SDIRK Time Integration and Variable Material Properties for Radiative Transfer

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1 Generic SDIRK

Consider a generic Butcher Tableaux:

Stage	c_i	a			
1	c_1	γ	0	$\dots$	0
2	c_2	a_{21}	γ	0	$\vdots$
i	c_i	a_{i1}	a_{i2}	$\ddots$	0
s	c_s	a_{c1}	a_{c2}	$\dots$	γ
		b_1	b_2	$\dots$	b_s

Let ψ be our variable of interest. Then $\psi_{n+1} = \psi(t_n + \Delta t)$ is:

$$\psi_{n+1} = \psi_n + \Delta t \sum_{i=1}^{s} b_i k_i \tag{1}$$

where k_i is defined as:

$$k_i = f\left(t_n + c_i \Delta t \ , \ \psi_n + \Delta t \sum_{j=1}^{i} c_{ij} k_j \right)$$

and

$$f(t,\psi) = \frac{\partial \psi}{\partial t}$$

Eq. (1) can also be interpreted as meaning:

$$\psi_i = \psi_n + \Delta t \sum_{j=1}^{i} a_{ij} f\left(t_n + \Delta t c_j, \psi_j\right) \tag{2}$$

2 Grey Equations

We know that the 1D, mono-energetic (grey) radiative transfer equations are:

$$\frac{1}{c}\frac{\partial I}{\partial t} + \mu \frac{\partial}{\partial x} I + \sigma_t I = \frac{\sigma_s}{4\pi}\phi + \frac{\sigma_a}{4\pi} acT^4 + S_I$$

$$C_v \frac{\partial T}{\partial t} = \sigma_a\left(\phi - acT^4\right) + S_T$$

Solving for the time derivatives:

$$\frac{\partial I}{\partial t} = c\left[\frac{\sigma_s}{4\pi}\phi + \frac{\sigma_a}{4\pi}acT^4 - \mu\frac{\partial I}{\partial x} - \sigma_t I + S_I\right]$$

$$\frac{\partial T}{\partial t} = \frac{1}{C_v}\left[\sigma_a\left(\phi - acT^4\right) + S_T\right]$$

Before going any further, we spatially discretize our equations, expanding in an arbitrary order Lagrangian basis. First, transforming to a generic reference element:

$$\begin{aligned} x &= x_i + \frac{\Delta x_i}{2}s \\ s &\in [-1,1] \\ \Delta x_i &= x_{i+1/2} - x_{i-1/2} \\ x_i &= \frac{x_{i+1/2} + x_{i-1/2}}{2} \end{aligned}$$

For generality, let $\widetilde{x}$ and $\widetilde{y}$ be unknowns represented using our finite element representation:

$$\widetilde{x}(s) = \vec{B}(s)\cdot\vec{x} = B_1(s)x_1 + B_2x_2 + \cdots + B_N x_{N_P}$$

where

$$\vec{B} = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_N \end{bmatrix},$$

$$B_i(s) = \prod_{\substack{k=1\\k\neq j}}^{N_P}\frac{s_k - s}{s_k - s_i},$$

and $N_P = P+1$ where P is the DFEM trial space degree. Further suppose we wish to take integrals of the form:

$$\int_{-1}^{1}\vec{B}\psi ds$$

we represent this result as

$$\mathbf{M}\vec{\psi}$$

where

$$\mathbf{M}_{ij} = \frac{\Delta x_c}{2}\int_{-1}^{1}B_i(s)B_j(s)ds \approx \frac{\Delta x_i}{2}\sum_{q=1}^{N_q}w_q B_i(s_q)B_j(s_q),$$

$\{s_q, q_q\}_{q=1}^{N_q}$ is a quadrature set, and

$$\psi \approx \widetilde{\psi} = \sum_{j=1}^{N_P}B_j(s)\psi_j .$$

If we want to calculate the integral:

$$\int_{-1}^{1}\vec{B}\widetilde{\psi}(s)\sigma(s)ds$$

we denote the result as:

$$\mathbf{R}_{\sigma}\vec{\psi}\,,$$

where

$$\mathbf{R}_{\sigma,ij} = \frac{\Delta x_c}{2}\int_{-1}^{1}\sigma(s)B_i(s)B_j(s)ds \approx \frac{\Delta x_c}{2}\sum_{q=1}^{N_q} w_q B_i(s_q)B_j(s_q)\sigma(s_q)\,.$$

Our fundamental unknowns will be expressed as:

$$\begin{aligned} I(s) &= \vec{B}(s)\cdot\vec{I} \\ T(s) &= \vec{B}(s)\cdot\vec{T}\,. \end{aligned}$$

The Planck function, $\widehat{B}(T)$ will be similarly expanded:

$$\widehat{B}(s) \approx \vec{B}(s)\cdot\vec{\widehat{B}}$$

where for the grey case:

$$\vec{\widehat{B}} = \frac{1}{4\pi}\begin{bmatrix} acT_1^4 \\ \vdots \\ acT_j^4 \\ \vdots \\ acT_N^4 \end{bmatrix}$$

Since the Planck function is a highly nonlinear function of T, we elect to linearize it about an arbitrary temperature, T^*:

$$\widehat{B}(T) \approx \widehat{B}(T^*) + \frac{\partial}{\partial T}\left[\widehat{B}(T^*)\right](T - T^*)$$

