Pierson Guthrey
pguthrey@iastate.edu

1 Important Fourier Transforms

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$$

General f		Distributions	
$f(x)$	$\hat{f}(k)$	1	$\sqrt{2\pi}\delta(k)$
$f(x-a)$	$e^{-ak}\hat{f}(k)$	$\delta(x)$	$\frac{1}{\sqrt{2\pi}}$
$e^{2\pi iax} f(x)$	$\hat{f}(k-2\pi a)$	e^{iax}	$\sqrt{2\pi}\delta(k-a)$
$f(ax)$	$\frac{1}{\lvert a\rvert}\hat{f}\left(\frac{k}{a}\right)$	$\cos(ax)$	$\frac{\sqrt{2\pi}}{2}\left(\delta(k-a)+\delta(k+a)\right)$
$\hat{f}(x)$	$f(-k)$	$\sin(ax)$	$\frac{\sqrt{2\pi}}{2i}\left(\delta(k-a)+\delta(k+a)\right)$
$\frac{d^n f(x)}{dx^n}$	$(ik)^n \hat{f}(k)$	x^n	$i^n\sqrt{2\pi}\delta^{(n)}(k)$
$x^n f(x)$	$i^n \frac{d^n \hat{f}(k)}{dk^n}$	$\frac{1}{x}$	$-\frac{i}{2}\sqrt{2\pi}\mathrm{sign}\,(k)$
$(f*g)(x)$	$\sqrt{2\pi}\hat{f}(k)\hat{g}(k)$	$\mathrm{sign}\,(x)$	$\frac{2}{\sqrt{2\pi}}\frac{1}{ik}$
$f(x)g(x)$	$\frac{(\hat{f}*\hat{g})(k)}{\sqrt{2\pi}}$	$H(x)$	$\frac{\sqrt{2\pi}}{2}\left(\frac{1}{i\pi k}+\delta(k)\right)$
$f(x)=\overline{f(x)}$ (real)	$\hat{f}(-k)=\overline{\hat{f}(k)}$		
$f(x)$ (real, even)	$\hat{f}(k)$ (real, even)		
$f(x)$ (real, odd)	$\hat{f}(k)$ (imaginary, odd)		
$\overline{f(x)}$	$\overline{\hat{f}(-k)}$		

2 Important Maclaurin Series

Trigonometric Functions

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ...$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ...$$

Hyperbolic Functions

$$\sinh(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} - ...$$

$$\cosh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} - ...$$

Exponential Function

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + ...$$

Natural Logarithm (for $|x| < 1$)

$$log(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n} = -x - \frac{x^2}{2} - \frac{x^3}{3} - ...$$

$$log(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - ...$$

Geometric Series (for $|x| < 1$)

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + ...$$

Binomial Series (for $|x| < 1$, $\alpha \in \mathbb{C}$)

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \left(\begin{array}{c} \alpha \\ n \end{array} \right) x^n, \qquad \left(\begin{array}{c} \alpha \\ n \end{array} \right) = \frac{\alpha(\alpha-1)...(\alpha-n+1)}{n!}$$

This includes the square root series for $\alpha = \frac{1}{2}$ and the infinite geometric series for $\alpha = -1$.

$$(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + ...$$

3 Calculus Techniques

3.0.1 Rapid Integration by Parts

This neat calculus trick is used in 1-D integration to shorten the process of using integration by parts over and over again. Consider attemping to integrate $\int_a^b f(x)g^{(n)}(x)dx$. Create a table similar to the one below

N	F	G	bdy
1	$f(x)$	$g^{(n)}(x)$	
2	$f'(x)$	$g^{(n-1)}(x)$	$+\left[f(x)g^{(n-1)}(x)\right]_0^1$
3	$f''(x)$	$g^{(n-2)}(x)$	$-\left[f'(x)g^{(n-2)}(x)\right]_0^1$
4	$f'''(x)$	$g^{(n-3)}(x)$	$+\left[f''(x)g^{(n-3)}(x)\right]_0^1$
$\vdots$			
$n-1$	$f^{(n-1)}(x)$	$g'(x)$	$(-1)^{n-2}\left[f^{(n-2)}g'(x)\right]_0^1$
n	$f^{(n)}(x)$	$g(x)$	$(-1)^{n-1}\left[f^{(n-1)}g(x)\right]_0^1$

