Basic in Applied Mathematics: Project List is Stochastics

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The binomial market model

The simplest example of a discrete-time financial market is given by the *binomial model*. We assume that trading takes place at discrete times $n = 0, 1, \dots, N$.

We consider first a risk-free asset (bond) B whose dynamics are given by

$$B_n = (1 + r)^n, \qquad n = 0, \dots, N, \tag{1}$$

where the short interest rate r is assumed to be constant.

In addition, we assume that there exists a single risky asset S, whose price process evolves according to

$$S_n = S_{n-1}(1 + \mu_n), \qquad n = 1, \dots, N, \tag{2}$$

where $(\mu_n)_{n \geq 1}$ is a sequence of independent and identically distributed random variables.

We assume that the random variables μ_n take only two possible values, namely

$$1 + \mu_n = \begin{cases} u, & \text{with probability } p, \\ d, & \text{with probability } 1 - p, \end{cases}$$

where $p \in (0, 1)$ and $0 < d < u$.

Equivalently, the law of μ_n is given by the discrete probability measure

$$\mathbb{P}_\mu = p\, \delta_{u-1} + (1 - p)\, \delta_{d-1},$$

where δ_x denotes the Dirac measure at the point x.

By construction, after n time steps the stock price can be written as

$$S_n = u^k d^{n-k} S_0,$$

where k denotes the number of upward movements of the stock price up to time n.

Since the increments are independent, the distribution of S_n is given by

$$\mathbb{P}\big(S_n = u^k d^{n-k} S_0\big) = \binom{n}{k} p^k (1-p)^{n-k}, \qquad 0 \le k \le n \le N. \tag{3}$$

A graphical representation of the stock price evolution in the binomial model for three time steps is shown below.

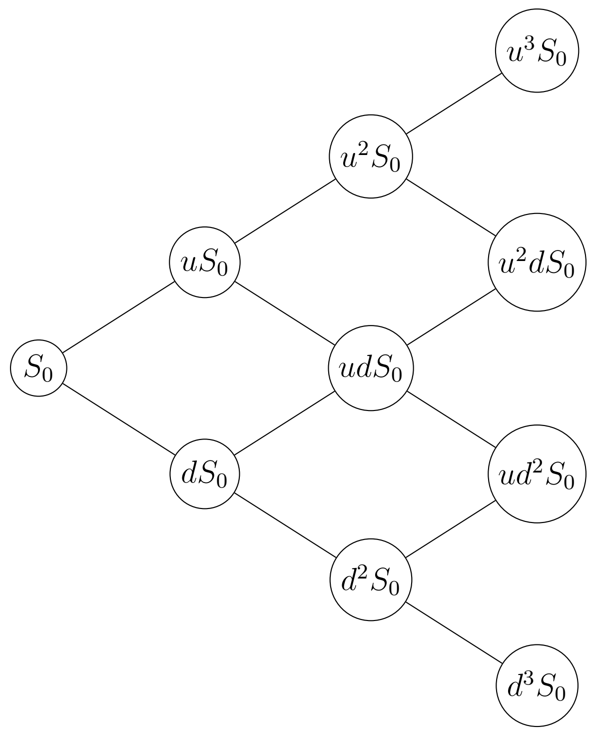

Figure 1: Recombining binomial tree with three time steps. At each period the stock price moves up by a factor u or down by a factor d.

In the binomial model, the condition

$$d < 1 + r < u \tag{4}$$

is equivalent to the existence and uniqueness of a martingale measure $\mathbb{Q}$.

Under condition (4), we define

$$q := \frac{1 + r - d}{u - d}. \tag{5}$$

The probability measure $\mathbb{Q}$ is then characterized by

$$\mathbb{Q}(1 + \mu_n = u) = 1 - \mathbb{Q}(1 + \mu_n = d) = q, \tag{6}$$

where the random variables $\mu_1, \ldots, \mu_N$ are independent under $\mathbb{Q}$.

