THE APPLIED MATH SPELLBOOK

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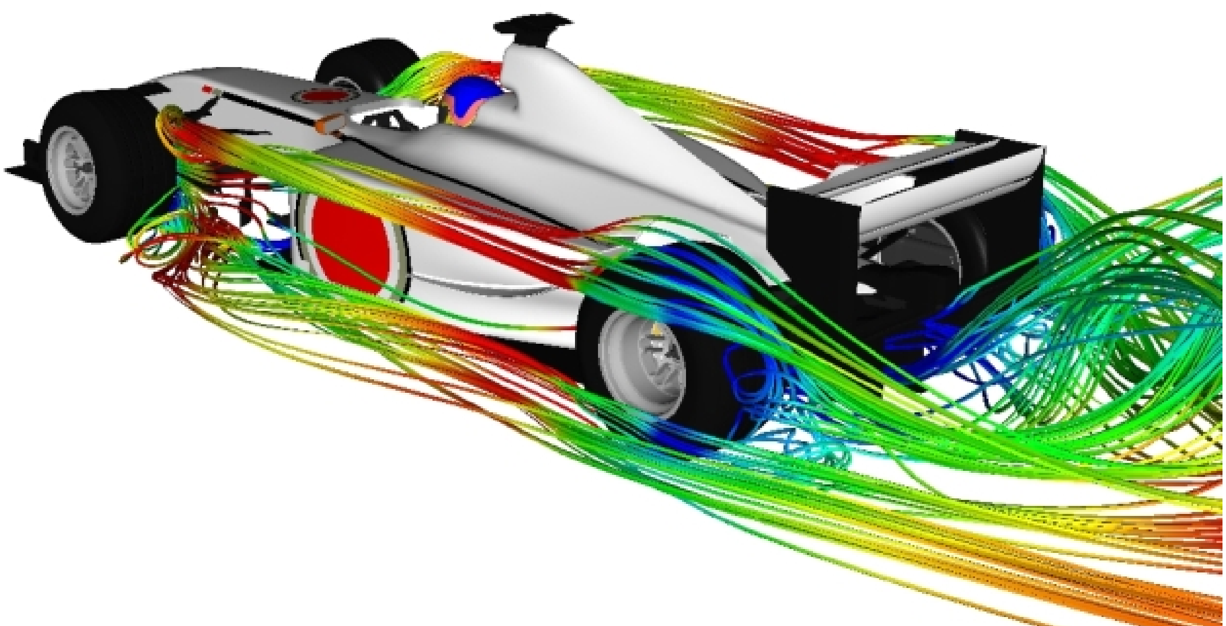

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Part I
Elementary Logic and Proofs

1 Logic

1.1 Logical Operations

Given propositions p and q,

- The **negation** of p is $\neg p$

p	$\neg p$
T	F
F	T

- The **disjunction** of p and q is $p \vee q$ and is read ”p or q”

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

- The **conjunction** of p and q is $p \wedge q$ and is read ”p and q”

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

- The **conditional** of p and q is $p \to q$ and is read ”if p then q”. p is the **hypothesis** and q is the **conclusion**. Whenever the hypothesis is false, $p \to q$ is said to be **vacuously true**

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	$T(vacuously)$
F	F	$T(vacuously)$

1.2 Truths, Implications, and Equivalences

- $q \implies p$ is the **converse** of $p \to q$
- $\neg q \implies \neg p$ is the **contrapositive** of $p \to q$
- A proposition is a **tautology** if it is true for all p,q
- A proposition is a **contradiction** if is is false for all p,q
- A proposition is a **contingency** if it is true for at least one pair p,q and false for at least one pair p,q
- A compound proposition is one that combines p,q with logical operations such as $P = p \vee q$

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- Given possibly compound propositions P and Q, if Q is true whenever P is true, $P \implies Q$ ("P implies Q") is a **logical implication**
- If both $P \implies Q$ and $Q \implies P$ then we say P and Q are a **logically equivalence**
- Any conditional proposition is logically equivalent to its contrapositive

1.3 Proofs

the following are equivalent:

$$A \implies B \qquad , \qquad \neg A \vee B \qquad , \qquad \neg B \implies \neg A$$

The following proofs are equivalent.
Direct Proof: assume A, prove B
Note: When proving "iff" ($\iff$) statements, show $A \implies B$ THEN show $B \implies A$
Contrapositive: assume $\neg B$, prove $\neg A$
Contradiction: assume A and $\neg B$, arrive at a contradiction

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Part II
Algebra

2 Structures and Properties

Algebraic structures refer to a carrier set and at least one finitary operation defined on that set, which obey certain algebraic properties.

2.1 Algebraic Properties

Closure

$$a + b \in \mathbb{R}, ab \in \mathbb{R} \tag{1}$$

Commutativity

$$a + b = b + a, ab = ba \tag{2}$$

Associativity

$$(a + b) + c = a + (b + c), (ab)c = a(bc) \tag{3}$$

Distributivity

$$(a + b)c = ac + bc \tag{4}$$

Identity

$$a + 0 = 0 + a = a, 1a = a1 = a \tag{5}$$

Invertbility

$$a + (-a) = 0, a(1/a) = 1 \tag{6}$$

Cancelation

$$a + x = a + y \Rightarrow x = y \tag{7}$$

Zero-Factor

$$a0 = 0a = 0 \tag{8}$$

Negation

$$-(-a) = a, (-a)b = a(-b) = -(ab), (-a)(-b) = ab \tag{9}$$

3 Morphisms

Morphisms preserve the group structure.
Homeomorphism: A homeomorphism $f : S \to T$ from a set S to a set T is a continuous bijecton with a continuous inverse.
A homeomorphism bijects the collection of open sets of M and the collection of open sets in N.

Isomorphism: An isomorphism is a homeomorphism in both directions.

4 Algebraic Structures

4.1 Groups

Sets A where you take $(a, b) \in A$ and perform an opertation to make a third element $c \in A$
Axioms: Closure, Associativity, Identity, Invertibility
Commutative groups are known as Abelian groups.

4.2 Ring

A ring is an Abelian group that defines the behavior of two operations.
Axioms: include group axioms, distributivity, and
Bilinearity

$$[ax + by, z] = a\,[x, z] + b\,[y, z]$$
$$[z, ax + by] = a\,[z, x] + b\,[z, y]$$

4.3 Fields

A field $\mathbb{F}$ is a commutative ring consisting of a set of elements and two operations (addition and multiplication) both with the following axioms:
Group properties (Closure, Associativity, Identity, Invertibility), Ring Properties (Commutativity, Distributivity), Cancelation, Zero-Factor, Negation
They also have the following properties

- Transitivity: $x < y < z \Rightarrow x < z$
- Trichotomy: Either $x < y$, $x > y$, or $x = y$, exclusively
- Translation: $x < y \Rightarrow x + z < y + z$

The Standard Fields are $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$

4.3.1 Archimedian Property

- Definition: The property of having no infinitely large or infinitely small elements
- **Infinitesimals** have been used to express the idea of objects so small that there is no way to see them or to measure them. The insight with exploiting infinitesimals was that objects could still retain certain specific properties, such as angle or slope, even though these objects were quantitatively small
- An algebraic structure in which any two non-zero elements are comparable, in the sense that neither of them is infinitesimal with respect to the other, is said to be **Archimedean**. A structure which has a pair of non-zero elements, one of which is infinitesimal with respect to the other, is said to be **non-Archimedean**.

4.4 Fundamental Theorem of Algebra

Every polynomial $p_n(x)$ over $\mathbb{C}$ of degree n has exactly n roots (allowing for multiples).

If $p_n(x)$ is real, then complex roots come in complex cojugate pairs.

Part III
Analysis

Equivalence Relations
An equivalence relation $\sim$ defines clusters of elements of A. For each $a \in A$, and equivalence of class a w.r.t to R

$$[a]_R = b \in A : aRb$$

An equivalence relation between A, B,and C satisfies three properties

- Identity: $A \sim A$
- Symmetry: $A \sim B \implies B \sim A$
- Transitivity: $A \sim B$ and $B \sim C \implies A \sim C$

5 Sets

5.1 Subsets

For a set A, B is a subset of A denoted $B \subseteq A$ if $\forall\, b \in B$ it is also true that $b \in A$

Superset: For a set B, A is a superset of B denoted $A \supseteq A$ if $\forall\, b \in b$ it is also true that $b \in A$

Proper subset: A proper subset A' of A, denoted $A' \subset A$, is a subset of A but is missing at least one element of A

Equivalent sets: For any two sets A and B, $A = B$ iff $A \subseteq B$ and $B \subseteq A$

Power Set: The power set $\mathcal{P}(S)$ of S is the collection of all subsets of S.

5.2 Collection of Sets

A collection of sets is denoted $\{E_\alpha\}_{\alpha \in A}$ where A is an indexing set and E_α is a set in the collection.

5.3 Union and Intersect

A union of two sets is

$$A \cup B = \{x : x \in A \vee x \in B\}$$

The intersection of two sets is

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

If $A \cap B = \emptyset$, A and B are **disjoint** Similiarly , A union of n sets is

$$\bigcup_{i=1}^{n} \{A_i\} = \{x : \exists\, i \text{ s.t. } x \in A_i\}$$

an intersection of n sets is

$$\bigcap_{i=1}^{n} \{A_i\} = \{x : x \in A_i\ \forall\, i\}$$

5.3.1 de Morgan's Laws

$$(E \cup F)^c = E^c \cap F^c$$
$$(E \cap F)^c = E^c \cup F^c$$

5.4 Sequences (x_n)

Sequences are functions $f : \mathbb{N} \to M$ with $p_n = f(n)$ denoted (x_n) or $(x_n)_{n=1}^{\infty}$
Repetition is allowed.
Sequences aren't in general surjections (Not everything in M needs to be used.)

5.4.1 Subsequences

Subsequences (q_k) of a sequence (p_n) are simply $q_k = p_{n_k}$, where $n_1 < n_2 <$ ... Every subsequence of a convergent sequence converges and it converges to the same limit as does the mother sequence.
There can be further sub-subsequences. (r_l) s.t $r_l = q_{k_l} = p_{n_{k_l}} = p_{n_{k(l)}}$

5.4.2 Nested Sequences

(A_n) is a nested sequence if $A_1 \supset A_2 \supset ... \supset A_n \supset ...$ and

$$\bigcap A_n = \{p : \ \forall\, n, p \in A_n\}$$

5.4.3 Limits $\lim\limits_{n\to\infty} p_n = p$

The sequence (a_n) converges to the limit p in M if

$$\forall\, \epsilon > 0\ \exists\, N \in \mathbb{N} \text{ s.t. } n \in \mathbb{N} \text{ and } n \geq N \implies d(p_n, p) < \epsilon$$

Otherwise it is said to diverge.
Limits of sequences are unique.

5.4.4 Cauchy Condition

A sequence is Cauchy iff $\forall\, \epsilon > 0\ \exists\, N \in \mathbb{N}$ s.t. $n, m \geq N \Rightarrow d(a_n, a_m) < \epsilon$
Every convergent sequence obeys a Cauchy Condition.

Cauchy Convergence Criterion: A sequence (a_n) in a complete metric space converges $\iff$ it obeys the Cauchy condition.

5.4.5 Series

Given a sequence (s_n) such that each is a partial sum $s_n = \sum\limits_{k=1}^{n} x_k$, if s_n converges to a limit s, we say the series $\sum x_k$ converges to s.
It is said to be **absolutely convergent** if $s_n = \sum\limits_{k=1}^{n} |x_k|$ converges.

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5.5 Open and Closed Sets

5.5.1 Limit Set

A point p is a limit point (or just limit) of S if there exists a sequence (p_n) in S that converges to it.
The limit set of a set S is the set

$$limS = \{p \in M : p \text{ is a limit of S}\}$$

$limS = \overline{S}$

5.5.2 Neighborhood

A neighborhood of a point p is any open set that contains p.

The r-neighbordhood of p is the set

$$M_r p = \{q \in M : d(p,q) < r\}$$

The following are equivalent:
a. p is a limit of S
b. $\forall\, r > 0$, $M_r(p) \cap S \neq 0$

Openness is dual to closedness: The complement of an open set is closed and the complement of a close set is open.

5.5.3 Inheritance principle

If $K \subset N \subset M$ where M is a metric space and N is a metric space then K is closed in N if and only if there is some subset L or M st L is closed in M and $K = L \cap N$. It is said that N inherits its closeds from M. Dually, a metric subspace inherits opens.

Assume that N is a metric subspace of M and also is a closed subset of M. A set $K \in N$ is closed in N if and only if it is closed in M. Dually,
Assume that N is a metric subspace of M and also is an open subset of M. A set $U \in N$ is open in N if and only if it is closed in M.

5.5.4 Closed Sets

A set K is closed if it contains all of its limit points.

- A closed set K has an open complement K^c.
- For the metric space $\mathbb{R}$, $[a, b]$ is closed for $a, b \in \mathbb{R}$.
- The intersection of any number of closed sets $\bigcap K$ is closed.
- A finite union of closed sets $\bigcup^n K$ is closed.
- K is closed if and only if $K' \subset K$
- The lub and glb of a nonempty bounded closed set K are in K.
- A countable collection of closed sets is known as a F_σ set

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5.5.5 Open Sets

A set is open if

$$\forall\, p \in S\; \exists\, r > 0 \text{ s.t. } d(p,q) < r \implies q \in S$$

or

$$\forall\, p \in S\; \exists\, r > 0 \text{ s.t. } \{q\} = \{q : d(p,q) < r\} \implies \{q\} \subset S$$

- An open set is a set whose complement is closed.
- For the metric space $\mathbb{R}$, (a,b) is open.
- The union of opens sets is open.
- A countable collection of open sets is known as a G_δ set
- Every open set can be uniquely expressed as a countable union of disjoint open intervals.
- Open subspaces of R are in one of the following forms: $(-\infty,b),(a,b),(a,\infty)$, and $\mathbb{R} = (-\infty,\infty)$

5.5.6 Neither closed nor open set

For the metric space $\mathbb{R}$, $(a,b]$ is neither closed nor open.

5.5.7 Clopen sets

A clopen set is both close and open. For the metric space $\mathbb{R}$, the empty set $\emptyset$ and the entire set of reals $\mathbb{R}$ are both clopen.

5.6 Closure, Interior, and Boundaries of Sets

5.6.1 Closure $\overline{S}$, cl(S)

The closure $\overline{S} = \bigcap K$ where K ranges through the collection of all closed sets that contains S.

$$\overline{S} = \{x \in M : \text{ if K is closed and } S \subset K \text{ then } x \in K\}$$

The closure is closed (intersect of closeds)
$\overline{S}$ is the smallest set that contains S.
$\overline{S} = limS$
When it is unclear, use $cl_M(S)$ to denote closure in M

5.6.2 Interior $int(S)$

The interior $int(S) = \bigcup U$ where U ranges through the collection of all open sets contained in S.

$$int(S) = \{x \in M : \text{ for some open } U \subset S, x \in U\}$$

When it is unclear, use $int_M(S)$ to denote boundary of S with respect to M

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5.6.3 Boundary ∂S

The boundary $\partial S = \overline{S} \cap \overline{S^c}$
or equivalently $\overline{S} \setminus int(S)$
The bounday is closed (intersect of closeds)
When it is unclear, use $\partial_M(S)$ to denote boundary within with respect to M

5.7 Unit Pieces

Unit Ball: $B^m = \{x \in \mathbb{R}^m : |x| \leq 1\}$
Unit Sphere: $S^{m-1} = \{x \in \mathbb{R}^m : |x| = 1\}$
Unit Cube: $[0,1]^m = [0,1] \times ... \times [0,1]$

5.8 Set Properties

5.8.1 Convexity

A set $E \subset \mathbb{R}^m$ is convex if $\forall\, x, y \in E$, the straight line segment between x and y is also in E. That is, E is convex iff

$$\forall\, x, y \in E, sx + (1-s)y \in E \text{ for } 0 \leq s \leq 1$$

Convex Combinations: Linear combinations of the form $z = sx + (1-s)y$ for $0 \leq s \leq 1$ are called convex combinations. The line segment formed from all of the convex combinations of x and y are denoted $[x, y]$

5.8.2 Set Diameter

of a nonempty set $S \subset M$ is $sup\,\{d(x, y)\}$, for $x, y \in S$

5.9 Zero Set

A set E is a zero set if for all $\epsilon > 0$ there exists a countable collection of open intervals $\{I_n = (a_n, b_n)\}_{n=1}^N$ such that $E \subset \bigcup\limits_{n=1}^N I_n$ and $\sum\limits_{n=1}^N (b_n - a_n) < \epsilon$ for $N \leq \infty$
For a zero set E,

- E has Lebesgue measure zero
- **Almost Everywhere (a.e.)**: Properties of a set that hold except possibly for points in a zero set are said to hold a.e.

The following are zero sets

- A countable union of zero sets
- A subset of a zero set
- A countable or finite set of points
- $\mathbb{Q}$
- The Cantor Set

6 Topology

Any property of a metric space or of a maping between metric spaces that can be described solely in terms of open sets is a topological property.
Namely: Countability, Connectedness, Compactness, Metrizability...and more

The **topology** of a set M is the collection of all open subsets of M. A topology is closed under union (unions of opens are open), finite intersection (finite intersections of open sets are open), and contains $\emptyset$ and M (both are open sets).

6.0.1 Topological Invariance

A property is topologically invariant if it is invariant under homeomorphisms. That is, if X has a topologically invariant property and X and Y are homeomorphic, then Y has that topological property.

6.0.2 Topological Equivalence

A homeomorphism $f : M \to N$ bijects topologies of M and N.

6.1 Intervals

For any open interval $I = (a, b)$

1. I Has no smallest element, either irrational or rational
2. There are strictly more irrational numbers in the interval than rational numbers.

6.2 ϵ - Principle

6.2.0.1 Form 1: If $a, b \in \mathbb{R}$ s.t. $a \leq b + \epsilon \;\; \forall \, \epsilon > 0$, then $a \leq b$

6.2.0.2 Form 2: If $a, b \in \mathbb{R}$ s.t. $|a - b| < \epsilon \;\; \forall \, \epsilon > 0$, then $a = b$

6.2.1 Preimage $f^{pre}(V)$

Let $f : M \to N$ be given. The pre-image of a set $V \subset N$ is

$$f^{pre}(V) = \{p \in M : f(p) \in V\}$$

6.3 Cardinality

If there exists a bijection between A and B, they are said to have equal cardinality.
Denoted $A \sim B$
$A \sim A$, $A \sim B \implies B \sim A$, $A \sim B \sim C \implies A \sim C$

6.3.1 Finite Set

A set A is finite if $A \; 1, 2, ..., n$ for some $n \in \mathbb{N}$.

