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Unconditionally Stable Diffusion-Synthetic Acceleration Methods for the Slab Geometry Discrete Ordinates Equations. Part I: Theory

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We consider the slab geometry discrete ordinates equations, with the weighted diamond, linear characteristic, linear discontinuous, and linear moments spatial differencing schemes. For each differencing scheme we derive a diffusion-synthetic, source-correction acceleration method which, for model (infinite medium, isotropic scattering, constant cross section, constant mesh spacing) problems, unconditionally reduces the spectral radius of the iteration method from the unaccelerated value of c (the scattering ratio) to $< c/3$.

I. INTRODUCTION

For most realistic neutron transport problems and all production transport codes, the transport equation must, as a practical matter, be solved iteratively.¹⁻⁴ The standard within-group iterative method, discussed below, is based solely on a sequence of transport iterations; with a starting guess of zero, the n 'th iterate describes the flux due to neutrons that have undergone at most n collisions. This method converges very slowly for problems with large regions and

scattering ratios near unity, where most of the neutrons undergo many collisions. To improve the convergence rate for this class of problems, various acceleration methods have been devised, which involve, within each single iteration, a transport calculation followed by some other type of calculation designed to improve the transport result.¹⁻¹⁸ However, stability difficulties with most of these methods have greatly limited their usefulness; these difficulties are well known and have been studied in the literature.⁷

¹W. A. RHOADES and F. R. MYNATT, "The DOT-III Two-Dimensional Discrete Ordinates Transport Code," ORNL TM-4280, Oak Ridge National Laboratory (1973).

²K. D. LATHROP and F. W. BRINKLEY, "TWOTRAN-II: An Interfaced, Exportable Version of the TWOTRAN Code for Two-Dimensional Transport," LA-4848-MS, Los Alamos National Laboratory (1973).

³T. R. HILL, "ONETRAN, A Discrete Ordinates Finite Element Code for the Solution of the One-Dimensional Multigroup Transport Equation," LA-5990-MS, Los Alamos National Laboratory (1973).

⁴T. J. SEED, W. F. MILLER, Jr., and F. W. BRINKLEY, Jr., "TRIDENT: A Two-Dimensional Multigroup Triangular Mesh Discrete Ordinates Explicit Neutron Transport Code," LA-6735-M, Los Alamos National Laboratory (1977).

⁵H. J. KOPP, *Nucl. Sci. Eng.*, **17**, 65 (1963).

⁶E. M. GELBARD and L. A. HAGEMAN, *Nucl. Sci. Eng.*, **37**, 288 (1969).

⁷W. H. REED, *Nucl. Sci. Eng.*, **45**, 245 (1971).

⁸R. C. ALLEN and G. M. WING, Jr., "A Study of the Synthetic Method as Applied to the Transport Equation in Slab Geometry—Some Analytical and Numerical Results," LA-UR-75-1104, Los Alamos National Laboratory (1975).

⁹E. E. LEWIS and W. F. MILLER, Jr., *Trans. Am. Nucl. Soc.*, **23**, 202 (1976).

¹⁰R. E. ALCOUFFE, *Trans. Am. Nucl. Soc.*, **23**, 203 (1976).

¹¹R. E. ALCOUFFE, *Nucl. Sci. Eng.*, **64**, 344 (1977).

¹²W. F. MILLER, Jr., *Nucl. Sci. Eng.*, **65**, 226 (1978).

¹³R. C. ALLEN and G. M. WING, Jr., *J. Optimization Th. Appl.*, **24**, 5 (1978).

In this paper we derive unconditionally stable diffusion-synthetic, source-correction acceleration methods appropriate to several spatial differencing schemes for the slab geometry discrete ordinates equations. To motivate our analysis, we begin with a review of previous work on diffusion-synthetic (or, as they were originally known, "synthetic") acceleration methods.

As originally described by Kopp,⁵ the synthetic method is a general iterative method for solving the transport equation. Gelbard and Hageman⁶ later specialized, analyzed, and tested the method as a tool for accelerating the standard discrete ordinates transport iteration scheme. These authors analyzed the continuous (i.e., undiscretized) equations and showed an improvement in the convergence rate, and they presented numerical results. We now rederive some of their analytical results from a fresh viewpoint.

The standard unaccelerated iterative method for solving the transport equation can be described as

$$\mu \frac{\partial \psi^{l+1/2}}{\partial x} + \psi^{l+1/2} = c\phi_0^l + S \quad (1)$$

and

$$\phi_0^{l+1} = \frac{1}{2} \int_{-1}^1 \psi^{l+1/2} d\mu \equiv \phi_0^{l+1/2}. \quad (2)$$

For an infinite medium, the spectral radius of this method is shown by the following Fourier analysis to be c . Let

$$\Psi^{l+1/2} = \psi^{l+1/2} - \psi^{l-1/2}$$

and

$$\Phi_0^{l+1} = \phi_0^{l+1} - \phi_0^l.$$

Then, Eqs. (1) and (2) give

$$\mu \frac{\partial \Psi^{l+1/2}}{\partial x} + \Psi^{l+1/2} = c\Phi_0^l$$

and

$$\Phi_0^{l+1} = \Phi_0^{l+1/2}.$$

¹⁴J. AULL, "The Stability of the Mesh-Cornered Synthetic Method of Diffusion Acceleration of the DOT-IV Transport Code," ORNL/TM-7097, Oak Ridge National Laboratory (1979).

¹⁵J. AULL, "Acceleration of the Inner Iteration of the DOT-IV Transport Code Using a New Source Correction Scheme," ORNL/TM-7404, Oak Ridge National Laboratory (1980).

¹⁶E. M. GELBARD, D. R. MCCOY, and E. W. LARSEN, *Trans. Am. Nucl. Soc.*, **39**, 462 (1981).

¹⁷E. W. LARSEN and D. R. MCCOY, *Trans. Am. Nucl. Soc.*, **39**, 469 (1981).

¹⁸J. E. MOREL, *Nucl. Sci. Eng.*, **82**, 34 (1982).

Introducing,

$$\Phi_0^l(x) = \exp(j\lambda x) \quad (j = \sqrt{-1}, -\infty < \lambda < \infty)$$

and

$$\Psi^{l+1/2}(x, \mu) = f(\mu) \exp(j\lambda x)$$

into these equations, we find that

$$f(\mu) = \frac{c}{1 + j\lambda\mu}$$

and

$$\Phi_0^{l+1} = \omega \exp(j\lambda x),$$

where

$$\omega = \frac{c}{2} \int_{-1}^1 \frac{d\mu}{1 + \lambda^2\mu^2}.$$

Thus, $\exp(j\lambda x)$ are eigenfunctions of this iteration scheme, with corresponding eigenvalues ω . The spectral radius (largest eigenvalue) is

$$\text{spr} \equiv \sup_{\lambda} |\omega| = c,$$

and so the reduction in error from one iteration to the next in the method [Eqs. (1) and (2)] is characterized by c . (We note that the eigenfunctions whose eigenvalues are close to c are those that correspond to $\lambda \approx 0$; these eigenfunctions are nearly linear functions of μ .) For values of c near unity, the solution found by this method can thus converge unacceptably slowly.

To get around this difficulty let us not, for each iteration, define ϕ_0^{l+1} by Eq. (2). [However, we insist that Eq. (2) holds upon convergence; i.e., whatever the new expression for ϕ_0^{l+1} is, this expression and $\phi_0^{l+1/2}$ must agree as $l \rightarrow \infty$.] We instead define ϕ_0^{l+1} by a new procedure, which we now develop.

To begin, we take the zero'th and first angular moments of Eq. (1), obtaining

$$\frac{d\phi_1^{l+1/2}}{dx} + \phi_0^{l+1/2} = c\phi_0^l + S \quad (3a)$$

and

$$\frac{2}{3} \frac{d\phi_2^{l+1/2}}{dx} + \frac{1}{3} \frac{d\phi_0^{l+1/2}}{dx} + \phi_1^{l+1/2} = 0. \quad (3b)$$

Here we have defined

$$\phi_n(x) = \frac{1}{2} \int_{-1}^1 P_n(\mu) \psi(x, \mu) d\mu,$$

with $P_n(\mu)$ the n 'th Legendre polynomial. Upon convergence, Eqs. (3) and (2) become

$$\frac{d\phi_1}{dx} + \phi_0 = c\phi_0 + S \quad (4a)$$

and

$$\frac{2}{3} \frac{d\phi_2}{dx} + \frac{1}{3} \frac{d\phi_0}{dx} + \phi_1 = 0, \quad (4b)$$

where the converged quantities are indicated without iteration superscripts.

Now we observe the following. If the angular flux ψ is a linear function of μ , which is nearly the case for the $\lambda \approx 0$ eigenfunctions, then $\phi_2 = 0$ and Eqs. (4) are two equations that exactly and uniquely determine the two unknowns ϕ_0 and ϕ_1 (provided the boundary conditions are correctly chosen, a subject we discuss below). We would like to devise a method for determining ϕ_0^{l+1} (and, if desired, ϕ_1^{l+1}) that preserves these features. That is, we seek an iterative version of the two equations, Eqs. (4), that has the following properties: The two unknowns are ϕ_0^{l+1} and ϕ_1^{l+1} , and all other terms depending on ψ vanish if the previous transport iterate is a linear function of μ . (From a Fourier-analytic point of view, this amounts to treating the $\lambda \approx 0$ eigenfunctions, which cause the poor convergence properties in the unaccelerated method, much more accurately.) An iterative form of Eqs. (4), which has these properties, is

$$\frac{d\phi_1^{l+1}}{dx} + (1 - c)\phi_0^{l+1} = S \quad (5a)$$

and

$$\frac{2}{3} \frac{d\phi_2^{l+1/2}}{dx} + \frac{1}{3} \frac{d\phi_0^{l+1}}{dx} + \phi_1^{l+1} = 0. \quad (5b)$$

In a sense, the derivation of Eqs. (5) from Eqs. (3) is the essential idea in this paper, since in the following sections we apply this same derivation to each of the various discretized forms of the transport equation.

To proceed, we subtract Eqs. (3) from Eqs. (5) and obtain the equivalent result

$$\begin{aligned} \frac{df_1^{l+1}}{dx} + (1 - c)f_0^{l+1} &= c(\phi_0^{l+1/2} - \phi_0^l), \\ \frac{1}{3} \frac{df_0^{l+1}}{dx} + f_1^{l+1} &= 0, \end{aligned}$$

where

$$\phi_n^{l+1} = \phi_n^{l+1/2} + f_n^{l+1}, \quad n = 0, 1,$$

or, eliminating f_1^{l+1} ,

$$-\frac{1}{3} \frac{d^2f_0^{l+1}}{dx^2} + (1 - c)f_0^{l+1} = c(\phi_0^{l+1/2} - \phi_0^l) \quad (6a)$$

and

$$\phi_0^{l+1} = \phi_0^{l+1/2} + f_0^{l+1}. \quad (6b)$$

Equations (1) and (6) comprise the synthetic iteration method of Gelbard and Hageman.⁶ The spectral radius of this method, obtained by the same Fourier analysis as described above for the unaccelerated method, satisfies for $0 \leq c \leq 1$,

$$\begin{aligned} \text{spr} &= \sup_{\lambda} \left[\frac{c\lambda^2}{\lambda^2 + 3(1 - c)} \right] \left| \int_{-1}^1 \frac{P_2(\mu)}{1 + \lambda^2\mu^2} d\mu \right| \\ &\leq \sup_{\lambda} c \left| \int_{-1}^1 \frac{P_2(\mu)}{1 + \lambda^2\mu^2} d\mu \right| \\ &< 0.23 c. \end{aligned} \quad (7)$$

[The bound (0.23) on the integral term was computed numerically.]

The above results [Eqs. (6) and (7)] were first derived by Gelbard and Hageman.⁶ The bound Eq. (7) indicates that in its continuous or undiscretized form the synthetic method has significantly better convergence properties for $c \approx 1$ than the standard method. The synthetic method (in its continuous form) has also been studied by Allen and Wing,^{8,13} who generalized the method to a wider class of mathematical problems and showed that it can be interpreted as a special case of the splitting method of matrix theory, applied to more general operators. For infinite medium transport problems, Eq. (7) provides a bound on the spectral radius; for finite medium transport problems, Allen and Wing^{8,13} derived similar, but much cruder, bounds.

Gelbard and Hageman⁶ presented numerical results, for some one-group, x,y geometry transport problems, which were in general agreement with the result, Eq. (7). These authors used diamond differencing (DD) for the transport equation and the PDQ-7 diffusion code,^{19,20} which employs a standard five-point vertex differencing scheme, for the diffusion equation.²¹ Reed⁷ later analyzed Eq. (1) with the DD method and Eq. (10) with a standard central differencing method. Using a combination of analytical and numerical results, he found the following:

1. The discretized unaccelerated method has $\text{spr} = c$ for all values of h (the width of a spatial cell), and thus is unconditionally stable.
2. The discretized synthetic method works well for small h , but the method has a spectral radius that depends on h , with the lower bound

$$\text{spr} \geq \frac{c}{\frac{4}{3h^2} + 1 - c}.$$

Thus, for $h > h^* = 2/[3(2c - 1)]^{1/2}$ one has $\text{spr} > 1$, which implies instability. For $c \approx 1$, $h^* \approx 2/\sqrt{3}$, and so Reed's discretized method here is stable only for spatial cell widths less than ~ 1 mfp.

