Secular Limit of Micromotion

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The motion of two ions in a linear Paul trap can be generated by solving the following Mathieu Equation, derived previously:

$$v_p''(\tau) = -\frac{(2\pi)^2}{\beta^2}\left(a_p - 2q_x \cos\left(\frac{4\pi\tau}{\beta}\right)\right) v_p(\tau) \tag{1}$$

Note that β has not been explicitly defined yet – it is just a factor involved in the non-dimensionalisation of the time. It is not the same as β_p, which arises in the solution to the Mathieu Equation below.

If we make the substitution $\xi = \frac{2\pi\tau}{\beta}$ (which is equivalent to $\xi = \frac{\omega_{RF}}{2}t$, as $\tau = \frac{\beta\omega_{RF}}{4\pi}t$), then the Mathieu Equation can be expressed more simply as:

$$v_p''(\xi) = -\left(a_p - 2q_x \cos(2\xi)\right) v_p(\xi) \tag{2}$$

The Mathieu Equation is one of many differential equations with periodic coefficients, the stable solutions of which can be found from Floquet theorem:

$$v_p(\xi) = Ae^{i\beta_p\xi}\sum_{n=-\infty}^{\infty} C_n e^{i2n\xi} + Be^{-i\beta_p\xi}\sum_{n=-\infty}^{\infty} C_n e^{-i2n\xi} \tag{3}$$

where β_p is a function of a_p and q_x. The contribution from the oscillating component of the trap is included in the summations – that is, the summations capture all contributions to the motion from the micromotion. The exponential terms out the front of the summations will determine the secular motion of the ions. If $\xi = \frac{\omega_{RF}}{2}t$, then we can see that the dimensional frequency of the secular motion (ie. in units of inverse dimensional time, $1/t$) for each mode will be given by:

$$\omega_{\text{sec},p} = \frac{\beta_p\omega_{RF}}{2} \tag{4}$$

By inserting the motion in (3) back into (2), one can obtain recurrence relations for the coefficients C_{2n}. With this substitution, the LHS of (2) becomes:

$$v_p''(\xi) = -\sum_{n=-\infty}^{\infty} C_n(\beta_p + 2n)^2(Ae^{2i(\frac{\beta_p}{2}+n)\xi} + Be^{-2i(\frac{\beta_p}{2}+n)\xi})$$

Accordingly, the RHS becomes:

$$-\left(a_p - 2q_x\cos(2\xi)\right)v_p(\xi) = -\sum_{n=-\infty}^{\infty} C_n(a_p - 2q_x\cos(2\xi))(Ae^{2i(\frac{\beta_p}{2}+n)\xi} + Be^{-2i(\frac{\beta_p}{2}+n)\xi}) \tag{5}$$

Equating the LHS and the RHS, we achieve:

$$-\sum_{n=-\infty}^{\infty} C_n(a_p - 2q_x\cos(2\xi))(Ae^{2i(\frac{\beta_p}{2}+n)\xi} + Be^{-2i(\frac{\beta_p}{2}+n)\xi}) = -\sum_{n=-\infty}^{\infty} C_n(\beta_p + 2n)^2(Ae^{2i(\frac{\beta_p}{2}+n)\xi} + Be^{-2i(\frac{\beta_p}{2}+n)\xi})$$

$$\implies \sum_{n=-\infty}^{\infty} C_n(a_p - 2q_x\cos(2\xi) - (\beta_p + 2n)^2)(Ae^{2i(\frac{\beta_p}{2}+n)\xi} + Be^{-2i(\frac{\beta_p}{2}+n)\xi}) = 0$$

$$\implies \sum_{n=-\infty}^{\infty} C_n\left(\frac{a_p - (\beta_p + 2n)^2}{q_x} - (e^{i2\xi} + e^{-i2\xi})\right)(Ae^{2i(\frac{\beta_p}{2}+n)\xi} + Be^{-2i(\frac{\beta_p}{2}+n)\xi}) = 0$$

where $\cos(2\xi) = \frac{1}{2}(e^{i2\xi} + e^{-i2\xi})$ has been used. If we denote $D_n = \frac{a_p - (\beta_p + 2n)^2}{q_x}$, then this further becomes:

$$\sum_{n=-\infty}^{\infty} C_n \left(D_n - e^{i2\xi} - e^{-i2\xi}\right) \left(Ae^{2i(\frac{\beta_p}{2}+n)\xi} + Be^{-2i(\frac{\beta_p}{2}+n)\xi}\right) = 0$$

$$\implies \sum_{n=-\infty}^{\infty} D_n C_n (Ae^{2i(\frac{\beta_p}{2}+n)\xi} + Be^{-2i(\frac{\beta_p}{2}+n)\xi}) - \sum_{n=-\infty}^{\infty} C_n (Ae^{2i(\frac{\beta_p}{2}+n+1)\xi} + Be^{-2i(\frac{\beta_p}{2}+n+1)\xi})$$
$$- \sum_{n=-\infty}^{\infty} C_n (Ae^{2i(\frac{\beta_p}{2}+n-1)\xi} + Be^{-2i(\frac{\beta_p}{2}+n-1)\xi}) = 0$$

Changing the index in the second sum to $n \to n+1$ and the index in the third sum to $n \to n-1$, we achieve:

$$\sum_{n=-\infty}^{\infty} (D_n C_n - C_{n-1} - C_{n+1})(Ae^{2i(\frac{\beta_p}{2}+n)\xi} + Be^{-2i(\frac{\beta_p}{2}+n)\xi}) = 0$$

Every Fourier component must be 0 in order to satisfy this condition, which leads to the following recurrence relation:

$$D_n C_n - C_{n+1} - C_{n-1} = 0 \tag{6}$$

This connects the coefficients to the known parameters a_p and q_x. It is most helpful to express the recurrence relation in continued fraction form, which can be done in either the increasing or decreasing (in n) direction.