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6.3.2 Infinite Set

A set is infinite if it is not finite.
An infinite set B is denumerable if $\exists f : A \to B$ s.t. f is a surjection from a denumerable set A.
Ex. $\mathbb{Q}^+$ is denumerable. $\exists f : \mathbb{N} \times \mathbb{N} \to \mathbb{Q}^+$, namely $f(n,m) = \frac{n}{m}$

6.3.3 Denumerable Sets

A set is denumerable if $A \sim \mathbb{N}$
Compositions of denumerable sets are denumerable (Ex. $\mathbb{N} \times \mathbb{N}$)
Infinite subsets of denumerable sets are deumerable
If $f : \mathbb{N} \to B$ is a surjection and B is infinite, then B is denumerable
Denumerable unions of denumerable sets are denumerable
$\mathbb{Q}$ is denumerable because $\exists f : \mathbb{N} \times \mathbb{N} \to \mathbb{Q}$ and f is surjective

6.3.4 Countable Sets

- A Set is countable if it is denumerable or finite.
- Any countable union of countable sets is countable. Ex. $\cup_{k=1}^{\infty} \mathbb{N}^k$ is countable
- $\mathbb{N}^k$ is countable
- $\mathbb{N}^k$ can be embedded into $\mathbb{N}^\infty$ as such: $(a_1, a_2, ...a_k) \to (a_1, a_2, ..., a_k, 0, 0, ...)$

6.3.5 Uncountable Sets

A set is uncountable if it is not countable.

- Every non-empty, perfect, complete metric space is uncountable.

6.4 Boundedness

A subset of a metric space M is bounded if for some $p \in M$ and some $r > 0$, $S \subset M_r(p)$
Otherwise, it is said to be unbounded.

- A bounded function $f : M \to N$ is one with a bounded range. $f(M) \subset N_r(q)$ for $q \in N$
- Graphs of bounded functons can be unbounded: $x \to sin(X)$ is bounded, $\{(x,y) \in \mathbb{R}^2 : y = sin(x)\}$ is not.
- **Bolzano Weierstrass Theorem** Any bounded sequence in $\mathbb{R}$ has a convergent subsequence.

Further, a space $A \subset M$ is **totally bounded** if for each $\epsilon > 0$ there exists a finite covering of A by ϵ-neigborhoods.

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6.4.1 Upper Bounds and Suprenums

Upper Bound: $M \in \mathbb{R}$ is an upper bound for $S \subset \mathbb{R}$ if $\exists\ s \in S$, $s \leq M$. Also, M is the Least Upper Bound or l.u.b. for S if M is less than all other Upper Bounds for S.

Least Upper Bound Property: If S is a non-empty supset of $\mathbb{R}$ and is bounded above then in $\mathbb{R}$ there exists a least upper bound for S.

Suprenum: The sup of S is it's least upper bound when it is bounded above and $+\infty$ otherwise.

6.4.2 Lower Bounds and Infimums

Lower Bound: $m \in \mathbb{R}$ is a Lower Bound for $S \subset \mathbb{R}$ if $\exists\ s \in S$, $s \geq m$. Also, m is the Greatest Lower Bound for S if m is greater than all other Lower Bounds for S.

Greatest Lower Bound Property: If S is a non-empty supset of $\mathbb{R}$ and is bounded below then in $\mathbb{R}$ there exists a Greatest Lower Bound for S.

Infimum: The inf of S is it's greatest lower bound it is bounded below and $-\infty$ otherwise.

6.5 Clustering and Condensing

6.5.1 Cluster Points S'

A set S clusters at p (and p is a cluster point of S, not necessarily in S) if each $M_r(p)$ contains infinitely many points of S
Fact: $S \cup S' = \bar{(}S)$
The following are equivalent conditions to S clustering at p:

1. There is a sequence of distinct points in S that converges to p
2. Each neighborhood of p contains infinitely many points of S
3. Each neighborhood of p contains at least two points of S
4. Each neighborhod of p contains at least one point of S other than p

6.5.2 Condensation Points

A set S condenses at p (and p is a condensation point of S) if each $M_r(p)$ contains uncountably many points of S

6.6 Compactness

A space M is covering compact if every open cover has a finite subcover.
A space is sequentially compact if every sequence (x_n) has a convergent subsequence (x_{n_k}).
A space is sequentially compact if and only if it is covering compact.
Every compact set is closed and bounded.
The following are compact:

- A closed subset of a compact set

- A box of closed intervals $[a_1, b_1] \times ... \times [a_m, b_m] \in \mathbb{R}^m$
- The Cartesian product of finitely many compact sets
- **Heine-Borel Theorem** Every closed and bounded subset of $\mathbb{R}^m$ is compact
- **Generalized Heine-Borel Theorem** Every closed and totally bounded subset of a complete metric space is compact
- A metric space is compact if and only if it is complete and totally bounded
- Any finite subset of a metric space
- The empty set
- The union of infinitely many compact sets
- The intersection of arbitarily many compact sets
- The boundary of a compact set (such as the unit 2-sphere in $\mathbb{R}^3$)
- The Cantor Set
- A nested sequence of non-empty compacts is compact and nonempty

6.6.1 Precompact

A subset A of a metric space X is pecompact if its closure in X is compact.
A set is compact if and only if it is closed and precompact.

- A bounded subset of $\mathbb{R}^n$ is precompact
- A function family $\mathcal{F} \subset C(\Omega)$ is precompact if is uniformly bounded and equicontinuous.

6.6.2 Nested Compacts

A nested sequence of non-empty compacts is compact and nonempty
A nested nonempty compact sequence with a diameter that tends to 0 as $n \to \infty$, then $A = \bigcap A_n$ is a single point

6.6.3 Continuous Transformation on Compacts

If $f : M \to N$ is continuous and A is a compact subset of M then $f(A)$ is a compact subset of N
The continuous image of a compact is compact.
Every continuous function defined on a compact is uniformly continuous.
A continuous real valued function defined on a compact set is bounded; it assumes minimum and maximum values.

If M is homeomorphic to N, then N is compact.
If M is compact then a continuous bijection $f : M \to N$ is a homeomorphism, its inverse $f^{-1} : M \to N$ is continuous.

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6.6.4 Embeddedness

$h : M \to N$ embeds M into N if h is a homeomorphism from M onto $h(M)$
Any property that holds for every embedded copy of M is an **absolute** or **intrinsic** property.
A compact is absolutely closed and absolutely bounded.

6.7 Connectedness

If M has a proper clopen subset A, M is disconnected
If M is disconnected, the **separation** of M is $M = A \sqcup A^c$
Otherwise, M is **connected**

- The continuous image of a connected is connected. If M is connected, $f : M \to N$ is continuous, and f is onto, then N is connected.
- If M is connected and M is homeomorphic to N then N is connected.

- **Generalized Intermediate Value Thm** Every continuous real valued function defined on a connected domain has the intermediate value property.
- $\mathbb{R}$ is connected.
- **Intermediate Value Thm for** $\mathbb{R}$ Every continuous function $\mathbb{R} \to \mathbb{R}$ has the intermediate value property.
- The following metric spaces are connected: (a, b), $[a, b]$, the circle

- If $S \subset T \subset \overline{S}$ and S is connected, then T is connected.
- The closure of a connected set is connected.

6.7.1 Totally Disconnected

A metric space M is **totally disconnected** if each point $p \in M$ has arbitrarily small clopen neighborhoods.
Given ϵ and $p \in M$, there exists a clopen set U

$$p \in U \subset M_\epsilon(p)$$

- A discrete space is totally disconnected
- $\mathbb{Q}$ and $\mathbb{N}$ is totally disconnected

6.7.2 Point Connected

The union of connected sets sharing at least one common point is connected.

6.7.3 Path Connected

A **path** joining p to q in a metric space M is a continuous function $f : [a,b] \to M$ such that $f(a) = p$ and $f(b) = q$. If each pair of points in M can be joined by a path M then M is **path connected**.

- Path connectedness implies connectedness, but the converse need not be true.

- All connected subsets of $\mathbb{R}$ are path connected
- Every open connected subset of $\mathbb{R}$ is path connected
- The **Topologist's Sine Curve** is connected yet not path connected

6.8 Perfect Sets

The following are equivalent: A set M is **perfect**

- if it has no isolated points. Given any point and a neighborhood of that point, there is another point within that neighborhood.
- if $M' = M$: Each $p \in M$ is a cluster point of M

Otherwise, a set is said to be **imperfect**. The imperfection of an imperfect set is the set of isolated points.

- $\mathbb{R}$ is a connected perfect space
- The Cantor set is perfect
- Every open subset of a perfect set is perfect
- The imperfection of $\mathbb{N}$ is itself
- Every non-empty, perfect, complete metric space is uncountable

6.9 Dense Sets

The following definitions are equivalent:
A subset A of a set X is **dense in** X...

- every point $x \in X$ either belongs to A or is a limit point of A
- if any neighborhood of every point $x \in X$ contains at least one point in A
- if every open cover of A is also an open cover of X.
- if and only if the only closed subset of X containing A is itself.
- if the closure of A is X: $\overline{A} = X$
- if the interior of the complement of A in X is empty

The **density** of a set X is the least cardinality of all dense subsets of X. Density is topologically invariant.

- $\mathbb{Q}$ is dense in $\mathbb{R}$ with countable density
- The Irrationals are dense in $\mathbb{R}$ yet disjoint.

- Every space without isolated points is dense in itself
- Every metric space is dense in its completion
- Transitivity: If $A \subseteq B \subseteq C$ and if A is dense in B and B is dense in C, then A is dense in C
- The image of a dense subset under a continuous surjections is dense.
- A space X with a subset A which is connected and dense in A must be connected

A subset is **somewhere dense** if

- There exists an open non-empty set $U \subset X$ such that $\overline{X \cap U} \supset U$
- $int\overline{X} \neq \emptyset$

A subset is **nowhere dense** if there is no neighborhood in X in which A is dense.

- A subset is nowhere dense if it is not somewhere dense
- A subset is nowhere dense if and only if the interior of its closure is empty
- The interior of the complement of a nowhere dense set is always dense
- The complement of a closed set that is nowhere dense is a dense open set

Related Notions

- A subset A of X is **meagre** if it can be expressed as the union of countably many nowhere dense subsets of X. $\mathbb{Q}$ are meagre as a subset of $\mathbb{R}$
- A space X with a countable subset that is dense in X is called **separable**
  $L^p(\Omega)$, $1 \leq p < \infty$ is separable but not for $p = \infty$
- A space is resolvable if it is the union of two disjoint subsets and k-resolvable if it contains k pairwise disjoint dense sets
- An embedding of a space as a dense subset of a compact space is a called a compactification of X
- A linear operator between vector spaces X and Y is densely defined if its domain is a dense subset of X and if its range is contained within Y

6.10 Coverings

A collection $\mathcal{U}$ of subsets of M covers $A \subset M$ if A is containted in the union of of the sets belonging to $\mathcal{U}$, and we say $\mathcal{U}$ is a **covering** or **cover** of A.

- The members of $\mathcal{U}$ are called **scraps** and are not themselves covers.
- The elements of U should be distinct but may overlap.
- A whole space forms an open cover of any set in it
- For covers that are intervals $\{(a_i, b_i)\}$, $\sum\limits_{n=1}^{N}(b_n - a_n)$ is the **total length** of the covering

Open covering : A open cover $\mathcal{U}$ is one whose sets (or scraps) are all open.

Subcoverings : $\mathcal{V}$ is a subcover of $\mathcal{U}$ if $\mathcal{U}$ and $\mathcal{V}$ both cover A and if $\mathcal{V} \subset \mathcal{U}$ ($\forall\, V \in \mathcal{V}, V \in \mathcal{U}$). We say $\mathcal{U}$ **reduces to** $\mathcal{V}$.

Covering Compact: A is covering compact if every open covering of A reduces to a finite subcovering. A set is covering compact if and only if it is sequentially compact.

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6.10.0.1 Lebesgue Number of a Covering The Lebesgue number $\lambda \in \mathbb{R}, \lambda \geq 0$ of a covering $\mathcal{U}$ of A is the number such that $\forall\, a \in A\ \exists\, U \in \mathcal{U}$ s.t. $M_\lambda(a) \subset U$, with λ independent of a.
That is, we know each point $a \in A$ is contained in some $U \in \mathcal{U}$

- Noncompact sets may have open covers with no positive Lebesgue number
- **Lebesgue Number Lemma**: Every open covering of a sequentially compact set has a Lebesgue number $\lambda > 0$

6.11 Topological Objects

6.11.1 Topologist's Sine Curve

The Topologist's Sine Curve is the set $M = G \cup Y$ where

$$G = \left\{ (x, y) \in \mathbb{R}^2 : y = \sin\left(\frac{1}{x}\right) \text{ and } 0 < x \leq \frac{1}{\pi} \right\}$$
$$Y = \left\{ (0, y) \in \mathbb{R}^2 : -1 \leq y \leq 1 \right\}$$

- M is a closed set

6.11.2 The Cantor Sets

The Cantor Ternary Set is constructed by removing the open middle portion if a set of line segments, beginning with $[0, 1]$ and continuing ad infinitum. The standard Cantor set is based on removing the middle thirds.

The Cantor Ternary Set is constructed by removing the open middle third if a set of line segments, beginning with $[0, 1]$ and continuing ad infinitum. Recursively,

$$C_n = \frac{C_{n-1}}{3} \cup \left(\frac{2}{3} + \frac{C_{n-1}}{3} \right)$$

so

$$C_0 \supset C_1 \supset C_2 \supset ... \supset C_n$$

The Cantor set contains all points not deleted at any step of the infinite process.

6.11.3 Explicit Representations

The Cantor set is

$$C = \bigcap_{m=1}^{\infty} C_n = \bigcap_{m=1}^{\infty} \bigcap_{k=0}^{3^{m-1}-1} \left(\left[0, \frac{3k+1}{3^m} \right] \cup \left[\frac{3k+2}{3^m}, 1 \right] \right)$$

or alternatively

$$C = [0, 1] \setminus \bigcup_{m=1}^{\infty} \bigcup_{k=0}^{3^{m-1}-1} \left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m} \right)$$

- The Cantor set is compact, nonempty, perfect, and totally disconnected.
- The Cantor set is uncountable.
- The Cantor set contains no interval
- The Cantor set is nowhere dense
- Anything with a 1 in its base 3 expansion if removed.

7 Real Analysis

7.1 Expansions

7.1.1 Binomial Expansion

For $a \in [0,1]$, $(a_1, ..., a_n)$ are chosen in $\{0,1\}$ s.t.
1.$a_1 = 1$ if $a \geq \frac{1}{2}$
2. Take $0 \leq x = 2^{n+1}(a - \sum\limits_{k=1}^{n} \frac{a_k}{2^k}) < 1$. $a_{n+1} = 1$ if $x \geq \frac{1}{2}$

This expansion is not unique because any numbr of the form $\frac{m}{2^k}$ for $m, k \in \mathbb{N}$ can be expressed equivalently as $(a_1, a_2, ..., a_n = 1, 0, 0...)$ or $(a_1, a_2, ..., a_n = 0, 1, 1...)$.

7.2 Metric Spaces (X, d)

A metrix space is a set X equipped with a metric d
A metric $d : X \times X \to \mathbb{R}^+$ on a nonempty set X takes two elements of some set X as input and outputs a non-negative real number with the properties that for every x, y, z in our set:

7.2.1 Definition

1. $d(x, y) \geq 0 \; \forall \; x, y \in X$, (distances are nonnegative)
   and $d(x, y) = 0 \iff x = y$ (the only point of distance zero from x is x itself)
2. $d(x, y) = d(y, x)$ (symmetry)
3. $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality)

7.2.2 Common metric spaces

- $\mathbb{R}$ is a metric space with metric $|x - y|$
  Triangle Inequality for $\mathbb{R}$: For all $x, y \in \mathbb{R}$, $|x + y| \leq |x| + |y|$
- $\mathbb{C}$ is a metric space with metric $|z - w|$
- $\mathbb{T}$, the unit circle, is a metric space with any metric on $\mathbb{R}^2$
- (X, d_p) is a metric space where $X \subset \mathbb{R}^n$ or $X \subset \mathbb{C}^n$ and $d_p(x, y) = \left(\sum\limits_{i=1}^{n} |x_i - y_i|^p \right)^{\frac{1}{p}}$ for p s.t. $1 \leq p \leq \infty$
  Triangle inequality proved using Holder Inequality
- $(C([a, b]), d_{max})$ is a metric space where f is continuous on $[a, b]$ and $d_{max}(x, y) = \max\limits_{a \leq x \leq b} |f(x) - g(x)|$
- $C^m(K)$, $m < \infty$ is a metric space with metric $d(f, g) = \sum\limits_{|\alpha| \leq m} \max\limits_{x \in X} |D^\alpha(f - g)|$

7.2.3 Common metric spaces???

- If (A, d) is a metric space and $B \subset A$, then (B, d) is a metric space for nonempty A, B
- d is an **ultrametric** if
  $$d(x, y) \leq \max \{d(x, z), d(z, y)\} \qquad \forall \; x, y, z \in X$$

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- d is a **pseudometric** if two distinct points can be zero. That is, $d(x, y) = 0$ does not imply $x = y$. Points governed by a pseudometric need not be distinguishable.

7.2.4 Common Metrics

- **Absolute Value Metric:** $d_1(\vec{\mathbf{x}}, \vec{\mathbf{y}}) = |x_1 - y_1| + ... + |x_m - y_m|$
- **Discrete Metric:** $d_0(\vec{\mathbf{x}}, \vec{\mathbf{y}}) = \begin{cases} 0 & \text{if } \vec{\mathbf{x}} = \vec{\mathbf{y}} \\ 1 & \text{if } \vec{\mathbf{x}} \neq \vec{\mathbf{y}} \end{cases}$
- **Euclidean Metric:** $d_2(\vec{\mathbf{x}}, \vec{\mathbf{y}}) = d_E(\vec{\mathbf{x}}, \vec{\mathbf{y}}) = \sqrt{(x_1 - y_1)^2 + ... + (x_m - y_m)^2}$
- **Maximum Metric:** $d_{max}(\vec{\mathbf{x}}, \vec{\mathbf{y}}) = d_\infty(\vec{\mathbf{x}}, \vec{\mathbf{y}}) = \max_i \{|x_i - y_i|\}$

7.2.5 Metric Subspaces

A metric $(Y, d|Y)$ is a metric subspace of X if $X \subset Y$ and we restrict the metric of (X, d) to Y.
$(\mathbb{R}, \|\cdot\|\ )$ is a metric subspace of $(\mathbb{C}, \|\cdot\|\ )$ and $(\mathbb{Q}, \|\cdot\|\ )$ is a metric subspace of $(\mathbb{R}, \|\cdot\|\ )$

7.2.6 Product Metrics

A product metric is a metric on a cartesian product $M = M_1 \times M_2 \times ... \times M_m$ of m metric spaces with metric $d_i(p_{i_1}, p_{i_2})$ on each M_i.