Moreover, under the martingale measure $\mathbb{Q}$ the stock price satisfies

$$\mathbb{Q}\big(S_n = u^k d^{n-k} S_0\big) = \binom{n}{k} q^k (1-q)^{n-k}, \qquad 0 \le k \le n \le N. \tag{7}$$

In particular, under the probability measure $\mathbb{Q}$ the discounted stock price process

$$\big((1+r)^{-n} S_n\big)_{n=0}^{N}$$

is a martingale, that is,

$$\mathbb{E}^{\mathbb{Q}}\big[S_n \,\big|\, S_{n-1}\big] = (1+r)S_{n-1}, \qquad n = 1, \dots, N.$$

Derivative pricing in the binomial model

By the fundamental theorems of asset pricing, under condition (4) the binomial market is free of arbitrage and complete (see the first chapter of [1], [2]). Consequently, the *arbitrage-free price* of a derivative X at time n is

$$H_n = \frac{1}{(1+r)^{N-n}} \, \mathbb{E}^{\mathbb{Q}}\big[X \,\big|\, S_0, S_1, \dots, S_n\big], \tag{8}$$

where H_n denotes the price of the derivative at time n.

In the case of a derivative written on the terminal stock price, i.e.

$$X = F(S_N),$$

we can write the price explicitly as

$$H_n = \frac{1}{(1+r)^{N-n}} \sum_{k=0}^{N-n} \binom{N-n}{k} q^k (1-q)^{N-n-k} F\big(u^k d^{N-n-k} S_n\big), \tag{9}$$

where q is the risk-neutral probability defined in (5).

Here, $X = F(S_N)$ is called a *derivative* because its payoff depends on the underlying stock price S_N at maturity. The quantity H_n represents the arbitrage-free price of this derivative at time n, computed as the discounted expected value under the risk-neutral measure $\mathbb{Q}$.

(a) Pricing European call and put options Consider a 4-period binomial market with the following parameters:

$$S_0 = 100, \quad u = 1.1, \quad d = 0.9, \quad r = 0.02.$$

We define a European *call* and a European *put* option on the stock with 5 different strike prices:

$$K \in \{90, 95, 100, 105, 110\}.$$

1. Using the binomial model, compute the arbitrage-free price H_n of each option at all time steps $n = 0, 1, 2, 3, 4$. You can use either the explicit formula (9) or backward induction on the recombining tree.
2. Plot the evolution of the option price over time for each strike. Produce one plot for the calls and one plot for the puts, showing how the price changes as time moves from $n = 0$ to $n = 4$.

Hints / guidance:

- Compute the risk-neutral probability q using

$$q = \frac{1 + r - d}{u - d}.$$

- The payoff at maturity is

$$F_{\text{call}}(S_4) = \max(S_4 - K, 0), \quad F_{\text{put}}(S_4) = \max(K - S_4, 0).$$

- Use a recombining binomial tree to compute H_n for $n = 3, 2, 1, 0$ recursively if you prefer backward induction.
- Label your plots clearly (time step on x-axis, option price on y-axis, different curves for different strikes).

(b) Hedging European call and put options In this task, you will compute the *replicating portfolio* for the European call and put options considered in part (a) and show that its evolution matches the option price over time.

We aim to construct a portfolio

$$V_{n-1} = \bar{\alpha}_n S_{n-1} + \bar{\beta}_n B_{n-1},$$

investing in the risky asset S and the risk-free bond B, which replicates the option payoff.

Suppose that S_{n-1} is the known stock price at time $n-1$. Then at the next step n, the stock can move either up or down:

$$S_n = \begin{cases} uS_{n-1}, \\ dS_{n-1}. \end{cases}$$

Denote by V_n^u and V_n^d the value of the replicating portfolio at time n in the up and down states, respectively. The portfolio must satisfy the system

$$\begin{cases} \bar{\alpha}_n \, uS_{n-1} + \bar{\beta}_n \, B_n = V_n^u, \\ \bar{\alpha}_n \, dS_{n-1} + \bar{\beta}_n \, B_n = V_n^d. \end{cases}$$

At the final time N, the portfolio must match the derivative payoff:

$$V_N = X = F(S_N),$$

so that backward induction can be used to compute $\bar{\alpha}_n$ and $\bar{\beta}_n$ at all previous times $n = N-1, N-2, \ldots, 0$.