¹⁹W. R. CADWELL, "PDQ-7 Reference Manual," WAPD-TM-678, Westinghouse Atomic Power Division (1967).

²⁰C. J. PFEIFER, "PDQ-7 Reference Manual II," WAPD-TM-947 (L), Westinghouse Atomic Power Division (1971).

²¹E. M. GELBARD, Argonne National Laboratory, Private Communication (1981).

Reed's results can be summarized as follows. The unaccelerated method has the disadvantage of slow convergence for $c \approx 1$, but the advantage of stability for all values of h . His synthetic method has the advantage of working well (accelerating) for small values of h , but the disadvantage of not working at all (instability) for $h > 2/\sqrt{3}$.

At this point, we remark that although Gelbard and Hageman did not report any numerical instabilities in their (x, y geometry) calculations, the spatial meshes in the problems they considered were not very large. Also, McCoy has shown that the five-point vertex differencing scheme used by Gelbard and Hageman for their diffusion calculation reduces, in slab geometry, to an edge differencing scheme that yields exactly the same spectral radius as that for Reed's method, which used central differencing.²² Therefore, the acceleration method used by Gelbard and Hageman will likely become unstable for large enough h .

The problem of obtaining an acceleration method that is stable for all values of h was finally solved by Alcouffe^{10,11} for the case of the DD transport equation. Alcouffe's method, which he termed "diffusion-synthetic," is not based on Eqs. (6), but rather on

$$-\frac{1}{3} \frac{d^2\phi_0^{l+1}}{dx^2} + (1-c)\phi_0^{l+1} = S - \frac{d}{dx} \left(\frac{1}{3} \frac{d\phi_0^{l+1/2}}{dx} + \phi_1^{l+1/2} \right), \quad (8)$$

which can be derived from Eqs. (3b) and (5). [Equations (6) and (8) are thus algebraically equivalent.⁹] Alcouffe's basic insight was that *the discretizations of the transport equation [Eq. (1)] and the diffusion equation [Eq. (8)] cannot be done independently; rather, one must begin with the discretized form of the transport equation and then derive the discretized form of the diffusion equation.* Alcouffe obtained the result

$$\text{spr} < c$$

for the discretized linear source-correction version of his iteration method, but more careful analyses^{16,17} have recently shown that for this method,

$$\text{spr} \leq 0.23c$$

for all values of h .

Subsequent to Alcouffe's work, Miller¹² derived an unconditionally stable "third moment" acceleration method, but he showed by numerical experiments that this method is not as effective in reducing the number of iterations as Alcouffe's. More recently, Alcouffe's linear method has been applied by Aull^{14,15}

to the synthetic equations, Eq. (6). (Alcouffe's linear method and Aull's synthetic method have been shown by Aull¹⁴ and McCoy²³ to be algebraically equivalent.) In addition, Morel¹⁸ has recently extended the method to accelerate both the cell-averaged fluxes and currents. [This is crucial for problems with highly anisotropic scattering. The main idea is that Eqs. (4) determine both ϕ_0 and ϕ_1 exactly if ψ is linear in μ .] All of the above work is based on the DD discrete ordinates equations.

Together, Alcouffe's and Reed's work emphasizes the importance of determining a discretization of the diffusion equation that is consistent with the discretization of the transport equation. This consistency requirement hints that a discretization of the diffusion equation, which produces an unconditionally stable acceleration method for a given spatial discretization of the transport equation, may *not* produce an unconditionally stable acceleration method for another discretization of the transport equation. In fact, this has been demonstrated^{17,24}; the spatial differencing of Eqs. (6) appropriate to the DD transport equation, Eq. (1), does not give unconditionally stable acceleration for the weighted diamond (WD) differenced transport equation. In this paper, therefore, we derive unconditionally stable acceleration methods appropriate to several linear non-diamond spatial differencing schemes for the slab geometry transport equation. (We do not consider nonlinear schemes, such as the exponential method,²⁵ because they are not amenable to the analysis as formulated here.) Our motivation is that certain of these schemes have been shown to be significantly more accurate than the diamond scheme²⁶ [and one, the linear discontinuous (LD) scheme, has been implemented in production transport codes^{3,4}]. Thus, for these schemes to achieve their full potential, acceleration methods for them, which are as efficient and reliable as Alcouffe's method for the diamond scheme, should be developed.

Our approach draws substantially from the work of Alcouffe^{10,11} and also somewhat from the analyses of Aull^{14,15} and Morel.¹⁸ From Alcouffe^{10,11} we use the idea that the diffusion differencing scheme must be derived from the transport differencing scheme in such a way that angular fluxes, which are linear in angle, are treated exactly. (We do not, however,

²³DONALD J. DUDZIAK, R. D. O'DELL, and R. E. ALCOUFFE, "Transport and Reactor Theory April 1-June 30, 1981," LA-9023-PR, Los Alamos National Laboratory (1981).

²⁴D. R. MC COY and E. W. LARSEN, *Nucl. Sci. Eng.*, **82**, 64 (1982).

²⁵P. BARBUCCI and F. Di PASQUANTONIO, *Nucl. Sci. Eng.*, **63**, 179 (1977).

²⁶R. E. ALCOUFFE, E. W. LARSEN, W. F. MILLER, JR., and B. R. WIENKE, *Nucl. Sci. Eng.*, **71**, 111 (1979).

²²DONALD J. DUDZIAK, R. D. O'DELL, and R. E. ALCOUFFE, "Transport and Reactor Theory July 1-September 30, 1981," LA-9141-PR, Los Alamos National Laboratory (1982).

follow Alcouffe's algebraic approach, which is oriented toward the DD scheme.) From Aull^{14,15} we use the idea that the numerical acceleration method, as formulated along the lines of Eqs. (6), always has a linear diffusion equation. [Alcouffe's method, which is based on Eq. (8), has a nonlinear diffusion equation if a negative flux fixup is used in the diffusion calculation. This nonlinearity, which is introduced as a correction term to ensure compatibility between the transport and diffusion calculations, can lead to numerical inefficiencies.¹⁶ Conceptually, Aull showed that this correction term can be introduced linearly, rather than nonlinearly.] Finally, from Morel¹⁸ we use the idea that for anisotropic scattering problems, the scalar flux and currents should both be accelerated.

For each transport differencing scheme treated in this paper, our derivation of the appropriate diffusion differencing scheme very closely follows the above derivation of Eqs. (6), but with the starting point the *discretized* form of the transport equation. (Also, we are as consistent as possible with the notation established above. Thus, for example, an " $l + \frac{1}{2}$ " iteration superscript refers to a quantity computed by a transport calculation, while an " l " or " $l + 1$ " iteration superscript refers to a quantity computed by a diffusion calculation.) We consider the one-group, discrete ordinates equations, with anisotropic scattering and the standard Gauss-Legendre angular quadrature sets.

The discretized forms of the transport and diffusion equations are given below in Secs. II through V. However, we display here the bounds on the spectral radii of the iteration schemes that we determined by Fourier analyses for infinite medium, isotropic scattering, constant cross section, and constant mesh spacing problems. All the unaccelerated methods have

$$\text{spr} = c , \quad 0 \leq c \leq 1 , \quad (9)$$

and all the accelerated methods have simple bounds of the form

$$\text{spr} \leq c K_N , \quad 0 \leq c \leq 1 ,$$

where K_N , which depends only on N (the order of the quadrature set), is tabulated in Table I. The result $K_N = 0$ for the S_2 -quadrature set is explained in Sec. II. For all cases, however, we have

$$K_N < \frac{1}{3} .$$

Thus, for these model problems, our acceleration methods are unconditionally stable, and they accelerate effectively for all values of N , h , and $c \leq 1$. This statement applies also to the numerical calculations (presented in a companion paper²⁴), which we performed for more realistic (finite medium, anisotropic scattering, variable cross section, and variable spatial mesh) problems. Taken together, our theoretical and

TABLE I
Values of K_N for Various Values of N and by
Various Differencing Schemes

N	Weighted Diamond	Linear Characteristic	Linear Discontinuous	Linear Moments
2	0	0	0	0
4	0.185	0.236	0.238	0.223
6	0.220	0.281	0.279	0.260
8	0.233	0.296	0.291	0.271
12	0.243	0.308	0.298	0.278
16	0.246	0.312	0.299	0.280
20	0.247	0.314	0.300	0.282
24	0.248	0.315	0.300	0.282

numerical results show that our acceleration methods can be used to reliably and significantly decrease the computational effort for computing nondiamond solutions of many discrete ordinates problems, in particular, problems characterized by large spatial regions and scattering ratios near unity.

An outline of the remainder of this paper follows. In Secs. II and III we treat the case of a general WD scheme. We do this not because these schemes are more accurate than the DD scheme, but because the analysis serves as a convenient model for the more complicated but more accurate schemes treated in Secs. IV and V. In Sec. IV we treat the linear characteristic (LC) scheme,^{26,27} and in Sec. V we treat the LD (Refs. 3 and 26) and linear moments^{28,29} (LM) schemes. (For these algebraically complex schemes, our method is not presented in detail, but the basic procedure very closely follows that in Secs. II and III.) We conclude with a brief discussion in Sec. VI.

II. WD: DISCRETIZED SYNTHETIC EQUATIONS

In slab geometry, unaccelerated WD schemes have the form

$$\begin{aligned} \frac{\mu_m}{h_i} (\psi_{m,i+1/2}^{l+1/2} - \psi_{m,i-1/2}^{l+1/2}) + \sigma_{Ti} \psi_{mi}^{l+1/2} \\ = \sigma_{S0i} \phi_{0i}^l + 3\sigma_{S1i} \mu_m \phi_{1i}^l + S_{mi} , \end{aligned} \quad (10a)$$

$$\psi_{mi}^{l+1/2} = \left(\frac{1 + \alpha_{mi}}{2} \right) \psi_{m,i+1/2}^{l+1/2} + \left(\frac{1 - \alpha_{mi}}{2} \right) \psi_{m,i-1/2}^{l+1/2} , \quad (10b)$$

²⁷D. V. GOPINATH, A. NATARAJAN, and V. SUN-DARARAMAN, *Nucl. Sci. Eng.*, **75**, 181 (1980).

²⁸R. VAIDYANATHAN, *Atomkernenergie*, **29**, 301 (1977).

²⁹R. VAIDYANATHAN, *Nucl. Sci. Eng.*, **71**, 46 (1979).

and

$$\phi_{ni}^{l+1} = \phi_{ni}^{l+1/2}, \quad n = 0, 1. \quad (11)$$

Equation (10a) is an iterative version of the transport balance equation. We assume a spatial mesh with cell edges $x_{i+1/2}$ and allow variable cell widths $h_i = x_{i+1/2} - x_{i-1/2}$. We allow the cross sections σ_T , σ_{S0} , and σ_{S1} to vary from cell to cell, although we require these quantities to be constant within each cell and to satisfy $|\sigma_{S1i}| < \sigma_{S0i} \leq \sigma_{Ti}$. For now we assume at most linearly anisotropic scattering, but later we discuss higher order generalizations. The order of the Gauss-Legendre quadrature set, N , is arbitrary but fixed.