- The default product metric on $\mathbb{R}^2$ is $d_{X \times Y}((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2)$

7.2.6.1 Common Product Metrics For $p_{i_1}, p_{i_2} \in M_i$, and let $p_1 = (p_{1_1}, p_{2_1}, ..., p_{m_1}) \in M$, and $p_2 = (p_{1_2}, p_{2_2}, ..., p_{m_2}) \in M$, the most common product metrics are:

$$\begin{aligned} d_{sum}(p_1, p_2) =& d_1(p_{1_1}, p_{1_2}) + d_2(p_{2_1}, p_{2_2}) + ... + d_m(p_{m_1}, p_{m_2}) \\ d_E(p_1, p_2) =& \sqrt{d_1(p_{1_1}, p_{1_2})^2 + d_2(p_{2_1}, p_{2_2})^2 + ... + d_m(p_{m_1}, p_{m_2})^2} \\ d_{max}(p_1, p_2) =& max\{d_1(p_{1_1}, p_{1_2}), d_2(p_{2_1}, p_{2_2}), ..., d_m(p_{m_1}, p_{m_2})\} \end{aligned}$$

7.3 Equivalent Metrics

Given two metric spaces (X, d_1) and (X, d_2) , d_1 and d_2 are equivalent if there exist nonzero constants c, C such that

$$cd_1(x_1, x_2) \leq d_2(x_1, x_2) \leq Cd_1(x_1, x_2) \qquad \forall\ x_1, x_2 \in X$$

- Equivalent metrics give rise to the same open sets hlCheck this
- Equivalent metrics form an equivalence relation?

The following are equivalent

For a sequence $p_n = (p_{1_n}, p_{2_n}, ..., p_{m_n})$ in $M = M_1 \times M_2 \times ... \times M_m$

1. (p_n) converges with respect to the metric d_{max}
2. (p_n) converges with respect to the metric d_E
3. (p_n) converges with respect to the metric d_{sum}
4. Each (p_{i_n}) converges in it's respective M_i

Fact: $d_{max} \leq d_E \leq d_{sum} \leq md_{max}$

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7.3.1 Completeness

A metric space M is complete if each Cauchy sequence in M converges to a limit in M.
$\mathbb{R}^m$ is complete
Every closed subset of a complete metric space is a complete metric space.
Every closed subset of Euclidean Space is a complete metric space.

7.3.1.1 Completions

A metric space $(\tilde{X}, \tilde{d})$ is called a **completion** of (X, d) if the following conditions are satisfied

1. $(\tilde{X}, \tilde{d})$ and (X, d) are isomorphic (there exists an isometry $\imath : X \to \tilde{X}$)
2. The image space $\imath(X)$ is dense in $\tilde{X}$
3. The space $(\tilde{X}, \tilde{d})$ is complete

- Every metric space has a completion
- Completions are unique up to isomorphism

7.4 Linear Spaces (or Vector Space)

A linear space X over the scalar field $\mathbb{C}$ is a set of points or vectors on which are defined operations of vector addition or scalar multiplication with the following properties.

- X is commutative with respect to vector addition
- There is a zero vector
- Each vector has a unique additive inverse
- Vector spaces are closed under linear combinations

7.5 Norms

A **Norm** on a vector space V is any function $\| \ \| : V \to \mathbb{R}$ with the properties for any given $v, w, \in V$ and $\lambda \in \mathbb{R}$

- Nonnegative: $\|v\| \geq 0$
- Positive Definite: $\|v\| = 0 \iff v = 0$
- Homogenous: $\|\lambda v\| = |\lambda| \|v\|$
- Triangle Inequality: $\|v + w\| \leq \|v\| + \|w\|$

Two norms $\|\cdot\|_a$ and $\|\cdot\|_b$ on a linear space X are **equivalent** if there are constants c and C such that

$$c\|\cdot\|_a \leq \|\cdot\|_b \leq C\|\cdot\|_a \ \forall \ x \in X$$

Equivalent norms generate the same topologies (same collection of open sets)

7.5.1 Seminorms

A function $X \to \mathbb{R}$ is a seminorm on X if:

- $p(x) \geq 0$ for all $x \in X$
- $p(x+y) \leq p(x) + p(y)$ for all $x, y \in X$
- $p(\lambda x) = |\lambda|\, p(x)$ for every $x \in X$ and $\lambda \in \mathbb{C}$
- However, $p(x) = 0$ need not imply $x = 0$

7.5.1.1 Seminorm Topology and Convergence

A countable or uncountable subset of seminorms $\{p_\alpha\}_{\alpha \in \mathcal{A}}$ indexed by a set $\mathcal{A}$ defined on a linear space X yields open neighborhoods defined in the following way:

$$\mathcal{N} = \{N_{\alpha_1,...,\alpha_n;\epsilon}(x) : x \in X, \alpha_1, ..., \alpha_n \in \mathcal{A}, \text{ and } \epsilon > 0\}$$

$$N_{\alpha_1,...,\alpha_n;\epsilon}(x) = \{y \in X : p_{\alpha_i}(x-y) < \epsilon \text{ for } i = 1, ..., n\}$$

A sequence (x_n) converges $x \in X$ in this topology if and only if $p_\alpha(x - x_n) \to 0$ as $n \to \infty$ for each $\alpha \in \mathcal{A}$

7.5.1.2 Hausdorff Topologies

A family of seminorms $\{p_\alpha\}_{\alpha \in \mathcal{A}}$ **separates points** if $p_\alpha(x) = 0$ for every α implies $x = 0$.
A topological linear space space whose topology may be derived from a family of seminorms that separates points is called a **locally convex space**
If the family of seminorms is finite and separates points, then

$$\|x\| \ = p_1(x) + ... + p_n(x)$$

defines a norm on X

7.5.1.3 Frechet space A locally convex space with X whose topology is generated by a countably infinite family of seminorms $\{p_n : n \in \mathbb{N}\}$ has a metrizeable topology with metric

$$d(x,y) = \sum_{n \in \mathbb{N}} \frac{1}{2^n} \frac{p_n(x-y)}{1 + p_n(x-y)}$$

7.5.2 Inner Product Spaces

An Inner Product Space is a vector space X equipped with an Inner Product.
An inner product on a complex linear space X is a continuous map

$$\langle \cdot, \cdot \rangle : X \times X \to \mathbb{C}$$

with the following properties for $x, z, y \in X$, $\alpha, \beta \in \mathbb{C}$

1. Linear in the First Argument: $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$
2. Hermitian symmetric: $\langle x, y \rangle = \overline{\langle y, x \rangle}$
3. Nonnegative: $\langle x, x \rangle \geq 0$
4. Positive Definite: $\langle x, x \rangle = 0$ if and only if $x = 0$

- Properties 1 and 2 imply $\langle\cdot,\cdot\rangle$ is **antilinear** (or **conjugate linear**) in the second argument

$$\langle x, \alpha y + \beta z\rangle = \overline{\alpha}\langle x, y\rangle + \overline{\beta}\langle x, z\rangle \qquad \text{(sesquilinear)}$$

- If $X \in \mathbb{R}$ then $\langle\cdot,\cdot\rangle$ is **billinear.** For $a, b, c, d \in \mathbb{R}$ and $x, y, z, w \in X$,

$$\langle ax + by, cz + dw\rangle = ac\langle x, z\rangle + bc\langle y, z\rangle + ad\langle x, w\rangle + bd\langle y, w\rangle \qquad \text{(bilinear)}$$

- A linear space with an inner product is an **inner product space** or **pre-Hilbert space**
- **Cauchy-Schwarz Inequality**: If $(X, \langle\cdot,\cdot\rangle)$ is an IPS, then

$$|\langle x, y\rangle| \le \|x\|\ \|y\| \qquad \forall\ x, y \in X$$

- The Cartesian product $X \times Y = \{(x, y) : x \in X, y \in Y\}$ of two inner product spaces $(X, \langle\cdot,\cdot\rangle_X)$ and $(Y, \langle\cdot,\cdot\rangle_Y)$ have an inner product and norm

$$\langle (x_1, x_2), (y_1, y_2)\rangle_{X\times Y} = \langle x_1, x_2\rangle_X + \langle y_1, y_2\rangle_Y$$

$$\|(x, y)\| = \sqrt{\|x\|^2 + \|y\|^2}$$

Examples

- $X = \mathbb{R}^n$, $\langle x, y\rangle = \sum_{j=1}^{n} x_j y_j$
- $X = \mathbb{C}^n$, $\langle x, y\rangle = \sum_{j=1}^{n} x_j \overline{y_j}$
- $X = L^2(\Omega)$, $\Omega \subset \mathbb{R}^n$, $\langle u, v\rangle = \int_\Omega u(x)\overline{v(x)}dx$

7.5.2.1 Normed Linear Spaces

If X is a vector/linear space with an inner product $\langle\cdot,\cdot\rangle$, then it is a normed linear space with norm and metric

$$\|x\| = \sqrt{\langle x, x\rangle} \qquad\qquad d(x, y) = \|x - y\|$$

- A normed linear space X is an inner product space with norm $\|x\| = \sqrt{\langle x, x\rangle}$ if and only if

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2 \qquad \text{(Parallelogram Law)}$$

- If a normed linear space that satisfies the paralleogram law, then there is at most one inner product that can exist because of the **polarization formula:**

$$\langle x, y\rangle = \frac{1}{4}\left(\|x + y\|^2 - \|x - y\|^2 - i\|x + iy\|^2 + i\|x - iy\|^2\right)$$

7.6 Banach Spaces

A Banach Space is normed linear space that is complete with respect to the metric $d(x, y) = \|x - y\|$

The following are Banach Spaces

- $C(\Omega) = \{f : f$ is continuous$\}$ when equipped with the sup norm

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- $C(K) = \{f : f$ is continuous on the compact set $K\}$ when equipped with the sup norm
- $C^k(\Omega)$ when equipped with the C^k norm $\|f\|_{C^k} = \|f\|_\infty + \|f'\|_\infty + ... + \left\|f^{(k)}\right\|_\infty$
- $L^p(\Omega)$ for $1 \leq p < \infty$
- $L^p(\Omega)$ for $p = \infty$ when $f \in L^\infty(\Omega)$ is **essentially bounded** (bounded on a subset of Ω whose complement has measure 0). The norm on f is the **essential suprenum**

$$\|f\|_\infty = \inf\{M : |f(x)| \leq M \text{ a.e. in [a,b]}\}$$

- For linear maps, boundedness is equivalent to continuity. A linear map is bounded if and only if it is continous.

7.6.1 Schauder Basis

A sequence $\{x_n\}_{n=1}^\infty \in X$ is a Schauder Basis of X if for evey $x \in X$ there is a unqiue sequence of scalars (c_n) such that $x = \sum_{n=1}^\infty c_n x_n$
It is necessary but insufficient that X is separable (since linear combinations of x_n with rational cofficients is dense and countable)

7.6.1.1 Isomorphisms

For a linear map $T : X \to Y$

- X and Y are **linearly isomorphic** if there is a one-to-one and onto (bijective) linear map T.
- If X and Y are normed linear spaces with both T and T^{-1} bounded then X and Y are **topologically isomorphic**.
- If T preserves norms such that $\|Tx\| = \|x\| \;\; \forall\, x \in X$ then X and Y are **isometrically isomorphic**.

7.7 Bounded Linear Maps

A linear map or linear operator T between spaces X and Y is a function $T : X \to Y$ such that

$$T(\alpha x + \beta y) = \alpha Tx + \beta Ty \qquad \forall\, \alpha, \beta \in \mathbb{R} \text{ and } x, y \in X$$

- Also known as a **linear transformation** of X or **linear operator** on X. We denote the set of all linear maps $T : X \to Y$ by $\mathcal{L}(X, Y)$, or $\mathcal{L}(X)$ when $X = Y$
- If T is one-to-one and onto (bijection) then it is **nonsingular** or **invertible** and we can define a linear map $T^{-1} : Y \to X$ with the properties $T^{-1}T = TT^{-1} = I$ and $T^{-1}y = x \iff y = Tx$
- A linear map is **bounded** if there is a constant M such that $\|Tx\| \leq M\|x\| \;\; \forall\, x \in X$. Otherwise, T is **unbounded**
- The **operator norm** or **uniform norm** $\|T\|$ of T by

$$\|T\| = \inf\{M : \|Tx\| \leq M\|x\| \;\; \forall\, x \in X\}$$

  We denote the set of all bounded linear maps $T : X \to Y$ by $\mathcal{B}(X, Y)$ or by $\mathcal{B}(X)$ when $X = Y$

- If T is a contraction, then $T^n \to \infty$ for $n \to \infty$

7.8 Hilbert Spaces

A **Hilbert Space** $\mathcal{H}$ is a complete inner product space with respect to the standard inner product space norm.

$$\|x\| = \sqrt{\langle x, x\rangle}$$

- Every Hilbert Space is a Banach space
- $\mathbb{R}^n$, $\mathbb{C}^n$, $L^2(\Omega), \ell^2$ are all Hilbert Spaces
- The standard inner product on $\mathbb{C}^n$ is
$$\langle x, y\rangle = \sum_{j=1}^{n} \overline{x_j} y_j$$
where $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$ with $x_j, y_j \in \mathbb{C}$.
This space is complete and therefore a finite dimensional Hilbert Space
- If $\mathcal{H}$ is real, the inner product is called the **dot product** and has a defined angle between vectors
$$\frac{\langle x, y\rangle}{\|x\| \; \|y\|} = cos(\theta)$$
- The inner product on $C(\Omega)$ is
$$\langle f, g\rangle = \int_{\Omega} \overline{f(x)} g(x) dx$$
and the completion with respect to the norm
$$\|f\| = \left(\int_{\Omega} |f(x)|^2 \, dx \right)^{\frac{1}{2}}$$
is $L^2(\Omega)$
- The bi-infinite complex sequences ℓ^2 is a Hilbert space
$$\ell^2(\mathbb{Z}) = \left\{ (z)_{n=-\infty}^{\infty} : \sum_{n=-\infty}^{\infty} |z_n|^2 < \infty \right\}$$
with inner product
$$\langle x, y\rangle = \sum_{n=-\infty}^{\infty} \overline{x_n} y_n$$
- **Solobev Spaces** $\mathcal{H}^k(\Omega) = W^{k,2}(\Omega)$ are Hilbert Spaces that are the completions of $C^k(\Omega)$ with norm
$$\|f\| = \left(\sum_{j=0}^{k} \int_{\Omega} \left| f^{(j)}(x) \right|^2 dx \right)^{\frac{1}{2}}$$

7.8.1 Orthogonality

Vectors $x, y \in \mathcal{H}$ are orthogonal, written $x \perp y$ if $\langle x, y\rangle = 0$

- **Pythagorean Theorem** If $x \perp y$, then $\left\|x^2 + y^2\right\| = \left\|x^2\right\| + \left\|y^2\right\|$
- Subsets A and B are orthogonal $A \perp B$ if $x \perp y$ for all $x \in A, y \in B$
- The **orthogonal complement** of a subset A, $A^{\perp}$, is the set of vectors orthogonal to A
$$A^{\perp} = \{x \in \mathcal{H} : x \perp y \; \forall \; y \in A\}$$
The orthogonal complement of a subset of a Hilbert space is always a closed linear subspace.

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7.8.1.1 Projections Denoted $\operatorname*{proj}_M x$ or $P_M x$

For each $\mathcal{M} \subset H$,

- There is single valued function that gives a unique closest point $y \in \mathcal{M}$ such that

$$\|x-y\| = \min_{z\in\mathcal{M}} \|x-z\| \qquad\qquad y = \operatorname*{argmin}_{z\in\mathcal{M}} \|x-z\|$$

- The point $y \in \mathcal{M}$ closest to $x \in \mathcal{H}$ is the unique element of $\mathcal{M}$ with the property that $(x-y) \perp \mathcal{M}$
- If $y \in M$, $z \in M^\perp$ and $x = y+z$ then $y = \operatorname*{proj}_M x$ and $z = \operatorname*{proj}_{M^\perp} x$
- Uniqueness. Let $x = y_1 + z_1 = y_2 + z_2$ with $y_1, y_2 \in M$ and $z_1, z_2 \in M^\perp$. This implies $y_1 - y_2 = z_2 - z_1 = w$ which implies $\exists\, w \in M \cap M^\perp$, but then $w = 0$ (the only element in that intersection), so $y_2 = y_1$ and $z_2 = z_1$
- Existence. Pick $u_m \in M$ such that $\|x-u_n\| = d_n \to d = \inf_{y\in M} \|x-y\|$. Use parallelogram law and reduce to $y = \operatorname*{proj}_M x$ and $z = \operatorname*{proj}_{M^\perp} x$ for $z \in M^\perp$.
- If $M = \mathcal{L}\{e_1, e_2, ..., e_n\}$, then $P_M(x) = \sum\limits_{j=1}^n \frac{\langle x, e_j\rangle}{\langle e_j, e_j\rangle} e_j$

7.8.1.2 Direct Sum

Given $\mathcal{M} \subset \mathcal{H}$ and $\mathcal{N} \subset \mathcal{H}$ are orthogonal closed linear subspaces ($\mathcal{M} \cap \mathcal{N} = 0$), the **orthogonal direct sum** or **direct sum** is defined by

$$\mathcal{M} \oplus \mathcal{N} = \{y + z : y \in \mathcal{M} \text{ and } z \in \mathcal{N}\}$$

7.8.1.3 Decomposition

If $\mathcal{M}$ is a closed subspace of a Hilbert space $\mathcal{H}$, then $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$

If $\mathcal{M}$ is not closed, then $\mathcal{H} = \overline{\mathcal{M}} \oplus \mathcal{M}^\perp$

7.8.2 Hilbert Bases

To check if a given sequence If $\{x_n\}$ is a basis, we check the following

1. $\{x_n\}$ is closed if the set of all finite linear combinations of them is dense in $\mathcal{H}$
2. $\{x_n\}$ is complete if and only if there is no nonzero vector orthogonal to all x_n, that is

$$\langle x, x_n\rangle = 0\ \forall\, n \iff x = 0$$

3. If $\{x_n\}$ are orthonormal they are **maximal orthonormal** if $\{x_n\}$ is not contianed in any strictly larger orthonormal set.

Let $\{e_n\}_{n=1}^\infty$ be orthonormal, the following are equivalent

1. $\{e_n\}$ is maximal orthonormal
2. $\{e_n\}$ is closed
3. $\{e_n\}$ is complete

4. $\{e_n\}$ is a basis of $\mathcal{H}$
5. Bessel's Inequality holds for all $x \in \mathcal{H}$
6. $\sum\limits_{n=1}^{\infty} \langle x, e_n \rangle \langle e_n, y \rangle = \langle x, y \rangle \;\forall\; x, y \in \mathcal{H}$

So we know a complete orthonormal set forms a basis, with $\langle x, e_n \rangle$ known as the n^{th} **Generalized Fourier Coefficient** of x with respect to $\{e_n\}$
A basis is only possible in a separable Hilbert space, though nonseparable Hilbert Spaces do exist.
The **Standard Basis** for a Hilbert Space is $e_n = (0, 0, ..., 1, ...)$ where the 1 appears in the n^{th} dimension.

