of the means and standard deviations of the predictive distributions of participant 30 and of a robot with stubbornness $\lambda = 0.1$

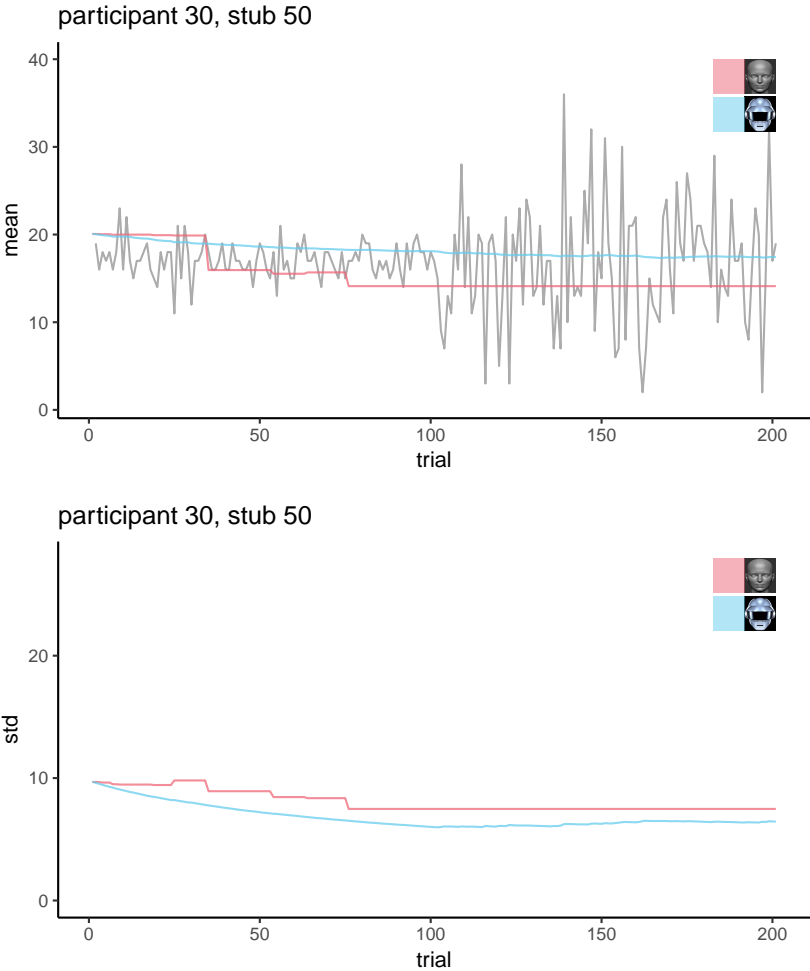


Figure 4 Comparison of the means and standard deviations of the predictive distributions of participant 30 and of a robot with stubbornness $\lambda = 50$

Only non-exchangeable or hierarchic models can exhibit a short evidence memory and be capable of believing that the underlying ‘mechanism’ has changed.

Some conclusions can be drawn from the properties of our model and from the examples:

- Participants who have great inertia against updating their predictions in view of the observations are *not* necessarily behaving at variance with the probability calculus. The latter says that they can be as stubborn as they please: larger Λ . If we judge such inertia as irrational, our judgement cannot be based on such a simple model; possibly it’s based on a hierarchic model where Λ is given a probability that depends on past experiences.
- The slots have a specific physical order, and from the way the ball falls into them it seems reasonable to assume that updates to the probability for one slot should affect those for nearby slots. The Johnson-Dirichlet model does not take this into account.
- A participant who, after observing outcome k , raises the bar under that slot *and nearby bars* is therefore not acting according to a Johnson-Dirichlet exchangeable model.
- Infinitely exchangeable priors are incapable of quickly adapting to changes in the empirical statistics of the outcomes.

6 Examples: robot’s surprise

A robot with low stubbornness quickly adapts its predictions to the observed outcomes, but as the outcomes accumulate its stubbornness increases. If there is a late change in the generation of the outcomes, at trial 101 for example, the robot will adapt its predictions more slowly.

