

zk-SNARKs

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From theory to practice...

zkSnark

Zero **K**nowledge **S**uccinct **N**on Interactive **A**rguments Of **K**nowledge

Use

Efficiently verify the correctness of computations without executing them

Applications

- Verify cloud computations (centralised, decentralised)
- Anonymous bitcoin (ZCash)

Application Model

- A client owns input u (e.g. query)
- A server owns a private input w (e.g. private DB)
- The client wishes to learn $z = f(u, w)$ for a function f known to both
- Client: computation correctness (integrity)
- Server: private input confidentiality

Client: its computing power should be confined to the bare minimum of sending u and receiving z

What zk-Snarks offer

- **Zero Knowledge:** The client (verifier \mathcal{V}) learns nothing but the validity of the computation
- **Succinct:** The proof is tiny compared to the computation
 - the proof size is constant $O_\lambda(1)$ (depends only on the security parameter λ)
 - verification time is $O_\lambda(|f| + |u| + |z|)$ and does not depend on the running time of f
- **Non Interactive:** The proofs are created without interaction with the verifier and are publicly verifiable strings
- **Arguments:** Soundness is guaranteed only against a computationally bounded server (prover \mathcal{P})
- **of Knowledge:** The proof cannot be constructed without access to a witness

Position in the complexity landscape...

- $NP = PCP[O(\log n), O(1)]$
- One-Way Functions $\Rightarrow NP \subseteq ZK$ (Goldreich, Micali, Wigderson) (ZKP for 3-COL)
- We can use PCP to construct ZK proofs (in theory)
- The proofs are hugely inefficient
- Can we construct better SNARKs without using PCPs?
- Yes, using QSPs and QAP - a better characterisation of NP and cryptographic assumptions

Main idea

- 1 Transform the verification of the computation to checking a relation between secret polynomials:

$$\text{computation validity} \leftrightarrow p(x)q(x) = s(x)r(x)$$

- 2 The verifier chooses a random evaluation point that must be kept secret:

$$p(x_0)q(x_0) = s(x_0)r(x_0)$$

- 3 Homomorphic Encryption to compute the evaluation of the polynomials at x_0 by using $\text{Enc}(x_0)$:

$$\text{Enc}(p(x_0))\text{Enc}(q(x_0)) = \text{Enc}(s(x_0))\text{Enc}(r(x_0))$$

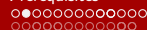
- 4 Randomise for ZK:

$$\text{Enc}(k + p(x_0))\text{Enc}(k + q(x_0)) = \text{Enc}(k + s(x_0))\text{Enc}(k + r(x_0))$$

ZK Proofs

- Shaffi Goldwasser, Silvio Micali and Charles Rackoff, 1985
- Interactive proof systems
 - Computation as a dialogue
 - Prover (\mathcal{P}): wants to prove that a string belongs to a language
 - Verifier (\mathcal{V}): wants to check the proof st:
 - A correct proof convinces \mathcal{V} with overwhelming probability
 - A wrong proof convinces \mathcal{V} with negligible probability
- Zero Knowledge Proofs
 - \mathcal{V} is convinced without learning anything else

A breakthrough with many theoretical and practical applications



An easy example

- \mathcal{V} is color blind
- $\mathcal{O} \mathcal{P}$ holds two identical balls of different color
- Can the \mathcal{V} be convinced of the different colors?
- Yes
 - \mathcal{P} hands the balls to \mathcal{V} (**commit**)
 - \mathcal{V} hides the balls behind his back, one in each hand
 - He **randomly** decides to switch hands or not
 - \mathcal{V} presents the balls to \mathcal{P} (**challenge**)
 - \mathcal{P} responds if the balls have switched hands (**response**)
 - \mathcal{V} accepts or not
 - Malicious \mathcal{P} : Cheating Probability 50%
 - **Repeat** to reduce

Definitions: Notation

- Language $\mathcal{L} \in \text{NP}$
- Polynomial Turing Machine \mathcal{M}
- $x \in \mathcal{L} \Leftrightarrow \exists w \in \{0, 1\}^{p(|x|)} : M(x, w) = 1$
- 2 PPT TM \mathcal{P}, \mathcal{V}
- $\langle \mathcal{P}(x, w), \mathcal{V}(x) \rangle$ is the interaction between \mathcal{P}, \mathcal{V} with common public input x and private \mathcal{P} input w .
- $out_{\mathcal{V}} \langle \mathcal{P}(x, w), \mathcal{V}(x) \rangle$ is the output of \mathcal{V} at the end of the protocol

Properties: Completeness and Soundness

Completeness

An honest \mathcal{P} , convinces an honest \mathcal{V} with certainty: If $x \in \mathcal{L}$ and $M(x, w) = 1$ then: $Pr[out_{\mathcal{V}} < \mathcal{P}(x, w), \mathcal{V}(x) > (x) = 1] = 1$

Properties: Soundness

A malicious \mathcal{P} (\mathcal{P}^*), only convinces an honest \mathcal{V} , with negligible probability. If $x \notin \mathcal{L} \quad \forall(\mathcal{P}^*, w)$:
 $Pr[out_{\mathcal{V}} < \mathcal{P}^*(x, w), \mathcal{V}(x) > (x) = 1] = \text{negl}(\lambda)$

Note:

Proof of Knowledge: \mathcal{P}^* is **not** PPT.