Expressing in vector/matrix form:

$$\vec{\widehat{B}} \approx \vec{\widehat{B}}^* + \widehat{\mathbf{D}}^*\left(\vec{T} - \vec{T^*}\right) \tag{3}$$

where: $\widehat{\mathbf{D}}^*$ is a $N \times N$ diagonal matrix with non zero elements, d_{jj}:

$$d_{jj} = \frac{\partial\widehat{B}(T_j^*}{\partial T}$$

Driving/manufactured solution sources are likely not to be polynomials, so we define the following source moments rather than expand S_I or S_T in the DFEM trial space:

$$\begin{aligned} \vec{S}_{I,j} &= \int_{-1}^{1} B_j S_I(s)\;ds \\ \vec{S}_{T,j} &= \int_{-1}^{1} B_j S_T(s)\;ds\,. \end{aligned}$$

Having defined all this notation, we give the spatially discretized equations for cell i:

$$\frac{\partial}{\partial t}\left[\mathbf{M}\vec{I}\right] = c\left[\frac{1}{4\pi}\mathbf{R}_{\sigma_s}\vec{\phi} + \mathbf{R}_{\sigma_a}\left(\vec{B}^* + \widehat{\mathbf{D}}^*\left(\vec{T} - \vec{T}^*\right)\right) - \mathbf{R}_{\sigma_t}\vec{I} - \mathbf{L}\vec{I} + \vec{f}I_{in} + \vec{S}_I\right] \tag{4}$$

$$\frac{\partial}{\partial t}\left[\mathbf{R}_{C_v^*}\vec{T}\right] = \mathbf{R}_{\sigma_a}\left[\vec{\phi} - 4\pi\vec{B}^* - 4\pi\mathbf{D}^*\left(\vec{T} - \vec{T}^*\right)\right] + \vec{S}_T \tag{5}$$

Solving for k_I and k_T we have:

$$k_I = c\mathbf{M}^{-1}\left[\frac{1}{4\pi}\mathbf{R}_{\sigma_s}\vec{\phi} + \mathbf{R}_{\sigma_a}\left(\vec{B}^* + \widehat{\mathbf{D}}^*\left(\vec{T} - \vec{T}^*\right)\right) - \mathbf{R}_{\sigma_t}\vec{I} - \mathbf{L}\vec{I} + \vec{f}I_{in} + \vec{S}_I\right] \tag{6}$$

$$k_T = \mathbf{R}_{C_v}^{-1}\left[\mathbf{R}_{\sigma_a}\left(\vec{\phi} - 4\pi\vec{B}^* - 4\pi\mathbf{D}^*\left(\vec{T} - \vec{T}^*\right)\right) + \vec{S}_T\right] \tag{7}$$

2.1 First RK step

We now look at Eq. (2) for step 1 of an arbitrary, diagonally implicit RK scheme:

$$\vec{I}_1 = \vec{I}_n + c\Delta t a_{11}\mathbf{M}^{-1}\left[\frac{1}{4\pi}\mathbf{R}_{\sigma_s,*}\vec{\phi}_1 + \mathbf{R}_{\sigma_a,*}\left(\vec{B}^* + \widehat{\mathbf{D}}^*\left(\vec{T}_1 - \vec{T}^*\right)\right) - \mathbf{R}_{\sigma_t,*}\vec{I}_1 - \widehat{\mathbf{L}}\vec{I}_1 + \vec{f}I_{in,1} + \vec{S}_I\right] \tag{8}$$

$$\vec{T}_1 = \vec{T}_n + \Delta t a_{11}\mathbf{R}_{C_v}^{-1}\left[\mathbf{R}_{\sigma_a}\left(\vec{\phi}_1 - 4\pi\vec{B}^* - 4\pi\mathbf{D}^*\left(\vec{T}_1 - \vec{T}^*\right)\right) + \vec{S}_T\right] \tag{9}$$

We now use Eq. (9) to eliminate the unknown temperature, $\vec{T}_1$ from Eq. (8). Solving Eq. (9) for $\vec{T}_1$:

$$\vec{T}_1 + 4\pi\Delta t a_{11}\mathbf{R}_{C_v}^{-1}\mathbf{R}_{\sigma_a}\mathbf{D}^*\vec{T}_1 = \vec{T}_n + \Delta t a_{11}\mathbf{R}_{C_v}^{-1}\left[\mathbf{R}_{\sigma_a *}\left(\vec{\phi}_1 - 4\pi\vec{B}^* + 4\pi\mathbf{D}^*\vec{T}^*\right) + \vec{S}_T\right]$$

$$\begin{aligned}\vec{T}_1 = \left[\mathbf{I} + 4\pi\Delta t a_{11}\mathbf{R}_{C_v}^{-1}\mathbf{R}_{\sigma_a}\mathbf{D}^*\right]^{-1}\left[\vec{T}_n + \Delta t a_{11}\mathbf{R}_{C_v}^{-1}\left[\mathbf{R}_{\sigma_a}\left(\vec{\phi}_1 - 4\pi\vec{B}^* + 4\pi\mathbf{D}^*\vec{T}^*\right) + \vec{S}_T\right]\right]\dots \\ + \left[\mathbf{I} + 4\pi\Delta t a_{11}\mathbf{R}_{C_v}^{-1}\mathbf{R}_{\sigma_a}\mathbf{D}^*\right]^{-1}\left[\vec{T}^* - \vec{T}^*\right]\end{aligned}$$

