χ-Binding Functions and Forbidden Induced Subgraphs

David Scholz

July 1, 2022

Introduction

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David Scholz

Department of Mathematics and Computer Science

University of Cologne

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David Scholz
Department of Mathematics and Computer Science
University of Cologne

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David Scholz

Department of Mathematics and Computer Science

University of Cologne

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Department of Mathematics and Computer Science

University of Cologne

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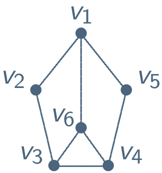

David Scholz

Department of Mathematics and Computer Science

University of Cologne

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Let G be a graph.

Definition 1

A graph H is an *induced subgraph* of G if $V(H) \subseteq V(G)$ and $(x, y) \in E(H)$ if and only if $(x, y) \in E(G)$ for all $x, y \in V(H)$.

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University of Cologne
Department of Mathematics and Computer Science

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University of Cologne

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Department of Mathematics and Computer Science

University of Cologne

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- A family $\mathcal{F}$ of graphs is $(H_1, \cdots, H_k)$-free if it contains all graphs that are $(H_1, \cdots, H_k)$-free.

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Department of Mathematics and Computer Science

University of Cologne

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- A family $\mathcal{F}$ which is closed under taking induced subgraphs is called *hereditary*.

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University of Cologne

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Department of Mathematics and Computer Science

University of Cologne

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- $\overline{G}$ is the *complement* of G if $V(\overline{G}) = V(G)$ and $(x, y) \in E(\overline{G}) \iff (x, y) \notin E(G)$ for all $x, y \in V(G)$.

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- $S \subseteq V(G)$ is called *stable* (independent) in G if $\overline{G}[S]$ is a complete graph.
- The cardinality of a largest stable set in G is called the *stable set number*, denoted by $\alpha(G)$.

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- The elements of $\{1, \cdots, k\}$ are called colors.
- A coloring is called *proper* if adjacent vertices receive different colors.
- The least k such that G admits a proper k-coloring is called the *chromatic number*, denoted by $\chi(G)$.

Perfect Graphs

Definition 2

A graph G is *perfect* if $\omega(H) = \chi(H)$ for every induced subgraph H of G.

David Scholz

Department of Mathematics and Computer Science

University of Cologne

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David Scholz
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University of Cologne

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A graph G is *perfect* if $\omega(H) = \chi(H)$ for every induced subgraph H of G.

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- A *minimal imperfect* graph is a graph that is not perfect but becomes perfect by deleting an arbitrary vertex.

David Scholz
Department of Mathematics and Computer Science
University of Cologne

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- Trivial lower bound: $\omega(G) \leq \chi(G)$.
- A *minimal imperfect* graph is a graph that is not perfect but becomes perfect by deleting an arbitrary vertex.
- Berge (1963) noticed that the minimal imperfect graphs are the induced odd cycles of length at least 5 (*odd holes*) and its complements (*odd antiholes*).

David Scholz

Department of Mathematics and Computer Science

University of Cologne

Perfect Graphs

Theorem 1 (Strong Perfect Graph Theorem, 2006)

A graph is perfect if and only if it does not contain an odd hole or odd antihole.

David Scholz

Department of Mathematics and Computer Science

University of Cologne

Perfect Graphs

Theorem 1 (Strong Perfect Graph Theorem, 2006)

A graph is perfect if and only if it does not contain an odd hole or odd antihole.

- Conjectured by Berge (1963), proven by Chudnovsky, Robertson, Seymour and Thomas (2006).

χ-Binding Functions

Definition 3 (Gyárfás, 1987)

A family $\mathcal{F}$ is χ-bounded with χ-binding function $f : \mathbb{N} \to \mathbb{N}$ if for all $G \in \mathcal{F}$ and every induced subgraph H of G, $\chi(H) \leq f(\omega(H))$.

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Theorem 2 (Erdős, 1959)

For any positive integers $k, l \geq 3$, there exists a graph G with girth $g(G) \geq l$ and chromatic number $\chi(G) \geq k$.

David Scholz

Department of Mathematics and Computer Science

University of Cologne

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Theorem 2 (Erdős, 1959)

For any positive integers $k, l \geq 3$, there exists a graph G with girth $g(G) \geq l$ and chromatic number $\chi(G) \geq k$.

$\implies$ There are graph families that are not χ-bounded.

David Scholz
Department of Mathematics and Computer Science
University of Cologne

χ-Binding Functions

Let $\mathcal{F}$ be a family of graphs.

David Scholz

Department of Mathematics and Computer Science

University of Cologne

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Let $\mathcal{F}$ be a family of graphs.

1. When is $\mathcal{F}$ χ-bounded?

David Scholz

Department of Mathematics and Computer Science

University of Cologne

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Let $\mathcal{F}$ be a family of graphs.