Argument of Knowledge: \mathcal{P}^* is PPT.

Properties:(Perfect) Zero Knowledge

\mathcal{V} does not gain any more knowledge than the validity of the \mathcal{P} 's claim.

For each \mathcal{V}^* there is a PPT \mathcal{S} :

If $x \in \mathcal{L}$ and $M(x, w) = 1$ the random variables:

$out_{\mathcal{V}^*} < \mathcal{P}(x, w), \mathcal{V}^*(x) > (x)$ and

$out_{\mathcal{V}^*} < \mathcal{S}(x), \mathcal{V}^*(x) > (x)$

follow the same distribution: We allow a **malicious verifier** that does not follow the protocol and cheats in order to learn w

Intuition

What ever the \mathcal{V} can learn after interacting with the \mathcal{P} , can be learnt by interacting with \mathcal{S} (disregarding \mathcal{P})

Constructing the simulator

A theoretical construction with practical applications

Reminder: \mathcal{S} does not have access to the witness

- \mathcal{S} take \mathcal{P} 's place during th interaction with \mathcal{V}
- We cannot distinguish between $\langle \mathcal{S}, \mathcal{V} \rangle$ and $\langle \mathcal{P}, \mathcal{V} \rangle$
- We allow rewinds:
- when \mathcal{V} sets a challenge that cannot be answered by \mathcal{S} then we stop and rewind it
- ZK if despite the rewind \mathcal{V} accepts at some point
- Why? Because he cannot distinguish between \mathcal{P} (with the witness) and \mathcal{S} (without the witness)
- As long as \mathcal{S} is PPT
- As a result \mathcal{V} extracts the same information from \mathcal{P} and \mathcal{S} (nothing to extract)

Cryptographic Applications

- Authentication without passwords
 - Proof that the user know the password
 - Transmission and processing is not needed
- Proof that a ciphertext contains a particular message
- Digital signatures
- Anti-Malleability
- In general: Proof that a player follows a protocol without releasing any private input

Σ - protocols

A 3 round protocol with an honest verifier and special soundness

- 1 **Commit** \mathcal{P} commits to a value
- 2 **Challenge** \mathcal{V} selects a random challenge uniformly from a challenge space (honest)
- 3 **Response** \mathcal{P} responds using the commitment, the witness and the random challenge.

Special Soundness

Two execution of the protocol with the same commitment reveal the witness

Knowledge of DLOG: Schnorr's protocol I

Protocol input

- **Public:** g is a generator of an order q subgroup of \mathbb{Z}_p^* with hard DLP and a random $h \in \mathbb{Z}_p^*$
- **Private:** \mathcal{P} knows a witness $x \in \mathbb{Z}_q^*$ st: $h = g^x \pmod{p}$

Goal

Proof of knowledge of x without releasing any more information

Knowledge of DLOG: Schnorr's protocol II

■ Commit ($\mathcal{P} \rightarrow \mathcal{V}$):

- Randomly Select $t \in_R \mathbb{Z}_q^*$
- Compute $y = g^t \bmod p$.
- Send y to \mathcal{V} .

■ Challenge ($\mathcal{V} \rightarrow \mathcal{P}$):

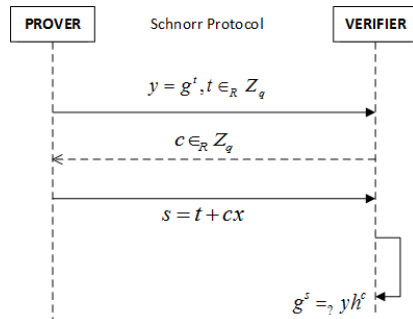
Select and challenge with
 $c \in_R \mathbb{Z}_q^*$

■ Response ($\mathcal{P} \rightarrow \mathcal{V}$):

\mathcal{P} computes $s = t + cx \bmod q$
 and sends it to \mathcal{V}

■ \mathcal{V} accepts iff

$$g^s = yh^c \pmod{p}$$



Properties I

■ Completeness

$$g^s = g^{t+cx} = g^t g^{cx} = y h^c \pmod{p}$$

- **Soundness** Probability that \mathcal{P}^* cheats an honest verifier: $\frac{1}{q}$ - negligible - repeat to decrease

- **Special soundness** Let (y, c, s) nad (y, c', s') be two successful protocol transcripts

$$\begin{aligned} g^s &= y h^c & g^{s'} &= y h^{c'} \Rightarrow g^s h^{-c} = g^{s'} h^{-c'} \Rightarrow \\ g^{s-xc} &= g^{s'-xc'} \Rightarrow s - xc = s' - xc' \Rightarrow x = \frac{c' - c}{s - s'} \end{aligned}$$

Since \mathcal{P} can answer these 2 questions he knows DLOG of h

Properties II

Zero knowledge: no

- A cheating verifier does not choose randomly
- but bases each challenge to the commitment received before \mathcal{S}
- In the simulated execution it will switch challenge
- \mathcal{S} will not be able to respond

How to add ZK:

- \mathcal{V} commits to randomness before the first message by \mathcal{P} or
- Challenge space $\{0, 1\}$
 - In this case \mathcal{V} has only two options.
 - As a result the \mathcal{S} can prepare for both.

