

Secular Limit of Micromotion

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The motion of two ions in a linear Paul trap can be generated by solving the following Mathieu Equation, derived previously:

$$v_p''(\tau) = -\frac{(2\pi)^2}{\beta^2} \left(a_p - 2q_x \cos\left(\frac{4\pi\tau}{\beta}\right) \right) v_p(\tau) \quad (1)$$

If we make the substitution $\xi = \frac{2\pi\tau}{\beta}$ (which is equivalent to $\xi = \frac{\omega_{RF}}{2}t$, as $\tau = \frac{\beta\omega_{RF}}{4\pi}t$), then the Mathieu Equation can be expressed more simply as:

$$v_p''(\xi) = -(a_p - 2q_x \cos(2\xi)) v_p(\xi) \quad (2)$$

The Mathieu Equation is one of many differential equations with periodic coefficients, the stable solutions of which can be found from Floquet theorem:

$$v_p(\xi) = A e^{i\beta\xi} \sum_{n=-\infty}^{\infty} C_{2n} e^{i2n\xi} + B e^{-i\beta\xi} \sum_{n=-\infty}^{\infty} C_{2n} e^{-i2n\xi} \quad (3)$$

The contribution from the oscillating component of the trap is included in the summations – that is, the summations capture all contributions to the motion from the micromotion. The exponential terms out the front of the summations will determine the secular motion of the ions. If $\xi = \frac{\omega_{RF}}{2}t$, then we can see that the dimensional frequency of the secular motion (ie. in units of inverse dimensional time, $1/t$) will be given by:

$$\omega_{\text{sec}} = \frac{\beta\omega_{RF}}{2} \quad (4)$$

By inserting the motion in (3) back into (2), one can obtain recurrence relations for the coefficients C_{2n} . With this substitution, the LHS of (2) becomes:

$$v_p''(\xi) = - \sum_{n=-\infty}^{\infty} C_{2n} (\beta + 2n)^2 (A e^{i(\beta+2n)\xi} + B e^{-i(\beta+2n)\xi})$$

Accordingly, the RHS becomes:

$$-(a_p - 2q_x \cos(2\xi)) v_p(\xi) = - \sum_{n=-\infty}^{\infty} C_{2n} (a_p - 2q_x \cos(2\xi)) (A e^{i(\beta+2n)\xi} + B e^{-i(\beta+2n)\xi}) \quad (5)$$

Equating the LHS and the RHS, we achieve:

$$\begin{aligned} - \sum_{n=-\infty}^{\infty} C_{2n} (a_p - 2q_x \cos(2\xi)) (A e^{i(\beta+2n)\xi} + B e^{-i(\beta+2n)\xi}) &= - \sum_{n=-\infty}^{\infty} C_{2n} (\beta + 2n)^2 (A e^{i(\beta+2n)\xi} + B e^{-i(\beta+2n)\xi}) \\ \implies \sum_{n=-\infty}^{\infty} C_{2n} (a_p - 2q_x \cos(2\xi) - (\beta + 2n)^2) (A e^{i(\beta+2n)\xi} + B e^{-i(\beta+2n)\xi}) &= 0 \\ \implies \sum_{n=-\infty}^{\infty} C_{2n} \left(\frac{a_p - (\beta + 2n)^2}{q_x} - (e^{i2\xi} + e^{-i2\xi}) \right) (A e^{i(\beta+2n)\xi} + B e^{-i(\beta+2n)\xi}) &= 0 \end{aligned}$$

where $\cos(2\xi) = \frac{1}{2}(e^{i2\xi} + e^{-i2\xi})$ has been used. If we denote $D_{2n} = \frac{a_p - (\beta+2n)^2}{q_x}$, then this further becomes:

$$\begin{aligned} & \sum_{n=-\infty}^{\infty} C_{2n} (D_{2n} - e^{i2\xi} - e^{-i2\xi}) (Ae^{i(\beta+2n)\xi} + Be^{-i(\beta+2n)\xi}) = 0 \\ \Rightarrow & \sum_{n=-\infty}^{\infty} D_{2n} C_{2n} (Ae^{i(\beta+2n)\xi} + Be^{-i(\beta+2n)\xi}) - \sum_{n=-\infty}^{\infty} C_{2n} (Ae^{i(\beta+2n+2)\xi} + Be^{-i(\beta+2n+2)\xi}) \\ & - \sum_{n=-\infty}^{\infty} C_{2n} (Ae^{i(\beta+2n-2)\xi} + Be^{-i(\beta+2n-2)\xi}) = 0 \end{aligned}$$

Changing the index in the second sum to $n \rightarrow 2n+2$ and the index in the third sum to $n \rightarrow 2n-2$, we achieve:

$$\sum_{n=-\infty}^{\infty} (D_{2n} C_{2n} - C_{2n-2} - C_{2n+2}) (Ae^{i(\beta+2n)\xi} + Be^{-i(\beta+2n)\xi}) = 0$$

Every Fourier component must be 0 in order to satisfy this condition, which leads to the following recurrence relation:

$$D_{2n} C_{2n} - C_{2n+2} - C_{2n-2} = 0 \quad (6)$$

This connects the coefficients to the known parameters a_p and q_x . It is most helpful to express the recurrence relation in continued fraction form, which can be done in either the increasing or decreasing (in n) direction.

