Generating Optimal 1-Planar Graphs for Automated Conjecture-Making

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Introduction

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- A conjecture is a formal statement which is initially open to formal proofs,
- Conjectures in form of inequalities: $I \leq J$, $I \leq J + K$, $I + J \leq K + L$ with I, J, K, L being invariants ($\chi \leq \Delta + 1$ (Brook's theorem)),
- Graph generation is the process of recursively applying some kind of rules in order to expand a smaller graph to a larger one without leaving the underlying class.

Optimal 1-planar graphs

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These are the plane graphs for which each face is a triangle (plane triangulations).

Optimal 1-planar graphs

Theorem 1 (Ringel)
Let G be a 1-planar graph on n vertices and m edges. Then $m \leq 4n - 8$.

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Now, if (v_1, v_2) and (v_3, v_4) in $E(\widetilde{G})$ cross, then consequently $\{(v_1, v_3), (v_1, v_4), (v_2, v_3), (v_2, v_4)\}$ are in $E(G)$ due to maximality of G.

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Hence, if we delete an edge from each pair of crossing edges, the resulting graph H is a plane triangulation on $n' = n$ vertices, m' edges and f' faces.

Optimal 1-planar graphs

By lemma 1 we have $m' \leq 3n' - 6$, therefore

$$n' - m' + f' \leq 2 \tag{1}$$

$$n' - 3n' + 6 + f' \leq 2 \tag{2}$$

$$f' \leq 2n' - 4 \tag{3}$$

□

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$$

Now, $m = m' + c \leq m' + \frac{f'}{2} \leq m' + \frac{2n'-4}{2} = m' + n' - 2 \leq 3n' - 6 + n' - 2 = 4n - 8$ $\square$

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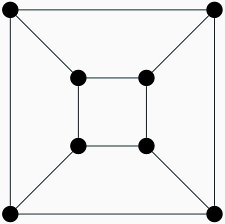
Definition 2
A (simple) quadrangulation of the sphere is a graph embedded on the sphere, such that every face is bounded by a 4-cycle.



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Given a simple quadrangulation $Q = (V, E)$, then for the average degree holds $\frac{1}{|V|} \sum_{v \in V} d(v) < 4$.

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$$\iff \sum_{v \in V} d(v) \geq 4n \tag{4}$$

$$\iff 2m \geq 4n \quad \text{(handshaking lemma)} \tag{5}$$

$$\iff \frac{1}{2}m \geq n \tag{6}$$

Optimal 1-planar graphs

Using Euler's formula ($n - m + f = 2$) yields to

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\frac{1}{2}m - m + f &\geq 2 & (7)\\
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$\square$

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Therefore, $\frac{1}{2}m \geq 2 + \frac{m}{2}$. A contradiction. $\square$

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Corollary 1
The connectivity of a simple quadrangulation cannot be more than 3.

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Given a simple quadrangulation $Q = (V, E)$ with $|V| = n$. Assume that for each $v \in V$, $d(v) = 4$. Then

$$\frac{1}{n}\sum_{v \in V} d(v) = \frac{4n}{n} = 4$$

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A contradiction to lemma 2. Hence, there must be a vertex $v' \in V$ with $d(v') \leq 3$. Its neighborhood form a cutset. $\square$

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- There are no optimal 1-planar graphs with 9 vertices.

Generating optimal 1-planar graphs

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Idea: Generate the simple 3-connected quadrangulations and add the crossing diagonals in each face.

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Definition 3
Given a class $\mathcal{C}$ of graphs and a set of expansions $\mathcal{P} := \{P_0, P_1, \cdots, P_k\}$. The relation $R(\mathcal{C}, \mathcal{P})$ is defined by

$$R(\mathcal{C}, \mathcal{P}) := \{(G, G') \in \mathcal{C} \times \mathcal{C} \mid G' \text{ can be obtained from G by applying some } P \in \mathcal{P}\}$$

Generating optimal 1-planar graphs

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Definition 4
A P_1-reduction consists of a contraction of a face (x, u, v, w) at $\{x, v\}$ whereby x has degree 3 and u, v, w each have degree at least 3.