Properties III

It provides **Honest Verifier Zero Knowledge**. Let \mathcal{S} without knowledge of the witness x and an honest \mathcal{V}

- \mathcal{S} follows the protocol and commits to $y = g^t, t \in_R \mathbb{Z}_q^*$
- \mathcal{V} selects $c \in_R \mathbb{Z}_q^*$
- If \mathcal{S} can answer (which occurs with negligible probability) the protocol resumes normally
- Else the \mathcal{V} is rewind (with the same random tape)
- \mathcal{V} selects the same $c \in_R \mathbb{Z}_q^*$ (because the random tape has not changed)
- \mathcal{S} sends $s = t$. \mathcal{V} will accept since $yh^c = g^t h^{-c} h^c = g^t = g^s$

The conversations $(t \in_R \mathbb{Z}_q; g^t h^{-c}, c \in_R \mathbb{Z}_q, t)$
 $(t, c \in_R \mathbb{Z}_q; g^t, c, t + xc)$ follow the same distribution

Removing interactivity

Question

Can we do away with \mathcal{V} ?

\mathcal{P} generates the proof by himself

The proof is verifiable by anyone

Fiat Shamir Transform

Replace the challenge with the output of a pseudorandom function on the commitment

In practice we use a hash function \mathcal{H}

Non-interactive Schnorr with the Fiat Shamir

Input

- **Public:** g is a generator of an order q subgroup of $(\mathbb{Z}_p^*$ with hard DLP and $h \in \mathbb{Z}_p^*$
- **Private:** \mathcal{P} has a witness $x \in \mathbb{Z}_q^*$ st: $h = g^x \bmod p$

The Prover:

- Randomly select $t \in_R \mathbb{Z}_q$,
- Compute $y = g^t \bmod p$
- **Compute** $c = \mathcal{H}(y)$ **where** \mathcal{H} **is a hash function in** \mathbb{Z}_q
- Compute $s = t + cx \bmod q$
- **Release** (h, c, s)
- **Anyone can verify that** $c = \mathcal{H}(g^s h^{-c})$

The common reference string

Both parties have access to a string of (random) data

This is created in a trusted way (e.g. through a secure multiparty computation protocol)

The prover simulates the verifier challenge by selecting data from the CRS

Homomorphic Encryption Schemes

Applying a function on the ciphertexts yields the encryption of a function on the plaintext

$$\text{Enc}(m_1) \otimes \text{Enc}(m_2) = \text{Enc}(m_1 \oplus m_2)$$

Multiplicative Homomorphism in El Gamal:

$$\begin{aligned} \text{Enc}(m_1) \cdot \text{Enc}(m_2) &= (g^{r_1}, m_1 h^{r_1}) \cdot (g^{r_2}, m_2 h^{r_2}) \\ &= (g^{r_1+r_2}, (m_1 \cdot m_2) h^{r_1+r_2}) \end{aligned}$$

Additive Homomorphism in El Gamal:

$$\begin{aligned} \text{Enc}(m_1) \cdot \text{Enc}(m_2) &= (g^{r_1}, g^{m_1} h^{r_1}) \cdot (g^{r_2}, g^{m_2} h^{r_2}) \\ &= (g^{r_1+r_2}, g^{m_1+m_2} h^{r_1+r_2}) \end{aligned}$$

Application - polynomials

Task

Let $\text{Enc}(x) = g^x$ where g is a suitable group generator and $p(x) = \sum_{i=0}^d a_i x^i$ a polynomial

Two parties with knowledge of x_0 and $p(x)$ respectively can compute $\text{Enc}(p(x_0))$

- The \mathcal{V} (the party that knows x_0) releases

$$\text{Enc}(x_0^0), \text{Enc}(x_0^1), \dots, \text{Enc}(x_0^d)$$

into the common reference string

- The \mathcal{P} (the party that knows the coefficients) computes:

$$\prod_{i=0}^d \text{Enc}(x_0^i)^{a_i} = \text{Enc}\left(\sum_{i=0}^d a_i x_0^i\right) = \text{Enc}(p(x_0))$$

Pairings I

In general

Functions that map elements from source groups $\mathcal{G}_1, \mathcal{G}_2$ or \mathcal{G}^2 to a destination group \mathcal{G}_T .

What is interesting: They transform difficult problems in \mathcal{G} to easy problems in \mathcal{G}_T .