7.8.2.1 Gram Schmidt

If $M = \mathcal{L}\{e_1, e_2, ..., e_n\}$ is linearly independent but not orthonormal, then create a new orthonormal basis $M = \mathcal{L}\{f_1, f_2, ..., f_n\}$ with

$$f_n = \frac{f_n'}{\|f_n'\|} \qquad f_n' = e_n - \sum_{i=1}^{n} \langle e_n, f_i \rangle f_i$$

7.8.2.2 Reisz Fischer Critertion

For an infinite dimensional basis,
$\sum\limits_{n=1}^{\infty} c_n e_n$ converges in $\mathcal{H}$ if and only if $\sum\limits_{n=1}^{\infty} |c_n|^2 < \infty$
This implies $\sum\limits_{n=1}^{\infty} c_n e_n$ converges if and only if $\{c_n\}_{i=1}^{\infty} \in \ell^2$

7.8.2.3 Riemann Lebesgue Lemma

7.8.2.4 Version 1

If given $\{e_n\}_{n=1}^{\infty}$ is ON and $x \in \mathcal{H}$ then $\sum\limits_{n=1}^{\infty} |\langle x, e_n \rangle|^2 < \infty$ implies $\lim\limits_{n\to\infty} \langle x, e_n \rangle = 0$

7.8.2.5 Version 2

Choose $\mathcal{H} = L^2(0, 2\pi)$ and $e_n = \frac{e^{inx}}{\sqrt{2\pi}}$. For any $f \in L^2(0, 2\pi)$, $\frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(x) e^{-inx} dx \to 0$ as $n \to \pm\infty$

7.8.2.6 Version 3

Replace n by λ_n in version 2. Same conclusion, yet different proof needed. (Use IBP, and the fact that f,f' are bounded)

7.8.2.7 Version 4

Use the fact that $C^1(0, 2\pi)$ is dense in $L^1(0, 2\pi)$. If $f \in L^1(0, 2\pi)$ pick $\epsilon > 0$ and $g \in C^1(0, 2\pi)$ with $\|f - g\|_{L^1} < \epsilon$ (by density). Then

$$\int_{\mathbb{T}} f(x) e^{-i\lambda x} dx = \int_{\mathbb{T}} (f(x) - g(x)) e^{-i\lambda x} dx + \int_{\mathbb{T}} g(x) e^{-i\lambda x} dx$$

So

$$\left| \int_{\mathbb{T}} f(x) e^{-i\lambda x} dx \right| \leq \|f(x) - g(x)\|_{L^1} \leq \epsilon$$

So

$$\limsup_{|\lambda|\to\infty} \int_0^{2\pi} f(x) e^{-i\lambda x} dx = 0$$

Smooth enough functions are dense in C^1

7.8.2.8 Version 5

Same conculsion works if limits of integration in version 4 are replaced by a, b or if we integrate over $\mathbb{R}$

7.8.2.9 Version 6

If f is real valued,

$$\int f(x)e^{-i\lambda x}dx = \int f(x)\cos(\lambda x)dx + i\int f(x)\sin(\lambda x)dx$$

Both terms go to zero.
One can replace f by $h + ig$ to get same conclusion for complex valued f

7.8.2.10 Bessel's Inequality For an infinite dimensional basis, $S_N = \sum \langle x, e_n\rangle e_n = P_{M_N}x$ where $M_N = \mathcal{L}\{e_1, e_2, ..., e_n\}$, which implies

$$\sum_{n=1}^{\infty} |\langle x, e_n\rangle|^2 \leq \|x\|^2$$

7.8.2.11 Weierstrass Approximation Theorem The set of polynomials is dense in $C([a, b])$, so for a given $\epsilon > 0$ and $f \in C([a, b])$ there exists a polynomial $P(x)$ such that

$$\|f(x) - P(x)\|_{L^\infty(a,b)} = \sup_{a\leq x\leq b} |f(x) - P(x)| < \epsilon$$

Proof. Weierstrass, Landau
Lemma: If $t > -1$ then $(1 + t)^n \geq (1 + nt)$. (Proven by induction) Lemma: If every $f \in C[0, 1]$ can be approximated by polynomials, then so can every $g \in C[a, b]$ since for the polynomial $\varphi = a + (b - a)x$, and a given $g \in [a, b]$, $g \circ \phi = f \in C[0, 1]$ so if $\exists\, P$ s.t. $\|f - P\|_\infty < \epsilon$ then $\left\|f \circ \varphi^{-1} - P \circ \varphi^{-1}\right\|_\infty < \epsilon$ since compositions of polynomials are polynomials.

7.8.3 Functionals on $\mathcal{H}$

A mapping $\ell : \mathcal{H} \to \mathbb{R}$ or $\mathbb{C}$ is a function on $\mathcal{H}$. If it is linear then

$$\ell(c_1x_1 + c_2x_2) = c_1\ell(x_1) + c_2\ell(x_2)$$

The most common linear functional is

$$\ell(x) = \langle x, y\rangle \text{ for some } y \in \mathcal{H}$$

A linear functional is bounded if

$$\exists\, c < \infty \text{ s.t. } \sup_{\substack{x\neq 0\\ x\in\mathcal{H}}} \|\ell\| = \sup_{\substack{x\neq 0\\ x\in\mathcal{H}}} \frac{|\ell(x)|}{\|x\|} = c$$

7.8.4 The Dual Space of $\mathcal{H}$

The set of bounded linear functionals is exactly $B(\mathcal{H}, \mathbb{C}) = \mathcal{H}^*$ which is the dual space of $\mathcal{H}$

7.8.4.1 Riesz Representation Theorem Given a Hilbert Space $\mathcal{H}$, let $\mathcal{H}^*$ denote its dual space consisting of allocntinous linear functionals $\ell : \mathcal{H} \to \mathbb{R}$ or $\ell : \mathcal{H} \to \mathbb{C}$. If $x \in \mathcal{H}$, then the function

$$\ell_x(y) = \langle x, y \rangle \qquad \forall\, y \in \mathcal{H}$$

is an element of $\mathcal{H}^*$. Define $\Phi : \mathcal{H} \to \mathcal{H}^*$ as $\Phi(x) = \ell_x$

- Every element in $\mathcal{H}^*$ can be written in this form
- If the target space is $\mathbb{R}$, Φ is an isometric isomorphism. If the target space is $\mathbb{C}$, Φ is an isometric anti-isomorphism.
  - Φ is bijective
  - $\|x\| \ = \|\Phi(x)\|$

If $\ell \in \mathcal{H}^*$ then $\exists\,!\, y \in \mathcal{H}$ such that $\ell(x) = \langle x, y \rangle$
The map $R : \ell \to y$, taking $\mathcal{H}^*$ onto $\mathcal{H}$ is also one-to-one (uniqueness), in a linear fasion, and it is an isometry.

$$\|y\| \ = \|Rx\| \ = \|x\|$$

Explain MORE?

- Every Banach Space has a dual space.
- Every Hilbert space is isometric to its dual space.

8 Functions

$f : X \to Y$ is a function if $\forall\, x \in X\ \exists\, y \in Y$ s.t. $y = f(x)$

8.0.4.2 Graph
The graph of f is the set of all ordered pairs (x, y) such that $x \in X, y \in Y$ and $y = f(x)$.

8.0.4.3 Linearity
A function $f : \mathbb{R} \to \mathbb{R}$ is linear if $f(x + y) = f(x) + f(y)$ and $f(ax) = af(x)$

- This implies $f(x) = a_1x_1 + ... + a_nx_n$
- $f(0) = 0$

8.0.4.4 Affine
A function $f : \mathbb{R} \to \mathbb{R}$ is affine if

$$f(x) + f(y) = f(x + y) - f(0)$$

That is, it describes a linear transformation followed by translation.

- This implies $f(x) = a_1x_1 + ... + a_nx_n - b$
- The coefficients $\{a_i\}$ can be scalars or dense or sparse matrices. The constant term is either a scalar or a column vector.

8.0.4.5 Homogeneity
A function f is homogenous of order κ if $f(ax) = a^\kappa f(x)$

8.0.4.6 Injective function (1-1 or one-to-one)
A function $f : \mathbb{R} \to \mathbb{R}$ is injective if $f(x) = f(y)$ implies that $x = y$. In other words, it passes both the horizontal and vertical line tests. Examples: x^3, $exp(x)$. $f(x) = x^2$ is not injective because although $f(-2) = f(2)$, $-2 \neq 2$. If f is injective and g is injective, $f \circ g$ is injective.
If $\exists$ an injection $f : A \to B$ and $\exists$ an injection $g : B \to A$, then $A \sim B$

8.0.4.7 Surjective function (onto)
A function $f : \mathbb{R} \to \mathbb{R}$ is injective if for every element $b \in B$ there exist at least one element $a \in A$ s.t. $f(a) = b$. In other words, for every element of the range there is an element in the domain s.t. f maps the element in the domain to the element in the range. Examples: x^3 and $2x$ are surjective but x^2 is not surjective because you can pick a negative range element.

8.0.4.8 Bijective
A function is bijective if it is both injective and surjective.
We write $A \sim B$ to mean that there exists a bijection between A and B.
For a bijection $f : A \to B$, the cardinality of each set is equal, that is $|A| = |B|$

8.0.4.9 Convex
A function is convex if no secant line lies below the function. That is, $\forall\, x_1, x_2 \in X$ and $t \in [0, 1]$,

$$f(x_1)(1-t) + f(x_2)(t) \geq f(x_2(t-1) + x_2(t))$$

8.0.4.10 Concave
A function is concave if no secant line lies above the function. That is, $\forall\, x_1, x_2 \in X$ and $t \in [0, 1]$,

$$(1-t)f(x_1) + (t)f(x_2) \leq f((t-1)x_2 + (t)x_2)$$

- **Schroeder-Bernstein Theorem** If A,B, are sets and $f : A \to B$, $g : B \to A$ are injections then $\exists\, h : A \to B$ which is a bijection.

8.0.4.11 Composition : If $f : A \to B$ and $g : B \to C$ then the composite is the map $g \circ f : A \to C$

- Compositions of injective functions are injective.
- Compositions of surjections are surjections.
- Compositions of bijections are bijections.

8.0.4.12 Preimage : The preimage of function $f : X \to Y$ is defined to be $f^{-1}(A) = \{x \in X | f(x) \in A\}$. It is all the elements which are mapped to A by f. Example: If $f(x) = x^2$ then $f^{-1}(4) = \pm 2$. A can be either a single element or a set. Note that inverses are functions which sometimes exist, but preimages are sets which always exist (they may be empty).

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8.1 Function Sequences

A sequence of real valued functions (f_n) with $f_n : X \to \mathbb{R}$ converges to f in one of two incompatible ways

- **Pointwise convergence**: $f_n \to f$ if $f_n(x) \to f(x)\ \forall\ x \in X$ (fails to preserve continuity)
- **Metric convergence**: $f_n \to f$ if $d(f_n(x), f(x)) \to 0$

8.1.0.13 Uniform Norm The default norm for metric convergence is $\|f\|_{\sup} = \|f\|_{L^\infty(\Omega)} = \sup\limits_{x\in X} |f(x)|$

- The norm is bounded if and ony if f is bounded.
- A sequence of bounded real valued functions (f_n) **converges uniformly** to to a function f on a metric space X if
$$\lim_{n\to\infty} \|f_n - f\|_\infty = 0 \qquad \text{(uniform convergence)}$$
- Uniform convergence implies pointwise convergence (but the converse need not be true)
- If f_n is continuous $\forall\ n$, and $f_n \to f$ uniformly, then f is continuous

This concept also applies to other function metric spaces

- For $f_k \in C^m(K)$, $d(f_k, f) \to 0 \iff \|f_k - f\|_{C_k} \to 0$ as $k \to \infty$
- For $f_k \in L^p(\Omega)$, $d(f_k, f) \to 0 \iff \int_\Omega |f_k - f|^p\, dx \to 0$ as $k \to \infty$
  $p = 2$ case known as **root mean square convergence**

8.2 Essential Suprenum, Essential Infimum

Given X and f,

Essential suprenum: The $\operatorname{ess\,sup} X = a$ s.t. $\{x \in S : f(x) > a\}$ is contained in a zero set. That is, a is the essential suprenum for X if $f(x) \le a$ for almost all $x \in X$

$$\operatorname{ess\,sup} f = \begin{cases} \inf\{a \in \mathbb{R} : \operatorname{meas}(\{x : f(x) > a\}) = 0\} & \{x : f(x) > a\} \neq \emptyset \\ \infty & \{x : f(x) > a\} = \emptyset \end{cases}$$

Essential infimum: The **essential lower bound**

$$\operatorname{ess\,inf} f = \begin{cases} \sup\{a \in \mathbb{R} : \operatorname{meas}(\{x : f(x) < a\}) = 0\} & \{x : f(x) < a\} \neq \emptyset \\ -\infty & \{x : f(x) < a\} = \emptyset \end{cases}$$

8.3 Continuity

The following definitions are equivalent.
Topological description for open sets: A function $f : X \to Y$ is continuous if, for every open set $V \subseteq Y$ the preimage $f^{-1}(V)$ is an open set in X.
Topological description for closed sets: A function $f : X \to Y$ is continuous if, for every closed set $V \subseteq Y$ the preimage $f^{-1}(V)$ is a closed set in X.
ϵ, δ **description :**

$$\forall\ \epsilon > 0 \text{ and } \forall\ p \in M\ \exists\ \delta > 0 \text{ s.t.}$$

$$q \in M \text{ and } d(p,q) < \delta \implies d(f(p), f(q)) < \epsilon$$

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Convergent Sequence description:
$f : M \to N$ is continuous if and only if it sends convergent sequences to convergent sequences, limits being sent to limits.
Facts of continuous functions f, g

- Compositions $(f \circ g)$ are continuous.
- Multiplying functions $(f \times g)$ is continuous.
- Division of functions (f/g) is continuous if $g \neq 0$ everywhere
- Addition/Subtraction $(f \pm g)$ is continuous.

8.3.1 Oscillation $osc_x(f)$

The oscillation of f at x is

$$\operatorname{osc}_x(f) = \limsup_{t\to x} f(t) - \liminf_{t\to x} f(t)$$

or equivalently

$$\operatorname{osc}_x(f) = \lim_{t\to 0} \operatorname{diam}\left(f([x-t, x+t])\right)$$

f is continuous at x if and only if $osc_x(f) = 0$

8.3.2 Discontinuity set of f

The set D of discontinuity points is the set

$$D = \{x \in [a,b] : f \text{ is discontinuous at the point x}\}$$

And the points at which f is continuous is D^C. This can be defined as

$$D = \bigcup_{k=1}^{\infty} \left\{ x \in [a,b] : \operatorname{osc}_x(f) \geq \frac{1}{k} \right\}$$

- $D(f \pm g) \subset D(f) \cup D(g)$
- $D(f * g) \subset D(f) \cup D(g)$
- $D\left(\frac{f}{g}\right) = D(f) \cup D\left(\frac{1}{g}\right) = D(f) \cup D(g) \cup (\operatorname{supp} g)^C$

8.3.3 Uniformly Continuous

Given (X, d_1) and (Y, d_2), a function $f : X \to Y$ is uniformly continuous if it is continuous for all $x \in X$. More precisely, if

$$\forall\, \epsilon > 0 \;\; \exists \;\; \delta \text{ s.t. } d_1(x_1, x_2) < \delta \implies d_2(f(x_1), f(x_2)) < \epsilon\; \forall\; x_1, x_2 \in X$$

- Uniform continuity implies pointwise continuity
- Every continuous function defined on a compact set is uniformly continuous

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8.3.4 Lipschitz Continuous

Given (X, d), a function $f : X \to \mathbb{R}$ is Lipschitz continuous or Lipschitz thre exists a constant C such that

$$|f(x_1) - f(x_2)| \leq Cd(x_1, x_2) \qquad \forall\, x_1, x_2 \in X$$

The Lipschitz constant $C = Lip(f)$ is defined as the maximum 'slope':

$$Lip(f) = \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x, y)}$$

or equivalently as the smallest possible C:

$$Lip(f) = \inf \{C : |f(x) - f(y)| \leq Cd(x, y)\ \forall\, x, y \in X\}$$

- Lipschitz continuity implies uniform continuity
- A continuously differentiable function with bounded partial derivatives is Lipschitz.
- A family of continuously differentiable functions with uniformly bounded partial derivatives is equicontinuous. If it is bounded, it is also precompact.

8.3.5 Absolutely Continuous

Absolute continuity is a smoothness property of functions that is stronger than continuity and uniform continuity.

8.3.5.1 Analysis Definiton Given an interval on the reals $I = [a, b]$, a function $f : I \to \mathbb{R}$ is absolutely continuous if for all $\epsilon > 0$ there exists $\delta > 0$ such that for all finite sequences of subintervals (x_k, y_k) on I,

$$\sum_k |y_k - x_k| < \delta \implies \sum_k |f(y_k) - f(x_k)| < \epsilon$$

We denote this $f \in AC(I)$

8.3.5.2 Calculus Definiton $f \in AC(I)$ if f has a derivative f' a.e. which is Lebesgue Integrable and

$$f(x) = f(a) + \int_a^x f'(t)dt \qquad \forall\, x \in I = [a, b]$$

8.3.5.3 Fundamental Theorem of Lebesgue Integral Calculus
If f there exists a Lebesgue Integrable function g and

$$f(x) = f(a) + \int_a^x g(t)dt \qquad \forall\, x \in I = [a, b]$$

Then $g = f'$ a.e., $f \in AC(I)$

8.3.5.4 Properties of $AC(I)$ Given $f, g \in AC(I)$,

- $f \pm g \in AC(I)$. $f * g \in AC(I)$ if I is closed. $\frac{f}{g} \in AC(I)$ if I is closed and g is nowhere zero on I.
- Absolute conintuity is stronger than uniform and Lipschitz continuity.

8.3.6 Equicontinuous Families

A function family $\mathcal{F}$ from a metric space (X, d_X) to a metric space (Y, d_Y) is **equicontinuous** if

$$\forall\, x \in X, \epsilon > 0\ \exists\, \delta > 0 \text{ s.t. } d(x_1, x_2) < \delta \implies d(f(x_1), f(x_2)) < \epsilon\ \forall\, f \in \mathcal{F}$$

- We require δ to not depend on f, but it may depend on x
- If δ does not depend on x, $\mathcal{F}$ is said to be uniformly equicontinuous
- Equicontinuous families on compact spaces are uniformly equicontinuous

8.3.7 Homeomorphisms

If $f : M \to N$ is a continuous bijection and the inverse $f^{-1} : N \to M$ is also a continuous bijection, the f is a homeomorphism and M and N are said to be homeomorphic, denoted $M \cong N$.
Geometrically, a homeomorphism bends, twists, stretches, and wrinkles the space M to make it coincide with N.
Homeomorphisms map the following

- $\partial M \to \partial N$
- $intM \to intN$
- $M^c \to N^c$

8.3.8 Isometries

An **isometry** or **isometric embedding** is a map $\imath : X \to Y$ that satisfies

$$d_Y(\imath(x_1), \imath(x_2)) = d_X(x_1, x_2) \qquad \forall\, x_1, x_2 \in X$$

- $\imath$ identifies a point $x \in X$ with its image $\imath(x) \in Y$ so that $\imath(X)$ is a copy of X embedded in Y
- Two metric spaces X and Y are isomorphic if there exists an isomorphism $\imath : X \to Y$. Isomorphic spaces are indistinguishable as metric spaces.
- Isometries are one-to-one and continuous
- An isometry that is onto is called a metric space isomorphism
- $\imath : \mathbb{C} \to \mathbb{R}^2$ defined by $\imath(x + iy) = (x, y)$ is a metric space isomorphism between $(\mathbb{C}, |\cdots|)$ and $(\mathbb{R}^2, \|\cdots\|\ )$. Since $\imath$ is linear, these are isomorphic as real normed linear spaces

8.3.9 Fundamental Theorem of Continuous Functions

Every continuous real valued functionof a real variable $x \in [a, b]$ is bounded, achieves maximum, intermediate, and maximum values, and is uniformly continuous.

