Resolving Full Motion in Presence of Micromotion

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The equation of motion we are interested in solving is:

$$v_p''(\tau) = -\frac{(2\pi)^2}{\beta^2}\left(a_p - 2q_x\cos\left(\frac{4\pi\tau}{\beta}\right)\right)v_p(\tau) \tag{1}$$

Or, if the substitution $\xi = \frac{2\pi\tau}{\beta}$ (which is equivalent to $\xi = \frac{\omega_{RF}}{2}t$, as $\tau = \frac{\beta\omega_{RF}}{4\pi}t$) is made:

$$v_p''(\xi) = -\left(a_p - 2q_x\cos(2\xi)\right)v_p(\xi) \tag{2}$$

The stable solutions of the Mathieu Equation can be found from Floquet's Theorem, which states that the Mathieu Equation admits a complex-valued solution of the form:

$$F(a_p, q_x, \xi) = e^{i\beta_p\xi}P(2\xi) + e^{-i\beta_p\xi}P(-2\xi) \tag{3}$$

Here, β_p is some complex number (which we have not yet defined), denoted as the Floquet exponent. $P(2\xi)$ is a complex-valued function of the same period as the periodic function in our differential equation, which is given in (2) (hence the dependency on 2ξ). That is, $P(2\xi)$ is a periodic function with π, which allows us to generate a general Fourier series for $P(2\xi)$:

$$P(2\xi) = \sum_{n=-\infty}^{\infty} C_n e^{i2n\xi} \tag{4}$$

Ultimately, this gives the stable solutions of the Mathieu Equation as:

$$v_p(\xi) = Ae^{i\beta_p\xi}\sum_{n=-\infty}^{\infty} C_n e^{i2n\xi} + Be^{-i\beta_p\xi}\sum_{n=-\infty}^{\infty} C_n e^{-i2n\xi} \tag{5}$$

β_p, the Floquet exponent, is a mode-specific value that is defined from the recurrence relations, and β is chosen as $\beta := \beta_1$ (see 'Secular Limit of Micromotion' document). If we now convert back to the non-dimensionalised time τ (which is related to ξ via $\xi = \frac{2\pi}{\beta}\tau$), this becomes:

$$v_p(\tau) = Ae^{i2\pi\frac{\beta_p}{\beta}\tau}\sum_{n=-\infty}^{\infty} C_n e^{in\frac{4\pi\tau}{\beta}} + Be^{-i2\pi\frac{\beta_p}{\beta}\tau}\sum_{n=-\infty}^{\infty} C_n e^{-in\frac{4\pi\tau}{\beta}} \tag{6}$$

By modifying A and B, this can be shifted to:

$$v_p(\tau) = A'e^{i2\pi\frac{\beta_p}{\beta}(\tau-\tau_0)}\sum_{n=-\infty}^{\infty} C_n e^{in\frac{4\pi\tau}{\beta}} + B'e^{-i2\pi\frac{\beta_p}{\beta}(\tau-\tau_0)}\sum_{n=-\infty}^{\infty} C_n e^{-in\frac{4\pi\tau}{\beta}} \tag{7}$$

Its corresponding time derivative is given by:

$$\begin{aligned} v_p'(\tau) = {} & A'\left(i2\pi\frac{\beta_p}{\beta}\right)e^{i2\pi\frac{\beta_p}{\beta}(\tau-\tau_0)}\sum_{n=-\infty}^{\infty} C_n e^{in\frac{4\pi\tau}{\beta}} + B'\left(-i2\pi\frac{\beta_p}{\beta}\right)e^{i2\pi\frac{\beta_p}{\beta}(\tau-\tau_0)}\sum_{n=-\infty}^{\infty} C_n e^{-in\frac{4\pi\tau}{\beta}} \\ & + A'e^{i2\pi\frac{\beta_p}{\beta}(\tau-\tau_0)}\sum_{n=-\infty}^{\infty} C_n\left(\frac{4\pi in}{\beta}\right)e^{in\frac{4\pi\tau}{\beta}} + B'e^{-i2\pi\frac{\beta_p}{\beta}(\tau-\tau_0)}\sum_{n=-\infty}^{\infty} C_n\left(-\frac{4\pi in}{\beta}\right)e^{-in\frac{4\pi\tau}{\beta}} \end{aligned} \tag{8}$$

We can solve for A' and B' by equating the motion to initial conditions. Note that A' and B' will need to be complex coefficients in order for the motion as a whole, $v_p(\tau)$, to be purely real. If we impose the initial condition $v_p(\tau_0) = x_0$, then this generates:

$$x_0 = A'\sum_{n=-\infty}^{\infty} C_n e^{in\frac{4\pi\tau_0}{\beta}} + B'\sum_{n=-\infty}^{\infty} C_n e^{-in\frac{4\pi\tau_0}{\beta}} \tag{9}$$

$$= (A_r' + iA_i')(f_c(\tau_0) + if_s(\tau_0)) + (B_r' + iB_i')(f_c(\tau_0) - if_s(\tau_0)) \tag{10}$$

where:

$$f_c(\tau) = \sum_{n=-\infty}^{\infty} C_n \cos\left(\frac{4n\pi\tau}{\beta}\right) \tag{11}$$

$$f_s(\tau) = \sum_{n=-\infty}^{\infty} C_n \sin\left(\frac{4n\pi\tau}{\beta}\right) \tag{12}$$

