

Enumerative Combinatorics

Lecture Notes

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Abstract

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1 Introduction

We start by discussing fundamental combinatorial constructions such as words of a fixed alphabet, permutations of finite sets and the number of subsets with a given number of elements. While we give some examples of how some combinatorial problems naturally arise, our main goal in this first section is to provide an intuitive introduction to the topic of Enumerative Combinatorics.

1.1 Words

The first definition of this lecture is the definition of a word over a finite alphabet of symbols.

Definition 1.1 (word). Let A be a finite set, called *alphabet*. A *word* is a finite sequence of elements of A . The *size* of A is its cardinality $|A|$. The *size* or *length* of a word is its number of elements from A .

We might abuse language to a certain extent and call the elements of A *letters*.

Example 1.2. Let $A = \{1, 2, 3, 4\}$, then 112, 132, 1234, 111, 2345, and so on are all words over A . Notice that the length of the words is not fixed.

In the study of combinatorics a very classical question is as follows. Given an alphabet A of length $|A| = n$, how many words over A of length k exist? This question leads us to the following theorem.

Theorem 1.1. The number of words of length k in an alphabet consisting of n letters is equal to n^k .

Proof. There are n possibilities for the first letter. The same holds for all other letters. Thus, the total number of words equals

$$\underbrace{n \cdot n \cdot n \cdots n}_{k\text{-times}} = n^k$$

□

The above proof is fairly simple, but there is a different possible interpretation of this problem. Let X and Y be sets. We might ask: what is the total number of maps of the form $X \rightarrow Y$ without restrictions (an element in Y can be hit multiple times)? It appears that this question can be solved using Theorem 1.1. For seeing this, we number the elements of X and Y using natural numbers. Given a map $f : X \rightarrow Y$, the sequence $f(1), f(2), \dots, f(k)$ forms a word of length k . The number of words is therefore equivalent to the number of such maps.

1.2 Permutations

A notion that is closely related to the definition of a word, is the definition of a *permutation*. First of all, the basic idea is to count maps, but usually with certain properties. Let X and Y be sets with $|X| = k$ and $|Y| = n$. We might ask: how many bijections $f : X \rightarrow Y$ are there? Recall that a map is bijective if it is surjective and injective. In other words, given two elements $x, y \in X$ such that $f(x) = f(y)$, then it is implied that $x = y$ (the injective property), and every element of Y appears in the preimage of a certain element of X (the surjective property). It follows that the cardinalities of X and Y must be equal. These observations lead us to the definition of a permutation.

Definition 1.3 (permutation). A *permutation* is a bijection from $\{1, 2, 3, \dots, n\}$ to itself.

In that sense the number of permutations is the number of bijections from $\{1, 2, \dots, n\}$ to itself. Equivalently, a permutation might be interpreted as a total linear ordering of the set $\{1, 2, 3, \dots, n\}$. If so, we need to arrange them in a certain way. How many ways to do this are there? The following theorem provides one possible solution.

Theorem 1.2. The number of permutations of $\{1, 2, \dots, n\}$ is equal to $n! = 1 \cdot 2 \cdot 3 \cdots n$.¹

Proof. There are n choices for the first element. For the second element there are $n - 1$ choices since one element is already occupied. For the k -th element there are $n - k + 1$ choices. For the last element there is one choice. Thus, the total number of permutations is given by $n(n - 1)(n - 2) \cdots 2 \cdot 1 = n!$. This completes the proof. \square

In the following, we will generalize the above result to injective maps. Let $k \leq n$. In how many ways can we produce a collection of k elements out of n ? Such a collection is called a k -permutation.

Definition 1.4 (k -permutation). A k -permutation is a total linear ordering on a k -element subset of $\{1, 2, \dots, n\}$.

Example 1.5. Let $\{1, 2, 3\}$. A 1-permutation is just a 1-element subset, thus there is nothing order. The set of 2-permutations is given by $\{\{1, 2\}, \{1, 3\}, \{2, 1\}, \{2, 3\}, \{3, 1\}, \{3, 2\}\}$. Thus, there are six 2-permutations in total.