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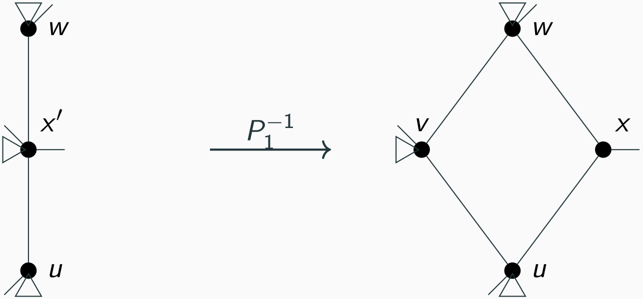

Definition 5
A P_3-reduction is a sequence of four contractions. We have faces $(u, v, w, x), (a, b, v, u), (b, c, w, v), (c, d, x, w)$ and (d, a, u, x). The vertices u, v, w, x all have degree 3, and we assume that a, b, c, d all have degree of at least 4. A P_3-reduction consists of a face contraction at $\{a, v\}$, followed by one at $\{b, w\}$, followed by one at $\{c, x\}$, followed by one at $\{d, u\}$.

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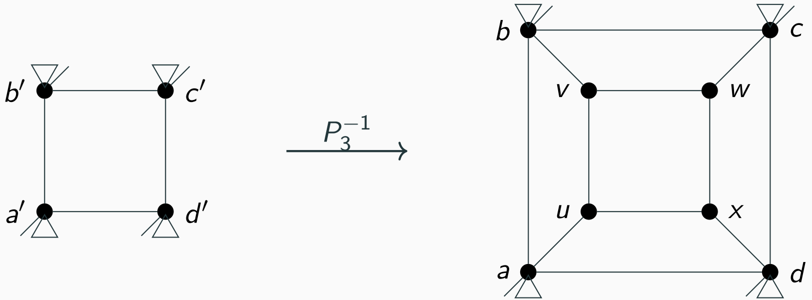

Definition 6
Given a simple quadrangulation Q with $n \geq 8$ vertices, whereby n is even. Q consists of a cycle $v_0 v_1 \cdots v_{n-3}$, as well as a vertex which is adjacent to $v_0, v_2, \cdots, v_{n-4}$ and a vertex which is adjacent to $v_1, v_3, \cdots, v_{n-3}$. We call Q a pseudo-double wheel. The smallest one is a cube.

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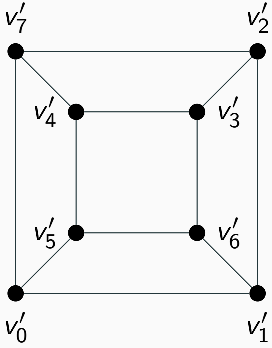
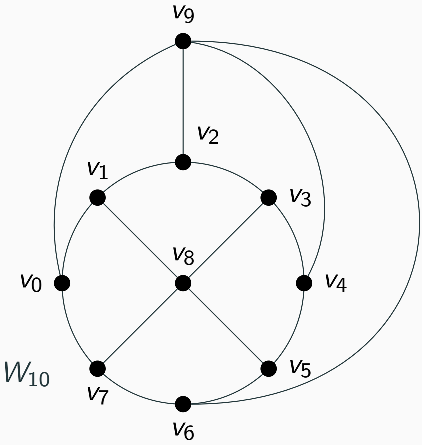


Generating optimal 1-planar graphs



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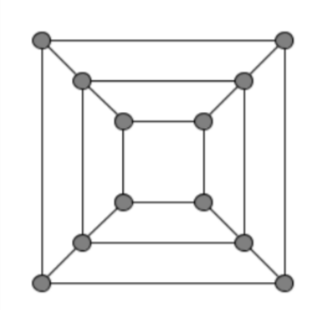

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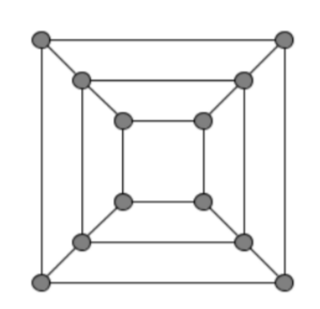

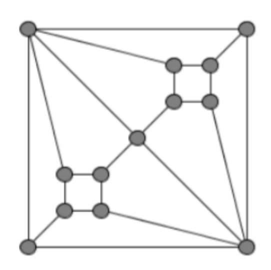

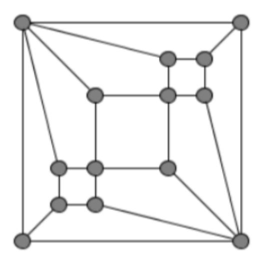

Generating optimal 1-planar graphs

Theorem 2
The class of all 3-connected quadrangulations is generated from the pseudo-double wheels by the P_1- and the P_3-expansions.