8.3.10 Mean Value Theorem

For a continuous $f : [a, b] \to \mathbb{R}$ which is differentiable on (a, b) there exists a c s.t.

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

8.4 Properties of Functions

8.4.1 Support $supp(f)$

The support of a function is the set of points where the function is not zero-valued, or the closure of that set. In other words, the support Y of $f : X \to \mathbb{C}$ is

$$Y = \overline{\{y \in X : f(y) \neq 0\}}$$

- A function supported in Y will vanish in $X \setminus Y$.
- A function with domain X has **finite support** if Y is finite.
- A function has **compact support** if Y is a compact subset of X.
- Equivalently, A function $f : \mathbb{R}^n \to \mathbb{R}$ has **compact support** if $\exists\, R > 0$ such that $f(x) = 0\, \forall\, x$ with $\|x\| \ > R$. That is, $\lim_{\|x\| \ \to +\infty} f(x) = 0$

8.4.2 Other Properties

- A fixed point of a function $f : X \to X$ is a point such that $f(x) = x$
- The diagonal of $X \times X$ is the set of all pairs $(x, x) \in X \times X$
- The $\operatorname*{argmax}_{x \in X} f(x)$ is the $x \in X$ where there maximum of $f(x)$ is achieved

8.5 Special Types of Functions

- **Step Functions** are constant except at a finite number of points where it is discontinuous.
- **Characteristic function** or **Indicator Function** χ_E of a set $E \subset \mathbb{R}$ is defined as

$$\chi_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}$$

- **Rational Ruler Function** is defined as

$$f(x) = \begin{cases} \frac{1}{q} & x = \frac{p}{q} \text{ for some p,q in lowest terms} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

- The **Sign Funtion** is defined as

$$\operatorname{sign}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$

8.6 Function Spaces

Function spaces are infinite-dimensional linear normed spaces.
The domain of a function Ω is the set such that $f : \Omega \to \mathbb{R}$.
We say $x \in \Omega$ if $\forall\, r > 0\ B_r(x) \cap \Omega \neq \emptyset$
We say $x \notin \Omega$ if $\forall\, r > 0\ B_r(x) \cap \Omega^c \neq \emptyset$
Ω can be

- Open if it is an open set and closed if it is a closed set. Similiarly it can be clopen or neither closed nor open.
- Bounded if is contained in some $B_r(0)$ with r finite
- Compact if it is closed and bounded

8.6.0.1 Set Properties of Function Spaces

- If all Cauchy sequences of functions are convergent, then the metric space is **complete**. Otherwise it is **incomplete**, and may be completed

8.7 Function Classes

8.7.1 Continuity Classes $C(\Omega)$

The following function classes real linear spaces and are closed under pointwise combination. That is

$$\text{If } f, g \in C(\Omega), \alpha, \beta \in \mathbb{R}, \text{ then } \alpha f + \beta g \in C(\Omega)$$

- $C(\Omega) = \{f : f \text{ is continuous on } \Omega\}$
- $C(K) = \{f : f \text{ is continuous on the compact set } K\}$
This is a complete normed linear space when equipped with the uniform norm so it is a Banach Space
Arzela-Ascoli Theorem : A subset of $C(K)$ is compact if and only if it is closed, bounded, and equicontinuous.
- $C_b(\Omega) = \{f : f \text{ is continuous and bounded on } \Omega\}$ (countinous functions that are bounded)
This is a complete normed linear space when equipped with the uniform norm so it is a Banach Space
- $C_k(\Omega) = \{f : f \text{ is continuous and has compact support} \Omega\}$
Note that $C_k(\Omega) \subset C_b(\Omega)$, but it need not be open, so it need not be a Banach Space.
C^k norm: recalling the sup norm $\|\cdot\|_\infty$, $C_k(\Omega)$ is complete with respect to the norm:
$$\|f\|_{C^k} = \|f\|_\infty + \|f'\|_\infty + ... + \left\|f^k\right\|_\infty$$
if $f \in C_0^\infty$
- $C_0(\Omega) = \left\{\overline{C_c} \text{ in } C_b\right\}$ is closed and is therefore a Banach space.

Note that
$$C(\Omega) \supset C_b(\Omega) \supset C_0(\Omega) \supset C_k(\Omega)$$
And if Ω is compact then these are equal.

8.7.2 Bounded Function Classes $B(\Omega)$

The following function classes real linear spaces and are closed under pointwise combination. That is

$$\text{If } f, g \in B(\Omega), \alpha, \beta \in \mathbb{R}, \text{ then } \alpha f + \beta g \in B(\Omega)$$

- $B(\Omega) = \{f : f \text{ is bounded on } \Omega\}$
- $CB(\Omega) = \{f : f \text{ is continuous and bounded on } \Omega\}$

8.7.3 Differentiability Classes $C^m(\Omega)$

- $C^m(\Omega) = \{f : D^\alpha$ is continuous on $\Omega\ \forall\ |\alpha| \leq m\}$
- $C^\infty(\Omega) = \{f : D^\alpha$ is continuous on $\Omega\ \forall\ |\alpha|\}$ **(Smooth Functions)**
- $C^\omega(\Omega) = \{f : f \in C^\infty$ and f equals the Taylor series expansion in a neighborhood around every point in $\Omega\}$ **(Analytic Functions)**

Note: $C^m(\Omega) \subset C^\omega(\Omega) \subset C^\infty(\Omega)$

8.7.4 Integrability Classes $\mathcal{R}$,L^p

- $\mathcal{R}(\Omega) = \{f : f$ is Riemann Integrable on $\Omega\}$, Riemann Integrable functions Note $\mathcal{R}(\Omega) \subset B(\Omega)$
- $L^p(\Omega) = \left\{f : \left(\int_\Omega |f|^p \, dx\right)^{\frac{1}{p}} < \infty\right\}$, Lebesgue Integrable functions

9 Calculus

9.1 Differentiation

The function $f : (a, b) \to \mathbb{R}$ is differentiable at x if there is a real number L such that

$$\lim_{t\to x} \frac{f(t) - f(x)}{t - x} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = L$$

and

$$\forall\, \epsilon > 0\ \exists\, \delta > 0 \text{ s.t. } |t - x| < \delta \implies \left|\frac{f(t) - f(x)}{t - x} - L\right| < \epsilon$$

We denote the derivative of f at x $f'(x)$.

- Differentiability of f implies continuity of f
- Note that if f is differentiable at x, $\frac{\Delta f}{\Delta x} = f'(x) + \sigma(\delta x)$ for some $\sigma(\delta x)$
- If a function is Lipschitz with constant M, then $|f(x)| \leq M$

If f, g are differentiable, then the following are diferetiable with the indicated derivative

- $(f + g)'(x) = f'(x) + g'(x)$
- $(f \times g)'(x) = f'(x)g(x) + f(x)g'(x)$
- If $f = c$ for some constant c, then $f' = 0$
- If $g(x) \neq 0$, $\left(\frac{f}{g}\right)' = \left(\frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}\right)$
- $(f \circ g)'(x) = g'(f(x)) \times f'(x)$

9.1.1 Mean Value Theorem (MVT)

A continuous function $f : [a, b] \to \mathbb{R}$ that is differentiable on the interval (a, b) has the mean value property there exists a point $\theta \in (a, b)$ such that

$$f(b) - f(a) = f'(\theta)(b - a)$$

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9.1.2 Ratio Mean Value Theorem (RMVT)

Continuous functions $f, g : [a, b] \to \mathbb{R}$ that are differentiable on the interval (a, b) has the mean value property there exists a point $\theta \in (a, b)$ such that

$$(f(b) - f(a))\, g'(\theta) = (g(b) - g(a))\, f'(\theta)$$

9.1.3 L'Hospital's Rule

Gven continuous functions $f, g : [a, b] \to \mathbb{R}$ that are differentiable on the interval (a, b), if f, g tend to 0 at b, and if $\frac{f'}{g'}$ tends to a finite limit L at b then $\frac{f}{g}$ also tends to L at b. (assuming $g(x) \neq 0, g'(x) \neq 0$)

9.1.4 Intermediate Value Theorem for Derivatives

Give a continuous function $f : [a, b] \to \mathbb{R}$ that is differentiable on the interval (a, b), its derivative $f'(x)$ has the intermediate value theorem.
The derivative of a differentiable function never has a jump discontinuity, since that would violate IVT for Derivatives.

9.1.5 Inverse Function Theorem

If $f : (a, b) \to (c, d)$ is a differetiable surjection and $f'(x) \neq 0 \;\forall\, x \in (a, b)$,then f is a homeomorphism with a differentiable inverse with derivative:

$$f^{-1}(y) = \frac{1}{f'(x)} \qquad \text{for } y = f(x)$$

Further, if both $f : (a, b) \to (c, d)$ is a homeomorphism of class C^r for $1 \leq r \leq \infty$ and $f'(x) \neq 0 \;\forall\, x$, then f is a C^r **diffeomorphism**

9.2 Higher Derivatives and More Dimensions

- $(f')'(x) = f''(x) = \lim\limits_{t \to x} \frac{f'(t) - f'(x)}{t - x}$
- Higher derivatives are denoted $f^{(r)} = (f^{(r-1)})'$
- If $f^{(r)}$ exists f is r^th order differentiable, with $f^{(0)} = f(x)$
  If f is r^th order differentiable, then $f, f', ..., f^{(r-1)}$ are continuous
- If $f^{(r)}$ exists for all r, f is **infinitely differentiable** also known as **smooth**
  If f is smooth then $f^{(r)}$ is continuous and smooth for all r

9.2.1 Leibniz Product Rule

Given r^th order ($r \geq 1$) differentiable functions $f, g : (a, b) \to \mathbb{R}$,

$$(f \times g)^{(r)}(x) = \sum_{k=0}^{r} \left(\begin{array}{c} r \\ k \end{array} \right) f^{(k)}(x) g^{(r-k)}(x)$$

For multiple dimensions,

$$\partial^\alpha (fg) = \sum_{\beta + \gamma = \alpha} \frac{\alpha!}{\beta! \gamma!} (\partial^\beta f)(\partial^\gamma g)$$

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9.3 Analytic Functions C^ω

A function $f : (a, b) \to \infty$ is analytic if it can be expressed locally as as a convergent power series. That is, for each x, there exists a $\delta > 0$ such that if $|h| < \delta$ then the series

$$f(x+h) = \sum_{r=0}^{\infty} a_r h^r$$

converges.

- Note that this implies
$$f^{(r)}(x) = r! a_r$$
Which gives rise to the uniquness of a power series expansion.
- $C^\omega \subset C^\infty$, and is a proper subset since smooth functions need not be analytic.

9.3.1 Taylor Approximation Theorem

The r^{th} order **Taylor polynomial** of an $r^t h$ order differentiable function f at x is

$$P(h) = f(x) + f'(x)h + \frac{f''(x)}{2}h^2 + ... + \frac{f^{(r)}(x)}{r!}h^r = \sum_{k=0}^{r} \frac{f^{(k)}(x)}{k!}h^k$$

Note that

$$P(0) = f(x) \qquad P'(0) = f'(x) \qquad P''(0) = f''(x) \qquad P^{(r)}(0) = f^{(r)}(x)$$

- P approximates f to order r at x in the sense that the remainder
$$R(h) = f(x+h) - P(h)$$
is r^{th} order flat at $h = 0$: $\frac{R(h)}{h^r} \to 0$ as $h \to 0$
- The Taylor polynomial is the only polynomial of degree $\leq r$ with this approximation property
- In addition, if f is $(r+1)^{th}$ order differentiable on (a, b) then for some θ between x and $x + h$,
$$R(h) = \frac{f^{(r+1)}(\theta)}{(r+1)!}h^{r+1}$$

9.4 Differentiability Classes $C^m(\Omega)$

- $C^m(\Omega) = \{f : D^\alpha$ is continuous on $\Omega \; \forall \; |\alpha| \leq m\}$
- $C^\infty(\Omega) = \{f : D^\alpha$ is continuous on $\Omega \; \forall \; |\alpha|\}$ **(Smooth Functions)**
- $C^\omega(\Omega) = \{f : f \in C^\infty$ and f equals the Taylor series expansion in a neighborhood around every point in $\Omega\}$ **(Analytic Functions)**
- $C^1 \subset C^2 \subset ... \subset C^\omega \subset C^\infty$

with proper subsets

9.5 Integration

The integral of $f : [a, b] \to \mathbb{R}$ for $f(x) > 0$ is

$$\int_a^b f(x)dx = \text{ area of } \mathbb{U}$$

Where $\mathbb{U}$ is the **undergraph** of f:

$$\mathbb{U} = \{(x, y) : a \leq x \leq b \text{ and } 0 \leq y \leq f(x)\}$$

9.5.1 Partitions

A **partition pair** $P, T \subset [a, b]$ is of the form $P = \{x_0, ..., x_n\}$ and $T = \{t_1, ..., t_n\}$ interlaced as

$$a = x_0 \leq t_1 \leq x_1 \leq t_2 \leq \ldots \leq x_{n-1} \leq t_n \leq x_n = b$$

And if we let $\Delta x_i = x_i - x_{i-1}$, they define a Riemmann sum

$$R(f, T, P) = \sum_{i=1}^{n} f(t_i)\Delta x_i = f(t_1)\Delta x_1 + \ldots + f(t_n)\Delta x_n$$

which is the area o the rectangles which approximate the area uder the graph of f.

- Each t_i are known as **sample points** of f
- The **mesh** of P is $\max_i \{\Delta x_i\}$
- A **coarse** partition is one with a large mesh, and a **fine** partition is one with a small mesh
- A partition P' **refines** P if $P' \supset P$

9.5.2 Riemann Integration

$I \in \mathbb{R}$ is the **Riemann integral** of f over $[a, b]$ if it satisfies:

$$\forall\, \epsilon > 0\; \exists\, \delta > 0 \text{ s.t. } \text{mesh}P < \delta \implies |R - I| < \epsilon$$

If it exists, I is unique, and we say f is **Riemann integrable**. We denote I:

$$I = \lim_{\text{mesh}P \to 0} R(f, P, T) = \int_a^b f(x)dx$$

- If f is Riemann integrable then it is bounded
- $\mathcal{R}$ is the set of all functions that are Riemann Integrable over $[a, b]$
- $\mathcal{R}$ is a vector space and $f \to \int_a^b f(x)dx$ is a linear map $\mathcal{R} \to \mathbb{R}$

  $$R(\alpha f + \beta g, P, T) = \alpha R(f, P, T) + \beta R(g, P, T)$$

  and so

  $$\int_a^b (\alpha f(x) + \beta g(x))dx = \alpha \int_a^b f(x)dx + \beta \int_a^b g(x)dx$$

- The constant function $h(x) \in \mathcal{R}$ where $h(x) = k$ for some k, and its integral is $k(b - a)$

- Any polynomial $P(x) \in \mathcal{R}$
- The Rational Ruler function is in $\mathcal{R}$
- Zeno's Staircase is in $\mathcal{R}$
- $\chi_{\mathbb{Q}} \notin \mathcal{R}$
- **Monotonicity of the Integral (Theorem):** If $f, g \in \mathcal{R}$, and $f(x) \leq g(x) \; \forall \; x$, then

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx$$

Corollary: If $|f(x)| \leq M \; \forall \; x$, then $\left| \int_a^b f(x)dx \right| \leq (b-a)M$

9.5.3 Darboux Integration

The **upper sum** and **lower sum** of $f : [a, b] \to [-M, M]$ with repsect to a partition P of $[a, b]$ is

$$U(f, P) = \sum_{i=1}^{n} M_i \Delta x_i \qquad \text{and} \qquad L(f, P) = \sum_{i=1}^{n} m_i \Delta x_i$$

where

$$M_i = \inf \{f(t) : x_{i-1} \leq t \leq x_i\} \qquad \text{and} \qquad m_i = \sup \{f(t) : x_{i-1} \leq t \leq x_i\}$$

which lead to the **upper integral** and **lower integral** over all partitions P

$$\overline{I} = \inf_P U(f, P) \qquad \text{and} \qquad \underline{I} = \sup_P L(f, P)$$

And f is assumed to be bounded so that $m_i, M_i \in \mathbb{R} \; \forall \; i$. We see for all f, P, T,

$$L(f, P) \leq R(f, P, T) \leq U(f, P)$$

And f is **Darboux Integrable** if $\underline{I} = \overline{I} = I$. That is, a bounded function is Darboux integrable if and only if

$$\forall \; \epsilon > 0 \; \exists \; P \text{ s.t. } U(f, P) - L(f, P) < \epsilon$$

- Darboux Integrability and Riemann Integrability are equivalent, which gives rise to **Riemann's Integrability Criterion:** f is Riemann Integrable if and only if

$$\forall \; \epsilon > 0 \; \exists \; P \text{ s.t. } U(f, P) - L(f, P) < \epsilon$$

- **Refinement Principle** Refining a partition causes the lower sum to increase and the upper sum to decrease
- For a refinement P' of P, let the **Common Refinement** be $P^* = P \cup P'$, then refinement principle says

$$L(f, P) \leq L(f, P^*) \leq U(f, P^*) \leq U(f, P)$$

9.5.4 Riemann-Lebesgue Theorem:

Let f be a bounded function defined on $[a,b]$ then f is Riemann integrable if and only if the set of point at which f is discontinuous is a zero set.
Corollaries: The following are Riemann integrable

- Every continuous function and every piecewise continuous function
- The characteristic function of $S \subset [a,b]$ if and only if ∂S is a zero set
- Every monotone funtion
- The product of Riemann integrable functions
- For $f : [a,b] \to [c,d]$ s.t. $f \in \mathcal{R}$, and $g : [c,d] \to \mathbb{R}$ s.t. $g \in C^1([c,d])$, then $(g \circ f)(x) = g(f(x)) \in \mathcal{R}$
- For $f : [a,b] \to \mathbb{R}$ s.t. $f \in \mathcal{R}$, and $g : [c,d] \to [a,b]$ is a bijecton whose inverse is Lipschitz then $(f \circ g)(x) = f(g(x)) \in \mathcal{R}$
- For $f : [a,b] \to \mathbb{R}$ s.t. $f \in \mathcal{R}$, and $g : [c,d] \to [a,b]$ is a C^1 diffeomorphism then $(f \circ g)(x) = f(g(x)) \in \mathcal{R}$
- If $f \in \mathcal{R}$ then $|f| \in \mathcal{R}$
- If $f : [a,b] \in \mathcal{R}$ and $a < c < b$, then its restrictions to $[a,c]$ and $[c,b]$ are Riemann integrable and
$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$
- If $f : [a,b] \to [0,M]$ s.t. $f \in \mathcal{R}$ for some $M > 0$ and $I = 0$, then $f(x) = 0$ at every continuity point x of f. This implies $f(x) = 0$ a.e.

