zk-SNARKs

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From theory to practice...

zkSnark

Zero **K**nowledge **S**uccinct **N**on **I**nteractive **A**rguments Of **K**nowledge

Use

Efficiently verify the correctness of computations without executing them

Applications

- Verify cloud computations (centralised, decentralised)
- Anonymous bitcoin (ZCash)

Application Model

- A client owns input u (e.g query)
- A server owns a private input w (e.g. private DB)
- The client wishes to learn $z = f(u, w)$ for a function f known to both
- Client: computation correctness (integrity)
- Server: private input confidentiality

Client: its computing power should be confined to the bare minimum of sending u and receiving z

What zk-Snarks offer

- **Z**ero **K**nowledge: The client (verifier $\mathcal{V}$) learns nothing but the validity of the computation
- **S**uccinct: The proof is tiny compared to the computation
  - the proof size is constant $O_\lambda(1)$ (depends only on the security parameter λ)
  - verification time is $O_\lambda(|f| + |u| + |z|)$ and does not depend on the running time of f
- **N**on **I**nteractive: The proofs are created without interaction with the verifier and are publicly verifiable strings
- **A**rguments: Soundness is guaranteed only against a computationally bounded server (prover $\mathcal{P}$)
- of **K**nowledge: The proof cannot be constructed without access to a witness

Position in the complexity landscape...

- $NP = PCP[O(logn), O(1)]$
- $NP \subseteq ZK$ (Goldreich, Micali, Wigderson)
- We can use PCP to construct ZK proofs (in theory)
- The proofs are hugely inefficient
- Can we construct better SNARKs without using PCPs?
- Yes, using QSPs and QAP - a better characterisation of NP and cryptographic assumptions

Main idea

1. Transform the verification of the computation to checking a relation between secret polynomials:

$$\text{computation validity} \leftrightarrow p(x)q(x) = s(x)r(x)$$

2. The verifier chooses a random evaluation point that must be kept secret:

$$p(x_0)q(x_0) = s(x_0)r(x_0)$$

3. Homomorphic Encryption to compute the evaluation of the polynomials at x_0 by using $\mathtt{Enc}(x_0)$:

$$\mathtt{Enc}(p(x_0))\mathtt{Enc}(q(x_0)) = \mathtt{Enc}(s(x_0))\mathtt{Enc}(r(x_0))$$

4. Blinding for ZK:

$$\mathtt{Enc}(p(x_0))\mathtt{Enc}(q(x_0))k = \mathtt{Enc}(s(x_0))\mathtt{Enc}(r(x_0))k$$

ZK Proofs

- Shaffi Goldwasser, Silvio Micali and Charles Rackoff, 1985
- Interactive proof systems
  - Computation as a dialogue
  - Prover ($\mathcal{P}$): wants to prove that a string belongs to a language
  - Verifier ($\mathcal{V}$): wants to check the proof st:
    - A correct proof convinces $\mathcal{V}$ with overwhelming probability
    - A wrong proof convinces $\mathcal{V}$ with negligible probability
- Zero Knowledge Proofs
  - $\mathcal{V}$ is convinced without learning anything else

A breakthrough with many theoretical and practical applications

An easy example

- $\mathcal{V}$ is color blind
- O $\mathcal{P}$ holds two identical balls of different color
- Can the $\mathcal{V}$ be convinced of the different colors?
- Yes
  - $\mathcal{P}$ hands the balls to $\mathcal{V}$ (commit)
  - $\mathcal{V}$ hides the balls behind his back, one in each hand
  - He randomly decides to switch hands or not
  - $\mathcal{V}$ presents the balls to $\mathcal{P}$ (challenge)
  - $\mathcal{P}$ responds if the balls have switched hands (response)
  - $\mathcal{V}$ accepts or not
  - Malicious $\mathcal{P}$: Cheating Probability 50%
  - Repeat to reduce

Definitions: Notation

- Language $\mathcal{L} \in \texttt{NP}$
- Polynomial Turing Machine $\mathcal{M}$
- $x \in \mathcal{L} \Leftrightarrow \exists w \in \{0,1\}^{p(|x|)} : M(x,w) = 1$
- 2 PPT TM $\mathcal{P}$, $\mathcal{V}$
- $< \mathcal{P}(x,w), \mathcal{V}(x) >$ is the interaction between $\mathcal{P}$, $\mathcal{V}$ with common public input x and private $\mathcal{P}$ input w.
- $out_{\mathcal{V}} < \mathcal{P}(x,w), \mathcal{V}(x) >$ is the output of $\mathcal{V}$ at the end of the protocol

Properties: Completeness and Soundness

Completeness

An honest $\mathcal{P}$, convinces an honest $\mathcal{V}$ with certainty: If $x \in \mathcal{L}$ and $M(x, w) = 1$ then: $Pr[out_{\mathcal{V}} < \mathcal{P}(x, w), \mathcal{V}(x) > (x) = 1] = 1$

Properties:Soundness

A malicious $\mathcal{P}$ ($\mathcal{P}^*$), only convinces an honest $\mathcal{V}$, with negligible probability. If $x \notin \mathcal{L}$ $\forall(\mathcal{P}^*, w)$:
$Pr[out_{\mathcal{V}} < \mathcal{P}^*(x, w), \mathcal{V}(x) > (x) = 1] = negl(\lambda)$

Note:
Proof of Knowledge: $\mathcal{P}^*$ is not PPT.
Argument of Knowledge: O $\mathcal{P}^*$ is PPT.