Tasks:

1. Solve the linear system above at each node to find $\bar{\alpha}_n$ and $\bar{\beta}_n$ (you can solve it analytically or numerically).
2. Compute the evolution of the replicating portfolio V_n over all time steps for each strike price used in part (a).
3. Plot V_n for calls and puts and compare with the option prices H_n obtained in part (a). You should observe that the portfolio exactly replicates the option value at each time step.

Hints:

- Use the recombining binomial tree from part (a) to organize your calculations.
- Make sure to check your portfolio values against the option prices at each node to confirm replication.

Black–Scholes model Recall that in the Black–Scholes model the risk-neutral stock price follows the geometric Brownian motion

$$dS_t = rS_t\,dt + \sigma S_t\,dW_t, \qquad S_0 \text{ given},$$

where r is the risk-free rate, σ is the volatility, and $(W_t)_{t\geq 0}$ is a standard Brownian motion.

The price of a European call option with strike K and maturity T is given by

$$C_0 = S_0\,N(d_1) - Ke^{-rT}N(d_2),$$

where

$$d_1 = \frac{\ln(S_0/K) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}, \qquad d_2 = d_1 - \sigma\sqrt{T},$$

and $N(\cdot)$ denotes the cumulative distribution function of the standard normal distribution.

(c) Convergence to the Black–Scholes model (Optional) In this task, you will numerically illustrate how the binomial model converges to the Black–Scholes price as the number of time steps increases.

Instructions:

1. Consider a European call option with the following parameters:

$$S_0 = 100, \quad K = 100, \quad r = 0.05, \quad \sigma = 0.2, \quad T = 1 \text{ year}.$$

2. For a sequence of increasing numbers of time steps

$$N \in \{5, 10, 20, 50, 100, 1000\},$$

compute the binomial price of the option at $t = 0$ by setting

$$u = e^{\sigma\sqrt{T/N}}, \qquad d = e^{-\sigma\sqrt{T/N}} = \frac{1}{u}.$$

3. Compute the Black–Scholes price for the same option using volatility σ.
4. Plot the binomial prices as a function of N together with the Black–Scholes price as a horizontal line, to visually show convergence. Alternatively, plot the error (difference between binomial and Black–Scholes price) and show that it converges to zero as N increases.

Hint: The risk-neutral probability in the above binomial tree is

$$q = \frac{e^{rT/N} - d}{u - d}.$$

Remark: Students may also experiment with alternative parametrizations of the up and down factors, for instance

$$u = e^{(r-\sigma^2/2)\Delta t + \sigma\sqrt{\Delta t}}, \qquad d = e^{(r-\sigma^2/2)\Delta t - \sigma\sqrt{\Delta t}}, \qquad \Delta t = T/N,$$

to see how the convergence to the Black–Scholes price is affected.

Kalman Filter in Discrete Time

The Kalman filter provides a recursive method to estimate an unobservable state vector from noisy observations in a linear Gaussian system.

Model dynamics Let $(X_n)_{n\geq 0}$ be the unobserved state and $(Y_n)_{n\geq 1}$ the observations. We assume the linear Gaussian dynamics:

$$X_0 \sim \mathcal{N}(\hat{x}_0, P_0), \tag{10}$$

$$X_n = AX_{n-1} + W_n, \qquad W_n \sim \mathcal{N}(0, Q), \tag{11}$$

$$Y_n = HX_n + V_n, \qquad V_n \sim \mathcal{N}(0, R), \tag{12}$$

where W_n and V_n are independent white Gaussian noises, and A, H, Q, R are known matrices (or scalars in the 1D case).