9.5.5 Fundamental Theorem of Calculus

For a continuous real valued fuction $f : [a,b] \to \mathbb{R}$ s.t. $f \in \mathcal{R}$, let its **indefinite integral** F be defined on $[a,b]$ by
$$F(x) = \int_a^x f(t)dt$$
F is continuous on $[a,b]$, differentiable on (a,b), and for all $x \in (a,b)$ such that $f(x)$ is continuous,
$$F'(x) = f(x)$$

Theorem The values of a continuous function defined on an interval $[a,b]$ form a bounded subset of $\mathbb{R}$. There exist $m, M \in \mathbb{R}$ s.t. $\forall\, x \in [a,b]$, $m \le f(x) \le M$

- The derivative of an indefinite integral exists a.e. and equals the integrand a.e.

9.5.6 Antiderivative

If $f = g'$, then g is the **Antiderivative** of f When G is an antiderivatve of $g : [a,b] \to \mathbb{R}$, we have
$$G'(x) = g(x) \qquad \forall\, x \in [a,b]$$
Notice that this is for all x, not merely almost all x

- Every continuous function has an antiderivative
- **Antiderivative Theorem:** If $f \in \mathcal{R}$ then its antiderivative, if it exists, differs from the indefinite integral by at most a constant.

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9.5.7 Integration Techniques

9.5.7.1 Integration by Substitution

If $f:[a,b]\to\mathbb{R}$ s.t. $f\in\mathcal{R}$, and $g:[c,d]\to[a,b]$ is a C^1 diffeomorphism (continuously differentiable bijection with $g'(x)>0\;\forall\;x$) then

$$\int_a^b f(y)dy=\int_c^d f(g(x))g'(x)dx$$

9.5.7.2 Integration by Parts

If $f,g:[a,b]\to\mathbb{R}$ are differetiable and $f',g'\in\mathcal{R}$, then

$$\int_a^b f(x)g'(x)=f(b)g(b)-f(a)g(a)\int_a^b f'(x)g(x)dx$$

9.5.8 Improper Integrals

If $a<c<b$ and $[a,c]\subset[a,b)$ and $f:[a,c]\to\mathbb{R}$ s.t. $f\in\mathcal{R}$, where either $\limsup_{x\to b}|f(x)|=\infty$ or $b=\infty$, then we have the **Improper Riemann Integral**

$$\int_a^b f(x)dx=\lim_{c\to b}\int_a^c f(x)dx$$

Or equivalently for either or both sides of the integral.

9.6 Higher Dimensional Calculus

- Let $\Omega\in\mathbb{R}^N$ be a bounded open set with a 'nice' boundary
- Define $\vec{\mathbf{n}}=\vec{\mathbf{n}}(x)$ as the unit outward normal derivative for $x\in\partial\Omega$
- $\vec{\nabla}=\left(\frac{\partial}{\partial x_1},...,\frac{\partial}{\partial x_N}\right)$ is known as the dell operator
- $\vec{\nabla}^2=\Delta$ is known as the Laplacian

9.6.1 Divergence Theorem

$$\int_\Omega \vec{\nabla}\cdot\vec{\mathbf{g}}dx=\int_{\partial\Omega}\vec{\mathbf{g}}\cdot\vec{\mathbf{n}}ds$$

The first integral is an integral over Ω, the second integral is a line integral around the boundary of Ω

9.6.2 Green's Identity

$$\int\int_{\Omega^N}(f\Delta g-g\Delta f)dx=\int_{\partial\Omega}\left(f\frac{\partial h}{\partial n}-h\frac{\partial f}{\partial n}\right)ds$$

The first integral is an integral over Ω, the second integral is a line integral around the boundary of Ω

9.6.3 Derivations

Starting with a basic formula:

$$\int_\Omega f\frac{\partial g_i}{\partial x_i}dx = -\int_\Omega g_i\frac{\partial f}{\partial x_i} + \int_{\partial\Omega} f g_i n_i ds$$

Summing over i gives

$$\int_\Omega f\sum_{i=1}^N \frac{\partial g_i}{\partial x_i}dx = -\int_\Omega \sum_{i=1}^N g_i\frac{\partial f}{\partial x_i} + \int_{\partial\Omega} f\sum_{i=1}^N g_i n_i ds$$

Which by various calculus definitions is

$$\int_\Omega f\vec{\nabla}\cdot\vec{\mathbf{g}}dx = -\int_\Omega \vec{\mathbf{g}}\cdot\vec{\nabla}f + \int_{\partial\Omega} f\vec{\mathbf{g}}\cdot\vec{\mathbf{n}}ds$$

Choosing $f = 1$ yields divergence theorem. Alternatively, choosing $\vec{\mathbf{g}} = \vec{\nabla}h$ for some scalar h means $\vec{\nabla}\cdot\vec{\mathbf{g}} = \vec{\nabla}^2 h = \Delta h$, so the above becomes

$$\int_\Omega f\nabla h dx = -\int_\Omega \vec{\nabla}h\cdot\vec{\nabla}f + \int_{\partial\Omega} f\frac{\partial h}{\partial n}ds$$

Reversing f, h and then subtracting that from the original gives Green's Identity

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Part IV
Differential Equations

10 Solutions

10.1 General Solutions

General Solutions are the set of all solutions to a DE. Generally, a n^{th} order D's general solution has n arbitrary constants.

Normalized Solutions: The solution set (for example $y(x) = c_1 y_1(x) + c_2 y_2(x)$ to a DE such that when $y(x=0) = 0$ and $y'(x=0) = 1$.

10.1.1 Well-Posed Problems

A problem is well posed if

- There is one solution (existence)
- The solution is unique (uniqueness)
- The solution depends continuously on the data (stability condition)
  Small changes in the intial or boundary conditions lead to small changes in the solution

Wronksian: The determinant of the Fundamental Matrix of a set of solutions to a differential equation. A set of solutions to a DE are linearly independent if the Wronskian identically vanishes for all $x \in I$. Note that $W \equiv 0$ does not imply linear dependence.

For f, g, $W(f,g) = fg' - gf'$. For n real or complex valued functions $f_1, f_2, ..., f_n$ which are $n-1$ times differentiable on an interval I, the Wronksian $W(f_1, ..., f_n)$ as a function on I is defined by

$$W(f_1, ..., f_n)(x) = \begin{vmatrix} f_1(x) & \dots & f_n(x) \\ f_1'(x) & \dots & f_n'(x) \\ \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & \dots & f_n^{(n-1)}(x) \end{vmatrix} x \in I$$

11 Ordinary Differential Equations

An n^{th} order ODE for $y = y(t)$ has the form

$$F(t, y', y'', .., y^{(n)}) = 0 \qquad \text{(Implicit Form)}$$

Usually it can be written

$$y^{(n)} = f(t, y', y'', .., y^{(n-1)}) \qquad \text{(Explicit Form)}$$

A solution y is defined on $y : I \to \mathbb{R}$ with $y \in C^n(I)$ for some $I \subseteq \mathbb{R}$ such that

$$y^{(n)}(t) = f(t, y'(t), y''(t), .., y^{(n-1)}(t)) \qquad \forall\, t \in I$$

- If the solution is defined on whole of $\mathbb{R}$ then we call it a **global solution**
- If the solution is defined on a subinterval of $\mathbb{R}$ then we call it a **local solution**

11.1 Senses of Solutions

11.1.0.1 Classical Solution

$u' = f$ in a **classical sense** if $u \in C^1$ and $u'(x) = f(x) \; \forall \; x$

11.1.0.2 Weak Solution

$u' = f$ in a **weak sense** if $u \in L^1_{\text{loc}}$ and $u' = f$ in $\mathcal{D}'$ sense.
Classical solutons are always also weak solutions

11.1.0.3 Distributional Solution

$u' = f$ in a **distributional sense** if $u \in \mathcal{D}'$ and $u' = f$ in $\mathcal{D}'$ sense.
Classical solutions and weak solutions are always also distributional solutions

11.1.0.4 Regularity of Solutions For $u' = 0$ all solutions are classical, weak, and distributional solutions.
For $xu' = 0$ the solution $u = \delta$ is neither classical nor weak.
Thus, the regularity of the solution depends on the DE.

11.1.1 Initial Value Problems (IVP)

A problem is an IVP if it is given in the form

$$\begin{gathered} y^{(n)} = f(t, y', y'', .., y^{(n-1)}) \\ y(a) = \gamma_0 \\ y'(a) = \gamma_1 \\ \vdots \\ y^{(n-1)} = \gamma_{n-1} \end{gathered}$$

where a is the lower boundary of the domain.

- Linear IVPs have a unique solution.
- **Existence of Solution**
  - **Local Existence Theorem or Peano Existence Theorem**: If f is continuous on $\mathbb{R}^n$, then every $(t_0, u_0, ..., u_0^{(n-1)})$ there exists an open interval $(t_0 - \epsilon, t_0 + \epsilon) = I \subset \mathbb{R}$ with $\epsilon > 0$ that contains t_0 and there exists a continuously dfferentiable function $u : I \to \mathbb{R}$ that satisfies the IVP.
  - **Local Existence Theorem** If f is continuous in a neighborhood of $(a, \gamma_0, ..., \gamma_{n-1})$ there exists an open interval $(t_0 - \epsilon, t_0 + \epsilon) = I \subset \mathbb{R}$ with $\epsilon > 0$ that contains t_0 and there exists a continuously dfferentiable function $u : I \to \mathbb{R}$ that satisfies the IVP.
- **Uniqueness of Solution**
  - **Uniqueness by Continuous Differentiability of f**: If ∇f is continuous (if f is continuously differentiable), then the solution is unique.
  - **Uniqueness by Lipschitz**: If $f(u, t)$ is Lipschitz continuous in u then the solution is unique.
- **Gronwall's Inequality**: For $u(t) \geq 0$ continous and $\phi(t) \geq 0$ continous defined on $0 \leq t \leq T$ and $u_0 \geq 0$ is a constant, if $u(t)$ satisfies

$$u(t) \leq u_0 + \int_0^t \phi(s) u(s) ds \qquad \text{for } t \in [0, T]$$

$$\text{then,} \qquad u(t) \leq u_0 exp\left(\int_0^t \phi(s) ds\right) \qquad \text{for } t \in [0, T]$$

11.1.2 Boundary Value Problems (BVP)

- A BVP with **separated conditions** affect multiple endpoints such as in the form
$$g_a(y(a)) = 0, \; g_b(y(b)) = 0$$
- A BVP with **unseparated conditons** affect the endpoints simultaneously, such as in the periodic conditions
$$y(a) - y(b) = 0$$

11.1.3 Systems of ODEs

One could also consider solutions to systems of ODEs.
Any ODE can be converted into a first order system of ODEs. Example:

$$x''' = t + \cos(x'')e^x \qquad \text{becomes} \qquad y' = \begin{pmatrix} x' \\ x'' \\ x''' \end{pmatrix} = \begin{pmatrix} y_2 \\ y_3 \\ t + \cos y_3 e^{y_2} \end{pmatrix}$$

11.2 Linear Equations

Linear equations are linear in y, and have the form
$$\sum_{j=0}^{n} a_j(t) D^{(j)} y(t) = g(t)$$
$$Ly = g(t)$$
Otherwise the ODE is **nonlinear** for some $a_0, a_1, ..., a_n, g$ and $D^k = \frac{d^k}{(dt)^k}$. Note that $a_j(t)$ need not be linear.

11.2.1 General solutions

For ($a_1, a_2, ..., a_n$ continuous; g continuous; $a_n \neq 0$), linear ODEs have infinitely many solutions of the form
$$y(t) = y_p(t) + \sum_{j=1}^{n} c_j y_j(t)$$
Where $(y_1, y_2, ..., y_n)$ are linearly independent solutions to $Ly = 0$ and $y_p(t)$ is a particular solution to $Ly = g$.

- Linear IVPs have a unique solution.

11.2.1.1 Linear Systems of ODEs Any system of linear ODEs can be viewed as the matrix equation
$$\vec{\mathbf{y}}' = A\vec{\mathbf{y}}$$
with solution
$$\vec{\mathbf{y}} = e^{At}\vec{\mathbf{c}} \qquad \text{where } x(t) = y_1(t)$$
For a vector of arbitrary constants c determined by intial or boundary values.

- If A is diagonalizable , $A = VDV^{-1}$ with its eigenvectors V, then $e^{At}c = Ve^{Dt}V^{-1}\vec{\mathbf{c}}$. Since $V = (\vec{\mathbf{v}}_1 \; \dots \; \vec{\mathbf{v}}_n)$ and we can define arbitrary constants $\vec{\mathbf{d}} = V^{-1}\vec{\mathbf{c}}$, this becomes
$$\vec{\mathbf{y}}(t) = e^{At}c = d_1 e^{\lambda_1 t}\vec{\mathbf{v}}_1 + d_2 e^{\lambda_2 t}\vec{\mathbf{v}}_2 + ... + d_n e^{\lambda_n t}\vec{\mathbf{v}}_n$$
Alternatively you can simply evaluate $Ve^{Dt}V^{-1}\vec{\mathbf{c}}$ and take the first component.

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11.3 Nonlinear Equations

Nonlinear equations such as

$$y' = y^2 \text{ with } u(t_0) = u_0$$

May have a unique solution, but usually only local solution.

11.3.1 Exactly Solvable Cases

First Order Linear Equations

$$y' + p(t)y = q(t)$$

The general solution is

$$y = \frac{1}{M(t)} \int_{t_0}^{t} q(u)M(u)du + \frac{C}{M(t)}$$

for $M(t) = e^{\int_{s_0}^{t} p(s)ds}$ and any constant C.

Linear Equations with Constant Coefficients

$$Ly = \sum_{j=0}^{n} a_j D^j y = 0$$

Solutions exist in the form

$$y(t) = e^{\lambda t}$$

where λ is a root of the characteristic polynomial

$$p(\lambda) = \sum_{j=0}^{n} a_j \lambda^j$$

If roots are repeated, the solutions associated with the same root must take on forms that are orthogonal to one another, such as

$$y(t)_1 = e^{\lambda t} \ , \ y(t)_2 = te^{\lambda t} \ , \ y(t)_3 = t^2 e^{\lambda t}$$

The pair of solutions associated with a pair of complex roots must be real, and so for a pair of roots $(\lambda \pm (\alpha + i\beta))$

$$y_1(t) = e^{\alpha t} cos(\beta t) \qquad y_2(t) = e^{\alpha t} sin(\beta t)$$

Euler Type Equations

$$Ly = \sum_{j=0}^{n} a_j (t - t_0)^j D^j y = 0$$

Solutions exist in the form

$$y(t) = (t - t_0)^{\lambda} \qquad t \neq t_0$$

Where λ can be found by using this solution form in the equation, which forms the **indicial equation**

$$\sum_{j=0}^{n} A_j t^j = 0$$

Where $A_n = a_n$ but the other coefficients depend on the nature of the ODE. If the indicial equation has...

- two roots, then
$$y(t) = c_1(t-t_0)^{\lambda_1} + c_2(t-t_0)^{\lambda_2}$$
- one root, then one must perform reduction of order. However, solutions typically have a solution that looks something like
$$y(t) = c_1(t-t_0)^{\lambda} + c_2(t-t_0)^{\lambda}\ln|t-t_0|$$
For higher algebraic muliplicities of the root, you will have additional solutions $\{(t-t_0)^\lambda(\ln|t-t_0|)^2, ..., (t-t_0)^\lambda(\ln|t-$
- a complex pair of roots $r = \lambda \pm i\omega$, one must solve for the real solutions. Typically you end up with a solution that looks something like
$$y(t) = c_1(t-t_0)^{\lambda}\cos(\omega\ln|t-t_0|) + c_2(t-t_0)^{\lambda}\sin(\omega\ln|t-t_0|)$$
For higher algebraic multiplicities you can solve for real valued solutions of the form
$$(t-t_0)^{\lambda}\cos(\omega\ln|t-t_0|)\ln|t-t_0|, (t-t_0)^{\lambda}\sin(\omega\ln|t-t_0|)\ln|t-t_0|, ...$$
$$..., (t-t_0)^{\lambda}\cos(\omega\ln|t-t_0|)(\ln|t-t_0|)^{m-1}, (t-t_0)^{\lambda}\sin(\omega\ln|t-t_0|)(\ln|t-t_0|)^{m-1}$$

11.3.1.1 Example

$$ax^2y'' + bxy' + cy = 0$$

Yields the indicial equation
$$a\lambda^2 + (b-a)\lambda + c = 0$$

Say $a = 1, b = -6, c = 10$. Then $\lambda_{1,2} = 2, 5$ and
$$y(t) = c_1x^2 + c_2x^5$$

Say $a = 1, b = -9, c = 25$. Then $\lambda = 5$ and we must additionally solve
$$y(x) = x^5u(x) \qquad v = u'$$

which has the solution
$$y(x) = x^5(c_1\ln|x| + c_2)$$

Say $a = 1, b = -3, c = 20$. Then $\lambda = 2 \pm 4i$
$$y(x) = c_1x^2\cos(4\ln|x|) + c_2x^2\sin(4\ln|x|)$$

11.3.2 Relation between Euler Equations and Constant Coefficient Equations

Let $y : (t_0, \infty) \to \mathbb{R}$ for and $Y : (-\infty, \infty) \to \mathbb{R}$ be functions of t and x respectively. Assume they are related by a substitution $x = e^t$. That is, $y(t) = Y(x)$. Then the Euler equation for y can be related to the constant coefficient equation for Y.