Properties:(Perfect) Zero Knowledge

$\mathcal{V}$ does not gain any more knowledge than the validity of the $\mathcal{P}$'s claim .

For each $\mathcal{V}^*$ there is a PPT $\mathcal{S}$:

If $x \in \mathcal{L}$ and $M(x, w) = 1$ the random variables:

$out_{\mathcal{V}^*} < \mathcal{P}(x, w), \mathcal{V}^*(x) > (x)$ and

$out_{\mathcal{V}^*} < \mathcal{S}(x), \mathcal{V}^*(x) > (x)$

follow the same distribution:

We allow a malicious verifier that does not follow the protocol and cheats in order to learn w

Intuition

What ever the $\mathcal{V}$ can learn after interacting with the $\mathcal{P}$, can be learnt by interacting with $\mathcal{S}$ (disregarding $\mathcal{P}$)

Constructing the simulator

A theoretical construction with practical applications

Reminder: $\mathcal{S}$ does not have access to the witness

- $\mathcal{S}$ take $\mathcal{P}$'s place during th interaction with $\mathcal{V}$
- We cannot distinguish between $<\mathcal{S}\ ,\mathcal{V}\ >$ and $<\mathcal{P}\ ,\mathcal{V}\ >$
- We allow rewinds:
- when $\mathcal{V}$ sets a challenge that cannot be answered by $\mathcal{S}$ then we stop and rewind it
- ZK if despite the rewind $\mathcal{V}$ accepts at some point
- Why? Because he cannot distinguish between $\mathcal{P}$ (with the witness) and $\mathcal{S}$ (without the witness)
- As long as $\mathcal{S}$ is PPT
- As a result $\mathcal{V}$ extracts the same information from $\mathcal{P}$ and $\mathcal{S}$ (nothing to extract)

Cryptographic Applications

- Authentication without passwords
  - Proof that the user know the password
  - Transmission and processing is not needed
- Proof that a ciphertext contains a particular message
- Digital signatures
- Anti-Malleability
- In general: Proof that a player follows a protocol without releasing any private input

Σ - protocols

A 3 round protocol with an honest verifier and special soundness

1. **Commit** $\mathcal{P}$ commits to a value
2. **Challenge** $\mathcal{V}$ selects a random challenge uniformly from a challenge space (honest)
3. **Response** $\mathcal{P}$ responds using the commitment, the witness and the random challenge.

Special Soundness

Two execution of the protocol with the same commitment reveal the witness

Knowledge of DLOG:Schnorr's protocol I

Protocol input

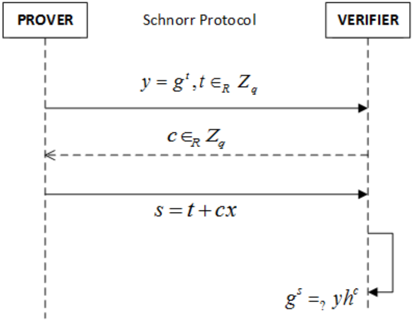
- **Public:** g is a generator of an order q subgroup of $\mathbb{Z}_p^*$ with hard DLP and a random $h \in \mathbb{Z}_p^*$
- **Private:** $\mathcal{P}$ knows a witness $x \in \mathbb{Z}_q^*$ st: $h = g^x \pmod{p}$

Goal

Proof of knowledge of x without releasing any more information

Knowledge of DLOG:Schnorr's protocol II

- **Commit** ($\mathcal{P} \rightarrow \mathcal{V}$):
  - Randomly Select $t \in_R \mathbb{Z}_q^*$
  - Compute $y = g^t \bmod p$.
  - Send y to $\mathcal{V}$.
- **Challenge** ($\mathcal{V} \rightarrow \mathcal{P}$): Select and challenge with $c \in_R \mathbb{Z}_q^*$
- **Response** ($\mathcal{P} \rightarrow \mathcal{V}$): $\mathcal{P}$ computes $s = t + cx \bmod q$ and sends it to $\mathcal{V}$
- $\mathcal{V}$ accepts iff $g^s = yh^c \pmod{p}$



Properties I

- **Completeness**

$$g^s = g^{t+cx} = g^t g^{cx} = yh^c \pmod{p}$$

- **Soundness** Probability that $\mathcal{P}^*$ cheats an honest verifier: $\frac{1}{q}$ - negligible - repeat to decrease
- **Special soundness** Let (y, c, s) nad (y, c', s') be two successful protocol transcripts

$$g^s = yh^c \quad g^{s'} = yh^{c'} \Rightarrow g^s h^{-c} = g^{s'} h^{-c'} \Rightarrow$$

$$g^{s-xc} = g^{s'-xc'} \Rightarrow s - xc = s' - xc' \Rightarrow x = \frac{c' - c}{s - s}$$

Since $\mathcal{P}$ can answer these 2 questions he knows DLOG of h

Properties II

Zero knowledge: no

- A cheating verifier does not choose randomly
- but bases each challenge to the commitment received before $\mathcal{S}$
- In the simulated execution it will switch challenge
- $\mathcal{S}$ will not be able to respond

How to add ZK:

- $\mathcal{V}$ commits to randomness before the first message by $\mathcal{P}$ or
- Challenge space $\{0, 1\}$
  - In this case $\mathcal{V}$ has only two options.
  - As a result the $\mathcal{S}$ can prepare for both.