Equation (10b) is the supplementary WD relation. We take the weights α_{mi} to be not greater than one in magnitude, to have the same sign as μ_m , and to be antisymmetric. In other words,

$$|\alpha_{mi}| \leq 1, \quad (12a)$$

$$\mu_m \alpha_{mi} \geq 0, \quad (12b)$$

and

$$\mu_k = -\mu_m \rightarrow \alpha_k = -\alpha_m. \quad (12c)$$

Certain of these constraints can be relaxed, but our goal here is not total generality. Consistent with the above constraints, however, is the usual diamond scheme,

$$\alpha_{mi} = 0,$$

the WD scheme with "fixed" weights,

$$\alpha_{mi} = a_i \frac{\mu_m}{|\mu_m|}, \quad 0 < a_i < 1,$$

the step scheme,

$$\alpha_{mi} = \frac{\mu_m}{|\mu_m|},$$

and the step characteristic scheme,²⁶

$$\alpha_{mi} = \frac{\exp(\epsilon_{mi}) + \exp(-\epsilon_{mi})}{\exp(\epsilon_{mi}) - \exp(-\epsilon_{mi})} - \frac{1}{\epsilon_{mi}},$$

$$\epsilon_{mi} = \frac{\sigma_{Ti} h_i}{2\mu_m}.$$

Equation (11) contains the definition

$$\phi_{ni}^{l+1/2} = \sum_{m=1}^N P_n(\mu_m) \psi_{mi}^{l+1/2} \omega_m \equiv L_n \psi_i^{l+1/2}, \quad (13)$$

where the angular quadrature weights are normalized so that

$$\sum_{m=1}^N \omega_m = 1.$$

Equations (10) are a discretized version of the transport equation (1), and Eq. (11) is a discretized

version of Eq. (2). To derive the discretized acceleration scheme, we shall, for each iteration, replace Eq. (11) by a discretized version of the synthetic equations (6) derived directly from the above discretized transport equation. [However, we insist that Eq. (11) holds as $l \rightarrow \infty$.] Our derivation is patterned on the procedure used in Sec. I to derive the continuous synthetic equations from the continuous transport equations. To carry out this derivation, we first define ρ_i , γ_{mi} , and β_{mi} as

$$\rho_i = L_1 \alpha_i = \sum_{m=1}^N \mu_m \alpha_{mi} \omega_m, \quad (14a)$$

$$\alpha_{mi} = 3\rho_i \mu_m + \gamma_{mi}, \quad (14b)$$

and

$$\mu_m \alpha_{mi} = \rho_i + \beta_{mi}. \quad (14c)$$

We note that $L_0 \gamma_i = L_1 \gamma_i = L_2 \gamma_i = 0$ and $L_0 \beta_i = L_1 \beta_i = 0$, and thus, if ψ_m is a linear function of μ_m ,

$$\psi_m(x) = \tilde{\psi}_0(x) + 3\mu_m \tilde{\psi}_1(x); \quad (15a)$$

then

$$L_0 \gamma_i \psi_i = L_0 \beta_i \psi_i = \phi_{2,i+1/2} = 0. \quad (15b)$$

To begin, we operate on Eqs. (10) by L_0 and L_1 and use the definitions, Eqs. (14), to obtain four equations:

$$\frac{1}{h_i} (\phi_{1,i+1/2}^{l+1/2} - \phi_{1,i-1/2}^{l+1/2}) + \sigma_{Ti} \phi_{0i}^{l+1/2} = \sigma_{S0i} \phi_{0i}^l + L_0 S_i, \quad (16a)$$

$$\begin{aligned} \frac{2}{3h_i} (\phi_{2,i+1/2}^{l+1/2} - \phi_{2,i-1/2}^{l+1/2}) + \frac{1}{3h_i} (\phi_{0,i+1/2}^{l+1/2} - \phi_{0,i-1/2}^{l+1/2}) \\ + \sigma_{Ti} \phi_{1i}^{l+1/2} = \sigma_{S1i} \phi_{1i}^l + L_1 S_i, \end{aligned} \quad (16b)$$

$$\begin{aligned} \phi_{0i}^{l+1/2} = \frac{1}{2} (\phi_{0,i+1/2}^{l+1/2} + \phi_{0,i-1/2}^{l+1/2}) + \frac{3}{2} \rho_i (\phi_{1,i+1/2}^{l+1/2} - \phi_{1,i-1/2}^{l+1/2}) \\ + \frac{1}{2} L_0 \gamma_i (\psi_{i+1/2}^{l+1/2} - \psi_{i-1/2}^{l+1/2}), \end{aligned} \quad (16c)$$

and

$$\begin{aligned} \phi_{1i}^{l+1/2} = \frac{1}{2} (\phi_{1,i+1/2}^{l+1/2} + \phi_{1,i-1/2}^{l+1/2}) + \frac{1}{2} \rho_i (\phi_{0,i+1/2}^{l+1/2} - \phi_{0,i-1/2}^{l+1/2}) \\ + \frac{1}{2} L_0 \beta_i (\psi_{i+1/2}^{l+1/2} - \psi_{i-1/2}^{l+1/2}). \end{aligned} \quad (16d)$$

These equations are automatically satisfied by the solution of Eqs. (10). Upon convergence ($l \rightarrow \infty$), they become

$$\frac{1}{h_i} (\phi_{1,i+1/2} - \phi_{1,i-1/2}) + \sigma_{Ti} \phi_{0i} = \sigma_{S0i} \phi_{0i} + L_0 S_i, \quad \text{etc.,} \quad (17)$$

where the converged quantities are indicated without iteration superscripts, and we have used the condition that Eq. (11) holds as $l \rightarrow \infty$.

We now observe the following: If ψ_m is a linear function of μ_m , then Eqs. (15) are satisfied and the full set of Eq. (17) reduces to four equations for four unknown functions (the cell-edge and cell-averaged fluxes and currents). [In a finite system of I spatial cells, there are $4I$ equations and $4I + 2$ unknowns ($2I$ cell-averaged quantities and $2I + 2$ cell-edge quantities).] Later we impose two boundary conditions and then the number of equations will be equal to the number of unknowns.] The resulting system of equations can then be solved to uniquely determine the cell-edge and cell-averaged fluxes and currents. We emphasize that this holds for any values of the cross sections and spatial mesh.

This suggests that stable acceleration equations are obtained by defining ϕ_{ni}^{l+1} as follows:

$$\frac{1}{h_i}(\phi_{1,i+1/2}^{l+1} - \phi_{1,i-1/2}^{l+1}) + (\sigma_{Ti} - \sigma_{Soi})\phi_{0i}^{l+1} = L_0 S_i , \quad (18a)$$

$$\begin{aligned} \frac{2}{3h_i}(\phi_{2,i+1/2}^{l+1/2} - \phi_{2,i-1/2}^{l+1/2}) + \frac{1}{3h_i}(\phi_{0,i+1/2}^{l+1} - \phi_{0,i-1/2}^{l+1}) \\ + (\sigma_{Ti} - \sigma_{S1i})\phi_{1i}^{l+1} = L_1 S_i , \end{aligned} \quad (18b)$$

$$\begin{aligned} \phi_{0i}^{l+1} = \frac{1}{2}(\phi_{0,i+1/2}^{l+1} + \phi_{0,i-1/2}^{l+1}) + \frac{3}{2}\rho_i(\phi_{1,i+1/2}^{l+1} - \phi_{1,i-1/2}^{l+1}) \\ + \frac{1}{2}L_0\gamma_i(\psi_{i+1/2}^{l+1/2} - \psi_{i-1/2}^{l+1/2}) , \end{aligned} \quad (18c)$$

and

$$\begin{aligned} \phi_{1i}^{l+1} = \frac{1}{2}(\phi_{1,i+1/2}^{l+1} + \phi_{1,i-1/2}^{l+1}) + \frac{1}{2}\rho_i(\phi_{0,i+1/2}^{l+1} - \phi_{0,i-1/2}^{l+1}) \\ + \frac{1}{2}L_0\beta_i(\psi_{i+1/2}^{l+1/2} - \psi_{i-1/2}^{l+1/2}) . \end{aligned} \quad (18d)$$

By the above remarks, if the $l + \frac{1}{2}$ quantities vanish, then Eqs. (18) can be manipulated to determine the $l + 1$ quantities exactly.

At this point, we have two options for manipulating these equations into a more computationally efficient form: We can begin operating with Eqs. (18) directly, or we can subtract Eqs. (16) from (18) and then manipulate the resulting equations. We choose this latter course for reasons discussed below. Upon performing the subtraction and defining

$$f_{n,i+1/2}^{l+1} = \phi_{n,i+1/2}^{l+1} - \phi_{n,i-1/2}^{l+1/2} , \quad n = 0, 1 , \quad (19)$$

we obtain

$$\begin{aligned} \frac{1}{h_i}(f_{1,i+1/2}^{l+1} - f_{1,i-1/2}^{l+1}) + (\sigma_{Ti} - \sigma_{Soi})(\phi_{0i}^{l+1} - \phi_{0i}^{l+1/2}) \\ = \sigma_{Soi}(\phi_{0i}^{l+1/2} - \phi_{0i}^l) , \end{aligned} \quad (20a)$$

$$\begin{aligned} \frac{1}{3h_i}(f_{0,i+1/2}^{l+1} - f_{0,i-1/2}^{l+1}) + (\sigma_{Ti} - \sigma_{S1i})(\phi_{1i}^{l+1} - \phi_{1i}^{l+1/2}) \\ = \sigma_{S1i}(\phi_{1i}^{l+1/2} - \phi_{1i}^l) , \end{aligned} \quad (20b)$$

$$\begin{aligned} \phi_{0i}^{l+1} &= \phi_{0i}^{l+1/2} + \frac{1}{2}(f_{0,i+1/2}^{l+1} + f_{0,i-1/2}^{l+1}) \\ &+ \frac{3}{2}\rho_i(f_{1,i+1/2}^{l+1} - f_{1,i-1/2}^{l+1}) , \end{aligned} \quad (21a)$$

and

$$\begin{aligned} \phi_{1i}^{l+1} &= \phi_{1i}^{l+1/2} + \frac{1}{2}(f_{1,i+1/2}^{l+1} + f_{1,i-1/2}^{l+1}) \\ &+ \frac{1}{2}\rho_i(f_{0,i+1/2}^{l+1} - f_{0,i-1/2}^{l+1}) . \end{aligned} \quad (21b)$$

Our strategy now is to obtain one equation for $f_{0,i+1/2}^{l+1}$ and three equations that express the remaining unknown functions in terms of $f_{0,i+1/2}^{l+1}$.

Using Eqs. (21) to eliminate ϕ_{ni}^{l+1} from Eqs. (20) we obtain, after simplification,

$$\begin{aligned} \left[\frac{1}{h_i} + \frac{3}{2}\rho_i(\sigma_{Ti} - \sigma_{Soi}) \right] (f_{1,i+1/2}^{l+1} - f_{1,i-1/2}^{l+1}) \\ + \frac{1}{2}(\sigma_{Ti} - \sigma_{Soi})(f_{0,i+1/2}^{l+1} + f_{0,i-1/2}^{l+1}) \\ = \sigma_{Soi}(\phi_{0i}^{l+1/2} - \phi_{0i}^l) \end{aligned} \quad (22a)$$

and

$$\begin{aligned} \frac{1}{3}\left[\frac{1}{h_i} + \frac{3}{2}\rho_i(\sigma_{Ti} - \sigma_{S1i})\right](f_{0,i+1/2}^{l+1} - f_{0,i-1/2}^{l+1}) \\ + \frac{1}{2}(\sigma_{Ti} - \sigma_{S1i})(f_{1,i+1/2}^{l+1} + f_{1,i-1/2}^{l+1}) \\ = \sigma_{S1i}(\phi_{1i}^{l+1/2} - \phi_{1i}^l) . \end{aligned} \quad (22b)$$

These are two equations for the two unknown functions f_0^{l+1} and f_1^{l+1} . If we define

$$\hat{\sigma}_{Ri} = \frac{\sigma_{Ti} - \sigma_{Soi}}{1 + \frac{3}{2}\rho_i(\sigma_{Ti} - \sigma_{Soi})h_i} , \quad (23a)$$

$$\hat{\sigma}_{Si} = \frac{\sigma_{Soi}}{1 + \frac{3}{2}\rho_i(\sigma_{Ti} - \sigma_{Soi})h_i} , \quad (23b)$$

$$g_{0i}^{l+1} = \hat{\sigma}_{Si}h_i(\phi_{0i}^{l+1/2} - \phi_{0i}^l) , \quad (23c)$$

$$D_i = \frac{1}{3(\sigma_{Ti} - \sigma_{S1i})} + \frac{1}{2}\rho_i h_i , \quad (23d)$$

$$a_i = \frac{\sigma_{S1i}}{\sigma_{Ti} - \sigma_{S1i}} , \quad (23e)$$

and

$$g_{1i}^{l+1/2} = a_i(\phi_{1i}^{l+1/2} - \phi_{1i}^l) , \quad (23f)$$

then Eqs. (22) can be written more compactly as

$$\begin{aligned} \frac{1}{2}(f_{1,i+1/2}^{l+1} - f_{1,i-1/2}^{l+1}) \\ = -\frac{1}{4}\hat{\sigma}_{Ri}h_i(f_{0,i+1/2}^{l+1} + f_{0,i-1/2}^{l+1}) + \frac{1}{2}g_{0i}^{l+1/2} \end{aligned} \quad (24a)$$

and

$$\frac{1}{2}(f_{1,i+1/2}^{l+1} + f_{1,i-1/2}^{l+1}) = -\frac{D_i}{h_i}(f_{0,i+1/2}^{l+1} - f_{0,i-1/2}^{l+1}) + g_{1i}^{l+1/2}. \quad (24b)$$

Adding and subtracting these two equations give

$$\begin{aligned} f_{1,i+1/2}^{l+1} &= -\frac{D_i}{h_i}(f_{0,i+1/2}^{l+1} - f_{0,i-1/2}^{l+1}) \\ &\quad - \frac{1}{4}\hat{\sigma}_{Ri}h_i(f_{0,i+1/2}^{l+1} + f_{0,i-1/2}^{l+1}) + \frac{1}{2}g_{0i}^{l+1/2} + g_{1i}^{l+1/2} \end{aligned} \quad (25)$$

and

$$\begin{aligned} f_{1,i+1/2}^{l+1} &= -\frac{D_{i+1}}{h_{i+1}}(f_{0,i+3/2}^{l+1} - f_{0,i+1/2}^{l+1}) \\ &\quad + \frac{1}{4}\hat{\sigma}_{R,i+1}h_{i+1}(f_{0,i+3/2}^{l+1} + f_{0,i+1/2}^{l+1}) \\ &\quad - \frac{1}{2}g_{0,i+1}^{l+1/2} + g_{1,i+1}^{l+1/2}, \end{aligned} \quad (26)$$

where we have replaced i by $i + 1$ in this second equation.