11.4 Nonlinear ODEs

$$y^{(n)} = f(t, y', y'', .., y^{(n-1)})$$
$$y(a) = \gamma_0$$
$$\vdots$$
$$y^{(n-1)}(a) = \gamma_{n-1}$$

11.4.1 Integraton Methods

11.4.1.1 Integrating Factor

Given

$$y'(x) + p(x)y(x) = q(x)$$

Multiplying by

$$y'(x)e^{\int pdx} + p(x)e^{\int pdx}y(x) = q(x)e^{\int pdx}$$

Integrating both sides is used with reverse product rule

$$y(x)e^{\int pdx} = \int q(x)e^{\int pdx}dx + c_1$$

11.4.1.2 Variation of Parameters

Given

$$y'' + q(t)y' + r(t)y = g(t)$$

We find the solutions to the associated homogenous equation ($y'' + q(t)y' + r(t)y = 0$)

$$y_c = (t) = c_1y_1(t) + c_2y_2(t)$$

And we want to find a particular solution to $y'' + q(t)y' + r(t)y = g(t)$ in the form

$$y_p = (t) = u_1(t)y_1(t) + u_2(t)y_2(t)$$

We let

$$u_1'(t)y_1(t) + u_2'(t)y_2(t) = 0 \qquad \text{Condition 1}$$

and so

$$y_p' = (t) = u_1'(t)y_1(t) + u_1(t)y_1'(t) + u_2'(t)y_2(t) + u_2(t)y_2'(t)$$
$$y_p' = u_1(t)y_1'(t) + u_2(t)y_2'(t)$$

Differentiating

$$y_p'' = (t) = u_1'(t)y_1'(t) + u_1(t)y_1''(t) + u_2'(t)y_2'(t) + u_2(t)y_2''(t)$$

Plugging this into the original equation and cancelling gives

$$u_1'y_1' + u_2'y_2' = g(t) \qquad \text{Condition 2}$$

Solving the system given by the two conditions gives

$$u_1' = -\frac{y_2g(t)}{y_1y_2' - y_2y_1'} \qquad u_2' = \frac{y_1g(t)}{y_1y_2' - y_2y_1'}$$

so

$$u_1 = -\int \frac{y_2g(t)}{y_1y_2' - y_2y_1'}dt \qquad u_2 = \int \frac{y_1g(t)}{y_1y_2' - y_2y_1'}dt$$

And so our particular solution is

$$y_p(t) = -y_1(t)\int \frac{y_2g(x)}{y_1y_2' - y_2y_1'}dx + y_2(t)\int \frac{y_1g(x)}{y_1y_2' - y_2y_1'}dx$$

So our general solution is

$$y = y_c(t) + y_p(t)$$

12 Integral Equations

For a domain $\Omega \subseteq \mathbb{R}^n$, we have a linear integral operator

$$L\{u\}(x) = \int_{\Omega} K(x,y)u(y)dy$$

with a kernel k defined for each $(x,y) \in \Omega$, i.e. on $\Omega \times \Omega$
L has associated IEs typically of the form

$$L\{u\}(x) = \lambda u(x) + g(x)$$
$$(\lambda - L)u(x) = g(x)$$

12.0.2 Solutions

As you solve the IE, any time you encounter

$$(\lambda - L)u(x) = g(x)$$

In the case of $\lambda = L$,

- $g(x) = 0$, then there are an infinite number of solutions
- $g(x) \neq 0$, then there are no solutions

In the case of $\lambda \neq L$,

$$u(x) = \frac{g(x)}{(\lambda - L)}$$

And you can continue to solve the IE

12.1 Fredholm IE of the First Kind; $\lambda = 0$

$$L\{u\}(x) = \int_{\Omega} K(x,y)u(y)dy = g(x)$$

12.2 Fredholm IE of the Second Kind; $\lambda \leq 0$

$$L\{u\}(x) = \int_{\Omega} K(x,y)u(y)dy = \lambda u(x) + g(x)$$

Note that you can often convert between IEs and other DEs.

12.2.0.1 Solutions via Fixed Point Method Fredholm IEs can be recast as fixed point problem

$$Tu(x) = u(x) \qquad \text{with} \quad Tu(x) = g(x) + \int_a^b k(x,y)u(y)dy$$

If $c = \sup_{a \leq x \leq b} \left(\int_a^b |k(x,y)|\, dy \right) < 1$, we know

$$\|Tu_1 - Tu_2\|_\infty = \sup_{a \leq x \leq b} \left| \left(\int_a^b k(x,y)(u_1(y) - u_2(y))dy \right) \right| \leq c\|u_1 - u_2\|_\infty$$

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so we know u is a unique solution and it can be obtained via the limit

$$u = \lim_{n\to\infty} T^n u_0$$

We express the inverse of the operator as the **Neumann series**

$$u = (I - L)^{-1} = \sum_{n=0}^{\infty} L^n$$

Using only partial sums is called the **Born approximation**

$$\left(\lambda I + L + L^2 + ...\right) u(x) = u(x) + \int_a^b k(x,y)u(y)dy + \int_a^b \int_a^b k(x,y)k(y,z)u(z)dydz + ...$$

The inverse of a differential operator A is an integral operator whose kernel is the Green's Function of A.

12.3 Special Kernels

Symmetric Kernels have the property $k(y,x) = \overline{k(y,x)}$
Convolution Kernels have the form $k(x,y) = k(x-y)$
Singular Kernels have the property $k(x,y) \to \infty$ for certain x, y (often $x = y$)

12.4 Volterra Equations

Equations s.t. $k(x,y) = 0$ for $y > x$ have the form

$$\int_a^x K(x,y)u(y)dy = \lambda u(x) + g(x)$$

12.4.1 Volterra IE of the First Kind $(\lambda = 0)$

$$\int_a^x K(x,y)u(y)dy = g(x)$$

by FTC,

$$u(x) = g'(x)$$

Solvability condition: $g(a) = 0$ and $g(x)$ is differentiable on Ω.

12.4.2 Volterra IE of the Second Kind $(\lambda \neq 0)$

$$\int_a^x K(x,y)u(y)dy = \lambda u(x) + g(x)$$

by FTC,

$$u(x) = \lambda u'(x) + g'(x)$$

which can be solved using ODE methods.
Solvability conditions: $u(a) = \frac{-g(a)}{\lambda}$, u' exists, g' exists

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12.5 Famous IEs

Fourier Transform has the form

$$Lu(x) = \int_{\mathbb{R}^n} e^{ixy} u(y) dy$$

Laplace Transform has the form

$$Lu(x) = \int_0^\infty e^{-xy} u(y) dy$$

Hilbert Transform has the form

$$Lu(x) = \frac{1}{\pi} \int_{-\infty}^\infty \frac{u(y)}{x-y} dy$$

Abel Integral Operator has the form

$$Lu(x) = \int_0^x \frac{u(y)}{\sqrt{x-y}} dy$$

13 Partial Differential Equations

13.1 Multiindex notation

A multiindex $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n) \in \mathbb{Z}_+^n$ denotes

$$D^\alpha = \frac{\partial^{|\alpha|}}{(\partial x_1)^{\alpha_1} (\partial x_2)^{\alpha_2} ... (\partial x_n)^{\alpha_n}}$$

$$x^\alpha = \prod_{i=1}^n x_i^{\alpha_i}$$

$$|x| = \sqrt{x_1^2 + ... + x_n^2} = \sqrt{\sum_{i=1}^n x_i^2}$$

$$x^\alpha f = \{y : y = x^\alpha f(x)\}$$

For multiindices α and β,

- $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n$
- $\alpha! \prod_{i=1}^n \alpha_i!$
- $\alpha + \beta = (\alpha_1 + \beta_1, ..., \alpha_n + \beta_n)$
- $\alpha \geq \beta$ if and only if $\alpha_i \geq \beta_i$ for $i = 1, ..., n$

13.2 Senses of Solution

Let $Lu = \sum_{|\alpha| \leq M} a_\alpha(x) D^\alpha u$ were $a_\alpha(x) \in C^\infty(\Omega)$ so $LT \in \mathcal{D}'(\Omega)$ for $T \in \mathcal{D}'(\Omega)$

Example: If $Lu = u_{tt} - u_{xx}$ then the general solution is $u(x,t) = F(x+t) + G(x-t)$

13.2.0.1 Classical Solution

$Lu = f$ in a **classical sense** if $u \in C^M(\Omega)$ and $u'(x) = f(x)\ \forall\ x \in \Omega$ Example: We require $F, G \in C^2(\Omega)$

13.2.0.2 Weak Solution

$Lu = f$ in a **weak sense** if $u \in L^1_{\text{loc}}$ and $Lu = f$ in $\mathcal{D}'$ sense.
Classical solutons are always also weak solutions
Example: We require $F, G \in L^1(\Omega)$

13.2.0.3 Distributional Solution

$Lu = f$ in a **distributional sense** if $u \in \mathcal{D}'$ and $Lu = f$ in $\mathcal{D}'$ sense.
Classical solutions and weak solutions are always also distributional solutions
Example: We require $F, G = \delta \in \mathcal{D}'(\Omega)$

13.3 Distributional Solutions

13.3.1 Fundamental Solutions

We say $E \in \mathcal{D}'(\mathbb{R}^N)$ is a **fundamental solution** of

$$L = \sum_{|\alpha| \leq M} a_\alpha(x) D^\alpha \qquad a_\alpha(x) \in C^\infty$$

if $LE = \delta$.

- Usually fundamental solutions can be found by taking the FT of $LE = \delta$ which is

$$\left(\sum_{|\alpha| \leq M} D^\alpha E \right)^\wedge = (\delta)^\wedge \to \sum_{|\alpha| \leq M} (ik)^\alpha \hat{E}(k) = \frac{1}{(2\pi)^{\frac{N}{2}}}$$

- Fundamental solutons are never unique since you can add solutions to the homogenous equation H and the equation $L(E + H) = f$ will still be valid

13.3.1.1 Translational Invariance If E is a fundamental solution to and $a_\alpha(x) = a_\alpha$ (constant coefficients) and the domain of E is all of $\mathbb{R}^N$, then $u = E * f$ is a solution formula for $Lu = f$, and E is **translationally invariant** which means it commutes with translation. That is, given $L : C_0^\infty(\mathbb{R}^N) \to C^\infty(\mathbb{R}^N)$,

$$\tau_h L\phi = L\tau_h\phi \qquad \forall\ \phi \in C_0^\infty,\ \forall\ h \in \mathbb{R}^N$$

Reason:

$$L\tau_h\phi = T(\tau_x(\check{\tau_h\phi})) = T(\tau_{x-h}\check{\phi}) = (T * \phi)(x - h) = L\phi(x - h) = \tau_h L\phi$$

13.3.1.1.1 Convergence If $L : C_0^\infty(\mathbb{R}^N) \to C^\infty(\mathbb{R}^N)$ such that L is translation invariant and continuous then there exists a unique $T \in \mathcal{D}'(\mathbb{R}^N)$ such that $L\phi = T * \phi$

13.3.2 Green's Function

Given a PDE, a fundamental solution that satisfies the boundary conditions is known as a **Green's Function** for that BVP. Given

$$L = \sum_{|\alpha| \leq M} a_\alpha(x) D^\alpha \qquad a_\alpha(x) \in C^\infty$$

Then $E = E(x,y)$ is a fundamental solution of $L_x E(x,y) = \delta(x-y)$ if

$$Lu(x) = \int_{\mathbb{R}^N} L_x E(x,y) f(y) dy = \int_{\mathbb{R}^N} \delta(x-y) f(y) dy = f(x)$$

$E(x,y)$ can be of the form $E(x-y)$ but need not be in general.

13.3.2.1 Example

$$K(x,y) = \begin{cases} y(x-1) & 0 \leq y \leq x \leq 1 \\ x(y-1) & 0 \leq x \leq y \leq 1 \end{cases}$$

IF $Lu = u''$ with $u(0) = u(1) = 0$ then $L_x K = \delta(x-y)$. If $E(x) = \frac{|x|}{2}$, then

$$LE = \delta(x) \qquad \text{and } LE(x-y) = \delta(x-y)$$

so they should differ by a solution to the homogenous equation. If $H(x,y) = K(x,y) - E(x-y)$ then

$$H(x,y) = \begin{cases} y(x-1) - \frac{1}{2}(x-y) & 0 \leq y \leq x \leq 1 \\ x(y-1) - \frac{1}{2}(y-x) & 0 \leq x \leq y \leq 1 \end{cases} = \begin{cases} \left(y - \frac{1}{2}\right) x - \frac{1}{2} y & 0 \leq x, y \leq 1 \end{cases}$$

so $L_x H = 0 \; \forall \; y$. If $u(x) = \int_0^1 K(x,y) f(y) dy$ then $u'' = f$ but also

$$u(0) = \int_0^1 K(0,y) f(y) dy = 0 \text{ and } u(1) = \int_0^1 K(1,y) f(y) dy = 0$$

so we have solved the BVP using the Green's function $K(x,y)$

13.3.2.2 Example:

For $N = 3$, $\Delta \left(\frac{-1}{4\pi|x|} \right) = \delta$.
Further, if $f \in C_0^\infty(\mathbb{R}^N)$ then let $u = E * f$ and so

$$Lu = L(E * f) = LE * f = \delta * f = f$$

Example: Let $E(x) = \frac{-1}{4\pi|x|}$ so $\Delta E(x) = \delta$ and so if $\Delta u = f$ then

$$u = (E * f)(x) = \frac{-1}{4\pi} \int_{\mathbb{R}^3} \frac{f(y)}{|x-y|} dy$$

which can be check to also be a classical solution since $E * f \in C^\infty$.

13.3.2.3 Example: In $\Omega = \mathbb{R}^2$,

$$Lu = u_{tt} - u_{xx} \qquad u(x,0) = h(x) \qquad u_t(x,0) = g(x)$$

Let $E(x,t) = \frac{1}{2}H\left(()\, t - |x|\right)$, where H is the Heaviside function. So

$$E(x,t) = \begin{cases} 0 & t < |x| \\ \frac{1}{2} & t > |x| \end{cases} \qquad \text{(Indicator function for the forward light cone)}$$

If $f \in C_0^\infty$, $u = E * f$ is a soluion of $Lu = f$.

$$u(x,t) = E * f = \int_{\mathbb{R}^2} E(x-y, t-s)dyds = \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(t-s-|x-y|)f(y,s)dyds$$

So

$$u(x,t) = \frac{1}{2}\int_{-\infty}^{t}\int_{x+t-s}^{x+s-t} f(y,s)dyds \qquad \text{(Indicator function for backward light cone)}$$

If we assume $f(x,t) = 0$ for $t < 0$, this becomes

$$u(x,t) = \frac{1}{2}\int_{0}^{t}\int_{x+t-s}^{x+s-t} f(y,s)dyds \qquad \text{(Integral of cone with vertex at } x, y \text{)}$$

Note $u(x,0) = u_t(x,0) = 0$

Also $(E \underset{(x)}{*} g) = \int_{-\infty}^{\infty} E(x-y,t)g(y)dy = \frac{1}{2}\int_{x-t}^{x+t} g(y)dy$ (Part of D'Alembert's Solution Formula)
and $\frac{\partial}{\partial t}(E \underset{(x)}{*} h)(x,t) = \frac{1}{2}(h(x+t) + h(x-t))$
so solution of PDE is

$$u(x,t) = (E * f)(x,y) + (E \underset{(x)}{*} g) + \frac{\partial}{\partial t}(E \underset{(x)}{*} h)(x,t)$$

13.3.3 Symbol of Operator L

Define the **Symbol of** L to be

$$P(k) = (2\pi)^{\frac{N}{2}} \sum_{|\alpha| \le M} (ik)^\alpha$$

with $P(k)\hat{E}(k) = 1$.

- The coefficients of the operator L yield a unique symbol and vice versa
- $P_m(k) = (2\pi)^{\frac{N}{2}} \sum\limits_{|\alpha| = M} (ik)^\alpha$ is the **principal symbol of** L and excludes lower order terms
  L is elliptic if $P_m(k) = 0$ if and only if $k = 0$ for $k \in \mathbb{R}^N$: that is $P_m(k)$ has no real roots except 0
- A fundamental solution should satisfy (assuming $P(k) \neq 0$)

  $$\hat{E}(k) = \frac{1}{P(k)} \implies E(x) = \frac{1}{(2\pi)^{\frac{N}{2}}}\int_{\mathbb{R}^N} \frac{e^{ikx}}{P(k)}dk$$

  But this is not clear if $\frac{1}{p} \notin \mathcal{S}$
- **Malgrange Ehrenpreis Theorem** If $L \neq 0$ everywere then a fundamental solution to $Lu = f$ exists.
- **Theorem** If L is elliptic, $f \in C^\infty(\mathbb{R}^N)$, $u \in \mathcal{D}'(\mathbb{R}^N)$ and $Lu = f$, then $u \in C^\infty(\mathbb{R}^N)$

13.3.3.1 Example: $L = \Delta$ then $P(k) = -(2\pi)^{\frac{N}{2}} \sum_{j=1}^{N} k_j^2$ so $P(k) = 0$ when $k_1 = k_2 = 0$ so is therefore elliptic

13.3.3.2 Example: $Lu = u_{tt} - u_{xx}$ then $P(k) = 2\pi(k_x^2 - k_t^2)$ so $P(k) = 0$ when $k_t = \pm k_x$ and is therefore not elliptic.

13.3.4 Duhamel's Principle

Duhamel's principle is a general method for obtaining solutions to inhomogeneous linear evolution equations (Ex: see heat equation) by convolution of a fundamental solution with the inhomogenous term.

13.3.4.1 Regularity of Solutions For $\Delta u = 0$ all solutions are classical, weak, and distributional solutions.
Thus, the regularity of the solution depends on the PDE.

13.4 Characteristic Curves

of a PDE are the lines on which the solution is constant.

13.5 Types of PDEs

13.5.1 Linear PDEs

A PDE of order m is **linear** if it can be expressed as

$$Lu(x) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha u(x) = f(x)$$

An order k linear PDE over $\mathbb{R}^n$ will have $\begin{pmatrix} n+k-1 \\ k-1 \end{pmatrix}$ distinct terms

13.5.2 Semilinear PDEs

A PDE of order m is **semilinear** if it can be expressed as

$$\sum_{|\alpha|=m} a_\alpha(x) D^\alpha u(x) + a_0 \left(D^{k-1}u, ..., Du, u, x\right) = f(x)$$

13.5.3 Quasilinear PDEs

A PDE of order m is **quasilinear** if it can be expressed as

$$\sum_{|\alpha|=m} a_\alpha \left(D^\alpha u(x), ..., Du, u, x\right) + a_0 \left(D^{k-1}u, ..., Du, u, x\right) = f(x)$$

13.5.4 Nonlinear PDEs

A PDE of order m is **nonlinear** if it depends nonlinearly upon the highest order derivatives.
PDEs are classified by order and type (elliptic, hyperbolic, parabolic)

13.6 First Order Equations $(m = 1)$

13.6.1 Linear Equations

13.6.1.1 Constant Coefficients

$$\vec{\nabla} u \cdot \vec{\theta} = \frac{\partial u}{\partial \vec{\theta}} = a u_x + b u_y = 0$$

A directional derivative vanishes for all points (x, y).
A characteristic curve though point $(x_0, 0)$ obeys $ay = b(x - x_0)$, so $x_0 = \frac{bx-ay}{b}$
If $u(x, 0) = f(x)$, then $u(x, y) = u(x_0, 0) = f(x_0) = f\left(\frac{bx-ay}{b}\right)$

13.6.1.2 Nonconstant Coefficients

$$a(x, y)u_x + b(x, y)u_y = 0$$

A directional derivative at each point vanishes.
A characteristic curve parameterized by $(x(t), y(t))$ satisfying the ODE system

$$\frac{dx}{dt} = a(x, y) \qquad \frac{dy}{dt} = b(x, y)$$

$$\frac{d}{dt} u(x(t), y(t)) = u_x \frac{dx}{dt} + u_y \frac{dy}{dt} = a u_x + b u_y = 0$$

u is a solution of the PDE $\implies$ the curves are characteristics $\implies$ u is constant along these curves
Let $f(x) = u(x, 0)$, if (x, y) and $(x_0, 0)$ lie on the same characteristic then $u(x, y) = u(x_0, 0) = f(x_0)$

13.6.1.3 Cauchy Problem

A solution u of the PDE is specified by $f(s)$ on a curve γ

- γ can be defined parametrically by $x = \phi(s)$ and $y = \psi(s)$ then $u(\phi(s), \psi(s)) = f(s)$
- γ can be nowhere characteristic nor can it be tangent to the characteristic direction $(x', y') = (a(x, y), b(x, y))$ because if $f = 0$ anywhere, then f touches a characteritic, which means it is a characteristic
- Solution: Existence isn't guaranteed. If γ is nowhere tangent to (a, b) and $(a, b) \neq (0, 0)$ then a unique solution exists locally (near γ)