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- How do we store the generated quadrangulations?

Let $p : E \to V^2$ define two additional mappings $s, t : E \to V$ by $(s(e), t(e)) := p(e), \forall e \in E$. We call p the incidence mapping of a graph G.

Definition 7
For each $e \in E$ we define two directed edges $e^+ := (s(e), t(e))$ and $e^- := (t(e), s(e))$ and call them darts of e. We denote the sets of all darts of a graph G by D_E and call it dart relation on G.

Definition 8
A rotation for a vertex $v \in V$ is a single cyclic permutation $\pi_k = (e_1^+ e_2^+ \cdots e_k^+)$ of darts with $k \in \{1, \cdots, deg(v)\}$ and v being the source of each e_k^+. We denote the rotation of v by $rot(v)$.

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A rotation system of G is a pair of permutations (α, σ) on D_E such that $\alpha = \Pi_{e \in E}(e^+ e^-)$ and $\sigma = \Pi_{v \in V} rot(v)$.

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Definition 10
Let (α, σ) be a rotation system for G and $\Phi = \alpha^{-1}\sigma^{-1}$. Then the cycles of Φ are called faces of G for this rotation system.

Conventions

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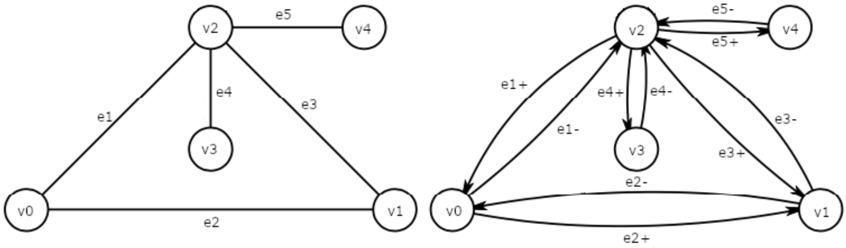
- Our rotations are given in cyclic clockwise order around its vertices.

Generating optimal 1-planar graphs



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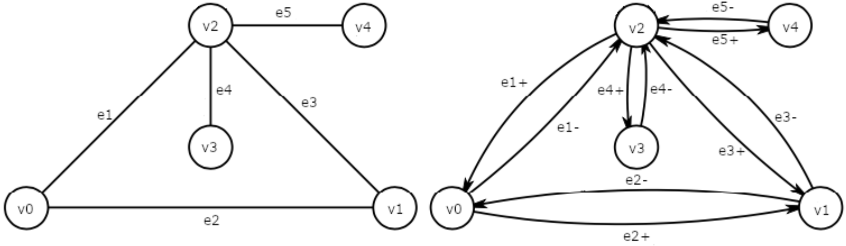

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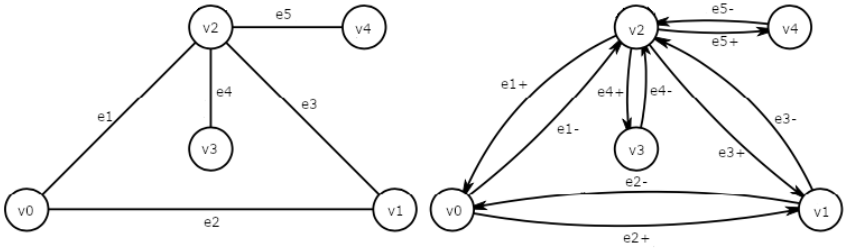

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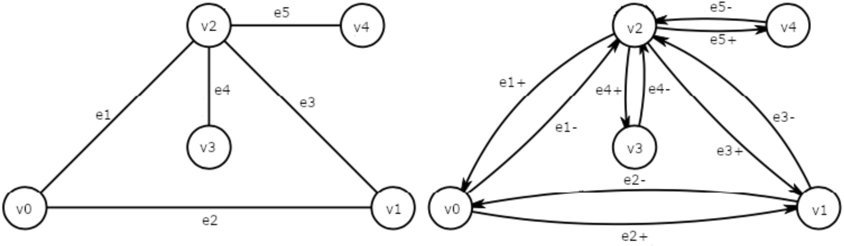

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Generating optimal 1-planar graphs

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- We list neighbours in clockwise order

$1 : 3, 4$	$2 : 3, 5$
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$5 : 2, 3, 4$	

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5   340   350   14520   1530   2340

Definition 11
Let $G = (V, E)$ and $G' = (V', E')$ be two graphs. We call G and G' *isomorphic*, and write $G \simeq G'$, if there exists a bijection $\varphi : V \to V'$ with $(x, y) \in E \iff (\varphi(x), \varphi(y)) \in E'$ $\forall x, y \in V$. Such a map φ is called an *isomorphism*.

