Resolving Full Motion in Presence of Micromotion

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The equation of motion we are interested in solving is:

$$v_p''(\tau) = -\frac{(2\pi)^2}{\beta^2}\left(a_p - 2q_x\cos\left(\frac{4\pi\tau}{\beta}\right)\right)v_p(\tau) \tag{1}$$

Or, if the substitution $\xi = \frac{2\pi\tau}{\beta}$ (which is equivalent to $\xi = \frac{\omega_{RF}}{2}t$, as $\tau = \frac{\beta\omega_{RF}}{4\pi}t$) is made:

$$v_p''(\xi) = -\left(a_p - 2q_x\cos(2\xi)\right)v_p(\xi) \tag{2}$$

The stable solutions of the Mathieu Equation can be found from Floquet's Theorem, which states that the Mathieu Equation admits a complex-valued solution of the form:

$$F(a_p, q_x, \xi) = e^{i\beta_p\xi}P(2\xi) + e^{-i\beta_p\xi}P(-2\xi) \tag{3}$$

Here, β_p is some complex number (which we have not yet defined), denoted as the Floquet exponent. $P(2\xi)$ is a complex-valued function of the same period as the periodic function in our differential equation, which is given in (2) (hence the dependency on 2ξ). That is, $P(2\xi)$ is a periodic function with π, which allows us to generate a general Fourier series for $P(2\xi)$:

$$P(2\xi) = \sum_{n=-\infty}^{\infty} C_n e^{i2n\xi} \tag{4}$$

Ultimately, this gives the stable solutions of the Mathieu Equation as:

$$v_p(\xi) = Ae^{i\beta_p\xi}\sum_{n=-\infty}^{\infty} C_n e^{i2n\xi} + Be^{-i\beta_p\xi}\sum_{n=-\infty}^{\infty} C_n e^{-i2n\xi} \tag{5}$$

β_p, the Floquet exponent, is a mode-specific value that is defined from the recurrence relations, and β is chosen as $\beta := \beta_1$ (see 'Secular Limit of Micromotion' document). If we now convert back to the non-dimensionalised time τ (which is related to ξ via $\xi = \frac{2\pi}{\beta}\tau$), this becomes:

$$v_p(\tau) = Ae^{i2\pi\frac{\beta_p}{\beta}\tau}\sum_{n=-\infty}^{\infty} C_n e^{in\frac{4\pi\tau}{\beta}} + Be^{-i2\pi\frac{\beta_p}{\beta}\tau}\sum_{n=-\infty}^{\infty} C_n e^{-in\frac{4\pi\tau}{\beta}} \tag{6}$$

By modifying A and B, this can be shifted to:

$$v_p(\tau) = A'e^{i2\pi\frac{\beta_p}{\beta}(\tau-\tau_0)}\sum_{n=-\infty}^{\infty} C_n e^{in\frac{4\pi\tau}{\beta}} + B'e^{-i2\pi\frac{\beta_p}{\beta}(\tau-\tau_0)}\sum_{n=-\infty}^{\infty} C_n e^{-in\frac{4\pi\tau}{\beta}} \tag{7}$$

Its corresponding time derivative is given by:

$$\begin{aligned} v_p'(\tau) = {} & A'\left(i2\pi\frac{\beta_p}{\beta}\right)e^{i2\pi\frac{\beta_p}{\beta}(\tau-\tau_0)}\sum_{n=-\infty}^{\infty} C_n e^{in\frac{4\pi\tau}{\beta}} + B'\left(-i2\pi\frac{\beta_p}{\beta}\right)e^{i2\pi\frac{\beta_p}{\beta}(\tau-\tau_0)}\sum_{n=-\infty}^{\infty} C_n e^{-in\frac{4\pi\tau}{\beta}} \\ & + A'e^{i2\pi\frac{\beta_p}{\beta}(\tau-\tau_0)}\sum_{n=-\infty}^{\infty} C_n\left(\frac{4\pi i n}{\beta}\right)e^{in\frac{4\pi\tau}{\beta}} + B'e^{-i2\pi\frac{\beta_p}{\beta}(\tau-\tau_0)}\sum_{n=-\infty}^{\infty} C_n\left(-\frac{4\pi i n}{\beta}\right)e^{-in\frac{4\pi\tau}{\beta}} \end{aligned} \tag{8}$$