Definition

A pairing is an efficiently calculable function $e : \mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G}_T$ st:

- Bilinear: $e(g^a, g^b) = e(g, g)^{ab}$ where $g \in \mathcal{G}$ $a, b \in \mathbb{Z}$
- Non-Degenerate: If $\mathcal{G} = \langle g \rangle$ then $\mathcal{G}_T = \langle e(g, g) \rangle$

Pairings II

In practice: $G = \mathcal{E}(\mathbb{F}_p)$ and $G_T = \mathbb{F}_{p^a}$

How to easily solve DDH

Input: (g, g^a, g^b, g^c)

Check if $g^c = g^{ab}$

Easily compute $e(g^a, g^b) = e(g, g)^{ab}$

Compare with $e(g, g^c) = e(g, g)^c$

but the CDH remains hard

Observation

The pairing allows us to do a multiplication between 'encrypted' values

Application - check the correct evaluation of polynomials I

- The \mathcal{V} that knows x_0 :
 - computes and publishes into the CRS:

$$\text{Enc}(x_0^0), \text{Enc}(x_0^1), \dots, \text{Enc}(x_0^d)$$

- selects a scaling factor b
- computes and publishes into the CRS:

$$\text{Enc}(bx_0^0), \text{Enc}(bx_0^1), \dots, \text{Enc}(bx_0^d)$$

- The \mathcal{P} that knows $p(x)$:
 - computes and publishes $\text{Enc}(p(x_0)), \text{Enc}(bp(x_0))$
- The secrets b, x_0 should be destroyed

Application - check the correct evaluation of polynomials II

Check:

- Use a pairing function e to compute:
 - $e(\text{Enc}(p(x_0)), \text{Enc}(b)) = e(g, g)^{bp(x_0)}$
 - $e(\text{Enc}(bp(x_0)), \text{Enc}(1)) = e(g, g)^{bp(x_0)}$

Observation

- The homomorphic combination of encrypted polynomials allows us to do additions
- plus the multiplication from the pairing

A 'new' security assumption I

Let \mathbb{G} a group of order q generated by g and $x \in_R \mathbb{Z}_q$. Let $h = g^x$

Knowledge of exponents (Damgard 1991)

For any adversary $\mathcal{A}(q, g, h)$ that outputs a value (c, y) such that $y = c^x$, there exists an extractor \mathcal{B} who on input $\mathcal{B}(q, g, h)$ outputs s : $c = g^s$

A 'new' security assumption II

Intuition

- The exponent in question is s
- Since $y = c^x$ and we do not know x the only way to have come up with (c, y) is through s
- That is: $c = g^s$ and $y = h^s$
- Between ZKP of DLOG equality and double DLOG knowledge
- Non standard, but cannot be derived from standard assumptions such as the DDH.

KoE Relation to zk-Snarks

There is no need to know x in order to validate knowledge of exponent:

$$e(h, c) = e(g, y) = e(g, g)^{sx}$$

The correspondence

$$C = \text{Enc}(p(x_0)) = g^{p(x_0)} \text{ and}$$

$$Y = \text{Enc}(bp(x_0)) = g^{bp(x_0)}$$

If it does not hold then a cheating prover might come up with Y without knowing $p(x_0)$

Remarks

- Is it sound?
- Answer: No - the prover can cheat by replacing p with any polynomial
- Is it zero knowledge?
- Answer: No - it allows the verifier to learn $\text{Enc}(p(x_0))$

Evaluate polynomials and check in ZK

ZK: \mathcal{V} must not even learn $\text{Enc}(p(x_0))$

- \mathcal{V} selects b, x_0 and computes:

$$\begin{aligned} & \text{Enc}(x_0^0), \text{Enc}(x_0^1), \dots, \text{Enc}(x_0^d) \\ & \text{Enc}(bx_0^0), \text{Enc}(bx_0^1), \dots, \text{Enc}(bx_0^d) \end{aligned}$$

- \mathcal{P} selects a and computes:

$$\begin{aligned} & \text{Enc}(a)\text{Enc}(p(x_0)) = \text{Enc}(a + p(x_0)) \\ & \text{Enc}(b)^a \text{Enc}(bp(x_0)) = \text{Enc}(ba)\text{Enc}(bp(x_0)) = \text{Enc}(b(a + p(x_0))) \end{aligned}$$

- Check the pairing step as before:

$$\begin{aligned} & e(\text{Enc}(a + p(x_0)), \text{Enc}(b)) = e(g, g)^{b(a + p(x_0))} \\ & e(\text{Enc}(b(a + p(x_0))), \text{Enc}(1)) = e(g, g)^{b(a + p(x_0))} \end{aligned}$$

R1CS

Definition

A system of rank-1 quadratic equations over \mathbb{F} is a set of constraints $\{(\mathbf{v}_j, \mathbf{w}_j, \mathbf{y}_j)\}_{i=1}^{N_c}$ and $n \in \mathbb{N}$ where:

- $\mathbf{v}_j, \mathbf{w}_j, \mathbf{y}_j \in \mathbb{F}^{1+N_v}$
- $n \leq N_v$

Satisfiability

A R1 system C is satisfiable on input $\mathbf{c} \in \mathbb{F}^n$ if there is a witness $\mathbf{s} \in \mathbb{F}^{N_v}$:

- $\mathbf{c} = (c_1, \dots, c_n)$
- $\forall j \in N_c : \mathbf{v}_j \cdot (1, \mathbf{c}) \times \mathbf{w}_j \cdot (1, \mathbf{c}) = \mathbf{y}_j \cdot (1, \mathbf{c})$