Now let us consider a system with I spatial cells and $I + 1$ cell edges. Then Eq. (25) defines $f_{1,i+1/2}^{l+1}$ for $1 \leq i \leq I$, while Eq. (26) defines $f_{1,i+1/2}^{l+1}$ for $0 \leq i \leq I - 1$. For $1 \leq i \leq I - 1$, both equations must hold; eliminating $f_{1,i+1/2}^{l+1}$ between these equations and rearranging, we obtain the consistency condition:

$$\begin{aligned} &-\frac{D_{i+1}}{h_{i+1}}(f_{0,i+3/2}^{l+1} - f_{0,i+1/2}^{l+1}) + \frac{D_i}{h_i}(f_{0,i+1/2}^{l+1} - f_{0,i-1/2}^{l+1}) \\ &\quad + \frac{1}{4}[\hat{\sigma}_{R,i+1}h_{i+1}(f_{0,i+3/2}^{l+1} + f_{0,i+1/2}^{l+1}) \\ &\quad + \hat{\sigma}_{Ri}h_i(f_{0,i+1/2}^{l+1} + f_{0,i-1/2}^{l+1})] \\ &= \frac{1}{2}(g_{0,i+1}^{l+1/2} + g_{0i}^{l+1/2}) - (g_{1,i+1}^{l+1/2} - g_{1i}^{l+1/2}). \end{aligned} \quad (27)$$

This is the desired discretization of the diffusion equation Eq. (6a). Using Eqs. (24) to eliminate f_1 from Eqs. (21), we obtain the following discretization of Eq. (6b):

$$\begin{aligned} \phi_{0i}^{l+1} &= \phi_{0i}^{l+1/2} + \left(\frac{1}{2} - \frac{3}{4}\rho_i\hat{\sigma}_{Ri}h_i\right)(f_{0,i+1/2}^{l+1} + f_{0,i-1/2}^{l+1}) \\ &\quad + \frac{3}{2}\rho_i g_{0i}^{l+1/2} \end{aligned} \quad (28a)$$

and

$$\phi_{1i}^{l+1} = \phi_{1i}^{l+1/2} + \left(\frac{1}{2}\rho_i - \frac{D_i}{h_i}\right)(f_{0,i+1/2}^{l+1} - f_{0,i-1/2}^{l+1}) + g_{1i}^{l+1/2}. \quad (28b)$$

To summarize, we solve Eqs. (10) for $\psi_{mi}^{l+1/2}$, then Eq. (27) for $f_{0,i+1/2}^{l+1}$, and finally, Eqs. (28) for ϕ_{ni}^{l+1} .

The various coefficients and sources in these equations are defined by Eqs. (23). [Actually, Eq. (27) holds for $I - 1$ values of i , and so two boundary conditions must be imposed to bring the number of equations equal to the number of unknowns ($I + 1$). These boundary conditions are discussed in Sec. III.]

The procedure described above gives one full iteration in our acceleration method for WD differencing schemes. This "recommended" method, Eqs. (10), (27), and (28), has the following computational advantages over the "original" method, based on Eqs. (10) and (18).

1. Storage in less. For the recommended method, five flux moment arrays ($f_{0,i+1/2}^{l+1}$; and $g_{ni}^{l+1/2}$, $\phi_{ni}^{l+1/2}$ for $n = 0, 1$) are needed, while for the original method, nine such arrays ($L_0\gamma_i\psi_{i+1/2}^{l+1/2}$, $L_0\beta_i\psi_{i+1/2}^{l+1/2}$, $\phi_{2,i+1/2}^{l+1/2}$; and ϕ_{ni}^l , ϕ_{ni}^l for $n = 0, 1$) are generally needed.

2. Upon convergence, the recommended diffusion Eq. (27) reduces to an identity, $0 = 0$, whereas the analogous original Eqs. (18a) and (18b) reduce to equations that are consistent with the converged transport Eq. (10) only if these transport equations hold in every cell. If we use negative flux fixups in the transport calculation, then Eq. (10b) will not hold in cells where fixups are used, and for such cells, Eqs. (18) must be modified to ensure that the transport and diffusion fluxes agree upon convergence. However, the recommended equation, Eq. (27), does not need to be modified. We note that Aull has pointed out this same advantage for his discretized synthetic method.^{14,15}

Up to now, we have assumed that the scattering order is at most linear. If higher order anisotropic scattering occurs, then the higher order flux moments, ϕ_{ni}^l for $n \geq 2$, should be included in the proper manner in S_{mi} in Eq. (10). The transport and diffusion calculations then proceed as described above. Because the diffusion calculation accelerates only ϕ_0 and ϕ_1 , it seems reasonable to define

$$\phi_{ni}^{l+1} = \phi_{ni}^{l+1/2}, \quad n \geq 2.$$

This procedure, which treats the higher order moments for the next transport calculation as unaccelerated, has the advantage of being linear, and it should perform satisfactorily for problems in which the higher order scattering terms are not too important (that is, for σ_{Sni} small compared to σ_{Ti} for $n \geq 2$). Morel,¹⁸ however, has studied this problem for DD, and he recommends a different (nonlinear) approach that we describe in Ref. 24.

This concludes our derivation of the discretized synthetic equations appropriate to WD differencing schemes. Four things remain to be done:

1. Prove that this acceleration method is stable, at least for a class of model problems.

2. Choose an appropriate way to start the iteration process.
3. Select boundary conditions for finite medium problems.
4. Test the acceleration method on realistic problems.

In Sec. III we first consider item 1; the class of model problems will have infinite media, isotropic scattering, constant cross section, and constant mesh spacing. Then we address items 2 and 3. In any case, the resolution of items 2 and 3 has no effect on the results of item 1 because the model problems have no boundaries and the acceleration method is linear, and so convergence rates and stability for these problems cannot depend on the choice of the initial iterate or boundary conditions. We consider item 4 in Ref. 24.

III. WD: STABILITY, STARTING ITERATE, AND BOUNDARY CONDITIONS

We first consider the stability and convergence properties of the acceleration method derived in Sec. II. For simplicity, we assume isotropic scattering ($\sigma_{S1i} = 0$). Equations (10) describe the transport calculation, and Eqs. (27) and (28a) describe the diffusion calculation. Conceptually, then, an iteration starts with ϕ_{0i}^l and ends with ϕ_{0i}^{l+1} . Since the iterative process is linear, we can write

$$\phi_{0i}^{l+1} = L\phi_{0i}^l + MS_i ,$$

and then

$$(\phi_{0i}^{l+1} - \phi_{0i}^l) = L(\phi_{0i}^l - \phi_{0i}^{l-1}) .$$

Thus, the iteration method converges if the spectral radius (largest eigenvalue) of L is < 1 , and it does not converge if the spectral radius is greater than or equal to one.

The eigenvalues of L can be computed for infinite medium, constant cross section, and uniform spatial mesh problems by the following Fourier procedure. In Eqs. (10), (27), and (28a) we set

$$S_{mi} = 0 , \quad \sigma_{S1i} = 0 ,$$

$$\phi_{0i}^l = \exp(j\lambda x_i) \quad (j = \sqrt{-1}, -\infty < \lambda < \infty) ,$$

$$\psi_{m,i+1/2}^{l+1/2} = t_m \exp(j\lambda x_{i+1/2}) ,$$

$$\psi_{mi}^{l+1/2} = u_m \exp(j\lambda x_i) ,$$

$$f_{0,i+1/2}^{l+1/2} = v \exp(j\lambda x_{i+1/2}) ,$$

and

$$\phi_{0i}^{l+1} = \omega \exp(j\lambda x_i) .$$

Then ω is the eigenvalue corresponding to the eigenfunction $\exp(j\lambda x_i)$. With no loss of generality, we can take distance to be measured in units of mean-free-

paths ($\sigma_T = 1$) and $\sigma_{S0} = c$. Then ω is a function of λ, c, h, N (the order of the angular quadrature set), and the weights α_m . The algebra leading to the expression for ω is straightforward, lengthy, and uninteresting, and so we just state the result, obtained after a considerable amount of manipulation:

$$\omega = c \left[\frac{\eta \sin^2 \xi}{\eta \sin^2 \xi + (1 - c) \cos^2 \xi} \right] \times \left[\frac{(2\theta_1 + 3h\theta_2)/(2 + 3h\rho) + \theta_3}{1 + \frac{3}{2} \rho h (1 - c)} \right] . \quad (29)$$

Here, we have introduced

$$\eta = \frac{4}{3} \left[\frac{1}{h} + \frac{3}{2} \rho (1 - c) \right] \left(\frac{1}{h} + \frac{3}{2} \rho \right) ,$$

$$\theta_1 = \sum_{m=1}^N (1 - 3\mu_m^2) \frac{\cos^2 \xi}{\cos^2 \xi + \left(\alpha_m + \frac{2\mu_m}{h} \right)^2 \sin^2 \xi} \omega_m ,$$

$$\theta_2 = \sum_{m=1}^N (\rho - \mu_m \alpha_m) \frac{\cos^2 \xi}{\cos^2 \xi + \left(\alpha_m + \frac{2\mu_m}{h} \right)^2 \sin^2 \xi} \omega_m ,$$

$$\theta_3 = \sum_{m=1}^N (\alpha_m - 3\rho\mu_m) \frac{\alpha_m + 2\mu_m/h}{\cos^2 \xi + \left(\alpha_m + \frac{2\mu_m}{h} \right)^2 \sin^2 \xi} \omega_m ,$$

and

$$\xi = \frac{\lambda h}{2} .$$

Since $0 \leq c \leq 1$ and $\eta > 0$, we can simplify Eq. (29) into the useful bound

$$|\omega| \leq c \left| \frac{2\theta_1 + 3h\theta_2}{2 + 3h\rho} + \theta_3 \right| ,$$

for which equality holds if $c = 1$. The above bound depends on h (for $0 < h < \infty$), ξ [for $0 \leq \xi \leq \pi/2$, since the bound is a periodic function of ξ , which takes all its values on $(0, \pi/2)$], N , and the set $\{\alpha_m\}$. Thus, we have

$$\text{spr} \leq c K_N ,$$

where

$$K_N = \sup \left| \frac{2\theta_1 + 3h\theta_2}{2 + 3h\rho} + \theta_3 \right| ,$$

the sup being taken over $0 < h < \infty$, $0 \leq \xi \leq \pi/2$, and all admissible choices of the weights $\{\alpha_m\}$.

At this point, we turn to computing K_N numerically. The results of a computer search are given in Table I and have been discussed in Sec. I. We find experimentally that K_N is maximized for $h = \infty$, $\xi = \pi/2$, and $|\alpha_m| = 1$. Introducing these values, we obtain explicitly

$$K_N = 1 - 3\rho^2$$

and

$$\rho = \sum_{m=1}^N \mu_m \alpha_m \omega_m = \sum_{m=1}^N |\mu_m| \omega_m .$$

Thus,

$$\lim_{N \rightarrow \infty} \rho = \frac{1}{2} \int_{-1}^1 |\mu| d\mu = \frac{1}{2} ,$$

which gives the interesting result

$$\lim_{N \rightarrow \infty} K_N = \frac{1}{4} .$$

At this point, we comment on the result $K_N = 0$ for the S_2 quadrature set. This seemingly anomalous result occurs because the acceleration method is designed to give the exact infinite medium transport flux in one iteration if ψ_m is a linear function of μ_m ; in the S_2 quadrature set, there are only two values of μ_m , so such a linear representation trivially holds. Our discretization thus is faithful to the well-known fact that in the spatially continuous form and the S_2 discrete ordinates approximation, transport theory is equivalent to diffusion theory. For a finite system, the numerical boundary conditions must be precisely chosen so that the exact S_2 transport results are obtained in one iteration; we derive these boundary conditions later in this section. Finally, we remark that the Fourier analysis described above, when applied to the unaccelerated iteration method, yields the result, Eq. (9).

This concludes our discussion of the stability of the acceleration method derived in Sec. II. We now address the question of how to begin the iteration process. The principle that guides us here, and in our determination of boundary conditions, is the following: If the exact numerical transport solution ψ_m is a linear function of μ_m , then the numerical process should generate this solution in the first iteration.