We can solve for A' and B' by equating the motion to initial conditions. Note that A' and B' will need to be complex coefficients in order for the motion as a whole, $v_p(\tau)$, to be purely real. If we impose the initial condition $v_p(\tau_0) = x_0$, then this generates:

$$x_0 = A'\sum_{n=-\infty}^{\infty} C_n e^{in\frac{4\pi\tau_0}{\beta}} + B'\sum_{n=-\infty}^{\infty} C_n e^{-in\frac{4\pi\tau_0}{\beta}} \tag{9}$$

$$= (A_r' + iA_i')(f_c(\tau_0) + if_s(\tau_0)) + (B_r' + iB_i')(f_c(\tau_0) - if_s(\tau_0)) \tag{10}$$

where:

$$f_c(\tau) = \sum_{n=-\infty}^{\infty} C_n \cos\left(\frac{4n\pi\tau}{\beta}\right) \tag{11}$$

$$f_s(\tau) = \sum_{n=-\infty}^{\infty} C_n \sin\left(\frac{4n\pi\tau}{\beta}\right) \tag{12}$$

If this is explicitly broken up into its real and imaginary parts, we generate:

$$x_0 = (A'_r + B'_r)f_c(\tau_0) + (B'_i - A'_i)f_s(\tau_0) \tag{13}$$

$$0 = (A'_i + B'_i)f_c(\tau_0) + (A'_r - B'_r)f_s(\tau_0) \tag{14}$$

If we also impose the initial condition $v'_p(\tau_0) = v_0$, then this generates:

$$\begin{aligned} v_0 &= A'\left(i2\pi\frac{\beta_p}{\beta}\right)\sum_{n=-\infty}^{\infty} C_n e^{in\frac{4\pi\tau_0}{\beta}} + B'\left(-i2\pi\frac{\beta_p}{\beta}\right)\sum_{n=-\infty}^{\infty} C_n e^{-in\frac{4\pi\tau_0}{\beta}} \\ &+ A'\sum_{n=-\infty}^{\infty} C_n\left(\frac{4\pi i n}{\beta}\right)e^{in\frac{4\pi\tau_0}{\beta}} + B'\sum_{n=-\infty}^{\infty} C_n\left(-\frac{4\pi i n}{\beta}\right)e^{-in\frac{4\pi\tau_0}{\beta}} \end{aligned} \tag{15}$$

$$\begin{aligned} &= (A'_r + iA'_i)\left(2\pi\frac{\beta_p}{\beta}\right)(if_c(\tau_0) - f_s(\tau_0)) + (B'_r + iB'_i)\left(2\pi\frac{\beta_p}{\beta}\right)(-if_c(\tau_0) - f_s(\tau_0)) \\ &+ (A'_r + iA'_i)(if'_s(\tau_0) + f'_c(\tau_0)) + (B'_r + iB'_i)(-if'_s(\tau_0) + f'_c(\tau_0)) \end{aligned} \tag{16}$$

If this is explicitly broken up into its real and imaginary parts, we generate:

$$v_0 = \frac{2\pi\beta_p}{\beta}[-(A'_r + B'_r)f_s(\tau_0) + (B'_i - A'_i)f_c(\tau_0)] + (A'_r + B'_r)f'_c(\tau_0) + (B'_i - A'_i)f'_s(\tau_0) \tag{17}$$

$$0 = \frac{2\pi\beta_p}{\beta}[-(A'_i + B'_i)f_s(\tau_0) + (A'_r - B'_r)f_c(\tau_0)] + (A'_i + B'_i)f'_c(\tau_0) + (A'_r - B'_r)f'_s(\tau_0) \tag{18}$$

Equations (13), (14), (17) and (18) now fully define A' and B'. The components of A' and B' can accordingly be solved as:

$$A'_r = -\frac{1}{2}\frac{f_s(\tau_0)v_0 - x_0\left(\frac{2\pi\beta_p}{\beta}f_c(\tau_0) + f'_s(\tau_0)\right)}{f_c(\tau_0)f'_s(\tau_0) - f'_c(\tau_0)f_s(\tau_0) + \frac{2\pi\beta_p}{\beta}(f_c(\tau_0)^2 + f_s(\tau_0)^2)} \tag{19}$$

$$B'_r = -\frac{1}{2}\frac{f_s(\tau_0)v_0 - x_0\left(\frac{2\pi\beta_p}{\beta}f_c(\tau_0) + f'_s(\tau_0)\right)}{f_c(\tau_0)f'_s(\tau_0) - f'_c(\tau_0)f_s(\tau_0) + \frac{2\pi\beta_p}{\beta}(f_c(\tau_0)^2 + f_s(\tau_0)^2)} \tag{20}$$

$$A'_i = -\frac{1}{2}\frac{f_c(\tau_0)v_0 - x_0\left(-\frac{2\pi\beta_p}{\beta}f_s(\tau_0) + f'_c(\tau_0)\right)}{f_c(\tau_0)f'_s(\tau_0) - f'_c(\tau_0)f_s(\tau_0) + \frac{2\pi\beta_p}{\beta}(f_c(\tau_0)^2 + f_s(\tau_0)^2)} \tag{21}$$

$$B'_i = \frac{1}{2}\frac{f_c(\tau_0)v_0 - x_0\left(-\frac{2\pi\beta_p}{\beta}f_s(\tau_0) + f'_c(\tau_0)\right)}{f_c(\tau_0)f'_s(\tau_0) - f'_c(\tau_0)f_s(\tau_0) + \frac{2\pi\beta_p}{\beta}(f_c(\tau_0)^2 + f_s(\tau_0)^2)} \tag{22}$$