Take all of the entries of column G except the first, and multiply them by the previous term in the F column, alternating signs each time. Evaluate these at the boundaries. Then take the last two terms and integrate, where the sign depends on n

$$\left[g^{(n-1)}f - f'g^{(n-2)} + f''g^{(n-3)} + (-1)^{(n-2)}f^{(n-2)}g' + (-1)^{n-1}f^{(n-1)}g\right]_a^b + (-1)^n \int_a^b f^{(n)}(x)g(x)dx$$

Example: $\int_0^1 xu(x)v''(x)dx$.

N	F	G
1	$xu(x)$	$v''(x)$
2	$u(x) + xu'(x)$	$v'(x)$
3	$2u'(x) + xu''(x)$	$v(x)$

Pierson Guthrey
pguthrey@iastate.edu

$$\int_0^1 xu(x)v''(x)dx = [xuv' - (u + xu')v]_0^1 + \int_0^1 (2u' + xu'')vdx$$

or

$$\int_0^1 xu(x)v''(x)dx = u(1)v'(1) - u(1)v(1) - u'(1)v(1) + u(0)v(0) + \int_0^1 (2u' + xu'')vdx$$

3.1 Coordinate Transformations

Making of linear transformation of coordinates

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix}$$

Involves the Jacobian

$$\frac{\partial(\alpha, \beta)}{\partial(x, t)} = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

so

$$dxdt = \left| \left(\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right)^{-1} \right| d\alpha d\beta$$

3.2 Integration and Derivatives

Mean Value Theorem for Integrals

$$\int_a^b f(x)dx = (b - a)f(\theta), \qquad \theta \in (a, b)$$

Directional Derivative

$$\nabla_{\vec{\mathbf{v}}} f(\vec{\mathbf{x}}) = \nabla f(\vec{\mathbf{x}}) \cdot \vec{\mathbf{v}}, \qquad \|v\|_2 = 1$$

This operator is linear, obeys the product rule $\nabla_{\vec{\mathbf{v}}}(fg) = f\nabla_{\vec{\mathbf{v}}} f + g\nabla_{\vec{\mathbf{v}}} g$ and obeys the chain rule $\nabla_{\vec{\mathbf{v}}}(h \circ g)(\vec{\mathbf{x}}) = h'(g(\vec{\mathbf{x}}))\nabla_{\vec{\mathbf{v}}} g(\vec{\mathbf{x}})$. Example: Unit normal vectors $\frac{df}{dn} = \nabla_{\vec{\mathbf{n}}} f$

3.3 Divergence Theorem

$$\int_\Omega \vec{\nabla} \cdot \vec{\mathbf{g}} dx = \int_{\partial\Omega} \vec{\mathbf{g}} \cdot \vec{\mathbf{n}} ds$$

The first integral is an integral over Ω, the second integral is a line integral around the boundary of Ω

3.4 Green's Identity

$$\int \int_{\Omega^N} (f\Delta g - g\Delta f)dx = \int_{\partial\Omega} \left(f\frac{\partial h}{\partial n} - h\frac{\partial f}{\partial n} \right) ds$$

The first integral is an integral over Ω, the second integral is a line integral around the boundary of Ω

Pierson Guthrey
pguthrey@iastate.edu

3.5 Laplace Operator

3.5.1 Polar Coordinates

$$\nabla^2 = \Delta = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}$$

If the system is radially symmetric, this becomes

$$\nabla^2 = \Delta = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)$$

3.5.2 Spherical Coordinates

$$\nabla^2 = \Delta = \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2\frac{\partial}{\partial \rho}\right) + \frac{1}{\rho^2 \sin(\theta)}\frac{\partial^2}{\partial \theta}\left(\sin(\theta)\frac{\partial}{\partial \theta}\right) + \frac{1}{\rho^2 \sin^2(\theta)}\frac{\partial^2}{\partial \phi^2}$$

If the system is radially symmetric, this becomes

$$\nabla^2 = \Delta = \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2\frac{\partial}{\partial \rho}\right)$$

3.6 Jacobian Factors

3.6.1 Polar Coordinates

$$\int_\Omega f(x,y)dxdy = \int_\Omega f(x,y,z)rdrd\theta$$

3.6.2 Spherical Coordinates

$$\int_\Omega f(x,y,z)dxdydz = \int_\Omega f(x,y,z)\rho^2 d\rho d\theta d\phi$$

3.7 Divergence and Curl

....

