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Numerische Mathematik

A historical note on Gauss–Kronrod quadrature

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Summary. The idea of Gauss–Kronrod quadrature, in a germinal form, is traced back to an 1894 paper of R. Skutsch.

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The idea of inserting $n+1$ nodes into an n-point Gaussian quadrature rule and choosing them and the weights of the resulting $(2n+1)$-point quadrature rule in such a manner as to maximize the polynomial degree of exactness is generally attributed to A.S. Kronrod [2], [3]. This is entirely justified, given that Kronrod developed the underlying theory and produced extensive numerical tables. The same idea, nevertheless, can be traced back at least to 1894, when R. Skutsch [5] pointed out the possibility of obtaining in this way a $(2n + 1)$-point formula of degree of exactness $3n + 1$ (resp. $3n + 2$ if n is odd). He also notes that the degree of exactness of the n-point Gauss formula cannot be improved by inserting fewer than $n+1$ points (*ibid.*, p. 81), a result proved later by Monegato [4, Lemma 1].

On p. 83, the paper also gives numerical results to 11 decimal digits for the integral $\int_{-1}^{1} \mathrm{d}x/(x + 3) = \ln 2$. The 7-point extension of the 3-point Gauss formula is compared in this example with, among others, the 3-point and 7-point Gauss formulae. Only end results are stated, none of the respective quadrature formulae. They are all correct to 11 digits, except for the result for the 7-point extension, which is off in the last two digits.

Could it be that Stieltjes knew about Skutsch's paper? In his last letter to Hermite [1, Vol. 2, p. 439], dated November 8, 1894, Stieltjes considered orthogonal polynomials relative to a sign-variable weight function (a Legendre polynomial), which are relevant to the Kronrod extension of Gauss formulae, but makes no reference to mechanical quadrature.

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