Properties III

It provides Honest Verifier Zero Knowledge. Let $\mathcal{S}$ without knowledge of the witness x and an honest $\mathcal{V}$

- $\mathcal{S}$ follows the protocol and commits to $y = g^t, t \in_R \mathbb{Z}_q^*$
- $\mathcal{V}$ selects $c \in_R \mathbb{Z}_q^*$
- If $\mathcal{S}$ can answer (which occurs with negligible probability) the protocol resumes normally
- Else the $\mathcal{V}$ is rewound (with the same random tape)
- $\mathcal{V}$ selects the same $c \in_R \mathbb{Z}_q^*$ (because the random tape has not changed)
- $\mathcal{S}$ sends $s = t$. $\mathcal{V}$ will accept since $yh^c = g^t h^{-c} h^c = g^t = g^s$

The conversations $(t \in_R \mathbb{Z}_q; g^t h^{-c}, c \in_R \mathbb{Z}_q, t)$
$(t, c \in_R \mathbb{Z}_q; g^t, c, t + xc)$ follow the same distribution

Removing interactivity

Question

Can we do away with $\mathcal{V}$?

$\mathcal{P}$ generates the proof by himself
The proof is verifiable by anyone

Fiat Shamir Transform

Replace the challenge with the output of a pseudorandom function on the commitment
In practice we use a hash function $\mathcal{H}$

Non-interactive Schnorr with the Fiat Shamir

Input

- **Public:** g is a generator of an order q subgroup of ($\mathbb{Z}_p^*$ with hard DLP and $h \in \mathbb{Z}_p^*$
- **Private:**$\mathcal{P}$ has a witness $x \in \mathbb{Z}_q^*$ st: $h = g^x \bmod p$

The Prover:

- Randomly select $t \in_R \mathbb{Z}_q$,
- Compute $y = g^t \bmod p$
- **Compute** $c = \mathcal{H}(y)$ **where** $\mathcal{H}$ **is a hash function in** $\mathbb{Z}_q$
- Compute $s = t + cx \bmod q$
- **Release** (h, c, s)
- **Anyone can verify that** $c = \mathcal{H}(g^s h^{-c})$

The common reference string

Both parties have access to a string of random data
This is created in a trusted way (e.g. through a secure multiparty computation protocol)
The prover simulates the verifier challenge by selecting random data

Homomorphic Encryption Schemes

Applying a function on the ciphertexts yields the encryption of a function on the plaintext

$$\mathtt{Enc}(m_1) \otimes \mathtt{Enc}(m_2) = \mathtt{Enc}(m_1 \oplus m_2)$$

Multiplicative Homomorphism in El Gamal:

$$\begin{aligned}\mathtt{Enc}(m_1) \cdot \mathtt{Enc}(m_2) &= (g^{r_1}, m_1 h^{r_1}) \cdot (g^{r_2}, m_2 h^{r_2}) \\ &= (g^{r_1+r_2}, (m_1 \cdot m_2) h^{r_1+r_2})\end{aligned}$$

Additive Homomorphism in El Gamal:

$$\begin{aligned}\mathtt{Enc}(m_1) \cdot \mathtt{Enc}(m_2) &= (g^{r_1}, g^{m_1} h^{r_1}) \cdot (g^{r_2}, g^{m_2} h^{r_2}) \\ &= (g^{r_1+r_2}, g^{m_1+m_2} h^{r_1+r_2})\end{aligned}$$

Application - polynomials

Task

Let $\texttt{Enc}(x) = g^x$ where g is a suitable group generator and $p(x) = \sum_{i=0}^{d} a_i x^i$ a polynomial
Two parties with knowledge of x_0 and $p(x)$ respectively can compute $\texttt{Enc}(p(x_0))$

- The $\mathcal{V}$ (the party that knows x_0) releases

$$\texttt{Enc}(x_0^0), \texttt{Enc}(x_0^1), \cdots, \texttt{Enc}(x_0^d)$$

  into the common reference string
- The $\mathcal{P}$ (the party that knows the coefficients) computes:

$$\prod_{i=0}^{d} \texttt{Enc}(x_0^i)^{a_i} = \texttt{Enc}(\sum_{i=0}^{d} a_i x_0^i) = \texttt{Enc}(p(x_0))$$

Pairings I

In general

Functions that map elements from source groups $\mathcal{G}_1, \mathcal{G}_2$ or $\mathcal{G}^2$ to a destination group $\mathcal{G}_T$.

What is interesting: They transform difficult problems in $\mathcal{G}$ to easy problems in $\mathcal{G}_T$.