3.8 Sequences and Series

3.8.1 Sequences

3.8.2 Series

The partial sum of a geometric series is given by ($r \neq 1$)

$$a + ar + ar^2 + ar^4 + ... + ar^{n-1} = \sum_{k=0}^{n-1} ar^k = a\frac{1-r^n}{1-r}$$

If and only if $|r| < 1$, then as $n \to \infty$,

$$a + ar + ar^2 + ar^4 + ... = \sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$$

Convergence A series S converges to a limit L if and only if the sequence of partial sums S_K converges to L.

- The p-series $\sum\limits_{n=1}^{\infty} \frac{1}{n^r}$ converges for $r > 1$ and diverges for $r \leq 1$.
- The harmonic series $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ diverges.
- If the sequence $\{b_n\}$ converges to the limit L as $n \to \infty$, then the telescoping series $\sum\limits_{n=1}^{\infty} (b_n - b_{n+1})$ converges to $b_1 - L$

For function series,

- A function series converges pointwise on Ω if it converges for each $x \in \Omega$. That is, pointwise convergence is defined as
$$S_N(x) = \sum_{n=1}^{N} f_n(x) \to S(x) = \sum_{n=1}^{\infty} f_n(x) \; \forall \; x \in \Omega$$
- A function series converges uniformly on Ω if it converges pointwise and remainder from the partial series sum converges to 0 as $n \to \infty$ independent of x. That is, it converges if
$$\forall \; \epsilon > 0, \; \exists \; N \text{ s.t. } n > N \implies |S_n(x) - f(x)| < \epsilon$$

4 Trigonometric Functions

$$\int_\Omega \cos(\alpha x) e^{\beta x} dx = \left(\beta^2 + \alpha^2\right)\left(\beta \cos(\alpha x) + \alpha \sin(\alpha x)\right) e^{\beta x}$$
$$\int_\Omega \sin(\alpha x) e^{\beta x} dx = \left(\beta^2 + \alpha^2\right)\left(\beta \sin(\alpha x) - \alpha \cos(\alpha x)\right) e^{\beta x}$$

$$e^{ix} = \cos(x) + i \sin(x) \qquad \cos(x) = \frac{1}{2}\left(e^{-ix} + e^{ix}\right) \qquad \sin(x) = \frac{i}{2}\left(e^{-ix} - e^{ix}\right)$$

4.1 Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1 \qquad 1 + \tan^2 x = \sec^2 x \qquad 1 + \cot^2 x = \csc^2 x$$

4.2 Sum- Difference Formulas

$$\sin(u \pm v) = \sin u \cos v \pm \cos u \sin v$$
$$\cos(u \pm v) = \cos u \cos v \mp \sin u \sin v$$
$$\tan(u \pm v) = \frac{\tan u \pm \tan v}{1 \mp \tan u \tan v}$$

4.3 Double Angle Formula

$$\sin(2u) = 2 \sin u \cos u$$
$$\cos(2u) = \cos^2 u - \sin^2 u = 2\cos^2 u - 1 = 1 - 2\sin^2 u$$
$$\tan(2u) = \frac{2 \tan u}{1 - \tan^2 u}$$

4.4 Sum to Product Formulas

$$\sin u \pm \sin v = 2\sin\left(\frac{u \pm v}{2}\right)\cos\left(\frac{u \mp v}{2}\right)$$
$$\cos u + \cos v = 2\cos\left(\frac{u + v}{2}\right)\cos\left(\frac{u - v}{2}\right)$$
$$\cos u - \cos v = -2\sin\left(\frac{u + v}{2}\right)\sin\left(\frac{u - v}{2}\right)$$