If this is explicitly broken up into its real and imaginary parts, we generate:

$$x_0 = (A'_r + B'_r)f_c(\tau_0) + (B'_i - A'_i)f_s(\tau_0) \tag{13}$$

$$0 = (A'_i + B'_i)f_c(\tau_0) + (A'_r - B'_r)f_s(\tau_0) \tag{14}$$

If we also impose the initial condition $v'_p(\tau_0) = v_0$, then this generates:

$$\begin{aligned} v_0 = {} & A'\left(i2\pi\frac{\beta_p}{\beta}\right)\sum_{n=-\infty}^{\infty} C_n e^{in\frac{4\pi\tau_0}{\beta}} + B'\left(-i2\pi\frac{\beta_p}{\beta}\right)\sum_{n=-\infty}^{\infty} C_n e^{-in\frac{4\pi\tau_0}{\beta}} \\ & + A'\sum_{n=-\infty}^{\infty} C_n\left(\frac{4\pi in}{\beta}\right)e^{in\frac{4\pi\tau_0}{\beta}} + B'\sum_{n=-\infty}^{\infty} C_n\left(-\frac{4\pi in}{\beta}\right)e^{-in\frac{4\pi\tau_0}{\beta}} \end{aligned} \tag{15}$$

$$\begin{aligned} & = (A'_r + iA'_i)\left(2\pi\frac{\beta_p}{\beta}\right)(if_c(\tau_0) - f_s(\tau_0)) + (B'_r + iB'_i)\left(2\pi\frac{\beta_p}{\beta}\right)(-if_c(\tau_0) - f_s(\tau_0)) \\ & + (A'_r + iA'_i)(if'_s(\tau_0) + f'_c(\tau_0)) + (B'_r + iB'_i)(-if'_s(\tau_0) + f'_c(\tau_0)) \end{aligned} \tag{16}$$

If this is explicitly broken up into its real and imaginary parts, we generate:

$$v_0 = \frac{2\pi\beta_p}{\beta}[-(A'_r + B'_r)f_s(\tau_0) + (B'_i - A'_i)f_c(\tau_0)] + (A'_r + B'_r)f'_c(\tau_0) + (B'_i - A'_i)f'_s(\tau_0) \tag{17}$$

$$0 = \frac{2\pi\beta_p}{\beta}[-(A'_i + B'_i)f_s(\tau_0) + (A'_r - B'_r)f_c(\tau_0)] + (A'_i + B'_i)f'_c(\tau_0) + (A'_r - B'_r)f'_s(\tau_0) \tag{18}$$

Equations (13), (14), (17) and (18) now fully define A' and B'. The components of A' and B' can accordingly be solved as:

$$A'_r = -\frac{1}{2}\frac{f_s(\tau_0)v_0 - x_0\left(\frac{2\pi\beta_p}{\beta}f_c(\tau_0) + f'_s(\tau_0)\right)}{f_c(\tau_0)f'_s(\tau_0) - f'_c(\tau_0)f_s(\tau_0) + \frac{2\pi\beta_p}{\beta}(f_c(\tau_0)^2 + f_s(\tau_0)^2)} \tag{19}$$

$$B'_r = -\frac{1}{2}\frac{f_s(\tau_0)v_0 - x_0\left(\frac{2\pi\beta_p}{\beta}f_c(\tau_0) + f'_s(\tau_0)\right)}{f_c(\tau_0)f'_s(\tau_0) - f'_c(\tau_0)f_s(\tau_0) + \frac{2\pi\beta_p}{\beta}(f_c(\tau_0)^2 + f_s(\tau_0)^2)} \tag{20}$$

$$A'_i = -\frac{1}{2}\frac{f_c(\tau_0)v_0 - x_0\left(-\frac{2\pi\beta_p}{\beta}f_s(\tau_0) + f'_c(\tau_0)\right)}{f_c(\tau_0)f'_s(\tau_0) - f'_c(\tau_0)f_s(\tau_0) + \frac{2\pi\beta_p}{\beta}(f_c(\tau_0)^2 + f_s(\tau_0)^2)} \tag{21}$$

$$B'_i = \frac{1}{2}\frac{f_c(\tau_0)v_0 - x_0\left(-\frac{2\pi\beta_p}{\beta}f_s(\tau_0) + f'_c(\tau_0)\right)}{f_c(\tau_0)f'_s(\tau_0) - f'_c(\tau_0)f_s(\tau_0) + \frac{2\pi\beta_p}{\beta}(f_c(\tau_0)^2 + f_s(\tau_0)^2)} \tag{22}$$