In the increasing direction, we generate:

$$\begin{aligned} \frac{C_{2n-2}}{C_{2n}} &= D_{2n} - \frac{C_{2n+2}}{C_{2n}} \\ \frac{C_{2n}}{C_{2n-2}} &= \frac{1}{D_{2n} - \frac{C_{2n+2}}{C_{2n}}} \end{aligned} \quad (7)$$

$$C_{2n} = \frac{C_{2n-2}}{D_{2n} - \frac{C_{2n+2}}{C_{2n}}} \quad (8)$$

$$= \frac{C_{2n-2}}{D_{2n} - \frac{1}{D_{2n+2} - \frac{1}{\dots}}} \quad (9)$$

where the last line has been generated by iteratively substituting (7) into (8) i times, with the index of (7) changed to $n \rightarrow n+i$. We can repeat the process in the decreasing direction, which generates:

$$\begin{aligned} \frac{C_{2n+2}}{C_{2n}} &= D_{2n} - \frac{C_{2n-2}}{C_{2n}} \\ \frac{C_{2n}}{C_{2n+2}} &= \frac{1}{D_{2n} - \frac{C_{2n-2}}{C_{2n}}} \end{aligned} \quad (10)$$

$$C_{2n} = \frac{C_{2n+2}}{D_{2n} - \frac{C_{2n-2}}{C_{2n}}} \quad (11)$$

$$= \frac{C_{2n+2}}{D_{2n} - \frac{1}{D_{2n-2} - \frac{1}{\dots}}} \quad (12)$$

Here, (10) has been iteratively substituted into (11) i times, with the index of (10) changed to $n \rightarrow n-i$. If $C_0 = 1$ is assumed without loss of generality, then (9) can be used to find the C_{2n} coefficients when n is increasing from 0. Similarly, (12) can be used to find the C_{2n} coefficients when n is decreasing from 0. Of course, we require the continued fraction expressions to be consistent for the same coefficient C_{2n} , which we enforce by deriving a condition on β .

If we substitute $n = 1$ into (9) and $n = 0$ into (12), then we generate the following two equations:

$$C_2 = \frac{C_0}{D_2 - \frac{1}{D_4 - \frac{1}{\dots}}} \quad (13)$$

$$C_0 = \frac{C_2}{D_0 - \frac{1}{D_{-2} - \frac{1}{D_{-4} - \frac{1}{\dots}}}} \quad (14)$$

Rearranging these for $\frac{C_2}{C_0}$ and equating them, we achieve:

$$\frac{1}{D_2 - \frac{1}{D_4 - \frac{1}{\dots}}} = D_0 - \frac{1}{D_{-2} - \frac{1}{D_{-4} - \frac{1}{\dots}}} \quad (15)$$

As $D_0 = \frac{a_p - \beta^2}{q_x}$, this can be easily rearranged for β^2 :

$$\beta^2 = a_p - q_x \left(\frac{1}{D_2 - \frac{1}{D_4 - \frac{1}{\dots}}} + \frac{1}{D_{-2} - \frac{1}{D_{-4} - \frac{1}{\dots}}} \right) \quad (16)$$

Numerical values for β can be extracted by numerically solving (16) once it has been truncated at a pre-specified value for n , then using this to solve for the C_{2n} coefficients up to that order iteratively.

We will consider the conditions necessary to reach the secular limit of the ion motion in (3) – that is, the micromotion can be reasonably neglected. It is important to note that approximately secular motion cannot be achieved by simply setting $\omega_{RF} \rightarrow \infty$, as such a limit would cause the oscillatory component of the trap to do less and less in comparison to the static component. Earnshaw's theorem dictates that a static potential is unable to trap a charged ion, so $\omega_{RF} \rightarrow \infty$ would accordingly be unable to confine the ion and no secular motion would be achievable. Rather, the secular limit can be achieved when the micromotion terms in (3) are not time-dependent, ie. the only non-zero micromotion coefficient is C_0 and all other terms for $n \neq 0$ vanish. In that case, the non-time-dependent terms can be absorbed into the constants A and B .

Via (9) and (12), we achieve rapidly-vanishing C_{2n} terms with increasing n when the D_{2n} terms also diverge rapidly with increasing n . Via the definition of the D_{2n} terms, we see that the D_{2n} terms will diverge quickly with $|n| \geq 1$ as long as q_x and a_p are both small. This indicates that for any given ω_{sec} , our condition to achieve the approximately secular motion will involve small q_x and a_p values.

In the lowest-order case for the ion trajectory ($a_p, q_x^2 \ll 1$), it is reasonable to assume that $C_{\pm 4} \approx 0$. This leads to the following approximation for β :

$$\beta \approx \sqrt{a_p + \frac{1}{2}q_x^2} \quad (17)$$

If we choose two small a_p, q_x values in the stability region for β – say, $a = 0.001$, $q = 0.001$ – and keep orders of n up to $n = 1$ (corresponding to $C_{\pm 4} \approx 0$), then β can be numerically solved as $\beta = 0.0316307$. This agrees with the approximation, which produces:

$$\beta \approx \sqrt{0.001 + \frac{1}{2}(0.001)^2} = 0.0316307$$

Importantly, keeping terms of up to $n = 2$ and $n = 3$ also produces the same value of β , indicating that the assumption $C_{\pm 4} \approx 0$ is appropriate here.