It is interesting to ask: is this prediction adaptation the same for a change from a narrow to a wide distribution, as for a change from a wide to a narrow one? or is there a difference in the adaptation speed?

The answer depends on how we measure such speed. One way could be this: starting from the trial in which the change occurs, we let a second robot with low stubbornness observe the new outcomes and make predictions, starting from frequency parameters equal to those reached by the first robot. The second robot will quickly adapt to the new observed outcomes, and we can use it as a touchstone for the first robot’s adaptation speed. The predictions of the second robot can also

be interpreted as if we had reset the stubbornness of the first robot to a low value.

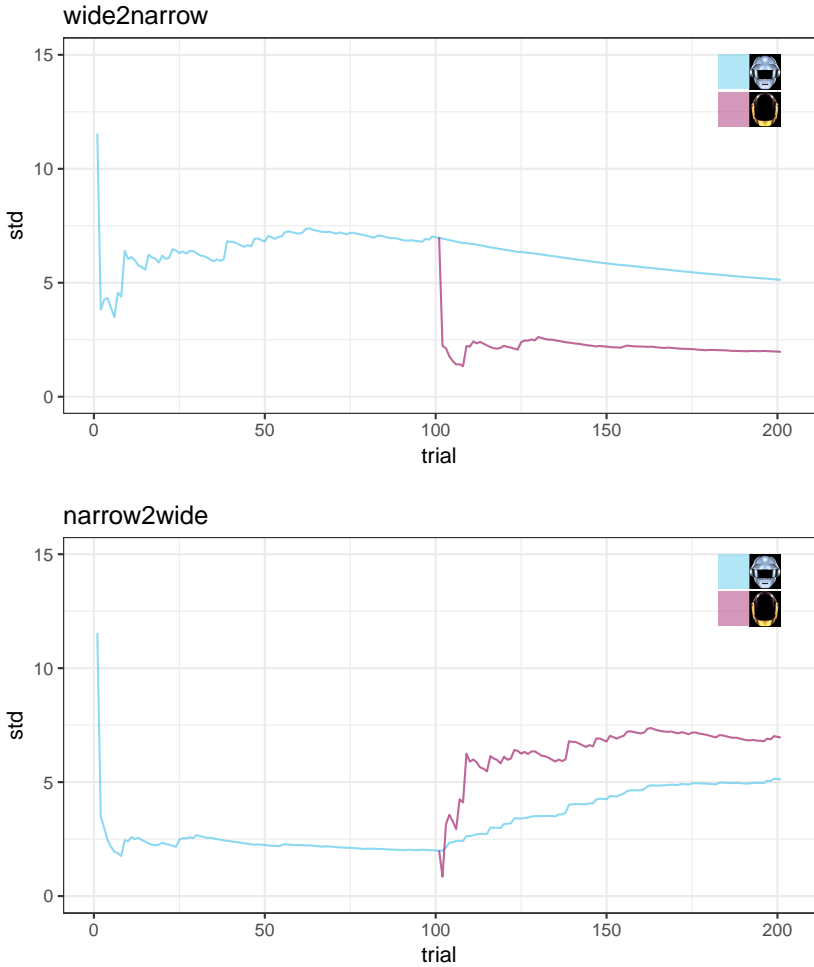


Figure 5 Adaptation speed of the standard deviation for a robot after a change in the generation of the outcomes, compared with that of a robot that starts learning after the change. The final relative entropy for the first robot relative to the second is 1.01 in the wide-to-narrow case, vs 0.260 in the narrow-to-wide; exchanging the distributions: 0.283 vs 0.211. The final normalized overlap is 0.949 for wide-to-narrow vs 0.823 for narrow-to-wide

The results of this comparison are shown in fig. 5. Looking at the standard deviations of the distributions it seems that a robot adapts more slowly in going from a wide to a narrow distribution than vice versa. This is true looking at the final relative entropy of the first robot relative to the second: 1.01 wide-to-narrow vs 0.260 narrow-to-wide; same if we exchange the distributions: 0.283 vs 0.211. The overlap seems to say the opposite: 0.106 wide-to-narrow vs 0.0512 narrow-to-wide; but the overlap is heavily influenced by the narrowness of the overlapping distributions, so it may not be a reliable measure in this case. If we use the normalized overlap (corresponding to the cosine of the angle between the distribution vectors) we find 0.949 vs 0.823, which agree with the first three measures.