$$\begin{aligned}\vec{T}_1 = \left[\mathbf{I} + 4\pi\Delta t a_{11}\mathbf{R}_{C_v}^{-1}\mathbf{R}_{\sigma_a}\mathbf{D}^*\right]^{-1}\left[\vec{T}_n + \Delta t a_{11}\mathbf{R}_{C_v}^{-1}\left[\mathbf{R}_{\sigma_a *}\left(\vec{\phi}_1 - 4\pi\vec{B}^*\right) + \vec{S}_T\right]\right]\dots \\ + \left[\mathbf{I} + 4\pi\Delta t a_{11}\mathbf{R}_{C_v}^{-1}\mathbf{R}_{\sigma_a}\mathbf{D}^*\right]^{-1}\left[\mathbf{I} + 4\pi\Delta t a_{11}\mathbf{R}_{C_v}^{-1}\mathbf{R}_{\sigma_a}\mathbf{D}^*\right]\vec{T}^*\dots \\ - \left[\mathbf{I} + 4\pi\Delta t a_{11}\mathbf{R}_{C_v}^{-1}\mathbf{R}_{\sigma_a}\mathbf{D}^*\right]^{-1}\vec{T}^*\end{aligned}$$

The "Temperature Update" for the first RK stage is the following:

$$\vec{T}_1 = \vec{T}^* + \left[\mathbf{I} + 4\pi\Delta t a_{11}\mathbf{R}_{C_v}^{-1}\mathbf{R}_{\sigma_a}\mathbf{D}^*\right]^{-1}\left[\vec{T}_n - \vec{T}^* + \Delta t a_{11}\mathbf{R}_{C_v}^{-1}\left[\mathbf{R}_{\sigma_a *}\left(\vec{\phi}_1 - 4\pi\vec{B}^*\right) + \vec{S}_T\right]\right] \tag{10}$$

Inserting Eq. (10) into Eq. (8):

$$\begin{aligned}\vec{I}_1 = \vec{I}_n + c\Delta t a_{11}\mathbf{M}^{-1}\left[\frac{1}{4\pi}\mathbf{R}_{\sigma_s,*}\vec{\phi}_1 + \mathbf{R}_{\sigma_a,*}\vec{B}^* - \mathbf{R}_{\sigma_t,*}\vec{I}_1 - \mathbf{L}\vec{I}_1 + \vec{f}I_{in,1} + \vec{S}_I\right]\dots \\ + c\Delta t a_{11}\mathbf{M}^{-1}\mathbf{R}_{\sigma_a,*}\widehat{\mathbf{D}}^*\left[\mathbf{I} + 4\pi\Delta t a_{11}\mathbf{R}_{C_v}^{-1}\mathbf{R}_{\sigma_a}\mathbf{D}^*\right]^{-1}\left[\vec{T}_n - \vec{T}^* + \Delta t a_{11}\mathbf{R}_{C_v}^{-1}\left[\mathbf{R}_{\sigma_a *}\left(\vec{\phi}_1 - 4\pi\vec{B}^*\right) + \vec{S}_T\right]\right]\end{aligned}$$

Multiply by $\frac{1}{c\Delta t a_{11}}\mathbf{M}$

$$\begin{aligned}\frac{1}{c\Delta t a_{11}}\mathbf{M}\vec{I}_1 = \frac{1}{c\Delta t a_{11}}\mathbf{M}\vec{I}_n + \left[\frac{1}{4\pi}\mathbf{R}_{\sigma_s,*}\vec{\phi}_1 + \mathbf{R}_{\sigma_a,*}\vec{B}^* - \mathbf{R}_{\sigma_t,*}\vec{I}_1 - \mathbf{L}\vec{I}_1 + \vec{f}I_{in,1} + \vec{S}_I\right]\dots \\ + \mathbf{R}_{\sigma_a,*}\widehat{\mathbf{D}}^*\left[\mathbf{I} + 4\pi\Delta t a_{11}\mathbf{R}_{C_v}^{-1}\mathbf{R}_{\sigma_a *}\mathbf{D}^*\right]^{-1}\left[\vec{T}_n - \vec{T}^* + \Delta t a_{11}\mathbf{R}_{C_v}^{-1}\left[\mathbf{R}_{\sigma_a *}\left(\vec{\phi}_1 - 4\pi\vec{B}^*\right) + \vec{S}_T\right]\right].\end{aligned}$$

Move some terms over to the LHS:

$$\mathbf{L}\vec{I}_1 + \left(\frac{1}{c\Delta t a_{11}}\mathbf{M} + \mathbf{R}_{\sigma_t,*}\right)\vec{I}_1 = \frac{1}{c\Delta t a_{11}}\mathbf{M}\vec{I}_n + \frac{1}{4\pi}\mathbf{R}_{\sigma_s,*}\vec{\phi}_1 + \mathbf{R}_{\sigma_a,*}\vec{B}^* + \vec{f}I_{in,1} + \vec{S}_I \ldots$$
$$+ \mathbf{R}_{\sigma_a,*}\widehat{\mathbf{D}}^*\left[\mathbf{I} + 4\pi\Delta t a_{11}\mathbf{R}_{C_v}^{-1}\mathbf{R}_{\sigma_a *}\mathbf{D}^*\right]^{-1}\left[\vec{T}_n - \vec{T}^* + \Delta t a_{11}\mathbf{R}_{C_v}^{-1}\left[\mathbf{R}_{\sigma_a *}\left(\vec{\phi}_1 - 4\pi\vec{B}^*\right) + \vec{S}_T\right]\right]$$