We can see that A and B are complex conjugates of one another, which ensures that our motion is indeed real. Substituting $A' = A'_r + iA'_i$ and $B' = A'_r - iA'_i$ (for simplicity) into (7), the motion becomes:

$$v_p(\tau) = 2(A'_r f_c(\tau) - A'_i f_s(\tau))\cos\left(2\pi\frac{\beta_p}{\beta}(\tau - \tau_0)\right) - 2(A'_i f_c(\tau) + A'_r f_s(\tau))\sin\left(2\pi\frac{\beta_p}{\beta}(\tau - \tau_0)\right) \tag{23}$$

Accordingly, by substituting the form of A'_r and A'_i from (19) and (21) respectively, we generate the motion:

$$\begin{aligned} v_p(\tau) = &\frac{1}{f'_c(\tau_0)f_s(\tau_0) - f_c(\tau_0)f'_s(\tau_0) - \frac{2\pi\beta_p}{\beta}(f_c(\tau_0)^2 + f_s(\tau_0)^2)}\Bigg(\cos\left(2\pi\frac{\beta_p}{\beta}(\tau - \tau_0)\right)\Big((f_s(\tau_0)v_0 - f'_s(\tau_0)x_0)f_c(\tau) \\ &+ (f'_c(\tau_0)x_0 - f_c(\tau_0)v_0)f_s(\tau) - \frac{2\pi\beta_p}{\beta}x_0(f_c(\tau_0)f_c(\tau) + f_s(\tau_0)f_s(\tau))\Big) + \sin\left(2\pi\frac{\beta_p}{\beta}(\tau - \tau_0)\right)\Big(f_s(\tau)(f'_s(\tau_0)x_0 \\ &- f_s(\tau_0)v_0 + \frac{2\pi\beta_p}{\beta}f_c(\tau_0)x_0) + f_c(\tau)(f'_c(\tau_0)x_0 - f_c(\tau)v_0 - \frac{2\pi\beta_p}{\beta}x_0 f_s(\tau))\Big)\Bigg) \end{aligned} \tag{24}$$

To provide a more convenient form of the motion, we can introduce the following constants:

$$\sigma_c = f_c(\tau_0) = \sum_{n=-\infty}^{\infty} C_n \cos\left(\frac{4n\pi\tau_0}{\beta}\right) \tag{25}$$

$$\sigma_s = f_s(\tau_0) = \sum_{n=-\infty}^{\infty} C_n \sin\left(\frac{4n\pi\tau_0}{\beta}\right) \tag{26}$$

$$\zeta_c = \frac{\beta}{4\pi} f_s'(\tau_0) = \sum_{n=-\infty}^{\infty} nC_n \cos\left(\frac{4n\pi\tau_0}{\beta}\right) \tag{27}$$

$$\zeta_s = -\frac{\beta}{4\pi} f_c'(\tau_0) = \sum_{n=-\infty}^{\infty} nC_n \sin\left(\frac{4n\pi\tau_0}{\beta}\right) \tag{28}$$

$$\rho = 4\pi\left(\sigma_c\left(\sigma_c + \frac{2}{\beta_p}\zeta_c\right) + \sigma_s\left(\sigma_s + \frac{2}{\beta_p}\zeta_s\right)\right) \tag{29}$$

This allows the motion to be written as:

$$\begin{aligned} v_p(\tau) = &\frac{2\beta}{\rho\beta_p} v_0 \sin\left(2\pi\frac{\beta_p}{\beta}(\tau-\tau_0)\right)\left(\sigma_c f_c(\tau-\tau_0) + \sigma_s f_s(\tau-\tau_0)\right) \\ &+ \frac{2\beta}{\rho\beta_p} v_0 \cos\left(2\pi\frac{\beta_p}{\beta}(\tau-\tau_0)\right)\left(\sigma_c f_s(\tau-\tau_0) - \sigma_s f_c(\tau-\tau_0)\right) \\ &+ \frac{2}{\rho\beta_p} 2\pi x_0 \sin\left(2\pi\frac{\beta_p}{\beta}(\tau-\tau_0)\right)\left((\beta_p\sigma_s + 2\zeta_s) f_c(\tau-\tau_0) - (\beta\sigma_c + 2\zeta_c) f_s(\tau-\tau_0)\right) \\ &+ \frac{2}{\rho\beta_p} 2\pi x_0 \cos\left(2\pi\frac{\beta_p}{\beta}(\tau-\tau_0)\right)\left((\beta_p\sigma_c + 2\zeta_c) f_c(\tau-\tau_0) + (\beta\sigma_s + 2\zeta_s) f_s(\tau-\tau_0)\right) \end{aligned} \tag{30}$$