Facts

BC to R1CS

Boolean circuit $C : \{0, 1\}^n \times \{0, 1\}^h \times \{0, 1\}$ with α wires and β (bilinear) gates \rightarrow R1CS with $N_v = \alpha$ and $N_c = \beta + h + 1$

AC to R1CS

Arithmetic circuit $C : \mathbb{F}^n \times \mathbb{F}^h \times \mathbb{F}^l$ with α wires and β (bilinear) gates \rightarrow R1CS with $N_v = \alpha$ and $N_c = \beta + l$



Quadratic Span Programs - QSP I

Definition

A QSP over a field \mathbb{F} for inputs of length n consists of

- 2 sets of source polynomials: $\mathcal{V} = \{v_0, \dots, v_m\}, \{w_0, \dots, w_m\}$
- the target polynomial: t
- an injective function $f: [n] \times \{0, 1\} \rightarrow [m]$

Quadratic Span Programs - QSP II

QSP Verification

An input $u \in \{0, 1\}^n$ is accepted by a QSP iff \exists tuples $a = (a_1, \dots, a_m)$, $b = (b_1, \dots, b_m) \in \mathbb{F}^m$:

- $a_k \wedge b_k = 1$, if $\exists i : k = f(i, u_i)$
- $a_k \wedge b_k = 0$, if $\exists i : k = f(i, 1 - u_i)$
- t divides the linear combination $v_a \cdot w_b$ where

$$v_a = v_0 + \sum_{i=1}^m a_i v_i,$$

$$w_b = w_0 + \sum_{i=1}^m b_i w_i$$

Quadratic Span Programs - QSP III

Remarks:

- Check if a target polynomial divides a linear combination of some given polynomials
- f restricts which polynomials can be used in the linear combination
- The NP witness is the pair a, b
- QSP Verification is NP-Complete
- In practice:
 - Find $h : th = v_a \cdot w_b \Leftrightarrow th - v_a \cdot w_b = 0$
 - Check that it is a zero polynomial
 - Evaluate at a single point $t(x_0)h(x_0) - v_a(x_0) \cdot w_b(x_0) = 0$
(The number of roots is tiny compared to the number of field elements)



Quadratic Arithmetic Programs I

Definition

A QAP \mathcal{Q} over a field \mathbb{F} is:

- 3 sets of source polynomials $\mathcal{V} = \{v_0, \dots, v_m\}$,
 $\mathcal{W} = \{w_0, \dots, w_m\}$, $\mathcal{Y} = \{y_0, \dots, y_m\}$
- the target polynomial t
- a function $f: \{0, 1\}^n \rightarrow \{0, 1\}^{n'}$

Quadratic Arithmetic Programs II

\mathcal{Q} computes f if: $(c_1, \dots, c_{n+n'}) \in \mathbb{F}^{n+n'}$ is a valid assignment of f 's inputs and outputs and there exist coefficients (c^{N+1}, \dots, c^m) such that $t(x)$ divides $p(x)$ where:

$$p(x) = (v_0(x) + \sum_{k=1}^m c_k v_k(x)) \cdot (w_0(x) + \sum_{k=1}^m c_k w_k(x)) \\ - (y_0(x) + \sum_{k=1}^m c_k y_k(x))$$

For simplicity: $v(x) = v_0(x) + \sum_{k=1}^m c_k v_k(x)$ etc.

From Code to QAP

Process

Code \rightarrow Algebraic Circuit \rightarrow R1CS \rightarrow QAP \rightarrow ZKSnark

```
def f(x):  
    y=x**3  
    return x+y+5
```

Task

Prove that you executed f with input = 3

Convert to circuit - Flattening

Convert code into a format that contains only commands of the form:

- $x=y$
- $x=y \text{ op } z$

As a result the function f becomes:

```
def f(x):
    sym_1 = x * x
    y = sym_1 * x
    sym_2 = y + x
    out = sym_2 + 5
```

Convert to R1CS

Rules

- Each command can be considered as a logic gate and represented as a relation between vectors
- The vectors have as many elements as the total number of variables in the command plus one (for constants)
- Mapping vector $[one, x, out, sym_1, y, sym_2]$
- Vector y is the left hand side
- Vector v, w are the right hand sides

Application to example commands

Command

$\text{sym}_1 = x * x$

Command

$y = \text{sym}_1 * x$

$$\begin{array}{l}
 [one, \quad x, out, \quad sym_1, y, \quad sym_2] \\
 \mathbf{v} = [0, \quad 1, 0, \quad 0, 0, \quad 0] \\
 \mathbf{w} = [0, \quad 1, 0, \quad 0, 0, \quad 0] \\
 \mathbf{y} = [0, \quad 0, 0, \quad 1, 0, \quad 0]
 \end{array}$$

Indeed $c = [1, 3, 0, 9, 0, 0]$
satisfies: $\mathbf{cv} \cdot \mathbf{cw} - \mathbf{cy} = 0$

$$\begin{array}{l}
 [one, \quad x, out, \quad sym_1, y, \quad sym_2] \\
 \mathbf{v} = [0, \quad 0, 0, \quad 1, 0, \quad 0] \\
 \mathbf{w} = [0, \quad 1, 0, \quad 0, 0, \quad 0] \\
 \mathbf{y} = [0, \quad 0, 0, \quad 0, 1, \quad 0]
 \end{array}$$