4.5 Differentiation

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\tfrac{d}{dx}\sin^{-1} x = \tfrac{1}{\sqrt{1-u^2}}\tfrac{du}{dx} \qquad \tfrac{d}{dx}\cos^{-1} x = \tfrac{-1}{\sqrt{1-u^2}}\tfrac{du}{dx} \qquad \tfrac{d}{dx}\tan^{-1} x = \tfrac{1}{1+u^2}\tfrac{du}{dx}$$
$$\tfrac{d}{dx}\csc^{-1} x = \tfrac{-1}{|u|\sqrt{u^2-1}}\tfrac{du}{dx} \qquad \tfrac{d}{dx}\sec^{-1} x = \tfrac{-1}{|u|\sqrt{u^2-1}}\tfrac{du}{dx} \qquad \tfrac{d}{dx}\cot^{-1} x = \tfrac{-1}{1+u^2}\tfrac{du}{dx}$$

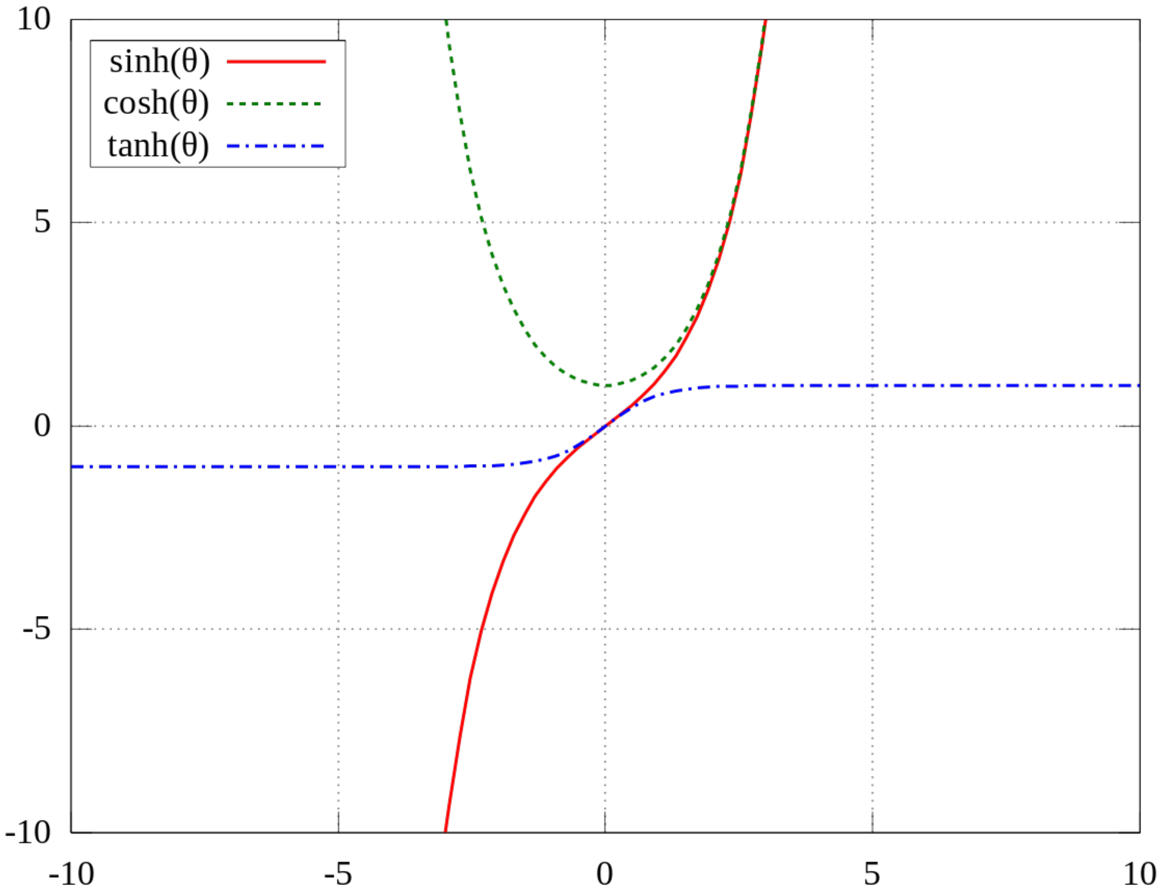
4.6 Integration

$$\int \sec^2 x = \tan x + C \qquad \int \csc^2 x = -\cot x + C$$

5 Hyperbolic Functions

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2} \qquad \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Pierson Guthrey
pguthrey@iastate.edu