1. When is $\mathcal{F}$ χ-bounded?
2. What is the smallest χ-binding function of $\mathcal{F}$?

David Scholz

Department of Mathematics and Computer Science

University of Cologne

χ-Binding Functions

Let $\mathcal{F}$ be a family of graphs.

1. When is $\mathcal{F}$ χ-bounded?
2. What is the smallest χ-binding function of $\mathcal{F}$?
3. Does there exist a polynomial χ-binding function for $\mathcal{F}$?

David Scholz
Department of Mathematics and Computer Science
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χ-Binding Functions

Let $\mathcal{F}$ be a family of graphs.

1. When is $\mathcal{F}$ χ-bounded?
2. What is the smallest χ-binding function of $\mathcal{F}$?
3. Does there exist a polynomial χ-binding function for $\mathcal{F}$?
4. Does there exist a linear χ-binding function for $\mathcal{F}$?

Erdős-Hajnal Conjecture

Definition 4

A hereditary family $\mathcal{F}$ of graphs has the Erdős-Hajnal property if there exists a constant $\epsilon > 0$ such that every $G \in \mathcal{F}$ contains a clique or a stable set of size at least $|V(G)|^{\epsilon}$.

David Scholz
University of Cologne
Department of Mathematics and Computer Science

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Conjecture 1 (Erdős and Hajnal, 1989)

For every graph H, the family of H-free graphs has the Erdős-Hajnal property.

David Scholz
Department of Mathematics and Computer Science
University of Cologne

Erdős-Hajnal Conjecture

There is a connection to χ-boundedness.

David Scholz

Department of Mathematics and Computer Science

University of Cologne

Erdős-Hajnal Conjecture

There is a connection to χ-boundedness.
Let $\mathcal{F}$ be a χ-bounded family of graphs with χ-binding function f.

David Scholz
University of Cologne
Department of Mathematics and Computer Science

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There is a connection to χ-boundedness.
Let $\mathcal{F}$ be a χ-bounded family of graphs with χ-binding function f.

- Let $G \in \mathcal{F}$ with chromatic number $\chi(G) = k$.

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Department of Mathematics and Computer Science

University of Cologne

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$$\implies |V(G)| = \left| \bigcup_{i=1}^{k} S_i \right| = \sum_{i=1}^{k} |S_i| \leq \sum_{i=1}^{k} \alpha(G) = k\alpha(G)$$

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Thus,

$$|V(G)| \leq \chi(G)\alpha(G) \leq f(\omega(G))\alpha(G) \iff \frac{|V(G)|}{\alpha(G)} \leq f(\omega(G))$$

David Scholz

Department of Mathematics and Computer Science

University of Cologne

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If f is a polynomial, then the Erdős-Hajnal property for $\mathcal{F}$ follows directly!

David Scholz

Department of Mathematics and Computer Science

University of Cologne

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- Let $G \in \mathcal{F}$ with $\alpha(G) = \alpha$, $\omega(G) = \omega$ and $|V(G)| = n$.

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- Let $G \in \mathcal{F}$ with $\alpha(G) = \alpha$, $\omega(G) = \omega$ and $|V(G)| = n$.
- Suppose that G is χ-bounded by a degree d polynomial f.

David Scholz

Department of Mathematics and Computer Science

University of Cologne

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David Scholz

Department of Mathematics and Computer Science

University of Cologne

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- Suppose that G is χ-bounded by a degree d polynomial f.
- Suppose that G does not satisfy the Erdős-Hajnal property.
- Then, $\alpha < n^{\epsilon}$ for all $\epsilon > 0$.

David Scholz

Department of Mathematics and Computer Science

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- Then, $\alpha < n^\epsilon$ for all $\epsilon > 0$.

$$\begin{aligned} \frac{n}{n^\epsilon} < \frac{n}{\alpha} &\leq f(\omega) \\ n^{1-\epsilon} &< f(\omega) \end{aligned}$$

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Department of Mathematics and Computer Science

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$$\frac{n}{n^{\epsilon}} < \frac{n}{\alpha} \leq f(\omega)$$
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Set $\epsilon = \frac{1}{d+1}$.

David Scholz

Department of Mathematics and Computer Science

University of Cologne

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$$\frac{n}{n^\epsilon} < \frac{n}{\alpha} \leq f(\omega)$$
$$n^{1-\epsilon} < f(\omega)$$

Set $\epsilon = \frac{1}{d+1}$. Then, $n^{\frac{d}{d+1} \cdot \frac{1}{d}} < \omega \iff n^{\frac{1}{d+1}} < \omega$.