Further manipulating to pull all of the $\vec{\phi}_1$ terms together:

$$\begin{aligned}
\mathbf{L}\vec{I}_1 + \left(\frac{1}{c\Delta t a_{11}}\mathbf{M} + \mathbf{R}_{\sigma_t,*}\right)\vec{I}_1 = \ldots \\
\frac{1}{4\pi}\mathbf{R}_{\sigma_s,*}\vec{\phi}_1 + \Delta t a_{11}\mathbf{R}_{\sigma_a,*}\widehat{\mathbf{D}}^*\left[\mathbf{I} + 4\pi\Delta t a_{11}\mathbf{R}_{C_v}^{-1}\mathbf{R}_{\sigma_a *}\mathbf{D}^*\right]^{-1}\mathbf{R}_{C_v}^{-1}\mathbf{R}_{\sigma_a *}\vec{\phi}_1 \ldots \\
+ \frac{1}{c\Delta t a_{11}}\mathbf{M}\vec{I}_n + \mathbf{R}_{\sigma_a,*}\vec{B}^* + \vec{f}I_{in,1}\vec{S}_I \ldots \\
+ \mathbf{R}_{\sigma_a,*}\widehat{\mathbf{D}}^*\left[\mathbf{I} + 4\pi\Delta t a_{11}\mathbf{R}_{C_v}^{-1}\mathbf{R}_{\sigma_a *}\mathbf{D}^*\right]^{-1}\left[\vec{T}_n - \vec{T}^* + \Delta t a_{11}\mathbf{R}_{C_v}^{-1}\left[\vec{S}_T - 4\pi\mathbf{R}_{\sigma_a *}\vec{B}^*\right]\right]
\end{aligned} \tag{11}$$

Though it does not look that familiar, Eq. (11) can be made to resemble the canonical mono-energetic neutron fission equation. Let us define the following terms:

$$\bar{\bar{\nu}} = 4\pi\Delta t a_{11}\mathbf{R}_{\sigma_a,*}\widehat{\mathbf{D}}^*\left[\mathbf{I} + 4\pi\Delta t a_{11}\mathbf{R}_{C_v}^{-1}\mathbf{R}_{\sigma_a *}\mathbf{D}^*\right]^{-1}\mathbf{R}_{C_v}^{-1} \tag{12a}$$

$$\begin{aligned}
\bar{\bar{\xi}}_d = \frac{1}{c\Delta t a_{11}}\mathbf{M}\vec{I}_n + \mathbf{R}_{\sigma_a,*}\vec{B}^* + \vec{S}_I \ldots \\
+ \mathbf{R}_{\sigma_a,*}\widehat{\mathbf{D}}^*\left[\mathbf{I} + 4\pi\Delta t a_{11}\mathbf{R}_{C_v}^{-1}\mathbf{R}_{\sigma_a *}\mathbf{D}^*\right]^{-1}\left[\vec{T}_n - \vec{T}^* + \Delta t a_{11}\mathbf{R}_{C_v}^{-1}\left[\vec{S}_T - 4\pi\mathbf{R}_{\sigma_a}\vec{B}^*\right]\right]
\end{aligned} \tag{12b}$$

$$\bar{\bar{\mathbf{R}}}_{\sigma_t} = \frac{1}{c\Delta t a_{11}}\mathbf{M} + \mathbf{R}_{\sigma_t,*} \tag{12c}$$

Inserting Eqs. (12) into Eq. (11) gives our final form:

$$\mathbf{L}\vec{I}_1 + \bar{\bar{\mathbf{R}}}_{\sigma_t}\vec{I}_1 = \frac{1}{4\pi}\mathbf{R}_{\sigma_s,*}\vec{\phi}_1 + \frac{1}{4\pi}\bar{\bar{\nu}}\mathbf{R}_{\sigma_a *}\vec{\phi}_1 + \vec{f}I_{in,1} + \bar{\bar{\xi}}_d \tag{13}$$

Having found $\vec{I}_1$ by solving Eq. (13), we use Eq. (10) to find $\vec{T}_1$. This is like an outre Newton iteration loop, with the update defined in Eq. (10). The following assume we have converged the outer Newton iteration loop:

$$k_{I,1} = c\mathbf{M}^{-1}\left[\frac{1}{4\pi}\mathbf{R}_{\sigma_s,1}\vec{\phi}_1 + \mathbf{R}_{\sigma_a,1}\widehat{\mathbf{B}}_1 - \mathbf{R}_{\sigma_t,1}\vec{I}_1 - \mathbf{L}\vec{I}_1 + \vec{f}I_{in,1} + \vec{S}_I\right]$$

$$k_{T,1} = \mathbf{R}_{C_v}^{-1}\left[\mathbf{R}_{\sigma_a}\left(\vec{\phi}_1 - 4\pi\widehat{\mathbf{B}}_1\right) + \vec{S}_T\right]$$