Definition

A pairing is an efficiently calculable function $e : \mathcal{G} \times \mathcal{G} \to \mathcal{G}_T$ st:

- Bilinear: $e(g^a, g^b) = e(g, g)^{ab}$ where $g \in \mathcal{G}$ $a, b \in \mathbb{Z}$
- Non-Degenerate:If $\mathcal{G} =< g >$ then $\mathcal{G}_T =< e(g, g) >$

Pairings II

In practice: $G = \mathcal{E}(\mathbb{F}_p)$ and $G_T = \mathbb{F}_{p^a}$

How to easily solve DDH

Input: (g, g^a, g^b, g^c)
Check if $g^c = g^{ab}$
Easily compute $e(g^a, g^b) = e(g, g)^{ab}$
Compare with $e(g, g^c) = e(g, g)^c$
but the CDH remains hard

Observation

The pairing allows us to do a multiplication between 'encrypted' values

Application - check the correct evaluation of polynomials I

- The $\mathcal{V}$ that knows x_0:
  - computes and publishes into the CRS:

$$\mathtt{Enc}(x_0^0), \mathtt{Enc}(x_0^1), \cdots, \mathtt{Enc}(x_0^d)$$

  - selects a scaling factor b
  - computes and publishes into the CRS:

$$\mathtt{Enc}(bx_0^0), \mathtt{Enc}(bx_0^1), \cdots, \mathtt{Enc}(bx_0^d)$$

- The $\mathcal{P}$ that knows $p(x)$:
  - computes and publishes $\mathtt{Enc}(p(x_0)), \mathtt{Enc}(bp(x_0))$
- The secrets b, x_0 should be destroyed

Application - check the correct evaluation of polynomials II

Check:

- Use a pairing function e to compute:
  - $e(\texttt{Enc}(p(x_0)), \texttt{Enc}(b)) = e(g, g)^{bp(x_0)}$
  - $e(\texttt{Enc}(bp(x_0)), \texttt{Enc}(1)) = e(g, g)^{bp(x_0)}$

Observation

- The homomorphic combination of encrypted polynomials allows us to do additions
- plus the multiplication from the pairing

A 'new' security assumption

Knowledge of exponents (Damgard 1991)

Let $\mathbb{G}$ a group of order q generated by g and $x \in_R \mathbb{Z}_q$. Let $h = g^x$
For any adversary $\mathcal{A}(q, g, h)$ that outputs a value (c, y) such that $y = c^x$, there exists an extractor $\mathcal{B}$ who on input $\mathcal{B}(q, g, h)$ outputs s: $c = g^s$

Intuition

- The exponent in question is s
- Since $y = c^x$ and we do not know x the only way to have come up with (c, y) is through s
- That is: $c = g^s$ and $y = h^s$
- Between ZKP of DLOG equality and double DLOG knowledge

KoE Relation to zk-Snarks

There is no need to know x in order to validate knowledge of exponent:

$$e(h, c) = e(g, y) = e(g, g)^{sx}$$

The correspondence

$C = \mathtt{Enc}(p(x_0)) = g^{p(x_0)}$ and
$Y = \mathtt{Enc}(bp(x_0)) = g^{bp(x_0)}$

If it does not hold then a cheating prover might come up with Y without knowing $p(x_0)$

Remarks

- Is it sound?
- Answer: No - the prover can cheat by replacing p with any polynomial
- Is it zero knowledge?
- Answer: No - it allows the verifier to learn $\texttt{Enc}(p(x_0))$

Evaluate polynomials and check in ZK

ZK: $\mathcal{V}$ must not even learn $\mathtt{Enc}(p(x_0))$

- $\mathcal{V}$ selects b, x_0 and computes:

$$\mathtt{Enc}(x_0^0), \mathtt{Enc}(x_0^1), \cdots \mathtt{Enc}(x_0^d)$$
$$\mathtt{Enc}(bx_0^0), \mathtt{Enc}(bx_0^1), \cdots \mathtt{Enc}(bx_0^d)$$

- $\mathcal{P}$ selects a and computes:

$$\mathtt{Enc}(a)\mathtt{Enc}(p(x_0)) = \mathtt{Enc}(a + p(x_0))$$
$$\mathtt{Enc}(b)^a\mathtt{Enc}(bp(x_0)) = \mathtt{Enc}(ba)\mathtt{Enc}(bp(x_0)) = \mathtt{Enc}(b(a + p(x_0)))$$

- Check the pairing step as before:

$$e(\mathtt{Enc}(a + p(x_0)), \mathtt{Enc}(b)) = e(g, g)^{b(a+p(x_0))}$$
$$e(\mathtt{Enc}(b(a + p(x_0))), \mathtt{Enc}(1)) = e(g, g)^{b(a+p(x_0))}$$

R1CS

Definition

A system of rank-1 quadratic equations over $\mathbb{F}$ is a set of constraints $C = \{(\boldsymbol{a}_j, \boldsymbol{b}_j, \boldsymbol{c}_j)\}_{i=1}^{N_c}$ and $n \in \mathbb{N}$ where:

- $\boldsymbol{a}_j, \boldsymbol{b}_j, \boldsymbol{c}_j \in \mathbb{F}^{1+N_v}$
- $n \leq N_v$

Satisfiability

A R1 system C is satisfiable on input $\boldsymbol{x} \in \mathbb{F}^n$ if there is a witness $s \in \mathbb{F}^{N_v}$:

- $\boldsymbol{x} = (s_1, \cdots, s_n)$
- $\forall j \in N_c : \boldsymbol{a}_j \cdot (1, \boldsymbol{s}) \times \boldsymbol{b}_j \cdot (1, \boldsymbol{s}) = \boldsymbol{c}_j \cdot (1, \boldsymbol{s})$

Facts

BC to R1CS

Boolean circuit $C : \{0,1\}^n \times \{0,1\}^h \times \{0,1\}$ with α wires and β (bilinear) gates $\rightarrow$ R1CS with with $N_v = \alpha$ and $N_c = \beta + h + 1$

AC to R1CS

Arithmetic circuit $C : \mathbb{F}^n \times \mathbb{F}^h \times \mathbb{F}^l$ with α wires and β (bilinear) gates $\rightarrow$ R1CS with with $N_v = \alpha$ and $N_c = \beta + l$

Quadratic Span Programs - QSP I

Definition

A QSP over a field $\mathbb{F}$ for inputs of length n consists of

- 2 sets of source polynomials: $\mathcal{V} = \{v_0, \cdots, v_m\}, \{w_0, \cdots, w_m\}$
- the target polynomial: t
- an injective function $f\colon [n] \times \{0,1\} \to [m]$

Quadratic Span Programs - QSP II

QSP Verification

An input $u \in \{0,1\}^n$ is accepted by a QSP iff $\exists$ tuples $a = (a_1, \cdots, a_m)$, $b = (b_1, \cdots, b_m) \in \mathbb{F}^m$:

- $a_k \wedge b_k = 1$, if $\exists i : k = f(i, u_i)$
- $a_k \wedge b_k = 0$, if $\exists i : k = f(i, 1 - u_i)$
- t divides the linear combination $v_a \cdot w_b$ where
  $v_a = v_0 + \sum_{i=1}^{m} a_i v_i$,
  $w_b = w_0 + \sum_{i=1}^{m} b_i w_i$

Quadratic Span Programs - QSP III

Remarks:

- Check if a target polynomial divides a linear combination of some given polynomials
- f restricts which polynomials can be used in the linear combination
- The NP witness are the a, b
- QSP Verification is NP-Complete
- In practice:
  - Find $h : th = v_a \cdot w_b \Leftrightarrow th - v_a \cdot w_b = \mathbf{0}$
  - Check that it is a zero polynomial
  - Evaluate at a single point $t(x_0)h(x_0) - v_a(x_0) \cdot w_b(x_0) = 0$ (The number of roots is tiny compared to the number of field elements)

Quadratic Arithmetic Programs I

Definition

A QAP $\mathcal{Q}$ over a field $\mathbb{F}$ is:

- 3 sets of source polynomials $\mathcal{V} = \{v_0, \cdots, v_m\}$, $\mathcal{W} = \{w_0, \cdots, w_m\}$, $\mathcal{Y} = \{y_0, \cdots, y_m\}$
- the target polynomial t
- a function $f \colon \{0,1\}^n \to \{0,1\}^{n'}$

Quadratic Arithmetic Programs II

$\mathcal{Q}$ computes f if: $(c_1, \cdots, c_{n+n'}) \in \mathbb{F}^{n+n'}$ is a valid assignment of f's inputs and outputs which happens if and only if there exist coefficients $(c^{N+1}, \cdots, c^m)$ such that $t(x)$ divides $p(x)$ where:

$$
\begin{aligned}
p(x) = (v_0(x) + \sum_{k=1}^{m} c_k v_k(x)) \cdot (w_0(x) + \sum_{k=1}^{m} c_k w_k(x)) \\
-(y_0(x) + \sum_{k=1}^{m} c_k y_k(x))
\end{aligned}
$$

From Code to QAP

Process

Code → Algebraic Circuit → R1CS → QAP → ZKSnark

```
def f(x):
    y=x**3
    return x+y+5
```

Task

Prove that you executed f with input $= 3$

Convert to circuit - Flattening

Convert code into a format that contains only commands of the form:

- `x=y`
- `x=y op z`

As a result the function *f* becomes:

```
def f(x):
    sym_1 = x * x
    y = sym_1 * x
    sym_2 = y + x
    out = sym_2 + 5
```

Convert to R1CS

Rules

- Each command can be considered as a logic gate and represented as a relation between vectors
- The vectors have as many elements as the total number of variables in the command plus one (for constants)
- Mapping vector $[one, x, out, sym_1, y, sym_2]$
- Vector $\boldsymbol{c}$ is the left hand side
- Vector $\boldsymbol{a}, \boldsymbol{b}$ are the right hand sides

Application to example commands

Command

$\text{sym}_1 = x * x$

Command

y = sym1 * x

$$
\begin{array}{rllll}
 & [one, & x, out, & sym_1, y, & sym_2] \\
\boldsymbol{a} = & [0, & 1, 0, & 0, 0, & 0] \\
\boldsymbol{b} = & [0, & 1, 0, & 0, 0, & 0] \\
\boldsymbol{c} = & [0, & 0, 0, & 1, 0, & 0]
\end{array}
$$

$$
\begin{array}{rllll}
 & [one, & x, out, & sym_1, y, & sym_2] \\
\boldsymbol{a} = & [0, & 0, 0, & 1, 0, & 0] \\
\boldsymbol{b} = & [0, & 1, 0, & 0, 0, & 0] \\
\boldsymbol{c} = & [0, & 0, 0, & 0, 1, & 0]
\end{array}
$$