David Scholz
University of Cologne
Department of Mathematics and Computer Science

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$$\frac{n}{n^\epsilon} < \frac{n}{\alpha} \leq f(\omega)$$
$$n^{1-\epsilon} < f(\omega)$$

Set $\epsilon = \frac{1}{d+1}$. Then, $n^{\frac{d}{d+1} \cdot \frac{1}{d}} < \omega \iff n^{\frac{1}{d+1}} < \omega$. But then G has a clique of size larger than n^ϵ as in the Erdős-Hajnal property. A contradiction.

David Scholz

Department of Mathematics and Computer Science

University of Cologne

Esperet's Conjecture

David Scholz

University of Cologne

Department of Mathematics and Computer Science

Esperet's Conjecture

Conjecture 2 (Esperet, 2012)

Let $\mathcal{F}$ be a χ-bounded family of graphs. Then, there exists a $c \in \mathbb{R}$, such that for any $G \in \mathcal{F}$, $\chi(G) \leq \omega(G)^c$.

David Scholz

Department of Mathematics and Computer Science

University of Cologne

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Theorem 3 (Briański, Davies, Walczak, 2022)

Let $f : \mathbb{N} \to \mathbb{N} \cup \{\infty\}$ be such that $f(1) = 1$ and $f(n) \geq \binom{3n+1}{3}$ for every $n \geq 2$. Then there exists a hereditary family of graphs $\mathcal{F}$ such that $\sup\{\chi(G) : G \in \mathcal{F} \text{ and } \omega(G) = n\} = f(n)$ for every $n \in \mathbb{N}$.

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Department of Mathematics and Computer Science

University of Cologne

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There exists a hereditary family of graphs which is χ-bounded by f and f is optimal for this family. In particular, f can be arbitrary in this case.

David Scholz

Department of Mathematics and Computer Science

University of Cologne

Esperet's Conjecture

Thus Briański, Davies and Walczak disprove the conjecture of Esperet by showing that there exist hereditary χ-bounded families of graphs not having a polynomial χ-binding function.

Thereby they motivate research on hereditary χ-bounded graph families, whether they are χ-bounded by a polynomial function!

Perfect Divisibility

Definition 5 (Hoàng, 2018)

A graph G is *perfectly divisible* if for every induced subgraph H of G, $V(H)$ can be partitioned into two sets A, B, such that $\omega(G[A]) < \omega(G)$ and $G[B]$ is perfect or $V(H)$ is a stable set.

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Observation 1 (Hoàng, 2018)

Let G be a perfectly divisible graph. Then G is χ-bounded with χ-binding function $f \in \mathcal{O}(\omega(G)^2)$.

David Scholz

Department of Mathematics and Computer Science

University of Cologne

Perfect Divisibility

Proof.

We claim that G is χ-bounded by $\binom{\omega(G)+1}{2}$.

David Scholz

Department of Mathematics and Computer Science

University of Cologne

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We claim that G is χ-bounded by $\binom{\omega(G)+1}{2}$. Induction on $\omega(G) = k$.

David Scholz

Department of Mathematics and Computer Science

University of Cologne

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We claim that G is χ-bounded by $\binom{\omega(G)+1}{2}$. Induction on $\omega(G) = k$. The base case is trivial.

David Scholz

Department of Mathematics and Computer Science

University of Cologne

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Proof.

We claim that G is χ-bounded by $\binom{\omega(G)+1}{2}$. Induction on $\omega(G) = k$. The base case is trivial. Suppose that a perfectly divisible graph with clique number $k-1$ is χ-bounded by $\binom{k}{2}$.

David Scholz

Department of Mathematics and Computer Science

University of Cologne

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We claim that G is χ-bounded by $\binom{\omega(G)+1}{2}$. Induction on $\omega(G) = k$. The base case is trivial. Suppose that a perfectly divisible graph with clique number $k-1$ is χ-bounded by $\binom{k}{2}$. Let G be a perfectly divisible graph with clique number k. Then there exists a partition of $V(G)$ into two sets A and B, such that $\omega(G[A]) < \omega(G)$ and $G[B]$ is perfect.

David Scholz

Department of Mathematics and Computer Science

University of Cologne

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We claim that G is χ-bounded by $\binom{\omega(G)+1}{2}$. Induction on $\omega(G) = k$. The base case is trivial. Suppose that a perfectly divisible graph with clique number $k-1$ is χ-bounded by $\binom{k}{2}$. Let G be a perfectly divisible graph with clique number k. Then there exists a partition of $V(G)$ into two sets A and B, such that $\omega(G[A]) < \omega(G)$ and $G[B]$ is perfect. By induction $\chi(G[A]) \leq \binom{k}{2}$ and $\chi(G[B]) = \omega(G[B]) \leq k$.

David Scholz

Department of Mathematics and Computer Science

University of Cologne

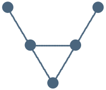
Perfect Divisibility

Proof.