2.2 i-th RK step

Moving on to the i-th RK step, we first write the equation for $\vec{I}_i$ and $\vec{T}_i$:

$$\vec{I}_i = \vec{I}_n + \Delta t\sum_{j=1}^{i-1} a_{ij}k_{I,j} + \Delta t a_{ii}c\mathbf{M}^{-1}\left[\frac{1}{4\pi}\mathbf{R}_{\sigma_s,*}\vec{\phi}_i + \mathbf{R}_{\sigma_a,*}\left(\vec{B}^* + \widehat{\mathbf{D}}^*\left(\vec{T}_i - \vec{T}^*\right)\right) - \mathbf{R}_{\sigma_t,*}\vec{I}_i - \mathbf{L}\vec{I}_i + \vec{f}I_{in,i} + \vec{S}_I\right] \tag{14}$$

$$\vec{T}_i = \vec{T}_n + \Delta t \sum_{j=1}^{i-1} a_{ij} k_{T,j} + \Delta t a_{ii} \mathbf{R}_{C_v}^{-1} \left[\mathbf{R}_{\sigma_a *} \left(\vec{\phi}_i - 4\pi \vec{B}^* - 4\pi \mathbf{D}^* \left(\vec{T}_i - \vec{T}^* \right) \right) + \vec{S}_T \right] \tag{15}$$

Proceeding in a similar fashion as before, we solve Eq. (15) for $\vec{T}_i$.

$$\vec{T}_i + 4\pi \Delta t a_{ii} \mathbf{R}_{C_v}^{-1} \mathbf{R}_{\sigma_a *} \mathbf{D}^* \vec{T}_i = \vec{T}_n + \Delta t \sum_{j=1}^{i-1} a_{ij} k_{T,j} + \Delta t a_{ii} \mathbf{R}_{C_v}^{-1} \left[\mathbf{R}_{\sigma_a *} \left(\vec{\phi}_i - 4\pi \vec{B}^* + 4\pi \mathbf{D}^* \vec{T}^* \right) + \vec{S}_T \right]$$

$$\left[\mathbf{I} + 4\pi \Delta t a_{ii} \mathbf{R}_{C_v}^{-1} \mathbf{R}_{\sigma_a} \mathbf{D}^* \right] \vec{T}_i = \vec{T}_n + \Delta t \sum_{j=1}^{i-1} a_{ij} k_{T,j} + \Delta t a_{ii} \mathbf{R}_{C_v}^{-1} \left[\mathbf{R}_{\sigma_a *} \left(\vec{\phi}_i - 4\pi \vec{B}^* + 4\pi \mathbf{D}^* \vec{T}^* \right) + \vec{S}_T \right]$$

$$\begin{aligned} \vec{T}_i = {} & \left[\mathbf{I} + 4\pi \Delta t a_{ii} \mathbf{R}_{C_v}^{-1} \mathbf{R}_{\sigma_a *} \mathbf{D}^* \right]^{-1} \left[\vec{T}_n + \Delta t \sum_{j=1}^{i-1} a_{ij} k_{T,j} \right] \ldots \\ & + \Delta t a_{ii} \left[\mathbf{I} + 4\pi \Delta t a_{ii} \mathbf{R}_{C_v}^{-1} \mathbf{R}_{\sigma_a *} \mathbf{D}^* \right]^{-1} \mathbf{R}_{C_v}^{-1} \left[\mathbf{R}_{\sigma_a *} \left(\vec{\phi}_i - 4\pi \vec{B}^* + 4\pi \mathbf{D}^* \vec{T}^* \right) + \vec{S}_T \right] \end{aligned}$$

$$\begin{aligned} \vec{T}_i = {} & \left[\mathbf{I} + 4\pi \Delta t a_{ii} \mathbf{R}_{C_v}^{-1} \mathbf{R}_{\sigma_a *} \mathbf{D}^* \right]^{-1} \left[\vec{T}_n + \Delta t \sum_{j=1}^{i-1} a_{ij} k_{T,j} \right] \ldots \\ & + \Delta t a_{ii} \left[\mathbf{I} + 4\pi \Delta t a_{ii} \mathbf{R}_{C_v}^{-1} \mathbf{R}_{\sigma_a *} \mathbf{D}^* \right]^{-1} \mathbf{R}_{C_v}^{-1} \left[\mathbf{R}_{\sigma_a *} \left(\vec{\phi}_i - 4\pi \vec{B}^* \right) + \vec{S}_T \right] \ldots \\ & + 4\pi \Delta t a_{ii} \left[\mathbf{I} + 4\pi \Delta t a_{ii} \mathbf{R}_{C_v}^{-1} \mathbf{R}_{\sigma_a *} \mathbf{D}^* \right]^{-1} \mathbf{R}_{C_v}^{-1} \mathbf{R}_{\sigma_a *} \mathbf{D}^* \vec{T}^* \end{aligned}$$

