

Enumerative Combinatorics

Lecture Notes

David Scholz

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Abstract

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1 Introduction

Combinatorics is the study of finite structures. Hallo Welt

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2.1 The cab driver problem

2.2 Balls in boxes and multisets

2.3 Integer compositions

2.4 Principle of inclusion and exclusion

2.5 The derangement problem

2.6 Exercises

3 Linear recurrences - The Fibonacci sequence

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3.1 Fibonacci's rabbit problem

3.2 Fibonacci numbers and the Pascal triangle

3.3 Domino tilings

3.4 Linear recurrence relations

3.5 The characteristic equation

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3.8 Linear recurrence relations of arbitrary order

3.9 The case of roots with multiplicities

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4.2 Recurrence relation for triangulations

4.3 The cashier problem

4.4 Dyck paths

4.5 Recurrence relations for Dyck paths

4.6 Reflection trick and a formula for Catalan numbers

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5 Generating functions: a unified approach to combinatorial problems - Solving linear recurrences

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- 5.3 When are formal power series invertible?
- 5.4 Derivation of formal power series
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6 Generating function of the Catalan sequence

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6.1 Composition of formal power series

6.2 Derivation and integration of formal power series

6.3 Chain rule - Inverse function theorem

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6.5 Binomial theorem for arbitrary exponents

6.6 Generating function for Catalan numbers

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7 Partitions - Euler's generating function for partitions and pentagonal formula

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7.1 Definition and first examples

7.2 Young diagrams

7.3 Generating function for partitions

7.4 Partition with odd and distinct summands

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7.6 Euler's pentagonal theorem

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