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Idea: Generate the optimal 1-planar graphs, pick one for each isomorphism class and reject all the others.

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  - McKay's algorithm is a search algorithm which finds the canonical hash faster.

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- Planar graphs can be represented using rotation systems,

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Summary

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Generating optimal 1-planar graphs

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Generating optimal 1-planar graphs

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- It is open source.

Idea: Extend plantri in a way such that the crossing diagonals are added after the face expansions.

Problems:

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- Let us contact the author for advice.

Generating optimal 1-planar graphs

Gunnar Brinkmann <Gunnar.Brinkmann@ugent.be>
an mich

Englisch > Deutsch Nachricht übersetzen

Dear David Scholz,

[...]

The only hint I can give you about the code is: DO NOT CHANGE IT.

[...]

Best wishes,

Gunnar

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    - Our conjecture-making program is also available as a sage package (hence we are staying in the same ecosystem).

Generating optimal 1-planar graphs

Plantri running times:

Graph count	Vertex count	CPU time (seconds)
1	8	0.00
1	10	0.00
1	11	0.00
3	12	0.00
3	13	0.00
11	14	0.00
18	15	0.00
58	16	0.00
139	17	0.00
451	18	0.00
1326	19	0.00
4461	20	0.01
14554	21	0.04
49957	22	0.15
171159	23	0.49
598102	24	1.70
2098675	25	5.87
7437910	26	20.67
26490072	27	72.51
94944685	28	255.02
341867921	29	904.37
1236864842	30	3253.92

Generating optimal 1-planar graphs

Prohibiting printing to stdout:

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1	8	0.00
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2098675	25	5.06
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26490072	27	61.60
94944685	28	216.31
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Generating optimal 1-planar graphs

Sage:

Graph count	Vertex count	CPU time (seconds)
1	8	0.01
1	10	0.01
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3	12	0.01
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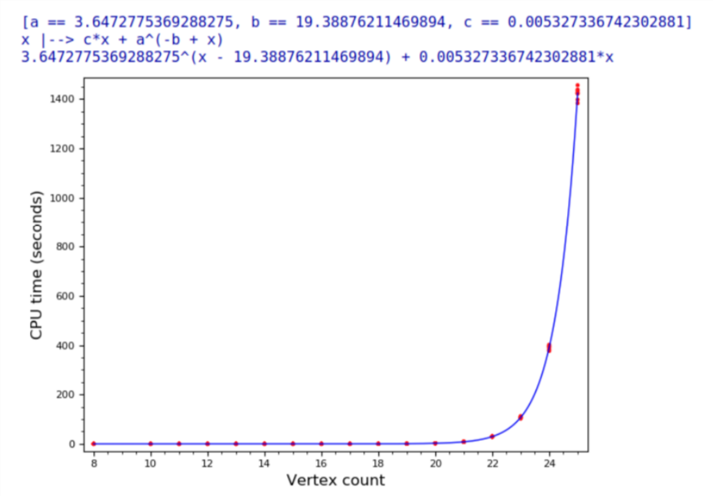
451	18	0.11
1326	19	0.33
4461	20	1.15
14554	21	3.94
49957	22	14.13
171159	23	50.79
598102	24	187.32
2098675	25	703.74
7437910	26	2491.38
26490072	27	9315.46
94944685	28	33326.80
341867921	29	127407.80
1236864842	30	–

Generating optimal 1-planar graphs



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  - Generality: True for a variety of objects (a whole class of graphs)
  - Fecundity: Proof attempts led to new concepts or new proof techniques.

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- Property-based conjectures ("Every 4-connected planar graph is hamiltonian")
- Conjectures in form of inequalities: $I \leq J$, $I \leq J + K$, $I + J \leq K + L$ with I, J, K, L being invariants ($\chi \leq \Delta + 1$ (Brook's theorem))

Automated conjecture-making

A program can produce an endless stream of conjectures in form of inequalities.