We have already remarked that if ψ_m is a linear function of μ_m , then Eqs. (18) exactly determine ϕ_{0i} and ϕ_{1i} . Thus, we propose to begin the iteration scheme with Eqs. (18), with $l = 0$ and $\psi^{1/2} = 0$. These equations (modulo boundary conditions, discussed below) satisfy the criteria stated above, and using the definitions, Eqs. (23), can be rewritten in the following simpler form:

$$\begin{aligned} & -\frac{D_{i+1}}{h_{i+1}} (\phi_{0,i+3/2}^1 - \phi_{0,i+1/2}^1) + \frac{D_i}{h_i} (\phi_{0,i+1/2}^1 - \phi_{0,i-1/2}^1) \\ & + \frac{1}{4} [\hat{\sigma}_{R,i+1} h_{i+1} (\phi_{0,i+3/2}^1 + \phi_{0,i+1/2}^1)] \\ & = (\mathcal{Q}_{0,i+1} + \mathcal{Q}_{0i}) - (\mathcal{Q}_{1,i+1} - \mathcal{Q}_{1i}) , \end{aligned} \quad (30)$$

$$\phi_{0i}^1 = \left(\frac{1}{2} - \frac{3}{4} \rho_i \hat{\sigma}_{Ri} h_i \right) (\phi_{0,i+1/2}^1 + \phi_{0,i-1/2}^1) + 3\rho_i \mathcal{Q}_{0i} , \quad (31a)$$

and

$$\phi_{1i}^1 = \left(\frac{1}{2} \rho_i - \frac{D_i}{h_i} \right) (\phi_{0,i+1/2}^1 - \phi_{0,i-1/2}^1) + \mathcal{Q}_{1i} . \quad (31b)$$

Here we have defined

$$\mathcal{Q}_{0i} = \frac{h_i (L_0 S_i)}{2 + 3\rho_i (\sigma_{Ti} - \sigma_{S0i}) h_i} \quad (32a)$$

and

$$\mathcal{Q}_{1i} = \frac{L_1 S_i}{\sigma_{Ti} - \sigma_{S1i}} . \quad (32b)$$

Boundary conditions for Eq. (30) are obtained by the following procedures. We first treat the left boundary, at $x = x_{1/2}$. If there is a prescribed incident flux on this boundary, $\psi_{m,1/2}$ for $\mu_m > 0$, which is a linear function of μ_m , then we consider

$$\psi_{m,1/2} = \phi_{0,1/2}^1 + 3\mu_m \phi_{1,1/2}^1 . \quad (33)$$

Multiplying by $\mu_m \omega_m$ and summing over $\mu_m > 0$, we obtain the condition

$$\sum_{\mu_m > 0} \mu_m \psi_{m,1/2} \omega_m = \left(\sum_{\mu_m > 0} \mu_m \omega_m \right) \phi_{0,1/2}^1 + \frac{1}{2} \phi_{1,1/2}^1 . \quad (34a)$$

Similarly, if there is a prescribed incident flux on the right boundary, $\psi_{m,I+1/2}$ for $\mu_m < 0$, we obtain the condition

$$\begin{aligned} & \sum_{\mu_m < 0} \mu_m \psi_{m,I+1/2} \omega_m \\ & = \left(\sum_{\mu_m < 0} \mu_m \omega_m \right) \phi_{0,I+1/2}^1 + \frac{1}{2} \phi_{1,I+1/2}^1 . \end{aligned} \quad (34b)$$

If the left (or right) boundary is reflecting, then by Eq. (33) we should set

$$\phi_{1,1/2}^1 = 0 \quad (\text{or } \phi_{1,I+1/2}^1 = 0) . \quad (35)$$

If the boundary conditions (at $x_{1/2}$ and $x_{I+1/2}$) are periodic, then

$$\phi_{n,1/2}^1 = \phi_{n,I+1/2}^1 , \quad n = 0 \text{ and } 1 . \quad (36)$$

The boundary conditions, Eqs. (34), (35), and (36), become explicit conditions on ϕ_0^1 by using the formulas

$$\begin{aligned} \phi_{1,1/2}^1 & = -\frac{D_1}{h_1} (\phi_{0,3/2}^1 - \phi_{0,1/2}^1) \\ & + \frac{1}{4} \hat{\sigma}_{R1} h_1 (\phi_{0,3/2}^1 + \phi_{0,1/2}^1) - \mathcal{Q}_{01} + \mathcal{Q}_{11} \end{aligned} \quad (37a)$$

and

$$\begin{aligned} \phi_{1,I+1/2}^1 & = -\frac{D_I}{h_I} (\phi_{0,I+1/2}^1 - \phi_{0,I-1/2}^1) \\ & - \frac{1}{4} \hat{\sigma}_{RI} h_I (\phi_{0,I+1/2}^1 + \phi_{0,I-1/2}^1) + \mathcal{Q}_{0I} + \mathcal{Q}_{1I} \end{aligned} \quad (37b)$$

It is easily shown that for the S_2 equations, these boundary conditions [Eqs. (34), (35), or (36)] produce the exact transport fluxes in one (diffusion) calculation.

Finally, we turn to the question of boundary conditions for Eq. (27). Upon convergence, $f_{0,i+1/2}^{l+1} \rightarrow 0$, and so any homogeneous boundary conditions (i.e., ones that admit the zero solution) are permissible. However, boundary conditions for f_0^{l+1} , which preserve certain features of the transport iterates, seem to be desirable. For example, if we define the diffusion edge angular flux Ψ^{l+1} as

$$\Psi_{m,i+1/2}^{l+1} = \phi_{0,i+1/2}^{l+1} + 3\mu_m \phi_{1,i+1/2}^{l+1},$$

then this quantity can be made to equal a prescribed incident current, to satisfy a reflecting boundary condition, or to satisfy a periodic boundary condition. If we require the diffusion boundary angular fluxes to satisfy the "same" boundary conditions as the transport flux, then we obtain the following conditions. For a prescribed incident flux at $x_{1/2}$,

$$0 = \left(\sum_{\mu_m > 0} \mu_m \omega_m \right) f_{0,1/2}^{l+1} + \frac{1}{2} f_{1,1/2}^{l+1}, \quad (38a)$$

and for a prescribed incident flux at $x_{I+1/2}$,

$$0 = \left(\sum_{\mu_m < 0} \mu_m \omega_m \right) f_{0,I+1/2}^{l+1} + \frac{1}{2} f_{1,I+1/2}^{l+1}. \quad (38b)$$

For a reflecting boundary at $x_{1/2}$,

$$0 = f_{1,1/2}^{l+1}, \quad (39a)$$

and at $x_{I+1/2}$,

$$0 = f_{1,I+1/2}^{l+1}. \quad (39b)$$

For periodic boundaries,

$$f_{n,1/2}^{l+1} = f_{n,I+1/2}^{l+1}, \quad n = 0, 1. \quad (40)$$

The above boundary conditions become explicit conditions on f_0^{l+1} by using Eqs. (26) and (25) to eliminate $f_{1,1/2}^{l+1}$ and $f_{1,I+1/2}^{l+1}$.

IV. LINEAR CHARACTERISTIC

The LC method has been derived elsewhere,^{26,27} but because we want to slightly reformulate the method, we rederive it here.

Let us consider the transport equation, for the i 'th cell, with a linear representation for the scattering source:

$$\mu_m \frac{d\psi_m^{l+1/2}}{dx} + \sigma_T \psi_m^{l+1/2} = \sigma_S \left[\phi_i^l + \xi_i^l \frac{2}{h} (x - x_i) \right]. \quad (41)$$

Here, x_i is the center of the cell, ϕ_i^l is the (l 'th) cell-averaged scalar flux, and ξ_i^l is the (l 'th) scalar flux "slope." In each cell calculation, we compute the

existing flux and a linear representation,

$$\psi_m^{l+1}(x) \approx \psi_{mi}^{l+1/2} + \xi_{mi}^{l+1/2} \frac{2}{h} (x - x_i), \quad (42)$$

which is used to construct the linear scattering source for the next iteration. In the LC method, we compute the exiting flux from the analytic solution of Eq. (41):

$$\begin{aligned} \exp(\epsilon_m) \psi_{m,i+1/2}^{l+1/2} - \exp(-\epsilon_m) \psi_{m,i-1/2}^{l+1/2} \\ = \frac{\sigma_S}{\sigma_T} \left(\phi_i^l [\exp(\epsilon_m) - \exp(-\epsilon_m)] \right. \\ \left. + \xi_i^l \left\{ \exp(\epsilon_m) + \exp(-\epsilon_m) \right. \right. \\ \left. \left. - \frac{1}{\epsilon_m} [\exp(\epsilon_m) - \exp(-\epsilon_m)] \right\} \right), \end{aligned} \quad (43a)$$

$$\epsilon_m = \frac{\sigma_T h}{2\mu_m}. \quad (43b)$$

We also compute the cell-averaged flux from the analytic solution (or, equivalently, from the balance equation) as

$$\frac{\mu_m}{h_i} (\psi_{m,i+1/2}^{l+1/2} - \psi_{m,i-1/2}^{l+1/2}) + \sigma_T \psi_{mi}^{l+1/2} = \sigma_S \phi_i^l, \quad (44)$$

while we define the flux slope as

$$\xi_{mi}^{l+1/2} = \frac{1}{2} (\psi_{m,i+1/2}^{l+1/2} - \psi_{m,i-1/2}^{l+1/2}). \quad (45)$$

[Alternatively, we could define

$$\xi_{mi}^{l+1/2} = \frac{6}{h^2} \int_{x_{i-1/2}}^{x_{i+1/2}} (x - x_i) \psi_m^{l+1/2}(x) dx, \quad (46)$$

and then the representation Eq. (42) would exactly preserve both the zero'th and first spatial moments of $\psi_m^{l+1/2}(x)$. This is the LM method,^{28,29} which we treat in Sec. V. In the LC method, we approximate the first spatial moment by Eq. (45).]

Eliminating ϕ_i^l between Eqs. (43) and (44) yields

$$\begin{aligned} \psi_{mi}^{l+1/2} = \left(\frac{1 + \alpha_m}{2} \right) \psi_{m,i+1/2}^{l+1/2} \\ + \left(\frac{1 - \alpha_m}{2} \right) \psi_{m,i-1/2}^{l+1/2} - \alpha_m \frac{\sigma_S}{\sigma_T} \xi_i^l \end{aligned} \quad (47a)$$

and

$$\alpha_m = \frac{\exp(\epsilon_m) + \exp(-\epsilon_m)}{\exp(\epsilon_m) - \exp(-\epsilon_m)} - \frac{1}{\epsilon_m}. \quad (47b)$$

Equations (44), (45), and (47) are three equations for the three unknowns: the cell-averaged flux, the flux slope, and the exiting flux. These equations are solved, and in the unaccelerated LC method we take

$$\phi_i^{l+1} = L_0 \psi_i^{l+1/2} \equiv \phi_i^{l+1/2}$$

and

$$\xi_i^{l+1} = L_0 \xi_i^{l+1/2} \equiv \xi_i^{l+1/2}$$

in order to proceed to the next iteration.

For the more general case of linearly anisotropic scattering, with variable cross sections and a linear inhomogeneous source $S_{mi} + T_{mi}(2/h_i)(x - x_i)$, the LC method in its unaccelerated form can be written as

$$\begin{aligned} \frac{\mu_m}{h_i} (\psi_{m,i+1/2}^{l+1/2} - \psi_{m,i-1/2}^{l+1/2}) + \sigma_{Ti} \psi_{mi}^{l+1/2} \\ = \sigma_{Soi} \phi_{0i}^l + 3\sigma_{S1i} \mu_m \phi_{1i}^l + S_{mi} , \end{aligned} \quad (48a)$$

$$\begin{aligned} \psi_{mi}^{l+1/2} = \left(\frac{1 + \alpha_{mi}}{2} \right) \psi_{m,i+1/2}^{l+1/2} + \left(\frac{1 - \alpha_{mi}}{2} \right) \psi_{m,i-1/2}^{l+1/2} \\ - \frac{\alpha_{mi}}{\sigma_{Ti}} (\sigma_{Soi} \xi_{0i}^l + 3\sigma_{S1i} \mu_m \xi_{1i}^l + T_{mi}) , \end{aligned} \quad (48b)$$

$$\xi_{mi}^{l+1/2} = \frac{1}{2} (\psi_{m,i+1/2}^{l+1/2} - \psi_{m,i-1/2}^{l+1/2}) , \quad (48c)$$

$$\phi_{ni}^{l+1} = L_n \psi_i^{l+1/2} \equiv \phi_{ni}^{l+1/2} , \quad \left. \right\} \quad n = 0, 1 , \quad (49a)$$

$$\xi_{ni}^{l+1} = L_n \xi_i^{l+1/2} \equiv \xi_{ni}^{l+1/2} , \quad \left. \right\} \quad n = 0, 1 , \quad (49b)$$

where

$$\alpha_{mi} = \frac{\exp(\epsilon_{mi}) + \exp(-\epsilon_{mi})}{\exp(\epsilon_{mi}) - \exp(-\epsilon_{mi})} - \frac{1}{\epsilon_{mi}} \quad (50a)$$

$$\epsilon_{mi} = \frac{\sigma_{Ti} h_i}{2 \mu_m} . \quad (50b)$$

As usual, to accelerate this method, we retain Eqs. (48) for the transport calculation, but we replace Eqs. (49) by a diffusion calculation derived below. [However, we insist that Eqs. (49) hold upon convergence.] We make full use of the fact that Eq. (48b) is the WD relation, Eq. (10b), with an extra term, and we also use the definitions, Eqs. (14).

