Enumerative Combinatorics

Lecture Notes

David Scholz

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Abstract

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1 Introduction

We start by discussing fundamental combinatorial constructions such as words of a fixed alphabet, permutations of finite sets and the number of subsets with a given number of elements. While we give some examples of how some combinatorial problems naturally arise, our main goal in this first section is to provide an intuitive introduction to the topic of Enumerative Combinatorics.

1.1 Words

The first definition of this lecture is the definition of a word over a finite alphabet of symbols.

Definition 1.1 (word)**.** Let A be a finite set, called *alphabet*. A *word* is a finite sequence of elements of A. The *size* of A is its cardinality $|A|$. The *size* or *length* of a word is its number of elements from A.

We might abuse language to a certain extent and call the elements of A *letters*.

Example 1.2. Let $A = \{1, 2, 3, 4\}$, then 112, 132, 1234, 111, 2345, and so on are all words over A. Notice that the length of the words is not fixed.

In the study of combinatorics a very classical question is as follows. Given an alphabet A of length $|A| = n$, how many words of length k consisting of n letters exist? This question leads us to the following theorem.

Theorem 1.1. The number of words of length k in an alphabet consisting of n letters is equal to n^k.

Proof. There are n possibilities for the first letter. The same holds for all other letters. Thus, the total number of words equals

$$\underbrace{n \cdot n \cdot n \cdots n}_{k\text{-times}} = n^k$$

□

The above proof is fairly simple, but there is a different possible interpretation of this problem. Let X and Y be sets. We might ask: what is the total number of maps of the form $X \to Y$ without restrictions (an element in Y can be hit multiple times)? It appears that this question can be solved using theorem 1.1. For seeing this, we number the elements of X and Y using natural numbers. Given a map $f : X \to Y$, the sequence $f(1), f(2), \cdots, f(k)$ forms a word of length k. The number of words is therefore equivalent to the number of such maps.

1.2 Permutations

A notion that is closely related to the definition of a word, is the definition of a *permutation*. First of all, the basic idea is to count maps, but usually with certain properties. Let X and Y be sets with $|X| = k$ and $|Y| = n$. We might ask: how many bijections $f : X \to Y$ are there? Recall that a map is bijective if it is surjective and injective. In other words, given two elements $x, y \in X$ such that $f(x) = f(y)$, then it is implied that $x = y$ (the injective property), and every element of Y appears in the preimage of a certain element of X (the surjective property). It follows that the cardinalities of X and Y must be equal. These observations lead us to the definition of a permutation.

Definition 1.3 (permutation)**.** A *permutation* is a bijection from $\{1, 2, 3, \cdots, n\}$ to itself.

In that sense the number of permutations is the number of bijections from $\{1, 2, \cdots, n\}$ to itself. Equivalently, a permutation might be interpreted as a total linear ordering of the set $\{1, 2, 3, \cdots, n\}$. If so, we need to arrange them in a certain way. How many ways to do this are there? The following theorem provides one possible solution.

Theorem 1.2. The number of permutations of $\{1, 2, \cdots, n\}$ is equal to $n! = 1 \cdot 2 \cdot 3 \cdots n$.[1]

Proof. There are n choices for the first element. For the second element there are $n - 1$ choices since one element is already occupied. For the k-th element there are $n - k + 1$ choices. For the last element there is one choice. Thus, the total number of permutations is given by $n(n-1)(n-2)\cdots 2 \cdot 1 = n!$. This completes the proof. □

In the following, we will generalize the above result to injective maps. Let $k \leq n$. In how many ways can we produce a collection of k elements out of n? Such a collection is called a k-permutation.

Definition 1.4 (k-permutation)**.** A k-permutation is a total linear ordering on a k-element subset of $\{1, 2, \cdots, n\}$.

Example 1.5. Let $\{1, 2, 3\}$. A 1-permutation is just a 1-element subset, thus there is nothing order. The set of 2-permutations is given by $\{\{1, 2\}, \{1, 3\}, \{2, 1\}, \{2, 3\}, \{3, 1\}, \{3, 2\}\}$. Thus , there are six 2-permutations in total.

[1] Recall that $0! = 1$.

The general case of the number of k-permutations is examined in the following theorem.

Theorem 1.3. The number of k-permutations of $\{1, 2, \cdots, n\}$ is equal to

$$n(n-1)\cdots(n-k+1).$$

Proof. There are n possible choices for the first entry of the subset. There are $(n-1)$ possible choices for the second element because the second element can occupy any entry except the one of the first element. For the k-th entry there are $n-k+1$ possible choices. All these possibilities are independent, hence the total number of k-permutations is equal to their product $n(n-1)\cdots(n-k+1)$. □

The number $n(n-1)\cdots(n-k+1)$ is sometimes called the k *decreasing power* and is denoted by $n^{\downarrow k}$.

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1.4 Binomial coefficients

1.5 The Pascal triangle

1.6 Exercises

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