6 Areas and Volumes

6.1 Two Dimensions

Shape	Area	Perimeter
Trapezoid	$\frac{b_1+b_2}{2}h$	sum of sides

6.2 Three Dimensions

For shapes with height h, base b, radius r,

Shape	Volume	Surface Area
Cone	$\frac{1}{3}\pi r^2$	$\pi r^2 + \pi r s = \pi r^2 \pi r \sqrt{r^2 + h^2}$
Pyramid	$\frac{1}{3}bh$	
Sphere	$\frac{4}{3}\pi r^3$	$4\pi r^2$

6.3 N Dimensions

Shape	Volume	Surface Area
Sphere	$\frac{\pi^{n\backslash 2}}{\Gamma\left(\frac{N}{2}+1\right)} r^N$	$\frac{N\pi^{n\backslash 2}}{\Gamma\left(\frac{N}{2}+1\right)} r^{N-1}$

7 Named Functions

7.1 Gamma Function

For a positive integer n,

$$\Gamma(n) = (n-1)!$$

This function is also defined for all complex numbers except negative integers and zero. For complex numbers with a positive real part, the Gamma function is defined as the improper integral

$$\Gamma(t) = \int_0^\infty x^{t-1} e^{-x} dx$$

8 Fourier

8.1 Parseval's Identity

Given the Fourier coefficients of f, c_n,

$$\sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx$$

8.2 Plancherel Theorem

If $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then $\hat{f} \in L^2(\mathbb{R})$, and the FT is an isometry wrt $\|\cdot\|_{L^2(\mathbb{R})}$

$$\int_{\mathbb{R}^N} |f(x)|^2 \, dx = \int_{\mathbb{R}^N} \left|\hat{f}(k)\right|^2 dk$$

8.3 Poisson Summation Formula

For $\phi \in \mathcal{S}$

$$\sqrt{2\pi} \sum_{n=-\infty}^{n} \phi(2\pi n) = \sum_{n=-\infty}^{n} \hat{\phi}(n)$$

9 Famous Inequalities

9.1 Jensen's Inequality

If ϕ is convex on $\mathbb{R}$ then

$$\phi\left(\frac{1}{b-a}\int_a^b f(t)dt\right) \leq \frac{1}{b-a}\int_a^b \phi(f(t))dt$$

If ϕ is concave on $\mathbb{R}$ then

$$\phi\left(\frac{1}{b-a}\int_a^b f(t)dt\right) \geq \frac{1}{b-a}\int_a^b \phi(f(t))dt$$

9.2 Normed Linear Space Inequalities

X is a vector space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|x\| = \sqrt{\langle x, x \rangle}$. Take any $x, y \in X$.

9.2.1 Cauchy Schwartz Inequality

$$|\langle x,y\rangle|^2 \le \langle x,x\rangle\langle y,y\rangle \text{ or equivalently, } |\langle x,y\rangle| \le \|x\|\,\|y\|$$

9.2.2 Parallelogram Law

X is an normed inner product space if and only if

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

9.2.3 Pythagorean Theorem

$$\left\|x^2+y^2\right\| = \left\|x^2\right\| + \left\|y^2\right\|$$

9.2.4 Bessel's Inequality

For an infinite dimensional basis, $S_N = \sum \langle x, e_n\rangle e_n = P_{M_N}x$ where $M_N = \mathcal{L}\{e_1, e_2, ..., e_n\}$, which implies

$$\sum_{n=1}^{\infty} |\langle x, e_n\rangle|^2 \le \|x\|^2$$

9.3 Young's Inequality

For $\epsilon, a, b > 0$, $1 \le p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then

$$ab \le \epsilon\frac{a^p}{p} + \epsilon^{\frac{-q}{p}}\frac{b^q}{q}$$

In particular, if $p = q = 2$,

$$ab \le \epsilon\frac{a^2}{2} + \frac{1}{\epsilon}\frac{b^2}{2}$$

Further, if $\epsilon = 1$,

$$ab \le \frac{a^2}{2} + \frac{b^2}{2}$$

9.4 Young's Convolution Inequality

If $\phi, \psi \in C_0^\infty$, then for $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ and $1 \le p, q, r \le \infty$

$$\|\phi * \psi\|_{L^r(\mathbb{R}^N)} \le \|\phi\|_{L^p(\mathbb{R}^N)}\|\psi\|_{L^q(\mathbb{R}^N)}$$