We can see that A and B are complex conjugates of one another, which ensures that our motion is indeed real. Substituting $A' = A'_r + iA'_i$ and $B' = A'_r - iA'_i$ (for simplicity) into (7), the motion becomes:

$$v_p(\tau) = 2(A'_r f_c(\tau) - A'_i f_s(\tau))\cos\left(2\pi\frac{\beta_p}{\beta}(\tau - \tau_0)\right) - 2(A'_i f_c(\tau) + A'_r f_s(\tau))\sin\left(2\pi\frac{\beta_p}{\beta}(\tau - \tau_0)\right) \tag{23}$$

Accordingly, by substituting the form of A'_r and A'_i from (19) and (21) respectively, we generate the motion:

$$\begin{aligned} v_p(\tau) = {} & \frac{1}{f'_c(\tau_0)f_s(\tau_0) - f_c(\tau_0)f'_s(\tau_0) - \frac{2\pi\beta_p}{\beta}(f_c(\tau_0)^2 + f_s(\tau_0))}\Bigg(\cos\left(2\pi\frac{\beta_p}{\beta}(\tau - \tau_0)\right)\Bigg((f_s(\tau_0)v_0 - f'_s(\tau_0)x_0)f_c(\tau) \\ & + (f'_c(\tau_0)x_0 - f_c(\tau_0)v_0)f_s(\tau) - \frac{2\pi\beta_p}{\beta}x_0(f_c(\tau_0)f_c(\tau) + f_s(\tau_0)f_s(\tau))\Bigg) + \sin\left(2\pi\frac{\beta_p}{\beta}(\tau - \tau_0)\right)\Bigg(f_s(\tau)(f'_s(\tau_0)x_0 \\ & - f_s(\tau_0)v_0 + \frac{2\pi\beta_p}{\beta}f_c(\tau_0)x_0) + f_c(\tau)(f'_c(\tau_0)x_0 - f_c(\tau)v_0 - \frac{2\pi\beta_p}{\beta}x_0 f_s(\tau))\Bigg)\Bigg) \end{aligned} \tag{24}$$

To provide a more convenient form of the motion, we can introduce the following constants:

$$\sigma_c = f_c(\tau_0) = \sum_{n=-\infty}^{\infty} C_n \cos\left(\frac{4n\pi\tau_0}{\beta}\right) \tag{25}$$

$$\sigma_s = f_s(\tau_0) = \sum_{n=-\infty}^{\infty} C_n \sin\left(\frac{4n\pi\tau_0}{\beta}\right) \tag{26}$$

$$\zeta_c = \frac{\beta}{4\pi} f_s'(\tau_0) = \sum_{n=-\infty}^{\infty} nC_n \cos\left(\frac{4n\pi\tau_0}{\beta}\right) \tag{27}$$

$$\zeta_s = -\frac{\beta}{4\pi} f_c'(\tau_0) = \sum_{n=-\infty}^{\infty} nC_n \sin\left(\frac{4n\pi\tau_0}{\beta}\right) \tag{28}$$

$$\rho = 4\pi\left(\sigma_c\left(\sigma_c + \frac{2}{\beta_p}\zeta_c\right) + \sigma_s\left(\sigma_s + \frac{2}{\beta_p}\zeta_s\right)\right) \tag{29}$$

This allows the motion to be written as:

$$\begin{aligned} v_p(\tau) = {} & \frac{2\beta}{\rho\beta_p} v_0 \sin\left(2\pi\frac{\beta_p}{\beta}(\tau-\tau_0)\right)(\sigma_c f_c(\tau-\tau_0) + \sigma_s f_s(\tau-\tau_0)) \\ & + \frac{2\beta}{\rho\beta_p} v_0 \cos\left(2\pi\frac{\beta_p}{\beta}(\tau-\tau_0)\right)(\sigma_c f_s(\tau-\tau_0) - \sigma_s f_c(\tau-\tau_0)) \\ & + \frac{2}{\rho\beta_p} 2\pi x_0 \sin\left(2\pi\frac{\beta_p}{\beta}(\tau-\tau_0)\right)((\beta_p\sigma_s + 2\zeta_s) f_c(\tau-\tau_0) - (\beta\sigma_c + 2\zeta_c) f_s(\tau-\tau_0)) \\ & + \frac{2}{\rho\beta_p} 2\pi x_0 \cos\left(2\pi\frac{\beta_p}{\beta}(\tau-\tau_0)\right)((\beta_p\sigma_c + 2\zeta_c) f_c(\tau-\tau_0) + (\beta\sigma_s + 2\zeta_s) f_s(\tau-\tau_0)) \end{aligned} \tag{30}$$