Homework 3

Problem 1

To compute the following by hand, we can consider the definition of a mod m as $a - \lfloor \frac{a}{m} \rfloor m$.

a.

$$3 \; mod \; 7 = 3 - \lfloor \frac{3}{7} \rfloor \cdot 7 = 3$$

b.

$$7 \; mod \; 7 = 7 - \lfloor \frac{7}{7} \rfloor \cdot 7 = 0$$

c.

$$10 \; mod \; 7 = 10 - \lfloor \frac{10}{7} \rfloor \cdot 7 = 3$$

d.

$$\begin{aligned} 10^9 \; mod \; 7 &= 10^9 - \lfloor \frac{10^9}{7} \rfloor \cdot 7 \\ &= 10^9 - \lfloor (\frac{10}{7^{1/9}})^9 \rfloor \cdot 7 \end{aligned}$$

The expression inside this sum corresponds to the Taylor Series expansion of the exponential function, where $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

$$\begin{aligned} e^{-\lambda} \sum_{k=0}^{\infty} \frac{(e^{it}\lambda)^k}{k!} &= e^{-\lambda} e^{e^{it}\lambda} \\ &= e^{\lambda(e^{it}-1)} \end{aligned}$$

(B) Given an Exponential random variable X with parameter $\lambda > 0$ on $[0, \infty]$, the probability density function (pdf) is described by

$$p_X(x) = \lambda e^{-\lambda x}$$

The characteristic function (for a continuous variable) can be derived as follows:

$$\begin{aligned} \psi_X(t) = E[e^{itX}] &= \int_0^{\infty} e^{itx} p_X(x) dx \\ &= \int_0^{\infty} e^{itx} \lambda e^{-\lambda x} dx \\ &= \lambda \left[\int_0^{\infty} e^{x(it-\lambda)} dx \right] \end{aligned}$$

Recognizing the integral as the form $\int_0^\infty e^{-ax}dx$ where $-a = \lambda - it$, we can evaluate as

$$\lambda\left[\int_0^\infty e^{x(it-\lambda)}\right] = \lambda\left[\frac{1}{\lambda - it}\right] = \frac{\lambda}{\lambda - it}$$

(C) X, Y are independent random variables with characteristic functions ψ_X and ψ_Y. $\psi_{(X,Y)}$ is the characteristic function from $\mathbb{R}^2 \to \mathbb{R}$ of the vector (X, Y):

$$\psi_{(X,Y)}(s,t) \;=\; \mathbb{E}\big[e^{\,i(s,t)\cdot(X,Y)}\big]$$

We must prove that

$$\psi_{(X,Y)}(s,t) \;=\; \psi_X(s)\,\psi_Y(t)$$

Given that X and Y are independent, by definition a joint expectation of a function on both variables can decompose into the product of the marginal expectations. We can say that

$$\mathbb{E}\big[e^{\,i(s,t)\cdot(X,Y)}\big] = E[e^{isX}]E[e^{isY}] = \psi_X(s)\,\psi_Y(t)$$

Problem 2