Example: $\|\phi * \psi\|_{L^p(\mathbb{R}^N)} \le \|\phi\|_{L^p(\mathbb{R}^N)}\|\psi\|_{L^1(\mathbb{R}^N)}$

9.5 Holder Inequality

Integral version
For u, v measurable, and $1 \le p, q \le \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$

$$\int_\Omega |u(x)v(x)|\,dx \le \left(\int_\Omega |u(x)|^p\,dx\right)^{\frac{1}{p}}\left(\int_\Omega |v(x)|^q\,dx\right)^{\frac{1}{q}}$$

$$\|u(x)v(x)\|_{L^1(\Omega)} = \|u(x)\|_{L^p(\Omega)}\|v(x)\|_{L^q(\Omega)}$$

Pierson Guthrey
pguthrey@iastate.edu

Sum Version
For $1 \leq p, q \leq \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$,

$$\sum |a_k b_K| \leq \left(\sum |a_k|^p\right)^{\frac{1}{p}} \left(\sum |b_k|^q\right)^{\frac{1}{q}}$$

9.6 Minkawski Inequality

For u, v measurable, and $1 \leq p \leq \infty$

$$\left(\int_\Omega |u(x) + v(x)|^p \, dx\right)^{\frac{1}{p}} \leq \left(\int_\Omega |u(x)|^p \, dx\right)^{\frac{1}{p}} + \left(\int_\Omega |v(x)|^p \, dx\right)^{\frac{1}{p}}$$

$$\|u + v\|_{L^p(\Omega)} \leq \|u\|_{L^p(\Omega)} + \|v\|_{L^p(\Omega)}$$

Sum Version
For $1 \leq p \leq \infty$

$$\left(\sum |a_k + b_k|^p\right)^{\frac{1}{p}} \leq \left(\sum |a_k|^p\right)^{\frac{1}{p}} + \left(\sum |b_k|^p\right)^{\frac{1}{p}}$$

10 Common Theorems

10.1 Stone-Weierstrass Theorem

(also known as Weierstrass Approximation Theorem)
Every continuous function on a closed interval can be uniformly approximated by a polynomial function.

10.2 Heine-Borel Theorem

Every closed and totally bounded subset of a complete matrix space is compact.

For a subset $S \subset \mathbb{R}^N$, the following are equivalent

- S is closed and bounded
- S is compact

10.3 Arzela Ascoli Theorem

Consider a sequence of real-valued continuous functions $\{f_n\}$ defined on a closed and bounded interval $[a, b]$ of the real line. There exists a subsequence $\{f_{n_k}\}$ that converges uniformly if and only if this sequence is uniformly bounded and equicontinuous.

10.4 Fubini's Theorem

Given measureable spaces A, B, and if f is $A \times B$ measureable, and if the integral with respect to a product measure satisfies

$$\int_{A \times B} |f(x, y)| \, d(x, y) < \infty$$

then the integral with respect to a product measure is equal to the iterated integrals

$$\int_{A \times B} f(x, y) d(x, y) = \int_A \left(\int_B f(x, y) dy\right) dx = \int_B \left(\int_A f(x, y) dx\right) dy$$

Corollary
If f satisfies the above conditions and additionally $f(x,y) = h(x)g(y)$, then

$$\int_{A\times B} f(x,y)d(x,y) = \int_A h(x)dx \int_B g(y)dy$$

10.5 Lax Milgram Theorem

If $a(\cdot,\cdot)$ be a bilinear form on $\mathcal{H}$ which is

- **bounded**: $|a(u,v)| \leq C\|u\|_{\mathcal{H}}\|v\|_{\mathcal{H}}$
- **coercive**: $|a(u,u)| \geq c\|u\|^2_{\mathcal{H}}$

then for any $f \in \mathcal{H}^*$ there is a unique solution $u \in \mathcal{H}$ to the equation $a(u,v) = \langle f, v\rangle$ and also $\|u\| \ \leq \frac{1}{c}\|u\|^2$.