We claim that G is χ-bounded by $\binom{\omega(G)+1}{2}$. Induction on $\omega(G) = k$. The base case is trivial. Suppose that a perfectly divisible graph with clique number $k-1$ is χ-bounded by $\binom{k}{2}$. Let G be a perfectly divisible graph with clique number k. Then there exists a partition of $V(G)$ into two sets A and B, such that $\omega(G[A]) < \omega(G)$ and $G[B]$ is perfect. By induction $\chi(G[A]) \leq \binom{k}{2}$ and $\chi(G[B]) = \omega(G[B]) \leq k$. Thus,
$\chi(G) \leq \chi(G[A]) + \chi(G[B]) \leq \binom{k}{2} + k = \binom{k+1}{2}$. □

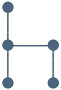
David Scholz
Department of Mathematics and Computer Science
University of Cologne

Perfect Divisibility



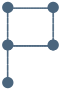
Bull

P_5

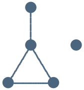


Chair

$3K_1$



Banner



co-Dart

David Scholz

Department of Mathematics and Computer Science

University of Cologne

Perfect Divisibility

Theorem 4 (Hoàng and McDiarmid, 2002)

A $3K_1$-free graph is perfectly divisible.

David Scholz

Department of Mathematics and Computer Science

University of Cologne

Perfect Divisibility

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Theorem 5 (Hoàng, 2018)

A (banner, odd hole)-free graph is perfectly divisible.

David Scholz
Department of Mathematics and Computer Science
University of Cologne

Perfect Divisibility

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Theorem 6 (Chudnovsky and Sivaraman, 2019)

A bull-free graph that is either odd-hole-free or P_5-free is perfectly divisible.

David Scholz

Department of Mathematics and Computer Science

University of Cologne

Perfect Divisibility

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Theorem 7 (Karthick, Kaufmann, Sivaraman, 2021)

A (chair, F)-free graph is perfectly divisible, when $F \in \{P_6, bull, co\text{-}dart\}$.

Hoàng's Conjecture

Conjecture 3 (Hoàng, 2018)

An odd-hole-free graph is perfectly divisible.

David Scholz

Department of Mathematics and Computer Science

University of Cologne

Hoàng's Conjecture

Conjecture 3 (Hoàng, 2018)

An odd-hole-free graph is perfectly divisible.

$\implies$ An odd-hole-free graph is χ-bounded by a quadratic function.

David Scholz

Department of Mathematics and Computer Science

University of Cologne

Hoàng's Conjecture

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An odd-hole-free graph is perfectly divisible.

$\Longrightarrow$ An odd-hole-free graph is χ-bounded by a quadratic function.

Theorem 8 (Scott and Seymour, 2015)

Let G be an odd-hole-free graph. Then $\chi(G) \leq 2^{2^{\omega(G)+2}}$.

David Scholz

Department of Mathematics and Computer Science

University of Cologne

Thank You!

References I

1. C. Berge, Perfect Graphs, Six Papers on Graph Theory, pp. 1-21. Indian Statistical Institute, Calcutta, 1963.
2. J.A. Bondy and U. S. R. Murty, Graph Theory, *Graduate Texts in Mathematics* **vol. 244**, Springer, New York, 2008.
3. M. Briański, J. Davies and B. Walczak, Separating polynomial χ-boundedness from χ- boundedness, *preprint* (2022), (Available at https://arxiv.org/abs/2201.08814).
4. M. Chudnovsky and V. Sivaraman, Perfect divisibility and 2-divisibility, *Journal of Graph Theory* **90** (2019), 54–60.
5. P. Erdős, Graph theory and probability, Can. J. Math. **11** (1959), 34-38.
6. P. Erdős and A. Hajnal, Ramsey-type theorems, Discrete Applied Mathematics **25** (1989), 37-52.

References II

7. A. Gyárfás, Problems from the world surrounding perfect graphs, *Zastosowania Matematyki Applicationes Mathematicae* **19** (1987), 413–441.
8. C. T. Hoàng, On the structure of (banner, odd hole)-free graphs, *Journal of Graph Theory* **89** (2018), 395–412.
9. C. T. Hoàng and C. McDiarmid, On the divisibility of graphs, *Discrete Mathematics* **242** (2002), 145–156.
10. T. Karthick, J. Kaufmann and V. Sivaraman, Coloring graph classes with no induced fork via perfect divisibility, *preprint* (2021). (Available at https://arxiv.org/abs/2104.02807)
11. I. Schiermeyer and B. Randerath, Polynomial χ-binding functions and forbidden induced subgraphs: A survey, *Graphs and Combinatorics* **35** (2019), 1–31.

References III

12. A. Scott and P. Seymour, A survey of χ-boundedness, *Journal of Graph Theory* **95** (2020), 473–504.

David Scholz

Department of Mathematics and Computer Science

University of Cologne