Adding nothing:

$$\begin{aligned} \vec{T}_i = {} & \left[\mathbf{I} + 4\pi \Delta t a_{ii} \mathbf{R}_{C_v}^{-1} \mathbf{R}_{\sigma_a *} \mathbf{D}^* \right]^{-1} \left[\vec{T}_n + \Delta t \sum_{j=1}^{i-1} a_{ij} k_{T,j} \right] \ldots \\ & + \Delta t a_{ii} \left[\mathbf{I} + 4\pi \Delta t a_{ii} \mathbf{R}_{C_v}^{-1} \mathbf{R}_{\sigma_a *} \mathbf{D}^* \right]^{-1} \mathbf{R}_{C_v}^{-1} \left[\mathbf{R}_{\sigma_a *} \left(\vec{\phi}_i - 4\pi \vec{B}^* \right) + \vec{S}_T \right] \ldots \\ & + \left[\mathbf{I} + 4\pi \Delta t a_{ii} \mathbf{R}_{C_v}^{-1} \mathbf{R}_{\sigma_a *} \mathbf{D}^* \right]^{-1} \left[4\pi \Delta t a_{ii} \mathbf{R}_{C_v}^{-1} \mathbf{R}_{\sigma_a *} \mathbf{D}^* \vec{T}^* \right] \ldots \\ & + \left[\mathbf{I} + 4\pi \Delta t a_{ii} \mathbf{R}_{C_v}^{-1} \mathbf{R}_{\sigma_a *} \mathbf{D}^* \right]^{-1} \left(\vec{T}^* - \vec{T}^* \right) \end{aligned}$$

simplifying

$$\begin{aligned} \vec{T}_i = {} & \left[\mathbf{I} + 4\pi \Delta t a_{ii} \mathbf{R}_{C_v}^{-1} \mathbf{R}_{\sigma_a *} \mathbf{D}^* \right]^{-1} \left[\vec{T}_n + \Delta t \sum_{j=1}^{i-1} a_{ij} k_{T,j} \right] \ldots \\ & + \Delta t a_{ii} \left[\mathbf{I} + 4\pi \Delta t a_{ii} \mathbf{R}_{C_v}^{-1} \mathbf{R}_{\sigma_a *} \mathbf{D}^* \right]^{-1} \mathbf{R}_{C_v}^{-1} \left[\mathbf{R}_{\sigma_a *} \left(\vec{\phi}_i - 4\pi \vec{B}^* \right) + \vec{S}_T \right] \ldots \\ & + \left[\mathbf{I} + 4\pi \Delta t a_{ii} \mathbf{R}_{C_v}^{-1} \mathbf{R}_{\sigma_a *} \mathbf{D}^* \right]^{-1} \left[\mathbf{I} + 4\pi \Delta t a_{ii} \mathbf{R}_{C_v}^{-1} \mathbf{R}_{\sigma_a *} \mathbf{D}^* \right] \vec{T}^* \ldots \\ & - \left[\mathbf{I} + 4\pi \Delta t a_{ii} \mathbf{R}_{C_v}^{-1} \mathbf{R}_{\sigma_a *} \mathbf{D}^* \right]^{-1} \vec{T}^* \end{aligned}$$

$$\vec{T}_i = \vec{T}^* + \left[\mathbf{I} + 4\pi\Delta t a_{ii}\mathbf{R}_{C_v}^{-1}\mathbf{R}_{\sigma_a *}\mathbf{D}^*\right]^{-1}\left[\vec{T}_n - \vec{T}^* + \Delta t\sum_{j=1}^{i-1} a_{ij}k_{T,j}\right]\dots$$
$$+ \Delta t a_{ii}\left[\mathbf{I} + 4\pi\Delta t a_{ii}\mathbf{R}_{C_v}^{-1}\mathbf{R}_{\sigma_a *}\mathbf{D}^*\right]^{-1}\mathbf{R}_{C_v}^{-1}\left[\mathbf{R}_{\sigma_a *}\left(\vec{\phi}_i - 4\pi\vec{B}^*\right) + \vec{S}_T\right] \quad (16)$$

Multiplying Eq. (14) by $\frac{1}{c\Delta t a_{ii}}\mathbf{M}$:

$$\frac{1}{c\Delta t a_{ii}}\mathbf{M}\vec{I}_i = \frac{1}{c\Delta t a_{ii}}\mathbf{M}\vec{I}_n + \frac{1}{c a_{ii}}\mathbf{M}\sum_{j=1}^{i-1} a_{ij}k_{I,j}\dots$$
$$+ \frac{1}{4\pi}\widehat{\mathbf{M}}_{\sigma_s,*}\vec{\phi}_i + \widehat{\mathbf{M}}_{\sigma_a,*}\left(\vec{B}^* + \widehat{\mathbf{D}}^*\left(\vec{T}_i - \vec{T}^*\right)\right) - \widehat{\mathbf{M}}_{\sigma_t,*}\vec{I}_i - \widehat{\mathbf{L}}\vec{I}_i + \vec{S}_I \quad (17)$$

Inserting Eq. (16) into Eq. (17):

$$\widehat{\mathbf{L}}\vec{I}_i + \left(\frac{1}{c\Delta t a_{ii}}\mathbf{M} + \mathbf{R}_{\sigma_t,*}\right)\vec{I}_i = \frac{1}{c\Delta t a_{ii}}\mathbf{M}\vec{I}_n + \frac{1}{c a_{ii}}\mathbf{M}\sum_{j=1}^{i-1} a_{ij}k_{I,j} + \frac{1}{4\pi}\mathbf{R}_{\sigma_s,*}\vec{\phi}_i + \mathbf{R}_{\sigma_a,*}\vec{B}^*\dots$$
$$+ \mathbf{R}_{\sigma_a,*}\widehat{\mathbf{D}}^*\Bigg\{\left[\mathbf{I} + 4\pi\Delta t a_{ii}\mathbf{R}_{C_v}^{-1}\mathbf{R}_{\sigma_a *}\mathbf{D}^*\right]^{-1}\left[\vec{T}_n - \vec{T}^* + \Delta t\sum_{j=1}^{i-1} a_{ij}k_{T,j}\right]\dots$$
$$+ \Delta t a_{ii}\left[\mathbf{I} + 4\pi\Delta t a_{ii}\mathbf{R}_{C_v}^{-1}\mathbf{R}_{\sigma_a *}\mathbf{D}^*\right]^{-1}\mathbf{R}_{C_v}^{-1}\left[\mathbf{R}_{\sigma_a *}\left(\vec{\phi}_i - 4\pi\vec{B}^*\right) + \vec{S}_T\right]\Bigg\}$$