Figure 6 shows that this phenomenon is even more striking if we average the sequences of standard deviations over 100 repetitions of such experiments. ✂ L: so good?! must recheck the script.

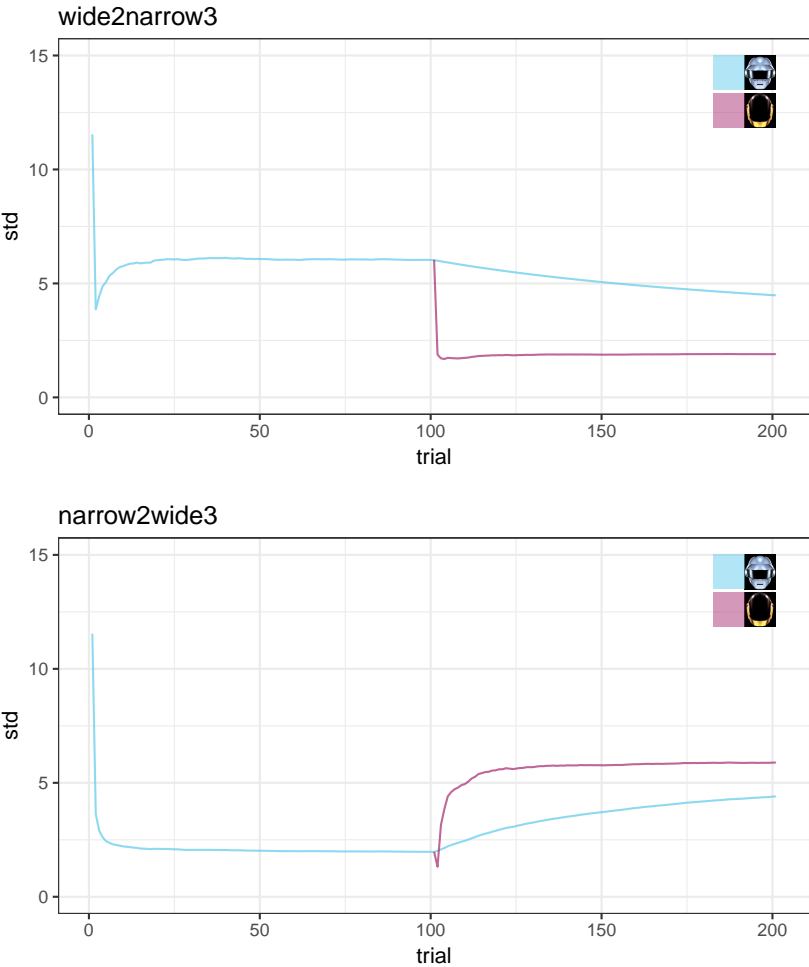


Figure 6 Adaptation speed of the standard deviation for a robot after a change in the generation of the outcomes, compared with that of a robot that starts learning after the change, averaged over 100 experiments. The normals have same mean 17 and standard deviations 6 and 1.9.