$c = [1, 3, 0, 9, 27, 0]$

Application to commands

Command
$\text{sym2} = y + x$

Command
$\text{out} = \text{sym2} + 5$

$$\begin{aligned}
 &[one, \quad x, out, \quad sym_1, y, \quad sym_2] \\
 \mathbf{v} &= [\quad 0, 1, \quad \quad 0, 0, \quad \quad 1, 0] \\
 \mathbf{w} &= [\quad 1, 0, \quad \quad 0, 0, \quad \quad 0, 0] \\
 \mathbf{y} &= [\quad 0, 0, \quad \quad 0, 0, \quad \quad 0, 1]
 \end{aligned}$$

$$\begin{aligned}
 &[one, \quad x, out, \quad sym_1, y, \quad sym_2] \\
 \mathbf{v} &= [5, \quad 0, 0, \quad \quad 0, 0, \quad \quad 1] \\
 \mathbf{w} &= [1, \quad 0, 0, \quad \quad 0, 0, \quad \quad 0] \\
 \mathbf{y} &= [1, \quad 0, 0, \quad \quad 0, 0, \quad \quad 0]
 \end{aligned}$$

Remark: addition is implied in
the dot product

$$\mathbf{c} = [1, 3, 0, 9, 27, 30]$$

The final R1CS

$$\mathbf{V} = \{[0, 1, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 1, 0, 0, 1, 0], [5, 0, 0, 0, 0, 1]\}$$

$$\mathbf{W} = \{[0, 1, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0]\}$$

$$\mathbf{Y} = \{[0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1], [1, 0, 0, 0, 0, 0]\}$$

The solution is the vector $\mathbf{c} = [1, 3, 35, 9, 27, 30]$

From Vectors To Polynomials

- Use Lagrange interpolation to transform the sets of m vectors with n elements into n polynomials of degree $m - 1$
- Construct polynomial v_j with values $v_j(i) = V[i][j]$ (value element of vector i in position j)
- For instance: $v_1(1) = 0, v_1(2) = 0, v_1(3) = 0, v_1(4) = 5$
- $v_1(x) = \frac{5}{6}x^3 - 5x^2 + \frac{55}{6}x - 5$
- $v_2(1) = 1, v_2(2) = 0, v_2(3) = 1, v_2(4) = 0$
- $v_2(x) = -\frac{2}{3}x^3 + 5x^2 + \frac{34}{3}x + 8$
- Repeat for w, y
- Finally add the polynomials together to obtain v, w, y

From Vectors To Polynomials - Why?

- Why? Because we can check all the constraints simultaneously!
- $cv(x) \cdot cw(X) = cy(x)$
- Define $t(x) = cv(x) \cdot cw(X) - cy(x)$
- This polynomial must be zero to all the points that correspond to the logic gates
- A multiple of the base polynomial $(x - 1)(x - 2)\dots$

Setup Phase I

- Non interactiveness - Public verifiability
- Fix the homomorphic encryption scheme, verifier, polynomials
- \mathcal{V} selects random field elements $x_0, b \in \mathbb{F}$
- computes and publishes in the CRS:
 - $\{\text{Enc}(x_0^k)\}_{k=0}^d$ (in reality: $d = 2 \cdot 10^6$)
 - $\{\text{Enc}(bx_0^k)\}_{k=0}^d$
 - $\{\text{Enc}(v_k(x_0)), \text{Enc}(bv_k(x_0))\}_{k=1}^m$
 - $\{\text{Enc}(w_k(x_0)), \text{Enc}(bw_k(x_0))\}_{k=1}^m$
 - $\{\text{Enc}(y_k(x_0)), \text{Enc}(by_k(x_0))\}_{k=1}^m$
 - $\text{Enc}(t(x_0)), \text{Enc}(bt(x_0))$

Setup Phase II

- selects random field values $\gamma, \beta_v, \beta_w, \beta_y$ in order to ensure soundness (i.e. that the correct polynomials were evaluated)
- computes and publishes in the CRS:
 - $\text{Enc}(\gamma), \text{Enc}(\beta_v \gamma), \text{Enc}(\beta_w \gamma), \text{Enc}(\beta_y \gamma)$
 - $\{\text{Enc}(\beta_v v_k(x_0))\}_{k=1}^m$
 - $\{\text{Enc}(\beta_w w_k(x_0))\}_{k=1}^m$
 - $\{\text{Enc}(\beta_y y_k(x_0))\}_{k=1}^m$
 - $\text{Enc}(\beta_v t(x_0)), \text{Enc}(\beta_w t(x_0)), \text{Enc}(\beta_y t(x_0))$

All computations in the proof must use only these elements

Performance: $O(|C|)$

The prover

- Evaluates the circuit for the function and obtains the output
- As a result the \mathcal{P} knows the values of c_i
- Solves for h

- Define:

■ I_{mid} : the indices that are not in IO of $f(\{N+1 \dots m\})$

■ $v_{mid}(x) = \sum_{k \in I_{mid}} c_k v_k(x)$

- Generate the proof (9 encrypted values):

■ $V_{mid} = \text{Enc}(v_{mid}(x_0))$, $W = \text{Enc}(w(x_0))$, $Y = \text{Enc}(y(x_0))$,
 $H = \text{Enc}(h(x_0))$

■ $V'_{mid} = \text{Enc}(bv_{mid}(x_0))$, $W' = \text{Enc}(bw(x_0))$, $Y' = \text{Enc}(by(x_0))$,
 $H' = \text{Enc}(bh(x_0))$

■ $K = \text{Enc}(\beta_v v_{mid}(x_0) + \beta_w w(x_0) + \beta_y y(x_0))$

- All these values can be computed by leveraging the homomorphic properties of the underlying cryptosystem from what is on the CRS
- Performance: $O(|C|) + O(|C| \log^2(|C|))$

The verifier

- Retrieves the values of c_i from the input u and the output
- Computes $\text{Enc}(v_{io}(x_0)) = \text{Enc}(\sum_{k \notin I_{mid}} c_k v_k(x_0))$
- Verifies the following equations using the pairing function:
 - $e(V'_{mid}, \text{Enc}(1)) = e(V_{mid}, \text{Enc}(b))$
 - $e(W', \text{Enc}(1)) = e(W, \text{Enc}(b))$,
 - $e(H', \text{Enc}(1)) = e(H, \text{Enc}(b))$
 - $e(Y', \text{Enc}(1)) = e(Y, \text{Enc}(b))$
 - For soundness check:

$$e(\text{Enc}(\gamma), K) = e(\text{Enc}(\beta_v \gamma), V_{mid}) \cdot e(\text{Enc}(\beta_w \gamma), W) \cdot e(\text{Enc}(\beta_y \gamma), Y)$$
 - Check the QAP relation:

$$\frac{e(\text{Enc}(v_0(x_0)) \cdot \text{Enc}(v_{io}(x_0)) \cdot V_{mid}, \text{Enc}(w_0(x_0)W))}{e(y_0(x_0)Y, \text{Enc}(1))} = e(H, \text{Enc}(t(x_0)))$$

Completeness

$$\begin{aligned}
 e(\text{Enc}(\gamma), K) &= \\
 e(\text{Enc}(\gamma), \text{Enc}(\beta_v v_{mid}(x_0) + \beta_w w(x_0) + \beta_y y(x_0))) &= \\
 e(g^\gamma, g^{\beta_v v_{mid}(x_0) + \beta_w w(x_0) + \beta_y y(x_0)}) &= \\
 e(g, g)^{\gamma \cdot (\beta_v v_{mid}(x_0) + \beta_w w(x_0) + \beta_y y(x_0))} &
 \end{aligned}$$

$$\begin{aligned}
 e(\text{Enc}(\beta_v \gamma), V_{mid}) \cdot e(\text{Enc}(\beta_w \gamma), W) \cdot e(\text{Enc}(\beta_y \gamma), Y) &= \\
 e(\text{Enc}(\beta_v \gamma, \text{Enc}(v_{mid}(x_0))) e(\text{Enc}(\beta_w \gamma), \text{Enc}(w(x_0))) e(\text{Enc}(\beta_y \gamma), \text{Enc}(y(x_0))) &= \\
 e(g, g)^{\beta_v \gamma v_{mid}(x_0)} \cdot e(g, g)^{\beta_w \gamma w(x_0)} \cdot e(g, g)^{\beta_y \gamma y(x_0)} &= \\
 e(g, g)^{\beta_v \gamma v_{mid}(x_0) + \beta_w \gamma w(x_0) + \beta_y \gamma y(x_0)} &
 \end{aligned}$$

Completeness for the QAP Relation I

The parts of the left hand pairings:

$$\begin{aligned} \text{Enc}(v_0(x_0))\text{Enc}(v_{io}(x_0))V_{mid} &= \text{Enc}(v_0(x_0))\text{Enc}(v_{io}(x_0))\text{Enc}(v_{mid}(x_0)) = \\ \text{Enc}(v_0(x_0) + v_{io}(x_0) + v_{mid}(x_0)) &= \text{Enc}(v_0(x_0) + \sum_{i=1}^m c_i v_i(x_0)) = \text{Enc}(v(x_0)) \end{aligned}$$

$$\begin{aligned} \text{Enc}(w_0(x_0))W &= \text{Enc}(w_0(x_0))\text{Enc}(w(x_0)) = \\ \text{Enc}(w_0(x_0) + \sum_{i=1}^m (c_i w_i(x_0))) &= \text{Enc}(w(x_0)) \end{aligned}$$

Completeness for the QAP Relation II

$$\text{Enc}(y_0(x_0)) Y = \text{Enc}(y_0(x_0)) \text{Enc}(y(x_0)) =$$

$$\text{Enc}(y_0(x_0) + \sum_{i=1}^m (c_i y_i(x_0))) = \text{Enc}(y(x_0))$$

Left hand side: $e(\text{Enc}(v(x_0)), \text{Enc}(w(x_0))) = e(g, g)^{v(x_0) \cdot w(x_0) - y(x_0)}$

Right hand side:

$$e(H, \text{Enc}(t(x_0))) = e(g^h(x_0), g^t(x_0)) = e(g, g)^{h(x_0)t(x_0)}$$

Intuition between soundness

The relation

$$e(\text{Enc}(\gamma), K) = e(\text{Enc}(\beta_v \gamma), V_{mid}) \cdot e(\text{Enc}(\beta_w \gamma), W) \cdot e(\text{Enc}(\beta_y \gamma), Y)$$

protects from a prover that tries to cheat by using another polynomial.