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- How can we decide which statements are significant and which are not?

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- If s and t are terms, the expression $t \leq s$ is a statement,
- We call a statement which holds for every object G in our data set a *candidate*.

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    - Truth test: $\alpha \leq s$ is true for all $G \in \mathcal{G}$
    - Significance test: at least one $G' \in \mathcal{G}$ for which $s(G') < t(G'), \forall t \in \mathcal{C}$.

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  - Search for counterexamples
  - Repeat conjecture generation.

Automated conjecture-making

Typical output:

1.	independence_number(x)	<=	ceil(arcsinh(log(size(x))))
2.	independence_number(x)	<=	1/2*min_degree(x)
3.	independence_number(x)	<=	-min_degree(x) + order(x)
4.	independence_number(x)	<=	floor(arcsinh(order(x)))
5.	independence_number(x)	<=	-minimum(num_skel_f(x), 1/4*min_degree(x)^2) + order(x)
6.	independence_number(x)	<=	crossing_number(x)/min_degree(x)
7.	independence_number(x)	<=	-floor(tan(size(x))) + num_skel_f(x)
8.	independence_number(x)	<=	ceil(log(crossing_number(x)))
9.	independence_number(x)	<=	floor(crossing_number(x)/min_degree(x))
10.	independence_number(x)	<=	2*(min_degree(x) - num_skel_f(x) + 1)^2
11.	independence_number(x)	<=	-size(x)/(max_degree(x) - order(x))
12.	independence_number(x)	<=	sqrt(max_degree(x) + order(x) - 1)
13.	independence_number(x)	<=	-1/4*max_degree(x) + 1/2*order(x)
14.	independence_number(x)	<=	-min_degree(x) + order(x)
15.	independence_number(x)	<=	-diameter(x) + 1/2*order(x)

Automated conjecture-making

Optimizations

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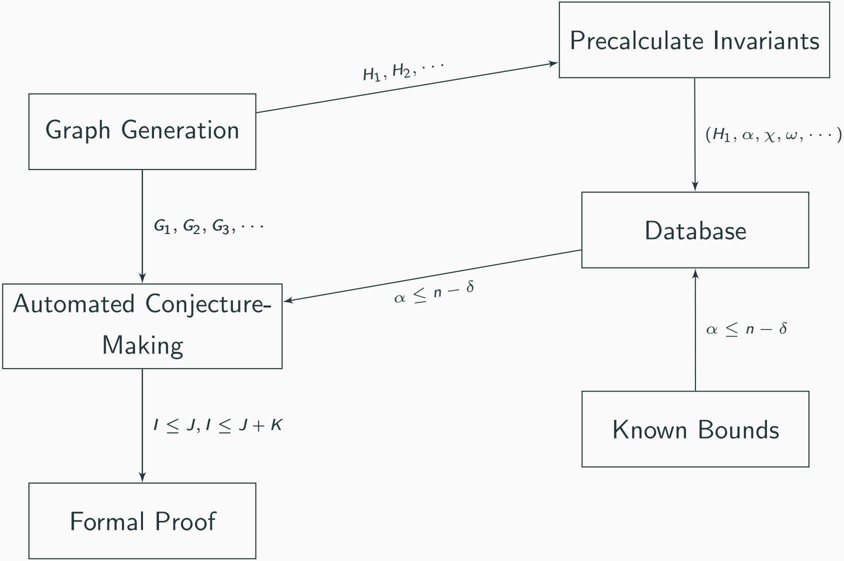
Optimizations

- Precalulate graphs and their invariants and store them in a database,
- Use known bounds from literature and store them in a database (Larson actually already published one),
- Use the project "House of Graphs" (database of the Ghent University containing large number of graphs),
- Speed up the generation process.

Automated conjecture-making



Discussion

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Conclusions

- We introduced the optimal 1-planar graphs,
- We have shown that they can be obtained by generating the simple quadrangulations and add the crossing diagonals in each face,
- We implemented a generation algorithm,
- We introduced the concept of automated conjecture-making and the Fajtlowicz-Dalmatian Heuristic
- We presented the program *conjecturing* and used it for calculating upper bounds for the independence number of the optimal 1-planar graphs.

Thank You!

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