To begin, we operate on Eqs. (48) by L_0 and L_1 to obtain six moment equations. For example, L_0 acting on Eq. (48b) gives

$$\begin{aligned} \phi_{0i}^{l+1/2} = \frac{1}{2} (\phi_{0,i+1/2}^{l+1/2} + \phi_{0,i-1/2}^{l+1/2}) + \frac{3}{2} \rho_i (\phi_{1,i+1/2}^{l+1/2} - \phi_{1,i-1/2}^{l+1/2}) \\ + \frac{1}{2} L_0 \gamma_i (\psi_{i+1/2}^{l+1/2} - \psi_{i-1/2}^{l+1/2}) \\ - \frac{1}{\sigma_{Ti}} (3\sigma_{S1i} \rho_i \xi_{1i}^l + L_0 \alpha_i T_i) . \end{aligned} \quad (51)$$

Next, we follow the procedure outlined in Sec. II and define acceleration equations by replacing the superscripts in $\phi_{n,i+1/2}^{l+1/2}$, $\phi_{ni}^{l+1/2}$, ϕ_{ni}^l , $\xi_{ni}^{l+1/2}$, and ξ_{ni}^l with $l+1$, for $n = 0$ and 1, in the aforementioned six equations. For example, Eq. (51) becomes

$$\begin{aligned} \phi_{0i}^{l+1} = \frac{1}{2} (\phi_{0,i+1/2}^{l+1} + \phi_{0,i-1/2}^{l+1}) + \frac{3}{2} \rho_i (\phi_{1,i+1/2}^{l+1} - \phi_{1,i-1/2}^{l+1}) \\ + \frac{1}{2} L_0 \gamma_i (\psi_{i+1/2}^{l+1/2} - \psi_{i-1/2}^{l+1/2}) \\ - \frac{1}{\sigma_{Ti}} (3\sigma_{S1i} \rho_i \xi_{1i}^{l+1} + L_0 \alpha_i T_i) . \end{aligned} \quad (52)$$

Now we subtract the full set of Eqs. (51) from the set (52) and use the definitions (19) to obtain

$$\begin{aligned} \frac{1}{h_i} (f_{1,i+1/2}^{l+1} - f_{1,i-1/2}^{l+1}) + (\sigma_{Ti} - \sigma_{Soi}) (\phi_{0i}^{l+1} - \phi_{0i}^{l+1/2}) \\ = \sigma_{Soi} (\phi_{0i}^{l+1/2} - \phi_{0i}^l) , \end{aligned} \quad (53a)$$

$$\begin{aligned} \frac{1}{3h_i} (f_{0,i+1/2}^{l+1} - f_{0,i-1/2}^{l+1}) + (\sigma_{Ti} - \sigma_{S1i}) (\phi_{1i}^{l+1} - \phi_{1i}^{l+1/2}) \\ = \sigma_{S1i} (\phi_{1i}^{l+1/2} - \phi_{1i}^l) , \end{aligned} \quad (53b)$$

$$\begin{aligned} \phi_{0i}^{l+1} = \phi_{0i}^{l+1/2} + \frac{1}{2} (f_{0,i+1/2}^{l+1} + f_{0,i-1/2}^{l+1}) \\ + \frac{3}{2} \rho_i (f_{1,i+1/2}^{l+1} - f_{1,i-1/2}^{l+1}) - 3\rho_i \frac{\sigma_{S1i}}{\sigma_{Ti}} (\xi_{1i}^{l+1} - \xi_{1i}^l) , \end{aligned} \quad (54a)$$

$$\begin{aligned} \phi_{1i}^{l+1} = \phi_{1i}^{l+1/2} + \frac{1}{2} (f_{1,i+1/2}^{l+1} + f_{1,i-1/2}^{l+1}) \\ + \frac{1}{2} \rho_i (f_{0,i+1/2}^{l+1} - f_{0,i-1/2}^{l+1}) - \rho_i \frac{\sigma_{Soi}}{\sigma_{Ti}} (\xi_{0i}^{l+1} - \xi_{0i}^l) , \end{aligned} \quad (54b)$$

$$\xi_{0i}^{l+1} = \xi_{0i}^{l+1/2} + \frac{1}{2} (f_{0,i+1/2}^{l+1} - f_{0,i-1/2}^{l+1}) , \quad (55a)$$

$$\xi_{1i}^{l+1} = \xi_{1i}^{l+1/2} + \frac{1}{2} (f_{1,i+1/2}^{l+1} - f_{1,i-1/2}^{l+1}) . \quad (55b)$$

This gives six equations for the six unknown functions, f_n^{l+1} , ϕ_n^{l+1} , and ξ_n^{l+1} for $n = 0$ and 1. These equations are solved as follows. We introduce Eqs. (55) into Eqs. (54) to eliminate ξ_{ni}^{l+1} , and then we introduce the resulting equations into Eqs. (53) to eliminate ϕ_{ni}^{l+1} . This gives two equations for $f_{n,i+1/2}^{l+1}$, which can be manipulated exactly as in Sec. II. Since this entire procedure straightforwardly follows that in Sec. II from Eqs. (21) onward, we simply state the final results.

The equations for ϕ_{ni}^{l+1} and ξ_{ni}^{l+1} are

$$\begin{aligned} -\frac{D_{i+1}}{h_{i+1}} (f_{0,i+3/2}^{l+1} - f_{0,i+1/2}^{l+1}) + \frac{D_i}{h_i} (f_{0,i+1/2}^{l+1} - f_{0,i-1/2}^{l+1}) \\ + \frac{1}{4} [\hat{\sigma}_{R,i+1} h_{i+1} (f_{0,i+3/2}^{l+1} + f_{0,i+1/2}^{l+1}) \\ + \hat{\sigma}_{Ri} h_i (f_{0,i+1/2}^{l+1} + f_{0,i-1/2}^{l+1})] \\ = \frac{1}{2} (g_{0,i+1}^{l+1/2} + g_{0i}^{l+1/2}) - (g_{1,i+1}^{l+1/2} - g_{1i}^{l+1/2}) , \end{aligned} \quad (56)$$

$$\begin{aligned}\phi_{0i}^{l+1} &= \phi_{0i}^{l+1/2} + \left[\frac{1}{2} - \frac{3}{4} \rho_i (\hat{\sigma}_{Ri} h_i) \left(1 - \frac{\sigma_{S1i}}{\sigma_{Ti}} \right) \right] \\ &\quad \times (f_{0,i+1/2}^{l+1} + f_{0,i-1/2}^{l+1}) \\ &\quad + 3\rho_i \left[\frac{1}{2} \left(1 - \frac{\sigma_{S1i}}{\sigma_{Ti}} \right) g_{0i}^{l+1/2} - \frac{\sigma_{S1i}}{\sigma_{Ti}} (\xi_{1i}^{l+1/2} - \xi_{0i}^l) \right],\end{aligned}\quad (57a)$$

$$\begin{aligned}\phi_{1i}^{l+1} &= \phi_{1i}^{l+1/2} + \left[\frac{1}{2} \rho_i \left(1 - \frac{\sigma_{S0i}}{\sigma_{Ti}} \right) - \frac{D_i}{h_i} \right] \\ &\quad \times (f_{0,i+1/2}^{l+1} - f_{0,i-1/2}^{l+1}) + g_{1i}^{l+1/2} \\ &\quad - \rho_i \frac{\sigma_{S0i}}{\sigma_{Ti}} (\xi_{0i}^{l+1/2} - \xi_{0i}^l),\end{aligned}\quad (57b)$$

$$\xi_{0i}^{l+1} = \xi_{0i}^{l+1/2} + \frac{1}{2} (f_{0,i+1/2}^{l+1} - f_{0,i-1/2}^{l+1}), \quad (58a)$$

$$\xi_{1i}^{l+1} = \xi_{1i}^{l+1/2} - \frac{1}{4} \hat{\sigma}_{Ri} h_i (f_{0,i+1/2}^{l+1} + f_{0,i-1/2}^{l+1}) + \frac{1}{2} g_{0i}^{l+1/2}. \quad (58b)$$

The various coefficients in these equations are defined by

$$\rho_i = \sum_{m=1}^N \mu_m \alpha_{mi} \omega_m, \quad (59a)$$

$$D_i = \frac{1}{3(\sigma_{Ti} - \sigma_{S1i})} + \frac{1}{2} \rho_i h_i \left(1 - \frac{\sigma_{S0i}}{\sigma_{Ti}} \right), \quad (59b)$$

$$\hat{\sigma}_{Ri} = \frac{\sigma_{Ti}(\sigma_{Ti} - \sigma_{S0i})}{\sigma_{Ti} + \frac{3}{2} \rho_i h_i (\sigma_{Ti} - \sigma_{S0i})(\sigma_{Ti} - \sigma_{S1i})}, \quad (59c)$$

and the sources are defined by

$$g_{0i}^{l+1/2} = \hat{\sigma}_{Si} h_i (\phi_{0i}^{l+1/2} - \phi_{0i}^l) + \tau_{0i} (\xi_{1i}^{l+1/2} - \xi_{1i}^l) \quad (60a)$$

and

$$g_{1i}^{l+1/2} = a_i (\phi_{1i}^{l+1/2} - \phi_{1i}^l) + \tau_{1i} (\xi_{0i}^{l+1/2} - \xi_{0i}^l), \quad (60b)$$

where

$$\hat{\sigma}_{Si} = \frac{\sigma_{Ti} \sigma_{S0i}}{\sigma_{Ti} + \frac{3}{2} \rho_i h_i (\sigma_{Ti} - \sigma_{S0i})(\sigma_{Ti} - \sigma_{S1i})} \quad (61a)$$

$$\tau_{0i} = \frac{3\sigma_{S1i} h_i \rho_i (\sigma_{Ti} - \sigma_{S0i})}{\sigma_{Ti} + \frac{3}{2} \rho_i h_i (\sigma_{Ti} - \sigma_{S0i})(\sigma_{Ti} - \sigma_{S1i})} \quad (61b)$$

$$a_i = \frac{\sigma_{S1i}}{\sigma_{Ti} - \sigma_{S1i}} \quad (61c)$$

$$\tau_{1i} = \frac{\sigma_{S0i} \rho_i}{\sigma_{Ti}}. \quad (61d)$$

The boundary conditions for Eq. (56) are given by Eqs. (38), (39), and (40), with

$$\begin{aligned}f_{1,1/2}^{l+1} &= -\frac{D_1}{h_1} (f_{0,3/2}^{l+1} - f_{0,1/2}^{l+1}) \\ &\quad + \frac{1}{4} \hat{\sigma}_{R1} h_1 (f_{0,3/2}^{l+1} + f_{0,1/2}^{l+1}) + g_{11}^{l+1/2} - \frac{1}{2} g_{01}^{l+1/2}\end{aligned}\quad (62a)$$

and

$$\begin{aligned}f_{1,I+1/2}^{l+1} &= -\frac{D_I}{h_I} (f_{0,I+1/2}^{l+1} - f_{0,I-1/2}^{l+1}) \\ &\quad - \frac{1}{4} \hat{\sigma}_{RI} h_I (f_{0,I+1/2}^{l+1} + f_{0,I-1/2}^{l+1}) \\ &\quad + g_{1I}^{l+1/2} + \frac{1}{2} g_{0I}^{l+1/2}.\end{aligned}\quad (62b)$$