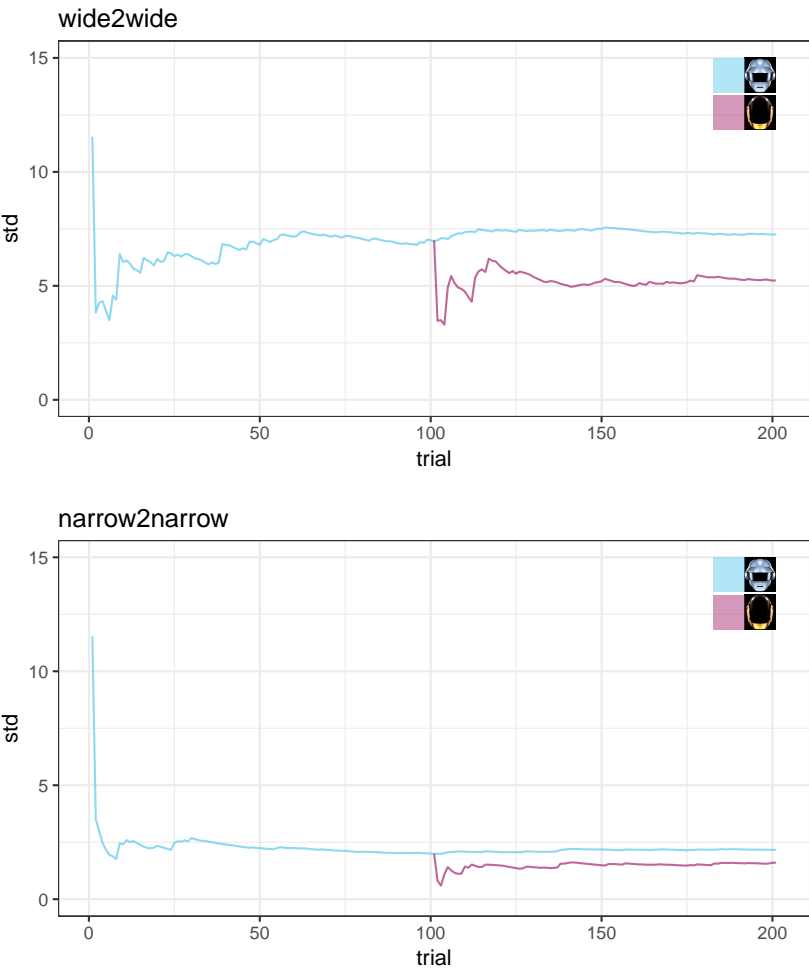


Figure 7 Adaptation speed of a robot after a change in the generation of the outcomes, compared with that of a robot that starts learning after the change.

7 How hierarchic models get updated

The probability calculus allows for inferences that learn from data in various degrees. We call ‘model’ a particular way of doing inference; each model is characterized by a particular capability of learning from data. These capabilities arise from a hierarchy of groups within groups of models.

At the bottom we have models that do not learn at all; we call them *independent*, because they assign independent probabilities to different data. For example, denoting a particular independent model by θ – which could be the value of a parameter identifying the model – and by I all other knowledge or assumptions besides this model, we have

$$p(d_1, d_2 | \theta, I) = p(d_1 | \theta, I) p(d_2 | \theta, I) \quad (12)$$

for any two data d_1, d_2 . This model does not learn because

$$p(d_2 | d_1, \theta, I) = p(d_2 | \theta, I), \quad (13)$$

that is, under this model one set of data is always irrelevant for the prediction of another set. The probability of an independent model given data d is

$$p(\theta | d, I) = \frac{p(d | \theta, I) p(\theta | I)}{\sum_{\theta} p(d | \theta, I) p(\theta | I)}, \quad (14)$$

where $p(\theta | I)$ is the probability over a range of such models based only on knowledge I , and $p(d | \theta, I)$ is the *likelihood* of the independent model given the data.

We can introduce the capability of learning from data by considering a collection $\{\theta\}$ of independent models, each having a probability, and letting the data influence the probabilities of these models, rather than the model themselves.