10.6 Fredholm Alternative

10.6.1 Operator Version

Given a compact integral operator K, a nonzero λ is either an eigenvalue of K of lies in the domain of the resolvent.

$$R_\lambda(K) = (K - \lambda I)^{-1}$$

10.6.2 Integral Equation Version

Let $K(x,y)$ be an kernel of the integral operator $Tu = \lambda u - \langle K, u\rangle$. If $K(x,y)$ yields a compact integral operator, then the following theorem holds: For any nonzero $\lambda \in \mathbb{C}$, either the integral equation

$$\lambda\phi(x) - \int_a^b K(x,y)\phi(y)dy = f(x)$$

has a solution for all $f(x)$ OR the associated homogenous case $f(x) = 0$ has only trivial solutions.
$K(x,y)$ being Hilbert Schmidt is a sufficient but not necessary condition.

10.6.3 Linear Algebra Version

For $A \in \mathbb{C}^{n\times m}$ and $b \in \mathbb{C}^{m\times 1}$,

- Either $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ has a solution $\vec{\mathbf{x}}$
- OR: $A^T\vec{\mathbf{y}} = 0$ has a solution $\vec{\mathbf{y}}$ with $\vec{\mathbf{y}}^T\vec{\mathbf{b}} \neq 0$.

That is, $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ has a solution if and only if for any $\vec{\mathbf{y}}$ s.t. $A^T\vec{\mathbf{y}} = 0$, $\vec{\mathbf{y}}^T\vec{\mathbf{b}} = 0$.

10.7 Riesz Representation Theorem

Given a Hilbert space $\mathcal{H}$ and its dual space $\mathcal{H}'$. For all $y \in \mathcal{H}'$, there exists a unique ϕ_y such that

$$\phi_y(x) = \langle x, y\rangle$$

10.8 Riemmann Lebesgue Lemma

The Fourier Transform of any L^1 function vanishes at infinity.
Let $f \in L^1(\mathbb{R})$ and since $f \in L^1$ there exists a smooth function (say g), compactly supported (say on $[a,b]$) that approximates f. Thus let $\|f-g\|_{L^1} < \epsilon$. Since g is smooth,

$$\hat{g}(k) = \int_a^b g(x)e^{-ixk}dx = \frac{g(b)e^{-ibk}}{-ik} - \frac{g(a)e^{-iak}}{-ik} + \int_a^b g'(x)e^{-ixk} - ikdx$$

So $|\hat{g}(k)| \to 0$ at at $k \to \pm\infty$. Then

$$\left|\hat{f}(k)\right| = \left|\int f(x)e^{-ixk}dx\right| \leq \left|\int (f(x)-g(x))e^{-ixk}dx\right| + |\hat{g}(k)| \leq \int |f(x)-g(x)|\,dx + |\hat{g}(k)| < \epsilon + |\hat{g}(k)|$$

So as $k \to \pm\infty$, $\limsup_{k\to\pm\infty} = 0$

10.9 Eigenfunction Expansion Theorem

Let K be a self adjoint compact operator and let (λ_k, e_k) be the set of eigenpairs for K where $\lambda_k \neq 0$ and e_k are the eigenfunctions orthonormalized to $\|e_k\| = 1$.
Any function in the range of K can be expanded in a Fourier series in the eigenfunctions of K corresponding to nonzero eigenvalues. There eigenfunctions form an orthonormal basis for $R(K)$ (but necessarily for $\mathcal{H}$). Thus, for all $f \in \mathcal{H}$,

$$Kf = \sum \langle Kf, e_k\rangle e_k = \sum \langle f, Ke_k\rangle e_k = \sum \lambda_k \langle f, e_k\rangle e_k$$

where equality is in the L^2 sense.
If we include the eigenfunctions for $\lambda = 0$, we have a basis for $\mathcal{H}$. If h is the projection if f onto the nullspace of K, then an arbitrary function can be decomposed uniquely as

$$f = h + \sum \langle f, e_k\rangle e_k$$