Re-arranging to isolate $\vec{\phi}_i$:

$$\widehat{\mathbf{L}}\vec{I}_i + \left(\frac{1}{c\Delta t a_{ii}}\mathbf{M} + \mathbf{R}_{\sigma_t,*}\right)\vec{I}_i = \frac{1}{4\pi}\mathbf{R}_{\sigma_s,*}\vec{\phi}_i\dots$$
$$+ \Delta t a_{ii}\widehat{\mathbf{M}}_{\sigma_a,*}\widehat{\mathbf{D}}^*\left[\mathbf{I} + 4\pi\Delta t a_{ii}\mathbf{R}_{C_v}^{-1}\mathbf{R}_{\sigma_a,*}\mathbf{D}^*\right]^{-1}\mathbf{R}_{C_v}^{-1}\mathbf{R}_{\sigma_a *}\vec{\phi}_i\dots$$
$$+ \frac{1}{c\Delta t a_{ii}}\mathbf{M}\vec{I}_n + \frac{1}{c a_{ii}}\mathbf{M}\sum_{j=1}^{i-1} a_{ij}k_{I,j} + \mathbf{R}_{\sigma_a,*}\vec{B}^*\dots$$
$$+\widehat{\mathbf{M}}_{\sigma_a,*}\widehat{\mathbf{D}}^*\left[\mathbf{I} + 4\pi\Delta t a_{ii}\mathbf{R}_{C_v}^{-1}\mathbf{R}_{\sigma_a *}\widehat{\mathbf{D}}^*\right]^{-1}\left[\vec{T}_n - \vec{T}^* + \Delta t\sum_{j=1}^{i-1} a_{ij}k_{T,j} + \Delta t a_{ii}\mathbf{R}_{C_v}^{-1}\left[\vec{S}_T - 4\pi\mathbf{R}_{\sigma_a *}\mathbf{B}^*\right]\right] \quad (18)$$

Make the following definitions:

$$\bar{\bar{\xi}}_{i,d} = \frac{1}{c\Delta t a_{ii}}\mathbf{M}\vec{I}_n + \frac{1}{c a_{ii}}\mathbf{M}\sum_{j=1}^{i-1} a_{ij}k_{I,j} + \widehat{\mathbf{M}}_{\sigma_a,*}\vec{B}^*\dots$$
$$+\mathbf{R}_{\sigma_a,*}\widehat{\mathbf{D}}^*\left[\mathbf{I} + 4\pi\Delta t a_{ii}\mathbf{R}_{C_v}^{-1}\mathbf{R}_{\sigma_a *}\widehat{\mathbf{D}}^*\right]^{-1}\left[\vec{T}_n - \vec{T}^* + \Delta t\sum_{j=1}^{i-1} a_{ij}k_{T,j} + \Delta t a_{ii}\mathbf{R}_{C_v}^{-1}\left[\vec{S}_T - 4\pi\mathbf{R}_{\sigma_a *}\mathbf{B}^*\right]\right] \quad (19a)$$

$$\bar{\bar{\nu}}_i = 4\pi\Delta t a_{ii}\widehat{\mathbf{M}}_{\sigma_a,*}\widehat{\mathbf{D}}^* \left[\mathbf{I} + 4\pi\Delta t a_{ii}\mathbf{R}_{C_v}^{-1}\mathbf{R}_{\sigma_a *}\mathbf{D}^*\right]^{-1}\mathbf{R}_{C_v}^{-1} \tag{19b}$$

$$\bar{\bar{\mathbf{M}}}_{\sigma_t,i} = \mathbf{M}_{\sigma_t,*} + \frac{1}{c\Delta t a_{ii}}\mathbf{M} \tag{19c}$$

This then gives us our final equation for the radiation intensity:

$$\widehat{\mathbf{L}}\vec{I}_i + \bar{\bar{\mathbf{M}}}_{\sigma_t,i}\vec{I}_i = \frac{1}{4\pi}\mathbf{M}_{\sigma_s,*}\vec{\phi}_i + \frac{1}{4\pi}\bar{\bar{\nu}}_i\widehat{\mathbf{M}}_{\sigma_a *}\vec{\phi}_i + \bar{\bar{\xi}}_{i,d} \tag{20}$$