- The values $\beta_v, \beta_w, \beta_y$ do not appear in the CRS in isolation
- The expression $\beta_v v_{mid}(x_0) + \beta_w w(x_0) + \beta_y y(x_0)$ can only be encrypted from the respected values in the CRS in encrypted form mixed with γ

Shifting for Zero Knowledge

The \mathcal{P} chooses $\delta_{mid}, \delta_w, \delta_y$.

Define

- $V_{\delta_{mid}} = \text{Enc}(v_{mid}(x_0) + \delta_{mid}t(x_0))$
- $w_{\delta}(x_0) = w(x_0) + \delta_w t(x_0)$
- $y_{\delta}(x_0) = y(x_0) + \delta_y t(x_0)$
- As a result V_{mid}, W, Y are randomised

The equation $v(x_0)w(x_0) - y(x_0) = h(x_0)t(x_0)$ must still hold

To achieve this we replace $H = \text{Enc}(h(x_0))$ in the CRS accordingly

vnTinyRAM

- zk-SNARKs for a general purpose CPU
- Circuit generator: Translate program execution into sequence of circuits
- Compose zk-SNARKs for these circuits
- Bound on the running time

Pinnocchio: A cloud based lie detector I

- General purpose computation validator
- Client: represents functions as a public evaluation key
- Client: provides input or ZKPoK of some property of the input
- Server: evaluates the computation and provides proof (signature)
- Compiler toolchain to use with C-programs
- Transforms to QAP, QSP
- Use:
 - Protect against malicious servers
 - Extra server feature (at a higher price)
- Performance
 - Setup: Linear in the size of the computation

Pinnocchio: A cloud based lie detector II

- Proof Size: constant (288 bytes)
 - Does not depend on function
 - Does not depend on input/output size
- Verification: Linear in the size of the input and output typically 10ms (5 - 7 orders of magnitude gain)
- Proof generation: up to 60 times fewer work

Bitcoin's problem I

Bitcoin is not anonymous

- All transactions are recorded in the blockchain
- Users use pseudonyms
- Deanonymization
 - The structure of the transaction graph
 - Real world information (value, dates, blockchain exit points)

Bitcoins are not fully fungible(?)

- In the protocol itself all coins have the same value

but...

Bitcoin's problem II

- Each coin has a history than can be traced
- This might have an effect on the ability to spend the coins or on their value (e.g. Wannacry ransomware)

A first solutions: mixes

- Users entrust their coins to a 'trusted' entity
- They receive coins with the same value but different origins
- Many problems (fees, delays, trust)

ZeroCoin

- A decentralised mix
- Two kinds of coins: base and anonymous
- Each anonymous transaction is accompanied by a ZK proof that the coin spent can be linked to a valid base coin
 - The base coin comes from a valid transaction
 - The base coin has not been spent
- Problems:
 - Performance bottleneck for ZK proofs
 - Functionality: Does not support all denominations etc.
 - Anonymity: Does not hide metadata

Transactions occur using the base coin and are periodically washed in the distributed mix

zCash=Zerocoin+SNARKs

■ Performance

- 288 byte proof
- 895MB CRS
- transaction < 1KB (vs 45KB in Zerocoin)
- 6ms verification (vs 450ms in Zerocoin)
- 40sec to make a transaction

zCash CRS generation ceremony I

Goal

- Generate x_0 in CRS: $g^{x_0^1}, \dots, g^{x_0^d}$
- No participant must learn the entire x_0
- All shares of x_0 must be later destroyed
- A single honest participant is required

zCash CRS generation ceremony II

The protocol

- Each participant generates a random s_i
- The first participant computes and publishes $g^{s_1}, \dots, g^{s_1^d}$
- The second participant computes $g^{s_1 s_2}, \dots, g^{s_1^d s_2^d}$
- ...
- The last participant computes $g^{s_1 s_2 \dots s_n}, \dots, g^{s_1^d s_2^d \dots s_n^d}$
- $x_0 = s_1 s_2 \dots s_n$

zCash CRS generation ceremony III

Validation

A participant might cheat by computing $g^{s_p \cdot s_i}$. validation can be done using pairings.

- $e(g^{s_i}, g^{s_i}) = e(g, g)^{s_i^2}$
- $e(g, g^{s_i^2}) = e(g, g)^{s_i^2}$

This check is repeated for all powers

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- 9 Alfred Menezes [An introduction to pairing based crypto](#)
- 10 [Zerocash parameter generation](#)