Indeed $s = [1, 3, 0, 9, 0, 0]$
satisfies: $\boldsymbol{sa} \cdot \boldsymbol{sb} - \boldsymbol{sc} = 0$

Application to commands

Command

sym2 = y+x

Command

out = sym2+5

$$\begin{array}{lcccc} & [one, & x, out, & sym_1, y, & sym_2] \\ \boldsymbol{a} = [& 0,1, & 0,0, & 1,0] \\ \boldsymbol{b} = [& 1,0, & 0,0, & 0,0] \\ \boldsymbol{c} = [& 0,0, & 0,0, & 0,1] \end{array}$$

$$\begin{array}{lcccc} & [one, & x, out, & sym_1, y, & sym_2] \\ \boldsymbol{a} = & [5, & 0,0, & 0,0, & 1] \\ \boldsymbol{b} = & [1, & 0,0, & 0,0, & 0] \\ \boldsymbol{c} = & [1, & 0,0, & 0,0, & 0] \end{array}$$

Remark: addition is implied in the dot product

The final R1CS

$$\boldsymbol{A} = \{[0,1,0,0,0,0],[0,0,0,1,0,0],[0,1,0,0,1,0],[5,0,0,0,0,1]\}$$
$$\boldsymbol{B} = \{[0,1,0,0,0,0],[0,1,0,0,0,0],[1,0,0,0,0,0],[1,0,0,0,0,0]\}$$
$$\boldsymbol{C} = \{[0,0,0,1,0,0],[0,0,0,0,1,0],[0,0,0,0,0,1],[1,0,0,0,0,0]\}$$

The execution is the vector $\boldsymbol{s} = [1,3,35,9,27,30]$

From Vectors To Polynomials

Why?

Because we can check all the constraints simultaneously!

- Use Lagrange interpolation to transform the sets of m vectors with n elements into n polynomials of degree $m-1$
- Construct polynomial $a_j(i) = A[i][j]$ (value element of vector i in position j)
- For instance: $a_1(1) = 0, a_1(2) = 0, a_1(3) = 0, a_1(4) = 5$
- $a_1(x) = \frac{5}{6}x^3 - 5x^2 + \frac{55}{6}x - 5$ or in vector form $\boldsymbol{a}_1 = [-5, \frac{55}{6}, -5, \frac{5}{6}]$
- Repeat for $\boldsymbol{B}, \boldsymbol{C}$
- Compute $t = A \cdot s \times B \cdot s - C \cdot s$
- Compute h from t, A, B, C using FFT
- Compute a_i, b_i from the input

Setup Phase I

- Public verifiability
- Non interactive
- Fix the homomorphic encryption scheme, verifier, polynomials
- $\mathcal{V}$ selects random field elements $x_0, b \in \mathbb{F}$
- computes and publishes in the CRS:
  - $\{\mathtt{Enc}(x_0^k)\}_{k=0}^{d}$
  - $\{\mathtt{Enc}(bx_0^k)\}_{k=0}^{d}$
  - $\{\mathtt{Enc}(v_k(x_0)), \mathtt{Enc}(bv_k(x_0))\}_{k=1}^{m}$
  - $\{\mathtt{Enc}(w_k(x_0)), \mathtt{Enc}(bw_k(x_0))\}_{k=1}^{m}$
  - $\{\mathtt{Enc}(y_k(x_0)), \mathtt{Enc}(by_k(x_0))\}_{k=1}^{m}$
  - $\mathtt{Enc}(t(x_0)), \mathtt{Enc}(bt(x_0))$

Setup Phase II

- selects random field values $\gamma, \beta_v, \beta_w, \beta_y$ in order to ensure soundness (i.e. that the correct polynomials were evaluated)
- computes and publishes in the CRS:
  - $\mathtt{Enc}(\gamma), \mathtt{Enc}(\beta_v\gamma), \mathtt{Enc}(\beta_w\gamma), \mathtt{Enc}(\beta_y\gamma)$
  - $\{\mathtt{Enc}(\beta_v v_k(x_0))\}_{k=1}^m$
  - $\{\mathtt{Enc}(\beta_w w_k(x_0))\}_{k=1}^m$
  - $\{\mathtt{Enc}(\beta_y y_k(x_0))\}_{k=1}^m$
  - $\mathtt{Enc}(\beta_v t(x_0)), \mathtt{Enc}(\beta_w t(x_0)), \mathtt{Enc}(\beta_y t(x_0))$

