We proof a needed Lemma first.

Lemma 1: Let $m, n, r \in \mathbb{Z}^+$ and $r \leq \min(m, n)$. Then

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$$

Proof: Let A and B be sets. Suppose $A \cap B = \emptyset$ and let $|A| = n$ and $|B| = m$, then clearly $|A \cup B| = |A| + |B| = m + n$. The number of ways to choose r elements from $A \cup B$ is equal to $\binom{m+n}{r}$.

Suppose that we choose $r \leq k$ elements from A, then we get $\binom{n}{k}$ possible choices. If we pick the remaining elements from B, then we get $\binom{m}{r-k}$ possible choices.

The number k is choosen from the set $\{1, 2, \cdots, r\}$, thus the total number of possible choices is given by

$$\sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$$

This completes the proof.

The following two corallaries follow directly from the above Lemma.

Corollary 1:

$$\sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k} = \sum_{k=0}^{r} \binom{m}{m-r+k}\binom{n}{k}$$

Proof: By the symmetry property of the binomial coefficient (compare with lecture 1)

$$\binom{n}{k} = \binom{n}{n-k}$$

the above equality follows directly.

Corollary 2: If $m = n$ and $r = n + 1$, then

$$\sum_{k=1}^{n} \binom{n}{k-1}\binom{n}{k} = \binom{2n}{n+1}$$

Proof: Replacing m with n and r with $n + 1$ in the formula given in Corollary 1, the claim follows.

Notice, that we have proven in Lecture 1, that for $1 \le k \le n$, we have $\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$ or equivalently $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$.

Theorem:

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

Proof: We proceed with induction on n. The base case is trivial. Assume that the claim holds for n. Then,

$$\begin{aligned}
\sum_{k=0}^{n+1} \binom{n+1}{k}^2 &= \binom{n+1}{0}^2 + \sum_{k=1}^{n+1} \binom{n+1}{k}^2 \\
&= 1 + \sum_{k=1}^{n} \binom{n+1}{k}^2 + \binom{n+1}{n+1}^2 \\
&= 1 + \sum_{k=1}^{n} \left[\binom{n}{k-1} + \binom{n}{k} \right]^2 + 1 \\
&= 2 + \sum_{k=1}^{n} \left[\binom{n}{k-1}^2 + 2\binom{n}{k-1}\binom{n}{k} + \binom{n}{k}^2 \right] \\
&= 2 + \sum_{k=1}^{n} \left[\binom{n}{k-1}^2 + \binom{n}{k}^2 \right] + 2\sum_{k+1}^{n} \binom{n}{k-1}\binom{n}{k} \\
&= 1 + \sum_{k=1}^{n} \binom{n}{k}^2 + 1 + \sum_{k=1}^{n} \binom{n}{k}^2 + 2\sum_{k=1}^{n} \binom{n}{k-1}\binom{n}{k} \\
&= \binom{n}{0} + \sum_{k=1}^{n} \binom{n}{k}^2 + \binom{n}{0} + \sum_{k=1}^{n} \binom{n}{k}^2 + 2\sum_{k=1}^{n} \binom{n}{k-1}\binom{n}{k} \\
&= \binom{2n}{n} + \binom{2n}{n} + 2\sum_{k=1}^{n} \binom{n}{k-1}\binom{n}{k} \\
&= 2\binom{2n}{n} + 2\sum_{k=1}^{n} \binom{n}{k-1}\binom{n}{k} \\
&= 2\binom{2n}{n} + 2\binom{2n}{n+1} && \text{(Corollary 2)} \\
&= 2\left(\binom{2n}{n} + \binom{2n}{n+1}\right) \\
&= 2\binom{2n+1}{n+1} \\
&= \binom{2(n+1)}{n+1}
\end{aligned}$$

This completes the proof.