Having found $\vec{I}$ for the i-th RK time step, we solve for $\vec{T}_i$ using Eq. (16). At this point, we again have the option to either iterate on $\vec{T}_i$, or we only solve for $\vec{T}_i$ once. Regardless of whether we iterate for $\vec{T}_i$ or not, we evaluate all material properties at the final value of $\vec{T}_i$, and apply the definitions of $k_{T,i}$ and $k_{I,i}$:

$$k_{I,i} = c\mathbf{M}^{-1}\left[\frac{1}{4\pi}\widehat{\mathbf{M}}_{\sigma_s,i}\vec{\phi}_i + \widehat{\mathbf{M}}_{\sigma_a,i}\widehat{\mathbf{B}}_i - \widehat{\mathbf{M}}_{\sigma_t}\vec{I}_i - \widehat{\mathbf{L}}\vec{I}_i + \vec{S}_I\right] \tag{21}$$

$$k_{T,i} = \mathbf{R}_{C_v}^{-1}\left[\mathbf{R}_{\sigma_a,i}\left(\vec{\phi}_i - 4\pi\widehat{\mathbf{B}}_i\right) + \vec{S}_T\right] \tag{22}$$

After completing all steps of the particular RK scheme, we advance $\vec{I}_n \to \vec{I}_{n+1}$ and $\vec{T}_n \to \vec{T}_{n+1}$:

$$\begin{aligned}
\vec{I}_{n+1} &= \vec{I}_n + \Delta t\sum_{i=1}^{s} b_i k_{I,i} \\
\vec{T}_{n+1} &= \vec{T}_n + \Delta t\sum_{i=1}^{s} b_i k_{T,i}
\end{aligned}$$

3 Multigroup Case

We previously considered the grey case, now we consider the spectrum of photon energies. We discretize the energy variable using the multigroup method. The multigroup method assumes a finite number of groups, G. Ideally, we would have:

$$\begin{aligned}
I_g &= \int_{E_{min}}^{E_{max}} I(E)dE \\
I_g\sigma_g &= \int_{E_{min}}^{E_{max}} I(E)\sigma(E)dE \\
\int_0^\infty \sigma(E)I(E)dE &= \sum_{g=1}^{G}\sigma_g I_g
\end{aligned}$$

where E_{min} and E_{max} are the minimum and maximum photon energy of each group, g. In practice though we are solving equations of the form:

$$\frac{1}{c}\frac{\partial I_g}{\partial t} + \mu\frac{\partial I_g}{\partial x} + \sigma_{t,g}I_g = \frac{\sigma_{s,g}}{4\pi}\phi_g + \sigma_{a,g}\widehat{B}_g \tag{23}$$

$$C_v\frac{\partial T}{\partial t} = \sum_{g=1}^{G}\sigma_{a,g}\left(\phi_g - 4\pi\widehat{B}_g\right) \tag{24}$$

where the σ_g and C_v are evaluated *a priori*. Adjusting our Planck function expansion for multigroup use, we define:

$$\widehat{\mathbf{B}}_g = \int_{E_{min,g}}^{E_{max,g}} \widehat{B}(E,T)dE$$

and $\mathbf{D}_g^*$ is a diagonal matrix with non-zero main diagonal elements $d_i i$:

$$d_{ii} = \int_{E_{min,g}}^{E_{max,g}} \frac{\partial \widehat{B}(E,T)}{\partial T} dE$$

giving our familiar linearization:

$$\widehat{\mathbf{B}}_g \approx \widehat{\mathbf{B}}_g^* + \widehat{\mathbf{D}}_g^* \left(\vec{T} - \vec{T}^*\right)$$

with this notation, our spatially discretized, temporally analytic equations are:

$$\frac{1}{c}\frac{\partial}{\partial t}\mathbf{M}\vec{I}_{d,g} + \widehat{\mathbf{L}}\vec{I}_{d,g} + \mathbf{R}_{\sigma_{t,g}^*}\vec{I}_{d,g} = \frac{1}{4\pi}\mathbf{R}_{\sigma_{s,g}^*}\vec{\phi} + \mathbf{R}_{\sigma_{a,g}^*}\left[\widehat{\mathbf{B}}_g^* + \widehat{\mathbf{D}}_g^*\left(\vec{T} - \vec{T}^*\right)\right] + \vec{S}_{I,g} \tag{25}$$

$$\frac{\partial}{\partial t}\left[\mathbf{R}_{C_v^*}\vec{T}\right] = \sum_{g=1}^{G}\mathbf{R}_{\sigma_{a,g}^*}\left\{\vec{\phi}_g - 4\pi\left[\widehat{\mathbf{B}}_g^* + \widehat{\mathbf{D}}_g^*\left(\vec{T} - \vec{T}^*\right)\right]\right\} + \vec{S}_T \tag{26}$$

solving for $k_{I,g}$ and k_T we have:

$$k_{I,g} = c\mathbf{M}^{-1}\left[\frac{1}{4\pi}\mathbf{R}_{\sigma_{s,g}^*}\vec{\phi}_g + \mathbf{R}_{\sigma_{a,g}^*}\left[\widehat{\mathbf{B}}_g^* + \widehat{\mathbf{D}}_g^*\left(\vec{T} - \vec{T}^*\right)\right] - \widehat{\mathbf{L}}\vec{I}_{d,g} - \mathbf{R}_{\sigma_{t,g}^*}\vec{I}_{d,g}\right] \tag{27}$$

$$k_T = \mathbf{R}_{C_v^*}^{-1}\left[\sum_{g=1}^{G}\mathbf{R}_{\sigma_{a,g}^*}\left\{\vec{\phi}_g - 4\pi\left[\widehat{\mathbf{B}}_g^* + \widehat{\mathbf{D}}_g^*\left(\vec{T} - \vec{T}^*\right)\right]\right\}\right] \tag{28}$$