All computations in the proof must use only these elements

The prover

- Evaluates the circuit for the function and obtains the output
- As a result the $\mathcal{P}$ knows the values of c_i
- Solves for h
- Define:
  - I_{mid}: the indices that are not in the image of f
  - $v_{mid}(x) = \sum_{k \in I_{mid}} c_k v_k(x)$
- Generate the proof (9 encrypted values):
  - $V_{mid} = \mathtt{Enc}(v_{mid}(x_0)),\ W = \mathtt{Enc}(w(x_0)),\ Y = \mathtt{Enc}(y(x_0)),$ $H = \mathtt{Enc}(h(x_0))$
  - $V'_{mid} = \mathtt{Enc}(bv_{mid}(x_0)),\ W' = \mathtt{Enc}(bw(x_0)),\ Y' = \mathtt{Enc}(by(x_0)),$ $H' = \mathtt{Enc}(bh(x_0))$
  - $K = \mathtt{Enc}(\beta_v v_{mid}(x_0) + \beta_w w(x_0) + \beta_y y(x_0))$
- All these values can be computed by leveraging the homomorphic properties of the underlying cryptosystem from what is on the CRS

The verifier

- Computes I_{mid} from the input u
- Computes $\mathtt{Enc}(v_{io}(x_0)) = \mathtt{Enc}(\sum_{k \notin I_{mid}} c_k v_k(x_0))$
- Verifies the following equations using the pairing function:
  - $e(V'_{mid}, \mathtt{Enc}(1)) = e(V_{mid}, \mathtt{Enc}(b))$
  - $e(W', \mathtt{Enc}(1)) = e(W, \mathtt{Enc}(b))$,
  - $e(H', \mathtt{Enc}(1)) = e(H, \mathtt{Enc}(b))$
  - $e(Y', \mathtt{Enc}(1)) = e(Y, \mathtt{Enc}(b))$
  - For soundness check:
    $e(\mathtt{Enc}(\gamma), K) =$
    $e(\mathtt{Enc}(\beta_v \gamma), V_{mid}) \cdot e(\mathtt{Enc}(\beta_w \gamma), W) \cdot e(\mathtt{Enc}(\beta_y \gamma), Y)$
  - Check the QAP relation:
    $\frac{e(\mathtt{Enc}(v_0(x_0)) \cdot \mathtt{Enc}(v_{in}(x_0)) \cdot V_{mid}, \mathtt{Enc}(w_0(x_0) W))}{e(y_0(x_0) Y, \mathtt{Enc}(1))} = e(H, \mathtt{Enc}(t(x_0)))$

Completeness

$$
\begin{aligned}
e(\mathtt{Enc}(\gamma), K) &= \\
e(\mathtt{Enc}(\gamma), \mathtt{Enc}(\beta_v v_{mid}(x_0) + \beta_w w(x_0) + \beta_y y(x_0))) &= \\
e(g^{\gamma}, g^{\beta_v v_{mid}(x_0)+\beta_w w(x_0)+\beta_y y(x_0)}) &= \\
e(g, g)^{\gamma\cdot(\beta_v v_{mid}(x_0)+\beta_w w(x_0)+\beta_y y(x_0))}
\end{aligned}
$$

$$
\begin{aligned}
e(\mathtt{Enc}(\beta_v\gamma), V_{mid}) \cdot e(\mathtt{Enc}(\beta_w\gamma), W) \cdot e(\mathtt{Enc}(\beta_y\gamma), Y) &= \\
e(\mathtt{Enc}(\beta_v\gamma, \mathtt{Enc}(v_{mid}(x_0)))e(\mathtt{Enc}(\beta_w\gamma), \mathtt{Enc}(w(x_0)))e(\mathtt{Enc}(\beta_y\gamma), \mathtt{Enc}(y(x_0))) &= \\
e(g, g)^{\beta_v\gamma v_{mid}(x_0)} \cdot e(g, g)^{\beta_w\gamma w(x_0)} \cdot e(g, g)^{\beta_y\gamma y(x_0)} &= \\
e(g, g)^{\beta_v\gamma v_{mid}(x_0)+\beta_w\gamma w(x_0)+\beta_y\gamma y(x_0)}
\end{aligned}
$$

Completeness for the QAP Relation I

The parts of the left hand pairings:

$$
\begin{aligned}
\mathtt{Enc}(v_0(x_0))\mathtt{Enc}(v_{in}(x_0))V_{mid} &= \\
\mathtt{Enc}(v_0(x_0))\mathtt{Enc}(v_{in}(x_0))\mathtt{Enc}(v_{mid}(x_0)) &= \\
\mathtt{Enc}(v_0(x_0) + v_{in}(x_0) + v_{mid}(x_0)) &= \\
\mathtt{Enc}(v_0(x_0) + \sum_{i=1}^{m} c_i v_i(x_0)) = \mathtt{Enc}(v(x_0))
\end{aligned}
$$

$$
\mathtt{Enc}(w_0(x_0))W = \mathtt{Enc}(w_0(x_0))\mathtt{Enc}(w(x_0)) =
$$

$$
\mathtt{Enc}(w_0(x_0) + \sum_{i=1}^{m}(c_i w_i(x_0))) = \mathtt{Enc}(w(x_0))
$$

Completeness for the QAP Relation II

$$\texttt{Enc}(y_0(x_0))\,Y = \texttt{Enc}(y_0(x_0))\texttt{Enc}(y(x_0)) =$$

$$\texttt{Enc}(y_0(x_0) + \sum_{i=1}^{m}(c_i y_i(x_0))) = \texttt{Enc}(y(x_0))$$

Completeness for the QAP Relation III

Left hand side:

$$e(\mathtt{Enc}(v_a(x_0)), \mathtt{Enc}(w_b(x_0))) = e(g, g)^{v(x_0) \cdot w(x_0) - y(x_0)}$$