Kalman filter recursion (prediction + update) Define

$$\hat{X}_{n|n-1} = \mathbb{E}[X_n \mid Y_1 = y_1, \ldots, Y_{n-1} = y_{n-1}]$$

and

$$\hat{X}_{n|n} = \mathbb{E}[X_n \mid Y_1 = y_1, \ldots, Y_n = y_n]$$

with corresponding covariances $P_{n|n-1}$ and $P_{n|n}$.

1. **Prediction step:**

$$\hat{X}_{n|n-1} = A\hat{X}_{n-1|n-1}, \tag{13}$$
$$P_{n|n-1} = AP_{n-1|n-1}A^\top + Q. \tag{14}$$

2. **Update (filtering) step:**

$$K_n = P_{n|n-1}H^\top(HP_{n|n-1}H^\top + R)^{-1}, \tag{15}$$
$$\hat{X}_{n|n} = \hat{X}_{n|n-1} + K_n(y_n - H\hat{X}_{n|n-1}), \tag{16}$$
$$P_{n|n} = (I - K_nH)P_{n|n-1}. \tag{17}$$

Here, K_n is the *Kalman gain*, $\hat{X}_{n|n}$ is the filtered estimate of the state at time n, and $P_{n|n}$ its conditional covariance (note it does not depend on the oberservations).

Kalman Filter Project Fix the system parameters (A, H, Q, R) and initial values $(\hat{x}_0, P_0)$, and simulate a realization of the state $(X_n)_{n\geq 0}$ and observations $(Y_n)_{n\geq 1}$ according to the linear Gaussian model.

Task: Implement the Kalman filter recursion to compute the filtered estimates $\hat{X}_{n|n}$ for $n = 1, \ldots, N$.

1. Plot the evolution of the filtered estimates $\hat{X}_{n|n}$ together with the true states X_n to visualize the performance of the filter.
2. Compute and plot the *mean square error (MSE)* of the filter:

$$\mathrm{MSE}_n = \mathbb{E}\left[\left(X_n - \hat{X}_{n|n}\right)^2\right].$$

   Since the true expectation is typically unknown, approximate it using the *Monte Carlo method*, i.e., simulate M independent realizations of the state and observation sequences, apply the Kalman filter to each, and compute the empirical mean:

$$\mathrm{MSE}_n \approx \frac{1}{M}\sum_{i=1}^{M}\left(X_n^{(i)} - \hat{X}_{n|n}^{(i)}\right)^2.$$

   This allows you to estimate the average squared error of the filter across different sample paths.

3. Observe that the Kalman filter provides the *minimum mean square error* estimate of the state given the observations, i.e., it is optimal in the Bayesian sense.

Estimating the Mean Return of an Asset As an example of a financial application, you can use the Kalman filter to estimate the (log) mean return of an asset, such as the MSCI World index.

Task:

1. Download historical daily prices for the MSCI World index (or another asset of your choice) from a public source such as Yahoo Finance.
2. Compute the daily *log returns* from the price data:

$$Y_n = \log \frac{S_n}{S_{n-1}}, \quad n = 1, \dots, N.$$

   Consider these log returns as your observations Y_n.

3. Assume a simple Gaussian random walk model for the (unobservable) mean return μ_n:

$$\mu_n = \mu_{n-1} + w_n, \qquad w_n \sim \mathcal{N}(0, Q),$$

$$Y_n = \mu_n + v_n, \qquad v_n \sim \mathcal{N}(0, R),$$

   where Q and R are the process and observation noise variances, respectively.

4. Apply the Kalman filter recursion to estimate $\hat{\mu}_{n|n}$, the filtered estimate of the mean log return at each day n.
5. Plot the estimated mean return $\hat{\mu}_{n|n}$ over time and discuss its evolution. Optionally, compare it with the simple sample mean of all observed returns up to each day.