Now we would like to determine the motion arising from a series of kicks at time $\{\tau_i\}$, which results in a corresponding change in velocity of $\{v_i\}$. (Note that this change in velocity is still referring to a change in velocity of a mode, not an ion). To determine the resulting motion, we can use the linearity of the equation of motion to write the solution as a sum of solutions which satisfy the change in velocity as initial conditions.

First we recall that v_p is defined as the linearised modes of the *displaced* ion positions from equilibrium (not the absolute ion positions). Thus, if we assume that the ions start (at $\tau = \tau_0$) at their equilibrium positions, then the linear combinations of the displaced positions from equilibrium that constitute v_p will also be 0 – this produces an initial modal displacement of $x_0 = 0$ at $\tau = \tau_0$. (30) therefore simplifies to:

$$\begin{aligned} v_p(\tau) = &\frac{2\beta}{\rho\beta_p} v_0 \sin\left(2\pi\frac{\beta_p}{\beta}(\tau-\tau_0)\right)\left(\sigma_c f_c(\tau-\tau_0) + \sigma_s f_s(\tau-\tau_0)\right) \\ &+ \frac{2\beta}{\rho\beta_p} v_0 \cos\left(2\pi\frac{\beta_p}{\beta}(\tau-\tau_0)\right)\left(\sigma_c f_s(\tau-\tau_0) - \sigma_s f_c(\tau-\tau_0)\right) \end{aligned} \tag{31}$$

Now, if we consider all the pulses independently of one another, then we would expect that the trajectory after each pulse will also satisfy (31), though now with initial velocity v_i at time $\tau = \tau_i$ (and corresponding changes to the constants that depend on the initial conditions, ie. σ_c, σ_s and ρ). Because the pulses do not instantaneously change the position of the ions (and therefore the modes), we can consider the initial condition of each pulse application to be $x_i = 0$[1]. That is, we can consider the application of each pulse as a new, independent trajectory with initial conditions as the instantaneous motional changes it enforces. Thus, the motion after pulse i (disregarding evolution due to previous and subsequent pulses) will be given by:

$$\begin{aligned} v_p(\tau) = &\frac{2\beta}{\rho_i\beta_p} v_i \sin\left(2\pi\frac{\beta_p}{\beta}(\tau-\tau_i)\right)\left(\sigma_{c,i} f_c(\tau-\tau_i) + \sigma_{s,i} f_s(\tau-\tau_i)\right) \\ &+ \frac{2\beta}{\rho\beta_p} v_i \cos\left(2\pi\frac{\beta_p}{\beta}(\tau-\tau_i)\right)\left(\sigma_{c,i} f_s(\tau-\tau_i) - \sigma_{s,i} f_c(\tau-\tau_i)\right) \end{aligned} \tag{32}$$

[1]This is in the same way that we consider the initial velocity for each pulse to be v_i, not $v(\tau_i - \delta) + v_i$ (where $v(\tau_i - \delta)$ is the velocity of the mode instantaneously before the application of the pulse). Hence, we do not need to account for the position instantaneously before the application of the pulse

where the constants $\sigma_{c,i}$, $\sigma_{s,i}$ and ρ_i are now specific to the time τ_i. Because the motion is linear, we can simply add the trajectories for all the pulses (treated individually) together. This gives the full motion as:

$$
\begin{aligned}
v_p(\tau) = &\sum_{i=0}^{n} \frac{2\beta}{\rho_i \beta_p} v_i \sin\left(2\pi \frac{\beta_p}{\beta}(\tau - \tau_i)\right) \left(\sigma_{c,i} f_c(\tau - \tau_i) + \sigma_{s,i} f_s(\tau - \tau_i)\right) \\
&+ \sum_{i=0}^{n} \frac{2\beta}{\rho_i \beta_p} v_i \cos\left(2\pi \frac{\beta_p}{\beta}(\tau - \tau_i)\right) \left(\sigma_{c,i} f_s(\tau - \tau_i) - \sigma_{s,i} f_c(\tau - \tau_i)\right)
\end{aligned}
\tag{33}
$$