¹Recall that $0! = 1$.

The general case of the number of k -permutations is examined in the following theorem.

Theorem 1.3. The number of k -permutations of $\{1, 2, \dots, n\}$ is equal to

$$n(n-1) \cdots (n-k+1).$$

Proof. There are n possible choices for the first entry of the subset. There are $(n-1)$ possible choices for the second element because the second element can occupy any entry except the one of the first element. For the k -th entry there are $n-k+1$ possible choices. All these possibilities are independent, hence the total number of k -permutations is equal to their product $n(n-1) \cdots (n-k+1)$. \square

The number $n(n-1) \cdots (n-k+1)$ is sometimes called the k decreasing power and is denoted by $n^{\downarrow k}$.

1.3 Merry-go-rounds and Fermat's little theorem

The problem which we are going to discuss next will illustrate a connection of combinatorics to algebra and number theory.

1.4 Binomial coefficients

Next, we consider the following important definition.

Definition 1.6 (binomial coefficient). Let $k \leq n$. The number of k -element subsets of $\{1, 2, \dots, n\}$ is called a *binomial coefficient* $\binom{n}{k}$ (" n choose k ").

How do we interpret this number? Suppose that we are hiring people to certain positions and suppose that we have n candidates and we have k identical positions. The number of selecting k people out of n is called $\binom{n}{k}$. Notice that this is different than the problem from problem about k -permutations. In our current case, the order of the k elements that we choose does not matter. Thus, choosing all k element subsets of a set and then ordering them in all possible ways is equal to the k -permutations that we have seen above. How do we count the number of k -element subsets of a set?

Theorem 1.4. The number of k -element subsets of a set with cardinality n is equal to

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1) \cdots (n-k+1)}{k!}.$$

Proof. We consider ordered collections of distinct elements of length k . There are n ways of choosing the first element, and $n-k+1$ ways of choosing the k -th element. Hence

there are

$$n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!}$$

ways such ordered collections. Since we are concerned with the number of unordered subsets and each subset has cardinality k , each subset can be rearranged in $k!$ ways. Therefore, the number of ways to choose a k -element subset of a set with cardinality n is given by

$$\frac{\frac{n!}{(n-k)!}}{k!} = \frac{n!}{k!(n-k)!}.$$

This completes the proof. \square

With the above theorem, we have shown something additionally that is not obvious at all and not necessarily easy to prove either. We have also shown that the number $\frac{n(n-1)\cdots(n-k+1)}{k!}$ is integer.

At this point we provide some properties of binomial coefficients. We leave the proof as an easy exercise to the reader.

- (i) $\binom{n}{0} = \binom{n}{n} = 1.$
- (ii) $\binom{n}{1} = \binom{n}{n-1} = n$
- (iii) $\binom{n}{1} = \binom{n}{n-k}.$
- (iv) $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$

Let us examine property (iv) in more detail. Despite its obvious algebraic proof, there is an intuitive combinatorial version as well. We provide a proof sketch.

Theorem 1.5. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

Proof-sketch. Given a set $A = \{1, \dots, n\}$ with n elements. The number of k -element subsets of A is $\binom{n}{k}$ according to theorem 1.4. Suppose that we pick the element 1 of A , then all k -element subsets of A are divided into two parts, some of them contain the element 1, and the others do not. Since we have already picked one element, we need to pick $k-1$ elements out of the remaining $n-1$ elements. There are $\binom{n-1}{k-1}$ subsets containing 1. Thus, the subsets of cardinality k that do not contain 1 is equal to $\binom{n-1}{k}$, because we want to pick a k -element subset out of $n-1$ elements since 1 element is forbidden.

Adding up the two parts are equal to $\binom{n}{k}$. This completes the proof. \square

Binomial coefficients can be arranged in a triangle which is called the Pascal triangle. Its rows are indexed by integers from 0 up to infinity and in each row we write down all binomial coefficients. For example, for $\binom{n}{k}$, $n = 4$ represents the fifth row of the Pascal triangle and since $k \leq n$ there are 5 binomial coefficients, namely $\binom{4}{0} = 1$, $\binom{4}{1} = 4$, $\binom{4}{2} = 6$, $\binom{4}{3} = 4$ and $\binom{4}{4} = 1$.