In the increasing direction, we generate:

$$\frac{C_{n-1}}{C_n} = D_n - \frac{C_{n+1}}{C_n}$$
$$\frac{C_n}{C_{n-1}} = \frac{1}{D_n - \frac{C_{n+1}}{C_n}} \tag{7}$$
$$C_n = \frac{C_{n-1}}{D_n - \frac{C_{n+1}}{C_n}} \tag{8}$$
$$= \frac{C_{n-1}}{D_n - \frac{1}{D_{n+1} - \frac{1}{\cdots}}} \tag{9}$$

where the last line has been generated by iteratively substituting (7) into (8) i times, with the index of (7) changed to $n \to n+i$. We can repeat the process in the decreasing direction, which generates:

$$\frac{C_{n+1}}{C_n} = D_n - \frac{C_{n-1}}{C_n}$$
$$\frac{C_n}{C_{n+1}} = \frac{1}{D_n - \frac{C_{n-1}}{C_n}} \tag{10}$$
$$C_n = \frac{C_{n+1}}{D_n - \frac{C_{n-1}}{C_n}} \tag{11}$$
$$= \frac{C_{n+1}}{D_n - \frac{1}{D_{n-1} - \frac{1}{\cdots}}} \tag{12}$$

Here, (10) has been iteratively substituted into (11) i times, with the index of (10) changed to $n \to n-i$. If $C_0 = 1$ is assumed without loss of generality, then (9) can be used to find the C_n coefficients when n is increasing from 0. Similarly, (12) can be used to find the C_n coefficients when n is decreasing from 0. Of course, we require the continued fraction expressions to be consistent for the same coefficient C_n, which we enforce by deriving a condition on β.

If we substitute $n = 1$ into (9) and $n = 0$ into (12), then we generate the following two equations:

$$C_1 = \frac{C_0}{D_1 - \frac{1}{D_2 - \frac{1}{\cdots}}} \tag{13}$$
$$C_0 = \frac{C_1}{D_0 - \frac{1}{D_{-1} - \frac{1}{D_{-2} - \frac{1}{\cdots}}}} \tag{14}$$

Rearranging these for $\frac{C_1}{C_0}$ and equating them, we achieve:

$$\frac{1}{D_1 - \frac{1}{D_2 - \frac{1}{\cdots}}} = D_0 - \frac{1}{D_{-1} - \frac{1}{D_{-2} - \frac{1}{\cdots}}} \tag{15}$$

As $D_0 = \frac{a_p - \beta_p^2}{q_x}$, this can be easily rearranged for β_p^2:

$$\beta_p^2 = a_p - q_x \left(\frac{1}{D_1 - \frac{1}{D_2 - \frac{1}{\cdots}}} + \frac{1}{D_{-1} - \frac{1}{D_{-2} - \frac{1}{\cdots}}} \right) \tag{16}$$

Numerical values for β_p can be extracted by numerically solving (16) once it has been truncated at a pre-specified value for n, then using this to solve for the C_n coefficients up to that order iteratively. It is clearly a mode-dependent value (hence the subscript p), as it depends on the mode-specific a_p. We can now define β from the above. β, the factor involved in the non-dimensionalisation of time, is *chosen* to be one of the β_p's – that is, we non-dimensionalise the time in terms of *one* of the secular mode periods, $\omega_{\text{sec},p}$ (given in (4)). We can choose this to be the first mode, and hence $\beta := \beta_1$.

We will consider the conditions necessary to reach the secular limit of the ion motion in (3) – that is, the micromotion can be reasonably neglected. It is important to note that approximately secular motion cannot be achieved by simply setting $\omega_{RF} \to \infty$, as such a limit would cause the oscillatory component of the trap to do less and less in comparison to the static component. Earnshaw's theorem dictates that a static potential is unable to trap a charged ion, so $\omega_{RF} \to \infty$ would accordingly be unable confine the ion and no secular motion would be achievable. Rather, the secular limit can be achieved when the micromotion terms in (3) are not time-dependent, ie. the only non-zero micromotion coefficient is C_0 and all other terms for $n \neq 0$ vanish. In that case, the non-time-dependent terms can be absorbed into the constants A and B.

Via (9) and (12), we achieve rapidly-vanishing C_n terms with increasing n when the D_n terms also diverge rapidly with increasing n. Via the definition of the D_n terms, we see that the D_n terms will diverge quickly with $|n| \geq 1$ as long as q_x and a_p are both small. This indicates that for any given ω_{sec}, our condition to achieve the approximately secular motion will involve small q_x and a_p values.

In the lowest-order case for the ion trajectory (a_p, $q_x^2 \ll 1$), it is reasonable to assume that $C_{\pm 4} \approx 0$. This leads to the following approximation for β_p:

$$\beta_p \approx \sqrt{a_p + \frac{1}{2} q_x^2} \tag{17}$$

If we choose two small a_p, q_x values in the stability region for β_p – say, $a = 0.001$, $q = 0.001$ – and keep orders of n up to $n = 4$ (corresponding to $C_{\pm 4} \approx 0$), then β_p can be numerically solved as $\beta_p = 0.0316307$. This agrees with the approximation, which produces:

$$\beta_p \approx \sqrt{0.001 + \frac{1}{2}(0.001)^2} = 0.0316307$$

Importantly, keeping terms of up to $n = 5$ and $n = 6$ also produces the same value of β_p, indicating that the assumption $C_{\pm 4} \approx 0$ is appropriate here.