This particular model, based on a collection of independent models, is usually called a *parametric* model. Let us denote a particular parametric model by μ . It is not independent because

$$\begin{aligned} p(d_1, d_2 | \mu, I) &= \sum_{\theta} p(d_1, d_2 | \theta, \mu, I) p(\theta | \mu, I) \\ &\equiv \sum_{\theta} p(d_1 | \theta, \mu, I) p(d_2 | \theta, \mu, I) p(\theta | \mu, I), \end{aligned} \quad (15)$$

which doesn't factorize unless $p(\theta | \mu, I)$ is a delta. The first equality comes from the law of total probability. Such a model learns because

$$p(d_2 | d_1, \mu, I) = \sum_{\theta} p(d_2 | \theta, \mu, I) p(\theta | d_1, \mu, I) \quad (16a)$$

with

$$p(\theta | d_1, \mu, I) = \frac{p(d_1 | \theta, \mu, I) p(\theta | \mu, I)}{\sum_{\theta} p(d_1 | \theta, \mu, I) p(\theta | \mu, I)}, \quad (16b)$$

where we see that data d_1 affect not the probability of d_2 directly, but the probability distribution for the various independent models. The probability of a parametric model given data d is

$$p(\mu | d, I) = \frac{p(d | \theta, I) p(\mu | I)}{\sum_{\mu} p(d | \mu, I) p(\mu | I)} \quad (17a)$$

with

$$p(d | \mu, I) = \sum_{\theta} p(d | \theta, \mu, I) p(\theta | \mu, I). \quad (17b)$$

where $p(\mu | I)$ is the probability over a range of parametric models based only on knowledge I . The last expression (17b) is the *likelihood* of the model given the data, and we see that it's given by a mixture of independent models.

Equations (17) and (15) show that a parametric model is constructed as an uncertainty over independent models, and eq. (16) shows that data affect this latter uncertainty. It is as if we were considering different ways of doing inference, and inferring which of such inferences is most probable. Each bottom inference is incapable to learn from data, but our inferences about these inferences can learn from data.

We can proceed analogously and consider a collection $\{\mu\}$ of parametric models, each having a probability, and letting the data influence this probability as well. The model constructed this way is usually called a one-level hierarchic model – even though we've seen that a parametric model can also be considered as hierarchic. Let us denote such a model by χ . The probability of the data is

$$\begin{aligned} p(d_1, d_2 | \chi, I) &= \sum_{\mu} p(d_1, d_2 | \mu, \chi, I) p(\mu | \chi, I) \\ &\equiv \sum_{\mu} \left\{ \sum_{\theta} \left[\prod_i p(d_i | \theta, \mu, \chi, I) \right] p(\theta | \mu, \chi, I) \right\} \times \\ &\quad p(\mu | \chi, I). \end{aligned} \quad (18)$$

Learning takes place this way:

$$p(d_2 | d_1, \chi, I) = \sum_{\mu} \left[\sum_{\theta} p(d_2 | \theta, \mu, \chi, I) p(\theta | d_1, \mu, \chi, I) \right] p(\mu | d_1, \chi, I) \quad (19a)$$

with

$$\begin{aligned} p(\theta | d_1, \mu, \chi, I) &= \frac{p(d_1 | \theta, \mu, \chi, I) p(\theta | \mu, \chi, I)}{\sum_{\theta} p(d_1 | \theta, \mu, \chi, I) p(\theta | \mu, \chi, I)} \\ &= \frac{p(d_1 | \theta, \mu, \chi, I) p(\theta | \mu, \chi, I)}{p(d_1 | \mu, \chi, I)}, \end{aligned} \quad (19b)$$

$$p(\mu | d_1, \chi, I) = \frac{p(d_1 | \mu, \chi, I) p(\mu | \chi, I)}{\sum_{\mu} p(d_1 | \mu, \chi, I) p(\mu | \chi, I)}. \quad (19c)$$

Note that the denominator of the update formula (19b) for θ is the updated probability (19c) for μ . Note also that the space of independent models $\{\theta\}$ can be different for different μ .

With a hierarchic model, it is as if we were considering different ‘super-inferences’ about ways of doing inferences, and inferring which of such super-inferences is the most probable.

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