The starting iterate is determined by

$$\begin{aligned}-\frac{D_{i+1}}{h_{i+1}} (\phi_{0,i+3/2}^1 - \phi_{0,i+1/2}^1) + \frac{D_i}{h_i} (\phi_{0,i+1/2}^1 - \phi_{0,i-1/2}^1) \\ + \frac{1}{4} [\hat{\sigma}_{R,i+1} h_{i+1} (\phi_{0,i+3/2}^1 + \phi_{0,i+1/2}^1) \\ + \hat{\sigma}_{R,i} h_i (\phi_{0,i+1/2}^1 + \phi_{0,i-1/2}^1)] \\ = (Q_{0,i+1} + Q_{0i}) - (Q_{1,i+1} - Q_{1i}),\end{aligned}\quad (63)$$

$$\xi_{0i}^1 = \frac{1}{2} (\phi_{0,i+1/2}^1 - \phi_{0,i-1/2}^1), \quad (64a)$$

$$\xi_{1i}^1 = -\frac{1}{4} \hat{\sigma}_{Ri} h_i (\phi_{0,i+1/2}^1 + \phi_{0,i-1/2}^1) + Q_{0i}, \quad (64b)$$

$$\begin{aligned}\phi_{0i}^1 &= \left[\frac{1}{2} - \frac{3}{4} \rho_i (\hat{\sigma}_{Ri} h_i) \left(1 - \frac{\sigma_{S1i}}{\sigma_{Ti}} \right) \right] (\phi_{0,i+1/2}^1 + \phi_{0,i-1/2}^1) \\ &\quad + 3\rho_i \left(1 - \frac{\sigma_{S1i}}{\sigma_{Ti}} \right) Q_{0i} - \frac{1}{\sigma_{Ti}} L_0 \alpha_i T_i,\end{aligned}\quad (64c)$$

and

$$\begin{aligned}\phi_{1i}^1 &= \left[\frac{1}{2} \rho_i \left(1 - \frac{\sigma_{S0i}}{\sigma_{Ti}} \right) - \frac{D_i}{h_i} \right] (\phi_{0,i+1/2}^1 - \phi_{0,i-1/2}^1) \\ &\quad + Q_{1i} - \frac{1}{\sigma_{Ti}} L_1 \alpha_i T_i,\end{aligned}\quad (64d)$$

where

$$Q_{0i} = \frac{[\sigma_{Ti}(L_0 S_i) + (\sigma_{Ti} - \sigma_{S0i})(L_0 \alpha_i T_i)] h_i}{2\sigma_{Ti} + 3\rho_i (\sigma_{Ti} - \sigma_{S0i})(\sigma_{Ti} - \sigma_{S1i}) h_i} \quad (65a)$$

$$Q_{1i} = \frac{1}{\sigma_{Ti} - \sigma_{S1i}} (L_1 S_i) + \frac{1}{\sigma_{Ti}} (L_1 \alpha_i T_i). \quad (65b)$$

The boundary conditions for Eq. (63) are given by Eqs. (34), (35), and (36) with

$$\begin{aligned}\phi_{1,1/2}^1 &= -\frac{D_1}{h_1} (\phi_{0,3/2}^1 - \phi_{0,1/2}^1) \\ &\quad + \frac{1}{4} \hat{\sigma}_{R1} h_1 (\phi_{0,3/2}^1 + \phi_{0,1/2}^1) + Q_{11} - Q_{01}\end{aligned}\quad (66a)$$

and

$$\begin{aligned}\phi_{0,I+1/2}^1 &= -\frac{D_I}{h_I}(\phi_{0,I+1/2}^1 - \phi_{0,I-1/2}^1) \\ &\quad - \frac{1}{4} \hat{\sigma}_{Ri} h_I (\phi_{0,I+1/2}^1 + \phi_{0,I-1/2}^1) + Q_{1I} + Q_{0I} .\end{aligned}\quad (66b)$$

The stability properties of this acceleration method, for infinite medium, isotropic scattering, constant cross section, and uniform mesh problems, are determined by a method similar to that described in Sec. III, and the results are given in Table I and discussed in Sec. I. The details of these calculations, however, are extremely lengthy, and we do not present them here. Applying this stability analysis to the unaccelerated iterative scheme [Eqs. (48) and (49)] yields the result Eq. (9).

V. LINEAR DISCONTINUOUS AND LINEAR MOMENTS

The LD and LM methods can be shown to satisfy very similar equations, and thus can be treated jointly. The LM method, developed by Vaidyanathan,^{28,29} is described in Sec. IV; for the cell problem Eq. (41), the method is defined by Eqs. (44), (46), and (47). Since Eqs. (44) and (46) state that the linear representation Eq. (42) exactly preserves the zero'th and first spatial moments of the analytic solution, the slope $\xi_{mi}^{l+1/2}$ can be defined, rather than by Eq. (46), by multiplying Eq. (41) by $x - x_i$ and integrating over the i 'th cell; this produces the first-order spatial balance equation:

$$\frac{3\mu_m}{h_i} (\psi_{m,i+1/2}^{l+1/2} + \psi_{m,i-1/2}^{l+1/2} - 2\psi_{mi}^{l+1/2}) + \sigma_T \xi_{mi}^{l+1/2} = \sigma_S \xi_{0i}^l .\quad (67)$$

Thus, the unaccelerated LM method can be defined by Eqs. (44), (47), and (67).

The LD method is also based on the cell problem (41), and the goal is also a representation of the flux in the form Eq. (42). However, in this method, the transport equation, Eq. (41), is not solved exactly. Instead, the zero'th and first-order spatial balance equations, Eqs. (44) and (67), together with the definitions

$$\xi_{mi}^{l+1/2} = \begin{cases} \psi_{m,i+1/2}^{l+1/2} - \psi_{mi}^{l+1/2} , & \mu_m > 0 , \\ \psi_{mi}^{l+1/2} - \psi_{m,i-1/2}^{l+1/2} , & \mu_m < 0 , \end{cases}\quad (68)$$

are used. Introducing Eq. (68) into Eq. (67) and rearranging, we obtain Eq. (47a) with

$$\alpha_m = \frac{\epsilon_m}{3 + |\epsilon_m|} ,\quad (69a)$$

$$\epsilon_m = \frac{\sigma_T h}{2\mu_m} .\quad (69b)$$

Thus, the LD method is defined by Eqs. (44), (47a), (67), and (69).

For linearly anisotropic scattering problems, with variable cross sections, variable mesh, and an inhomogeneous source $S_{mi} + T_{mi}(2/h_i)(x - x_i)$, the unaccelerated LD and LM methods are described by

$$\begin{aligned}\frac{\mu_m}{h_i} (\psi_{m,i+1/2}^{l+1/2} - \psi_{m,i-1/2}^{l+1/2}) + \sigma_{Ti} \psi_{mi}^{l+1/2} \\ = \sigma_{Soi} \phi_{0i}^l + 3\sigma_{S1i} \mu_m \phi_{1i}^l + S_{mi} ,\end{aligned}\quad (70a)$$

$$\begin{aligned}\frac{3\mu_m}{h_i} (\psi_{m,i+1/2}^{l+1/2} + \psi_{m,i-1/2}^{l+1/2} - 2\psi_{mi}^{l+1/2}) + \sigma_{Ti} \xi_{mi}^{l+1/2} \\ = \sigma_{Soi} \xi_{0i}^l + 3\sigma_{S1i} \mu_m \xi_{1i}^l + T_{mi} ,\end{aligned}\quad (70b)$$

$$\begin{aligned}\psi_{mi}^{l+1/2} &= \left(\frac{1 + \alpha_{mi}}{2} \right) \psi_{m,i+1/2}^{l+1/2} + \left(\frac{1 - \alpha_{mi}}{2} \right) \psi_{m,i-1/2}^{l+1/2} \\ &\quad - \frac{\alpha_{mi}}{\sigma_{Ti}} (\sigma_{Soi} \xi_{0i}^l + 3\sigma_{S1i} \mu_m \xi_{1i}^l + T_{mi}) ,\end{aligned}\quad (70c)$$

$$\phi_{ni}^{l+1} = L_n \psi_i^{l+1/2} \equiv \phi_{ni}^{l+1/2} \quad (71a)$$

$$\xi_{ni}^{l+1} = L_n \xi_i^{l+1/2} \equiv \xi_{ni}^{l+1/2} \quad (71b)$$

where

$$\alpha_{mi} = \begin{cases} \frac{\epsilon_{mi}}{3 + |\epsilon_{mi}|} & (\text{LD}) , \\ \frac{\exp(\epsilon_{mi}) + \exp(-\epsilon_{mi})}{\exp(\epsilon_{mi}) - \exp(-\epsilon_{mi})} - \frac{1}{\epsilon_{mi}} & (\text{LM}) , \end{cases}\quad (72a)$$

$$\epsilon_{mi} = \frac{\sigma_{Ti} h_i}{2\mu_m} .\quad (72b)$$

As usual, we retain Eqs. (70) for the transport calculation; we replace Eqs. (71) by a diffusion calculation for each iteration; and we require Eqs. (71) to hold upon convergence. To derive the diffusion calculation, we operate on Eqs. (70) by L_0 and L_1 to obtain six moment equations. For example, L_0 acting on Eq. (70c) gives, using Eqs. (14),

$$\begin{aligned}\phi_{0i}^{l+1/2} &= \frac{1}{2} (\phi_{0,i+1/2}^{l+1/2} + \phi_{0,i-1/2}^{l+1/2}) + \frac{3}{2} \rho_i (\phi_{1,i+1/2}^{l+1/2} - \phi_{1,i-1/2}^{l+1/2}) \\ &\quad + \frac{1}{2} L_0 \gamma_i (\psi_{i+1/2}^{l+1/2} - \psi_{i-1/2}^{l+1/2}) \\ &\quad - \frac{1}{\sigma_{Ti}} (3\sigma_{S1i} \rho_i \xi_{1i}^l + L_0 \alpha_i T_i) .\end{aligned}\quad (73)$$

Next, we follow the procedure outlined in Secs. II and IV and define acceleration equations by replacing the superscripts in $\phi_{n,i+1/2}^{l+1/2}$, $\phi_{ni}^{l+1/2}$, ϕ_{0i}^l , $\xi_{ni}^{l+1/2}$, and ξ_{0i}^l with $l + 1$ for $n = 0$ and 1, in the aforementioned six equations. For example, Eq. (73) becomes

$$\begin{aligned}\phi_{0i}^{l+1} &= \frac{1}{2} (\phi_{0,i+1/2}^{l+1} + \phi_{0,i-1/2}^{l+1}) + \frac{3}{2} \rho_i (\phi_{1,i+1/2}^{l+1} - \phi_{1,i-1/2}^{l+1}) \\ &\quad + \frac{1}{2} L_0 \gamma_i (\psi_{i+1/2}^{l+1/2} - \psi_{i-1/2}^{l+1/2}) \\ &\quad - \frac{1}{\sigma_{Ti}} (3\sigma_{S1i} \rho_i \xi_{1i}^{l+1} + L_0 \alpha_i T_i) .\end{aligned}\quad (74)$$

Now we subtract the full Eq. (73) from Eqs. (74) and use the definitions, Eq. (19) to obtain

$$\begin{aligned}\frac{1}{h_i} (f_{1,i+1/2}^{l+1} - f_{1,i-1/2}^{l+1}) + (\sigma_{Ti} - \sigma_{Soi}) (\phi_{0i}^{l+1} - \phi_{0i}^{l+1/2}) \\ = \sigma_{Soi} (\phi_{0i}^{l+1/2} - \phi_{0i}^l) ,\end{aligned}\quad (75a)$$

$$\begin{aligned}\frac{1}{3h_i} (f_{0,i+1/2}^{l+1} - f_{0,i-1/2}^{l+1}) + (\sigma_{Ti} - \sigma_{S1i}) (\phi_{1i}^{l+1} - \phi_{1i}^{l+1/2}) \\ = \sigma_{S1i} (\phi_{1i}^{l+1/2} - \phi_{1i}^l) ,\end{aligned}\quad (75b)$$

$$\begin{aligned}\frac{3}{h_i} [f_{1,i+1/2}^{l+1} + f_{1,i-1/2}^{l+1} - 2(\phi_{1i}^{l+1} - \phi_{1i}^{l+1/2})] + (\sigma_{Ti} - \sigma_{Soi}) \xi_{0i}^{l+1} \\ = \sigma_{Ti} \xi_{0i}^{l+1/2} - \sigma_{Soi} \xi_{0i}^l ,\end{aligned}\quad (76a)$$

$$\begin{aligned}\frac{1}{h_i} [f_{0,i+1/2}^{l+1} + f_{0,i-1/2}^{l+1} - 2(\phi_{0i}^{l+1} - \phi_{0i}^{l+1/2})] + (\sigma_{Ti} - \sigma_{S1i}) \xi_{1i}^{l+1} \\ = \sigma_{Ti} \xi_{1i}^{l+1/2} - \sigma_{S1i} \xi_{1i}^l ,\end{aligned}\quad (76b)$$