Right hand side:

$$e(H, \mathtt{Enc}(t(x_0))) = e(g^h(x_0), g^t(x_0)) = e(g, g)^{h(x_0) t(x_0)}$$

Intuition between soundness

The relation
$e(\mathtt{Enc}(\gamma), K) = e(\mathtt{Enc}(\beta_v\gamma), V_{mid}) \cdot e(\mathtt{Enc}(\beta_w\gamma), W) \cdot e(\mathtt{Enc}(\beta_y\gamma), Y)$
protects from a prover that tries to cheat by using another polynomial.

- The values $\beta_v, \beta_w, \beta_y$ do not appear in the CRS in isolation
- The expression $\beta_v v_{mid}(x_0) + \beta_w w(x_0)) + \beta_y y(x_0)$ can only be encrypted from the respected values in the CRS in encrypted form mixed with γ

Shifting for Zero Knowledge

The $\mathcal{P}$ chooses $\delta_{mid}, \delta_w, \delta_y$.
Define

- $V_{\delta mid} = \texttt{Enc}(v_{mid}(x_0) + \delta_{mid} t(x_0))$
- $w_\delta(x_0) = w(x_0) + \delta_w t(x_0)$
- $y_\delta(x_0) = y(x_0) + \delta_y t(x_0)$
- As a result V_{mid}, W, Y are randomised

The equation $v(x_0)w(x_0) - y(x_0) = h(x_0)t(x_0)$ must still hold
To achieve this we replace $H = \texttt{Enc}(h(x_0))$ in the CRS accordingly

vnTinyRAM

- zk-SNARKs for a general purpose CPU
- Circuit generator: Translate program execution into sequence of circuits
- Compose zk-SNARKs for these circuits
- Bound on the running time

Pinnochio: A cloud based lie detector I

- General purpose computation validator
- Client: represents functions as a public evaluation key
- Client: provides input or ZKPoK of some property of the input
- Server: evaluates the computation and provides proof (signature)
- Compiler toolchain to use with C-programs
- Transforms to QAP, QSP
- Use:
  - Protect against malicious servers
  - Extra server feature (at a higher price)
- Performance
  - Setup: Linear in the size of the computation

Pinnochio: A cloud based lie detector II

- Proof Size: constant (288 bytes)
  - Does not depend on function
  - Does not depend on input/output size
- Verification: Linear in the size of the input and output typically 10ms (5 - 7 orders of magnitude gain)
- Proof generation: up to 60 times fewer work

Bitcoin's problem I

Bitcoin is not anonymous

- All transactions are recorded in the blockchain
- Users use pseudonyms
- Deanonymization
  - The structure of the transaction graph
  - Real world information (value, dates, blockchain exit points)

Bitcoins are not fully fungible

- In the protocol itself all coins have the same value

but...

Bitcoin's problem II

- Each coin has a history than can be traced
- This might have an effect on the ability to spend the coins or on their value (e.g. Wanacry ransomware)

A first solutions: mixes

- Users entrust their coins to a 'trusted' entity
- They receive coins with the same value but different origins
- Many problems (fees, delays, trust)

ZeroCoin

- A decentralised mix
- Two kinds of coins: base and anonymous
- Each anonymous transaction is accompanied by a ZK proof that the coin spent can be linked to a valid base coin
  - The base coin comes from a valid transaction
  - The base coin is unspent
- Problems:
  - Performance bottleneck for ZK proofs
  - Functionality: Does not support all denominations etc.
  - Anonymity: Does not hide metadata

Transactions occur using the base coin and are periodically washed in the distributed mix

zCash=Zerocoin+SNARKs

- Performance
    - 288 byte proof
    - 895MB CRS
    - transaction $<$ 1KB (vs 45KB in Zerocoin)
    - 6ms verification (vs 450ms in Zerocoin)
    - 1min to make a transaction

zCash CRS generation ceremony I

Goal

- Generate x_0 in CRS: $g^{x_0^1}, \cdots, g^{x_0^d}$
- No participant must learn the entire x_0
- All shares of x_0 must be later destroyed
- A single honest participant is required

zCash CRS generation ceremony II

The protocol

- Each participant generates a random s_i
- The first participant computes and publishes $g^{s_1}, \cdots, g^{s_1^d}$
- The second partipant computes $g^{s_1 s_2}, \cdots, g^{s_1^d {s_2}^d}$
- $\cdots$
- The last participant computes $g^{s_1 s_2 \cdots s_n}, \cdots, g^{s_1^d s_2^d \cdots {s_n}^d}$
- $x_0 = s_1 s_2 \cdots s_n$

zCash CRS generation ceremony III

Validation

A partipant might cheat by computing $g^{s_p \cdot s_i^{`}}$. validation can be done using pairings.

- $e(g^{s_i}, g^{s_i}) = e(g, g)^{s_i^2}$
- $e(g, g^{s_i^2}) = e(g, g)^{s_i^2}$

This check is repeat for all powers

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