The relation given in Theorem 1.5 provides a way of constructing the Pascal triangle. It says that the sum of two neighbors, namely $\binom{n-1}{k-1}$ and $\binom{n-1}{k}$ is equal to $\binom{n}{k}$. Thus, taking two neighbors in one row and adding them up gives as the element below them (e.g. $3+3$ in row $n=3$ provides the element 6 in the row $n=4$).

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Proposition 1.1. $\binom{n}{0} + \binom{n}{k} + \cdots + \binom{n}{n} = 2^n$.

Proof. The binomial coefficient $\binom{n}{k}$ is the number of k -element subsets of a n -element set $A = \{1, \dots, n\}$. Hence,

$$\sum_{k=0}^n \binom{n}{k}$$

is equal to the number of all possible subsets of A . We can associate a subset of A with a binary number of length n whereby each digit in this binary number represents a specific element in A . Now, setting a digit to 1 means that we pick the element from A at this position for our subset. Setting it to 0 means that we do not pick it. This is equal to the number of maps $\{1, \dots, n\} \rightarrow \{0, 1\}$ and the total number of such maps is equal to 2^n or in other words, the number of binary numbers that can be represented with n digits is equal to 2^n . This completes the proof. \square

We call the above binomial coefficients, because they arise as coefficients in the binomial formulas $(a + b)^n$. Let us prove this.

Theorem 1.6. (Newton's Binomial Theorem)

$$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} \cdot b^k = a^n + n \cdot a^{n-1} \cdot b + \frac{n(n-1)}{2} a^{n-2} \cdot b^2 + \dots + n \cdot a \cdot b^{n-1} + b^n$$

Proof.

1.6 Exercises

1. Find the number of subsets of a 10-element set with an odd number of elements.
2. Compute the sum $\sum_{k=0}^n 2^k \binom{n}{k}$.
3. Find the number of ways to rearrange the letters of the word *EULER*.
4. Find the number of ways to rearrange the letters of the word *CATALAN*.
5. How many necklaces can be formed with 7 distinct beads? Necklaces can be rotated and/or flipped over. All beads must be used.

2 Permutations and binomial coefficients continued

Hello

2.1 The cab driver problem

2.2 Balls in boxes and multisets

2.3 Integer compositions

2.4 Principle of inclusion and exclusion

2.5 The derangement problem

2.6 Exercises

3 Linear recurrences - The Fibonacci sequence

Hallo

3.1 Fibonacci's rabbit problem

3.2 Fibonacci numbers and the Pascal triangle

3.3 Domino tilings

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3.8 Linear recurrence relations of arbitrary order

3.9 The case of roots with multiplicities

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4 A nonlinear recurrence: many faces of Catalan numbers

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4.2 Recurrence relation for triangulations

4.3 The cashier problem

4.4 Dyck paths

4.5 Recurrence relations for Dyck paths

4.6 Reflection trick and a formula for Catalan numbers

4.7 Binary trees

4.8 Exercises

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- 5.3 When are formal power series invertible?
- 5.4 Derivation of formal power series
- 5.5 Binomial theorem for negative integer exponents
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- 5.8 Solving linear recurrence relations: general case
- 5.9 Exercises

6 Generating function of the Catalan sequence

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6.1 Composition of formal power series

6.2 Derivation and integration of formal power series

6.3 Chain rule - Inverse function theorem

6.4 Logarithm - Logarithmic derivative

6.5 Binomial theorem for arbitrary exponents

6.6 Generating function for Catalan numbers

6.7 Exercises

7 Partitions - Euler's generating function for partitions and pentagonal formula

Hallo

7.1 Definition and first examples

7.2 Young diagrams

7.3 Generating function for partitions

7.4 Partition with odd and distinct summands

7.5 Sylvester's bijection

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- 8.3 Recurrence relation for q -binomial coefficients
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