The equations for ϕ_{ni}^{l+1} and ξ_{ni}^{l+1} are as follows:

$$\begin{aligned}-\frac{D_{i+1}}{h_{i+1}} (f_{0,i+3/2}^{l+1} - f_{0,i+1/2}^{l+1}) + \frac{D_i}{h_i} (f_{0,i+1/2}^{l+1} - f_{0,i-1/2}^{l+1}) + \frac{1}{4} [\hat{\sigma}_{R,i+1} h_{i+1} (f_{0,i+3/2}^{l+1} + f_{0,i+1/2}^{l+1}) + \hat{\sigma}_{Ri} h_i (f_{0,i+1/2}^{l+1} + f_{0,i-1/2}^{l+1})] \\ = \frac{1}{2} (g_{0,i+1}^{l+1/2} + g_{0i}^{l+1/2}) - (g_{1,i+1}^{l+1/2} - g_{1i}^{l+1/2}) ,\end{aligned}\quad (78)$$

$$\xi_{0i}^{l+1} = \xi_{0i}^l + \sigma_{Ti} \frac{3\rho_i (f_{0,i+1/2}^{l+1} - f_{0,i-1/2}^{l+1}) + \sigma_{Ti} h_i (\xi_{0i}^{l+1/2} - \xi_{0i}^l)}{6\rho_i \sigma_{Soi} + (\sigma_{Ti} h_i)(\sigma_{Ti} - \sigma_{Soi})} ,\quad (79a)$$

$$\xi_{1i}^{l+1} = \xi_{1i}^l + \sigma_{Ti} \frac{\left(-\frac{3}{2}\right) \rho_i (\hat{\sigma}_{Ri} h_i) (f_{0,i+1/2}^{l+1} + f_{0,i-1/2}^{l+1}) + \sigma_{Ti} h_i (\xi_{1i}^{l+1/2} - \xi_{1i}^l) + 3\rho_i g_{0i}^{l+1/2}}{6\rho_i \sigma_{S1i} + \sigma_{Ti} h_i (\sigma_{Ti} - \sigma_{S1i})} ,\quad (79b)$$

$$\phi_{0i}^{l+1} = \phi_{0i}^{l+1/2} + \left(\frac{1}{2} - \frac{3}{4} \rho_i \hat{\sigma}_{Ri} h_i\right) (f_{0,i+1/2}^{l+1} + f_{0,i-1/2}^{l+1}) - 3 \frac{\sigma_{S1i}}{\sigma_{Ti}} \rho_i (\xi_{1i}^{l+1} - \xi_{1i}^l) + \frac{3}{2} \rho_i g_{0i}^{l+1/2} ,\quad (80a)$$

$$\phi_{1i}^{l+1} = \phi_{1i}^{l+1/2} + \left(\frac{1}{2} \rho_i - \frac{D_i}{h_i}\right) (f_{0,i+1/2}^{l+1} - f_{0,i-1/2}^{l+1}) - \rho_i \frac{\sigma_{Soi}}{\sigma_{Ti}} (\xi_{0i}^{l+1} - \xi_{0i}^l) + g_{1i}^{l+1/2} .\quad (80b)$$

The various coefficients in these equations are defined by

$$\rho_i = \sum_{m=1}^N \mu_m \alpha_{mi} \omega_m ,\quad (81a)$$

$$D_i = \frac{1}{3(\sigma_{Ti} - \sigma_{S1i})} + \frac{\rho_i \sigma_{Ti} h_i^2 (\sigma_{Ti} - \sigma_{Soi})}{12\rho_i \sigma_{Soi} + 2(\sigma_{Ti} h_i)(\sigma_{Ti} - \sigma_{Soi})} ,\quad (81b)$$

$$\hat{\sigma}_{Ri} = \frac{\sigma_{Ti} - \sigma_{Soi}}{1 + \Delta_i} ,\quad (81c)$$

and

$$\Delta_i = \frac{3\rho_i(\sigma_{Ti}h_i)^2(\sigma_{Ti} - \sigma_{S0i})(\sigma_{Ti} - \sigma_{S1i})}{12\rho_i\sigma_{Ti}\sigma_{S1i} + 2(\sigma_{Ti}h_i)\sigma_{Ti}(\sigma_{Ti} - \sigma_{S1i})}, \quad (81d)$$

and the sources are defined by

$$g_{0i}^{l+1/2} = \hat{\sigma}_{Si}h_i(\phi_{0i}^{l+1/2} - \phi_{0i}^l) + \tau_{0i}(\xi_{1i}^{l+1/2} - \xi_{1i}^l) \quad (82a)$$

and

$$g_{1i}^{l+1/2} = a_i(\phi_{1i}^{l+1/2} - \phi_{1i}^l) + \tau_{1i}(\xi_{0i}^{l+1/2} - \xi_{0i}^l), \quad (82b)$$

where

$$\hat{\sigma}_{Si} = \frac{\sigma_{S0i}}{1 + \Delta_i} \quad (83a)$$

$$a_i = \frac{\sigma_{S1i}}{\sigma_{Ti} - \sigma_{S1i}} \quad (83b)$$

$$\tau_{0i} = \frac{3(\sigma_{Ti}h_i)^2\rho_i\sigma_{S1i}(\sigma_{Ti} - \sigma_{S0i})}{6\rho_i\sigma_{Ti}\sigma_{S1i} + (\sigma_{Ti}h_i)\sigma_{Ti}(\sigma_{Ti} - \sigma_{S1i}) + \frac{3}{2}\rho_i(\sigma_{Ti}h_i)^2(\sigma_{Ti} - \sigma_{S0i})(\sigma_{Ti} - \sigma_{S1i})} \quad (83c)$$

$$\tau_{1i} = \frac{\rho_i(\sigma_{Ti}h_i)\sigma_{S0i}}{6\rho_i\sigma_{S0i} + (\sigma_{Ti}h_i)(\sigma_{Ti} - \sigma_{S0i})}. \quad (83d)$$

The boundary conditions for Eq. (78) are given by Eqs. (38), (39), and (40), with

$$f_{1,1/2}^{l+1} = -\frac{D_1}{h_1}(f_{0,3/2}^{l+1} - f_{0,1/2}^{l+1}) + \frac{1}{4}\hat{\sigma}_{R1}h_1(f_{0,3/2}^{l+1} + f_{0,1/2}^{l+1}) + g_{11}^{l+1/2} - \frac{1}{2}g_{01}^{l+1/2} \quad (84a)$$

and

$$f_{1,I+1/2}^{l+1} = -\frac{D_I}{h_I}(f_{0,I+1/2}^{l+1} - f_{0,I-1/2}^{l+1}) - \frac{1}{4}\hat{\sigma}_{RI}h_I(f_{0,I+1/2}^{l+1} + f_{0,I-1/2}^{l+1}) + g_{1I}^{l+1/2} + \frac{1}{2}g_{0I}^{l+1/2}. \quad (84b)$$

The starting iterate is determined by

$$-\frac{D_{i+1}}{h_{i+1}}(\phi_{0,i+3/2}^1 - \phi_{0,i+1/2}^1) + \frac{D_i}{h_i}(\phi_{0,i+1/2}^1 - \phi_{0,i-1/2}^1) + \frac{1}{4}[\hat{\sigma}_{R,i+1}h_{i+1}(\phi_{0,i+3/2}^1 + \phi_{0,i+1/2}^1) + \hat{\sigma}_{Ri}h_i(\phi_{0,i+1/2}^1 + \phi_{0,i-1/2}^1)] \\ = (Q_{0,i+1} + Q_{0i}) - (Q_{1,i+1} - Q_{1i}), \quad (85)$$

$$\xi_{0i}^1 = \frac{3\rho_i\sigma_{Ti}(\phi_{0,i+1/2}^1 - \phi_{0,i-1/2}^1) + L_0(\sigma_{Ti}h_i - 6\mu\alpha_i)T_i}{6\rho_i\sigma_{S0i} + (\sigma_{Ti}h_i)(\sigma_{Ti} - \sigma_{S0i})}, \quad (86a)$$

$$\xi_{1i}^1 = \frac{-\frac{3}{2}\rho_i(\sigma_{Ti}h_i)\hat{\sigma}_{Ri}(\phi_{0,i+1/2}^1 + \phi_{0,i-1/2}^1) + 6\rho_i\sigma_{Ti}Q_{0i} + L_0(\sigma_{Ti}h_i\mu - 2\alpha_i)T_i}{6\rho_i\sigma_{S1i} + (\sigma_{Ti}h_i)(\sigma_{Ti} - \sigma_{S1i})}, \quad (86b)$$

$$\phi_{0i}^1 = \left(\frac{1}{2} - \frac{3}{4}\rho_i\hat{\sigma}_{Ri}h_i\right)(\phi_{0,i+1/2}^1 + \phi_{0,i-1/2}^1) - 3\rho_i\frac{\sigma_{S1i}}{\sigma_{Ti}}\xi_{1i}^1 + 3\rho_iQ_{0i} - \frac{1}{\sigma_{Ti}}L_0\alpha_iT_i, \quad (86c)$$

$$\phi_{1i}^1 = \left(\frac{1}{2}\rho_i - \frac{D_i}{h_i}\right)(\phi_{0,i+1/2}^1 - \phi_{0,i-1/2}^1) - \rho_i\frac{\sigma_{S0i}}{\sigma_{Ti}}\xi_{0i}^1 + Q_{1i} - \frac{1}{\sigma_{Ti}}L_1\alpha_iT_i, \quad (86d)$$

where

$$Q_{0i} = \frac{h_i}{2(1 + \Delta_i)} \left\{ L_0S_i + \frac{\sigma_{Ti} - \sigma_{S0i}}{6\rho_i\sigma_{S1i} + (\sigma_{Ti}h_i)(\sigma_{Ti} - \sigma_{S1i})} \right. \\ \times [3\rho_i(\sigma_{S1i}h_i)(L_1T_i) + (\sigma_{Ti} - \sigma_{S1i})h_i(L_0\alpha_iT_i)] \left. \right\} \quad (87a)$$

$$Q_{1i} = \frac{L_1S_i}{\sigma_{Ti} - \sigma_{S1i}} + \frac{1}{6\rho_i\sigma_{S0i} + (\sigma_{Ti}h_i)(\sigma_{Ti} - \sigma_{S0i})} [\rho_i(\sigma_{S0i}h_i)(L_0T_i) + (\sigma_{Ti} - \sigma_{S0i})h_i(L_1\alpha_iT_i)]. \quad (87b)$$

The boundary conditions for Eq. (85) are given by Eqs. (34), (35), and (36) with

$$\begin{aligned}\phi_{1,1/2}^1 &= -\frac{D_1}{h_1}(\phi_{0,3/2}^1 - \phi_{0,1/2}^1) + \frac{1}{4}\hat{\sigma}_{R1}h_1(\phi_{0,3/2}^1 + \phi_{0,1/2}^1) \\ &\quad + Q_{11} - Q_{01}\end{aligned}\quad (88a)$$

and

$$\begin{aligned}\phi_{1,I+1/2}^1 &= -\frac{D_I}{h_I}(\phi_{0,I+1/2}^1 - \phi_{0,I-1/2}^1) \\ &\quad - \frac{1}{4}\hat{\sigma}_{RI}h_I(\phi_{0,I+1/2}^1 + \phi_{0,I-1/2}^1) + Q_{1I} + Q_{0I}.\end{aligned}\quad (88b)$$

We have analyzed the stability and convergence properties of these acceleration methods by the procedure described in Sec. IV. The result, for infinite medium, isotropic scattering, constant cross section, and constant spatial mesh problems is the bound on the spectral radius

$$\text{spr} \leq cK_N,$$

where K_N is tabulated in Table I. The analysis leading to this bound is extremely lengthy and is not presented here. Finally, applying this analysis to the unaccelerated iterative scheme [Eqs. (70), (71), and (72)] yields the result, Eq. (9).

VI. DISCUSSION

The acceleration methods derived above apply to certain differencing schemes for the discrete ordinates equations in slab geometry. These methods are tested on finite region, anisotropic scattering, one-group problems with variable cross sections and a variable spatial mesh in our companion paper.²⁴

Our procedure for deriving acceleration methods should be straightforwardly applicable to any differencing scheme in slab geometry possessing linear relationships between the cell-edge and cell-averaged

quantities. We also expect our procedure to be applicable in the one-dimensional curvilinear geometries. For higher dimensions, the situation is not so clear. One can certainly apply our procedure and derive equations for the accelerated flux moments, but for ease of implementation, these equations need to be converted into an equivalent, single, vertex-differenced diffusion equation. Such a conversion is possible for the case of the DD method.¹¹ The question of whether the conversion is possible for other spatial differencing schemes is currently under investigation. We are also currently examining the adaptation of our methods, which are developed here for source problems, to the acceleration of eigenvalue problems.

In Sec. II above, we treated a general class of WD differencing schemes in which the weights are required to satisfy the conditions, Eqs. (12). We carried out this treatment because the analysis provides a relatively simple framework by which the more complicated schemes can be treated, not because the WD schemes contained in Sec. II are inherently important or widely used. In fact, apart from the pure diamond scheme where the weights are zero, the WD schemes, which are currently used, are generally employed only as a device to eliminate or reduce the occurrence of negative fluxes¹; the weights depend on the iteration index I and the transport flux $\psi^{I+1/2}$, and, in general, they do not satisfy Eq. (12c). We can generalize our analysis in Sec. II to include the possibility of weights that depend on I and that do not satisfy Eq. (12c); the resulting acceleration method should be more consistent with the transport scheme. This matter is also currently under investigation.

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