Vasicek Short-Rate Model

The Vasicek model describes the evolution of the short interest rate $(r_t)_{t\geq 0}$ as a *mean-reverting* stochastic process. Its dynamics are given by the stochastic differential equation (SDE)

$$dr_t = \theta(\mu - r_t)\, dt + \sigma\, dW_t, \quad r_0 \text{ given},$$

where:

- $\theta > 0$ is the *speed of mean reversion*, controlling how fast r_t tends to revert to its long-term level μ,
- μ is the *long-term mean* of the short rate,
- $\sigma > 0$ is the *volatility*, measuring the intensity of random fluctuations,
- $(W_t)_{t\geq 0}$ is a standard Brownian motion representing Gaussian noise.

Interpretation as ODE + Gaussian noise The Vasicek SDE can be viewed as a classical ordinary differential equation (ODE)

$$\frac{dr_t}{dt} = \theta(\mu - r_t)$$

driven by an additive Gaussian noise σdW_t. The deterministic part $\theta(\mu - r_t)$ ensures mean reversion, while the stochastic part σdW_t introduces randomness in the evolution of the interest rate.

This model is widely used in finance for simulating short-term interest rates and pricing interest-rate dependent securities (see, for instance, [3]).

(a): Monte Carlo Simulation and Mean Consider the Vasicek short-rate model

$$dr_t = \theta(\mu - r_t)\,dt + \sigma\,dW_t, \qquad r_0 \text{ given},$$

and discretize it using the Euler scheme with time step Δt:

$$r_{t+\Delta t} = r_t + \theta(\mu - r_t)\Delta t + \sigma\sqrt{\Delta t}\,Z_t, \quad Z_t \sim \mathcal{N}(0,1).$$

Task:

1. Simulate an increasing number of paths of r_t for $N \in \{10, 50, 100, 1000\}$ using the Euler discretization.
2. For each t, compute the *empirical mean* over the N paths:

$$\bar{r}_t = \frac{1}{N}\sum_{i=1}^{N} r_t^{(i)}.$$

   This is an example of a *Monte Carlo estimate*, i.e., approximating the expected value by the average over simulated samples.

3. Plot $\bar{r}_t$ together with the theoretical mean of the Vasicek process:

$$\mathbb{E}[r_t] = r_0 e^{-\theta t} + \mu(1 - e^{-\theta t}),$$

   and observe how the empirical mean converges to the theoretical curve as N increases.

Part (b): Zero-Coupon Bond Pricing A zero-coupon bond is a financial instrument that pays exactly \$1 at its maturity T. The price of the bond at time $t \le T$, denoted $P(t,T)$, depends on the current short rate r_t.

Affine form: In the Vasicek model, the bond price can be written as

$$P(t,T) = e^{A(t,T) - B(t,T) r_t},$$

where $A(t,T)$ and $B(t,T)$ are functions of time that satisfy the following ordinary differential equations (ODEs):

$$\frac{dB(t,T)}{dt} = 1 - \theta B(t,T), \qquad B(T,T) = 0,$$

$$\frac{dA(t,T)}{dt} = \frac{1}{2}\sigma^2 B(t,T)^2 - \theta\mu B(t,T), \qquad A(T,T) = 0.$$

Task:

1. Using fixed realistic parameters, for example:

   $$r_0 = 0.03, \quad \theta = 0.15, \quad \mu = 0.05, \quad \sigma = 0.01,$$

   solve the ODEs for $B(t,T)$ and $A(t,T)$ either analytically or numerically, for selected maturities $T \in \{1, 2, 5, 10\}$ years.

2. Plot the zero-coupon bond prices $P(0,T)$ as a function of T, showing the yield curve implied by the Vasicek model.

3. Optionally, you can simulate multiple short-rate paths using Euler discretization and compute the bond price via Monte Carlo to compare with the analytic/numerical solution.

References

[1] H. Föllmer and A. Schied, *Stochastic Finance: An Introduction in Discrete Time*, 4th ed., De Gruyter, Berlin, 2016.

[2] A. Pascucci and W. J. Runggaldier, *Financial Mathematics: Theory and Problems for Multi-period Models*, Springer, Milan, 2012.

[3] T. Björk, *Arbitrage Theory in Continuous Time*, 4th ed., Oxford University Press, Oxford, 2020.