

# Enumerative Combinatorics

Lecture Notes

David Scholz

January 28, 2022

## **Abstract**

Abstract

# Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
1.1	Words . . . . .	1
1.2	Permutations . . . . .	1
1.3	Merry-go-rounds and Fermat's little theorem . . . . .	1
1.4	Binomial coefficients . . . . .	1
1.5	The Pascal triangle . . . . .	1
1.6	Exercises . . . . .	1
<b>2</b>	<b>Permutations and binomial coefficients</b>	<b>2</b>
2.1	The cab driver problem . . . . .	2
2.2	Balls in boxes and multisets . . . . .	2
2.3	Integer compositions . . . . .	2
2.4	Principle of inclusion and exclusion . . . . .	2
2.5	The derangement problem . . . . .	2
2.6	Exercises . . . . .	2
<b>3</b>	<b>Linear recurrences - The Fibonacci sequence</b>	<b>3</b>
3.1	Fibonacci's rabbit problem . . . . .	3
3.2	Fibonacci numbers and the Pascal triangle . . . . .	3
3.3	Domino tilings . . . . .	3
3.4	Linear recurrence relations . . . . .	3
3.5	The characteristic equation . . . . .	3
3.6	Linear recurrence relations of order 2 . . . . .	3
3.7	The Binet formula . . . . .	3
3.8	Linear recurrence relations of arbitrary order . . . . .	3
3.9	The case of roots with multiplicities . . . . .	3
3.10	Exercises . . . . .	3
<b>4</b>	<b>A nonlinear recurrence: many faces of Catalan numbers</b>	<b>4</b>
4.1	Triangulations of a polygon . . . . .	4
4.2	Recurrence relation for triangulations . . . . .	4
4.3	The cashier problem . . . . .	4
4.4	Dyck paths . . . . .	4
4.5	Recurrence relations for Dyck paths . . . . .	4
4.6	Reflection trick and a formula for Catalan numbers . . . . .	4
4.7	Binary trees . . . . .	4
4.8	Exercises . . . . .	4

<b>5</b>	<b>Generating functions: a unified approach to combinatorial problems - Solving linear recurrences</b>	<b>5</b>
5.1	Generating functions: first examples . . . . .	5
5.2	Formal power series . . . . .	5
5.3	When are formal power series invertible? . . . . .	5
5.4	Derivation of formal power series . . . . .	5
5.5	Binomial theorem for negative integer exponents . . . . .	5
5.6	Solving the Fibonacci recurrence relation . . . . .	5
5.7	Generating functions of linear recurrence relations are rational . . . . .	5
5.8	Solving linear recurrence relations: general case . . . . .	5
5.9	Exercises . . . . .	5
<b>6</b>	<b>Generating function of the Catalan sequence</b>	<b>6</b>
6.1	Composition of formal power series . . . . .	6
6.2	Derivation and integration of formal power series . . . . .	6
6.3	Chain rule - Inverse function theorem . . . . .	6
6.4	Logarithm - Logarithmic derivative . . . . .	6
6.5	Binomial theorem for arbitrary exponents . . . . .	6
6.6	Generating function for Catalan numbers . . . . .	6
6.7	Exercises . . . . .	6
<b>7</b>	<b>Partitions - Euler's generating function for partitions and pentagonal formula</b>	<b>7</b>
7.1	Definition and first examples . . . . .	7
7.2	Young diagrams . . . . .	7
7.3	Generating function for partitions . . . . .	7
7.4	Partition with odd and distinct summands . . . . .	7
7.5	Sylvester's bijection . . . . .	7
7.6	Euler's pentagonal theorem . . . . .	7
7.7	Proof of Euler's pentagonal theorem . . . . .	7
7.8	Computing the number of partitions via the pentagonal theorem . . . . .	7
7.9	Exercises . . . . .	7
<b>8</b>	<b>Gaussian binomial coefficients</b>	<b>8</b>
8.1	Generating function for partitions inside a rectangle . . . . .	8
8.2	$q$ -binomial coefficients: definition and first properties . . . . .	8
8.3	Recurrence relation for $q$ -binomial coefficients . . . . .	8
8.4	Explicit formula for $q$ -binomial coefficients . . . . .	8
8.5	Euler's partition function . . . . .	8
8.6	$q$ -binomial coefficients in linear algebra . . . . .	8

8.7	$q$ -binomial theorem . . . . .	8
8.8	Exercises . . . . .	8

# 1 Introduction

We start by discussing fundamental combinatorial constructions such as words of a fixed alphabet, permutations of finite sets and the number of subsets

## 1.1 Words

## 1.2 Permutations

## 1.3 Merry-go-rounds and Fermat's little theorem

## 1.4 Binomial coefficients

## 1.5 The Pascal triangle

## 1.6 Exercises

## 2 Permutations and binomial coefficients

Hello

2.1 The cab driver problem

2.2 Balls in boxes and multisets

2.3 Integer compositions

2.4 Principle of inclusion and exclusion

2.5 The derangement problem

2.6 Exercises

## 3 Linear recurrences - The Fibonacci sequence

Hallo

3.1 Fibonacci's rabbit problem

3.2 Fibonacci numbers and the Pascal triangle

3.3 Domino tilings

3.4 Linear recurrence relations

3.5 The characteristic equation

3.6 Linear recurrence relations of order 2

3.7 The Binet formula

3.8 Linear recurrence relations of arbitrary order

3.9 The case of roots with multiplicities

3.10 Exercises



## 4 A nonlinear recurrence: many faces of Catalan numbers

Hallo

4.1 Triangulations of a polygon

4.2 Recurrence relation for triangulations

4.3 The cashier problem

4.4 Dyck paths

4.5 Recurrence relations for Dyck paths

4.6 Reflection trick and a formula for Catalan numbers

4.7 Binary trees

4.8 Exercises

## 5 Generating functions: a unified approach to combinatorial problems - Solving linear recurrences

Hallo

- 5.1 Generating functions: first examples
- 5.2 Formal power series
- 5.3 When are formal power series invertible?
- 5.4 Derivation of formal power series
- 5.5 Binomial theorem for negative integer exponents
- 5.6 Solving the Fibonacci recurrence relation
- 5.7 Generating functions of linear recurrence relations are rational
- 5.8 Solving linear recurrence relations: general case
- 5.9 Exercises

## 6 Generating function of the Catalan sequence

Hallo

6.1 Composition of formal power series

6.2 Derivation and integration of formal power series

6.3 Chain rule - Inverse function theorem

6.4 Logarithm - Logarithmic derivative

6.5 Binomial theorem for arbitrary exponents

6.6 Generating function for Catalan numbers

6.7 Exercises

## 7 Partitions - Euler's generating function for partitions and pentagonal formula

Hallo

7.1 Definition and first examples

7.2 Young diagrams

7.3 Generating function for partitions

7.4 Partition with odd and distinct summands

7.5 Sylvester's bijection

7.6 Euler's pentagonal theorem

7.7 Proof of Euler's pentagonal theorem

7.8 Computing the number of partitions via the pentagonal theorem

7.9 Exercises

## 8 Gaussian binomial coefficients

Hallo

- 8.1 Generating function for partitions inside a rectangle
- 8.2  $q$ -binomial coefficients: definition and first properties
- 8.3 Recurrence relation for  $q$ -binomial coefficients
- 8.4 Explicit formula for  $q$ -binomial coefficients
- 8.5 Euler's partition function
- 8.6  $q$ -binomial coefficients in linear algebra
- 8.7  $q$ -binomial theorem
- 8.8 Exercises