Solve the ODE for $(x(t, s), y(t, s))$, a fixed s which is a characteristic through $(\phi(s), \psi(s))$

$$\frac{dx}{dt} = a(x, y) \qquad x(0, s) = \phi(s)$$
$$\frac{dy}{dt} = b(x, y) \qquad y(0, s) = \psi(s)$$

Perform a transformation $(x(t, s), y(t, s)) \to (t(x, y), s(x, y))$
Which means $u(0, s) = f(s)$ becomes $u(x, y) = f(s(x, y))$
Solution is valid at a point (x, y) if it can be conncted to γ by a characteristic curve

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13.6.2 Semilinear Equations

$$a(x,y)u_x + b(x,y)u_y = c(x,y,u)$$

So we have an ODE system

$$\frac{dx}{dt} = a(x,y) \qquad \frac{dy}{dt} = b(x,y)$$

$$\frac{d}{dt}\left(u(x(t),y(t))\right) = a(x,y)u_x + b(x,y)u_y = c(x,y,u) = c(x(t),y(t),u(x(t),y(t)))$$

Parameterize γ (see Cauchy Problem), solve system to get $t(x,y)$ and $s(x,y)$ and solve

$$u' = c(x,y,u) \qquad u(s,0) = f(s)$$

using previously obtained $x(t,s)$ and $s(t,s)$ to set $u(t,s)$ and substitute to get $u = u(x,y)$

13.7 Second Order Equations $(m = 2)$

Typically made easier using a linear transformation of coordinates

Let $\eta = \phi(x,y)$, $\xi = \psi(x,y)$ be an invertible transformation $(x,y) \to (\eta,\xi)$. We need to find u_x and u_{xx} in terms of the new system. Using chain rule,

$$\begin{aligned}
u_x &= u_\eta \phi_x + u_\xi \psi_x \\
u_{xx} &= u_{\eta\eta}\phi_x^2 + u_\eta + u_{\xi\eta}\psi_x\phi_x + u_\xi + u_{\eta\xi}\phi_x\psi_x + u_\eta + u_{\xi\xi}\psi_x^2 + u_\xi \\
u_{xx} &= u_{\eta\eta}\phi_x^2 + 2u_{\xi\eta}\psi_x\phi_x + u_{\xi\xi}\psi_x^2 + 2u_\eta + 2u_\xi
\end{aligned}$$

Now we must find u_y and u_{yy} in the new system.

$$\begin{aligned}
u_y &= u_\eta \phi_y + u_\xi \psi_y \\
u_{yy} &= u_{\eta\eta}\phi_y^2 + u_\eta + u_{\xi\eta}\psi_y\phi_y + u_\xi + u_{\eta\xi}\phi_y\psi_y + u_\eta + u_{\xi\xi}\psi_y^2 + u_\xi \\
u_{yy} &= u_{\eta\eta}\phi_y^2 + 2u_{\xi\eta}\psi_y\phi_y + u_{\xi\xi}\psi_y^2 + 2u_\eta + 2u_\xi
\end{aligned}$$

Now we must compute the mixed derivative u_{xy} based on u_x

$$\begin{aligned}
u_{xy} &= u_{\eta\eta}\phi_y\phi_x + u_\eta + u_{\xi\eta}\phi_y\psi_x + u\eta + u_{\eta\xi}\psi_y\phi_x + u_\xi + u_{\xi\xi}\psi_y\psi_x + u_\xi \\
u_{xy} &= u_{\eta\eta}\phi_y\phi_x + u_{\xi\eta}\left(\phi_y\psi_x + \psi_y\phi_x\right) + u_{\xi\xi}\psi_y\psi_x + 2u_\eta + 2u_\xi
\end{aligned}$$

Adding everything together, a second order equation in the original coordinates

$$au_{xx} + 2bu_{xy} + cu_{yy} + du_x + fu_y + hu = g(x,y)$$

becomes in the new coordinate system:

$$Au_{\eta\eta} + Bu_{\eta\xi} + Cu_{\xi\xi} + Du_\eta + Fu_\xi + Hu = g(\eta,\xi)$$

where

$$\begin{aligned}
A(\eta,\xi) &= a\phi_x^2 + b\phi_x\phi_y + c\phi_y^2 \\
B(\eta,\xi) &= 2a\phi_x\psi_x + b(\phi_y\psi_x + \phi_x\psi_y) + 2c\psi_y\phi_y \\
C(\eta,\xi) &= a\psi_x^2 + b\psi_x\psi_y + c\psi_y^2 \\
D(\eta,\xi) &= d\phi_x + f\phi_y + 2a + 2b + 2c \\
F(\eta,\xi) &= d\psi_x + f\psi_y + 2a + 2b + 2c \\
H(\eta,\xi) &= h
\end{aligned}$$

If the transformation $\eta = \phi(x, y)$, $\xi = \psi(x, y)$ is linear, it can be expressed in the form

$$\begin{pmatrix} \eta \\ \xi \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

with $\alpha\delta - \beta\gamma \neq 0$ to ensure invertibility. This means $\phi_x = \alpha$, $\phi_y = \beta$, $\psi_x = \gamma$, $\psi_y = \delta$, so

$$\begin{aligned}
A(\eta, \xi) =& a\alpha^2 + b\alpha\beta + c\beta^2 \\
B(\eta, \xi) =& 2a\alpha\gamma + b(\beta\gamma + \alpha\delta) + 2c\beta\delta \\
C(\eta, \xi) =& a\gamma^2 + b\gamma\delta + c\delta^2 \\
D(\eta, \xi) =& d\alpha + f\beta + 2a + 2b + 2c \\
F(\eta, \xi) =& d\gamma + f\delta + 2a + 2b + 2c \\
H(\eta, \xi) =& h
\end{aligned}$$

13.7.1 Types of Equations

13.7.1.1 Parabolic Equations $b^2 - ac = 0$ With equations of this type, one can choose $b(\eta, \xi) = c(\eta, \xi) = 0$ to obtain

$$u_{\eta\eta} = 0$$

or equivalently $b(\eta, \xi) = a(\eta, \xi) = 0$ to get

$$u_{\xi\xi} = 0$$

13.7.1.2 Hyperbolic Equations $b^2 - ac > 0$ With equations of this type, one can choose $a(\eta, \xi) = c(\eta, \xi) = 0$ to obtain

$$u_{\xi\eta} + D^{|1|}u = 0$$

or equivalently $b(\eta, \xi) = 0$ to get

$$u_{\xi\xi} - u_{\eta\eta} + D^{|1|}u = 0$$

Where $D^{|1|}u$ is the lower order terms
If $D^{|1|}u = 0$, the solutions to these equations are of the form

$$u(\eta, \xi) = \Phi(\eta) + \Psi(\xi)$$

for some functions $\Phi(\eta), \Psi(\xi)$

13.7.1.3 Elliptic Equations $b^2 - ac < 0$ With equations of this type, one can choose $a(\eta, \xi) = c(\eta, \xi) = 0$ to obtain

$$u_{\xi\eta} + D^{|1|}u = 0$$

or equivalently $b(\eta, \xi) = 0$ to get

$$u_{\xi\xi} - u_{\eta\eta} + D^{|1|}u = 0$$

Where $D^{|1|}u$ is the lower order terms

13.7.2 Separation of Variables

Seeking separable solutions $u(x_1, ..., x_n) = X_1(x_1) \times ... \times X_n(x_n)$ works for many kinds of PDEs in multiple dimensions. $(0 \leq \theta < 2\pi, r \geq 0)$

13.7.3 Boundary Conditions

For a well-posed mth order PDE, you will have up to m side conditions, usually in the form of

$$F(u) = 0 \text{ on } \partial\Omega$$

Boundary conditions are said to be **homogenous** if they are closed under linear combinations, such as

$$u = 0 \text{ on } \partial\Omega$$

The most common boundary conditions are:

13.7.3.1 Dirichlet Conditions (First Type)

$$u = g \text{ on } \partial\Omega$$

13.7.3.2 Neumann (Second Type)

$$\frac{\partial u}{\partial n} = \nabla u \cdot \vec{\mathbf{n}} = g \text{ on } \partial\Omega$$

where $\vec{\mathbf{n}}$ is the unit outward normal

13.7.3.3 Robin (Third Type)

$$\frac{\partial u}{\partial n} + \sigma u = \nabla u \cdot \vec{\mathbf{n}} + \sigma u = g \text{ on } \partial\Omega$$

13.7.4 Common Well-Posed Problems

The following equations are well-posed and have classic solutions

13.7.4.1 Poisson's Equation

$$\Delta u = f \qquad \text{(Poisson's Equation)}$$

$$\Delta u = 0 \qquad \text{(Laplace's Equation)}$$

Solutions are harmonic functions, typically with Dirichlet, Neumann, or Robin conditions

$$u_{xx} + u_{yy} + u_{zz} = f \qquad \text{(rectangular coordinates)}$$

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = f \qquad \text{(polar coordinates)}$$

Requires product solution families to be 2π periodic in θ:

$$u_n(\theta, r) = \begin{cases} c_1 r^n \cos(n\theta) + c_2 r^n \sin(n\theta) + c_3 r^{-n}\cos(n\theta) + c_4 r^{-n}\sin(n\theta) & n = 1, 2, 3... \\ c_1 + c_2 log(r) & n = 0 \end{cases}$$

Coefficients are found using boundary conditions, initial conditions, and Fourier theory

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^2 u}{\partial z^2} = 0 \qquad \text{(cylindrical coordinates)}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2 sin(\theta)}\frac{\partial^2}{\partial \theta^2}\left(\sin(\theta)\frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2\sin^2(\theta)}\frac{\partial^2 u}{\partial \phi^2} = 0 \qquad \text{(spherical coordinates)}$$

13.7.4.1.1 Using Distributions

$u(x) = \frac{-1}{4\pi|x|}$, note $u \in L^1_{\text{loc}}(\mathbb{R}^3)$ since

$$\int_{\mathbb{R}} |u(x)|\, dx = \int_0^R \int_0^{2\pi} \int_{-\pi}^{\pi} \frac{1}{4\pi r} r^2 \sin(\theta) d\phi dr d\theta = \frac{C}{2} R^2$$

Claim: $\Delta u = \delta$. Reason:

$$u(\Delta\phi) = \int u\Delta\phi = \lim_{\epsilon \to 0} \int_{\epsilon < |x| < R} u\Delta\phi dx$$

Since ϕ has compact support and so for some R $\phi(|x| > R) = 0$. By Green's Identity this becomes

$$\lim_{\epsilon \to 0} \int_{\epsilon < |x| < R} u\Delta\phi dx = \lim_{\epsilon \to 0} \left(\int_{\epsilon < |x| < R} \phi(x)\Delta u dx + \int_{|x| = \epsilon} \phi \frac{\partial u}{\partial n} - u \frac{\partial \phi}{\partial n} ds \right)$$

Note that $\Delta u \propto \Delta \frac{1}{r} = 0$, and

$$\left| \int_{|x|=\epsilon} u \frac{\partial \phi}{\partial n} ds \right| \leq \frac{1}{4\pi\epsilon} \int_{|x|=\epsilon} \left| \frac{\partial \phi}{\partial n} \right| ds = \frac{1}{4\pi\epsilon} \max\left(\left| \frac{\partial \phi}{\partial n} \right| \right) 4\pi\epsilon^2 \leq C\epsilon \to 0$$

since the $4\pi\epsilon^2$ is the surface area of a sphere. We also see that since $\frac{\partial u}{\partial n} = -\frac{\partial u}{\partial r}$,

$$\int_{|x|=\epsilon} \phi \frac{\partial u}{\partial n} ds = -\int_{|x|=\epsilon} \phi \frac{\partial u}{\partial r} ds = -\int_{|x|=\epsilon} \phi \frac{1}{4\pi\epsilon^2} ds = \frac{1}{4\pi\epsilon^2} \int_{|x|=\epsilon} \phi ds$$

which is the average of ϕ over the sphere $|x| = \epsilon$ which must tend to the value at the center. That is, as $\epsilon \to 0$, $\frac{1}{4\pi\epsilon^2} \int_{|x|=\epsilon} \phi ds \to \phi(0)$, so

$$\Delta u(\phi) = u(\Delta\phi) = \phi(0) = \delta(\phi)$$

In higher dimensions,

$$u(x) = \begin{cases} \frac{-1}{2\pi} \log(x) & N = 2 \\ \frac{1}{A_N |x|^{N-2}} & N \geq 3 \end{cases}$$

Where A_N is the surface area of the N-Sphere.

13.7.4.2 Fundamental Solution

$$E(x) = \begin{cases} \frac{|x|}{2} & N = 1 \\ \frac{1}{2\pi} \log|x| & N = 2 \\ \frac{C_N}{|x|^{N-2}} & N \geq 3 \end{cases}$$

Where $C_N = \frac{???}{N(N-1)a_n}$ and a_n is the volume of $B_1(0)$ in $\mathbb{R}^N$ If Δ is replaced by the positive operator $-\Delta$, the the FS is $E_-(x) = -E(x)$

13.7.4.3 Wave Equation

$$u_{tt} - c^2 u_{xx} = 0$$

Typically with Dirichlet, Neumann, or Robin conditions in space and some

$$\begin{aligned} u(\vec{\mathbf{x}}, 0) &= f(\vec{\mathbf{x}}) \\ u_t(\vec{\mathbf{x}}, 0) &= g(\vec{\mathbf{x}}) \qquad x \in \Omega \end{aligned}$$

$\eta = x + ct$ and $\xi = x - ct$, then
$$u(x,t) = \Phi(x+ct) + \Psi(x-ct)$$
If initial conditions $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$ are given,
then $f(x) = \Phi(x) + \Psi(x)$ and $g(x) = \Phi'(x) + \Psi'(x)$ and so $\int_0^x g(y)dy = \Phi(x) + \Psi(x) + C$
Combining, $F(x) = \frac{1}{2}\left(f(x) + \int_0^x g(y)dy + C\right)$ giving us
D'Alembert's Solution Formula
$$u(x,t) = \frac{1}{2}\left(f(x+ct) + f(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)dy$$

13.7.4.4 Fundamental Solution

$$E(x,t) = \begin{cases} \frac{1}{2}H(t-|x|) & N=1 \\ \frac{1}{2}\frac{H(t-|x|)}{\sqrt{t^2-|x|^2}} & N=2 \\ \frac{\delta(t-|x|)}{4\pi|x|} & N=3 \end{cases}$$
Example: For $N = 3$
$$(E * f) = \int_{\mathbb{R}^4} \frac{\delta(s-|y|)}{4\pi|y|} f(x-y,t-s)dsdy = \int_{\mathbb{R}^3, s=|y|} \frac{1}{4\pi|y|} f(x-y,t-|y|)dy$$
Assume $f(x,t) < 0$ for $t < 0$, so
$$(E * f) = \int_{\mathbb{R}^3, |y|<t} \frac{1}{4\pi|y|} f(x-y,t-|y|)dy = \int_{B(x,t)} \frac{1}{4\pi|x-y|} f(y,t-|x-y|)dy$$
So the solution of $u_{tt} - \Delta u = f$, $u(x,0) = h(x)$, $u_t(x,0) = g(x)$ can be shown to be
$$u(x,t) = (E \underset{(x)}{*} g) + \frac{\partial}{\partial t}(E \underset{(x)}{*} h) + (E * f)$$

13.7.4.4.1 Uniqueness Showing the uniqueness of solutions to this problem is equivalent to showing that the homogenous problem
$$u_{tt} - c^2 u_{xx} = 0 \qquad u_t(\vec{\mathbf{x}},0) = u(\vec{\mathbf{x}},0) = 0$$
with homogenous boundary conditions has only a trivial solution.

13.7.4.4.2 Four Point Property for solutions to the wave equation $u_{xx} - u_{tt} = 0$ on $\Omega \subset \mathbb{R}^2$ containing the tilted rectangle with vertices $(x,t), (x+h-k, t+h+k), (x+h, t+h), u(x-k, t+k)$,
$$u(x,t) + u(x+h-k, t+h+k) = u(x+h, t+h) + u(x-k, t+k)$$

13.7.4.4.3 Using Distributions
Let $F \in L^1_{\text{loc}}(\mathbb{R})$, then $u(x,t) = F(x+t)$ is one solution to the wave equation in $\mathcal{D}'(\mathbb{R}^2)$ since
$$T(\phi) = (u_{tt} - u_{xx})(\phi) = u(\phi_{tt} - \phi_{xx}) = \int\int_{\mathbb{R}^2} F(x+t)(\phi_{tt}(x,t) - \phi_{xx}(x,t))dxdt$$
By change of coordinates, $\xi = x+t$, $\eta = x-t$, $\phi_{tt} - \phi_{xx} = -4\phi_{\xi\eta}$, $dxdt = \frac{\partial(x,t)}{\partial(\xi,\eta)} = -\frac{1}{2}d\xi d\eta$
$$= 2\int_{-\infty}^{\infty} F(\eta) \int_{-\infty}^{\infty} \phi_{\xi\eta} d\xi d\eta = 0$$
since ϕ has compact support

13.7.4.5 Heat Equation

$$u_t - \Delta u = 0 \qquad\qquad (\vec{\mathbf{x}}, t) \in \Omega \times [0, \infty)$$

Typically with Dirichlet, Neumann, or Robin conditions in space and some

$$u(\vec{\mathbf{x}}, 0) = f(\vec{\mathbf{x}}), x \in \Omega$$

Look for separable solutions of the form $u(t, x) = \Phi(t)\Psi(x)$ where

$$\Phi(t) = c_1 e^{kt} + c_2 e^{-kt} \qquad \Psi(x) = c_1 sin(kx) + c_2 cos(kx)$$

Solutions will usually involve $k = k_n$ so the set of product solutions is

$$\{\Phi_n(t)\Psi_n(x)\}_0^\infty = \left\{c_1 e^{k_n^2 t} sin(k_n x) + c_2 e^{k_n^2 t} cos(k_n x) + c_3 e^{-k_n^2 t} sin(k_n x) + c_4 e^{-k_n^2 t} sin(k_n x)\right\}_0^\infty$$

Where the $c_1, ..., c_4$ is chosen to satify the boundary and initial conditions using Fourier theory.

13.7.4.5.1 Uniqueness Showing the uniqueness of solutions to this problem is equivalent to showing that the homogenous problem

$$u_t - \Delta u = 0 \qquad u_t(\vec{\mathbf{x}}, 0) = u(\vec{\mathbf{x}}, 0) = 0$$

with homogenous boundary conditions has only a trivial solution.

13.7.4.6 Fundamental Solution

$$E(x, t) = \frac{H(t)}{(2\pi t)^{\frac{N}{2}}} e^{\frac{-|x|^2}{4t}}$$

13.7.4.7 Reason:

13.7.4.7.1 Using FT For $Lu = u_t - u_{xx}$ with $u(x, 0) = f(x)$ taking the partial FT in x yields

$$\hat{u}(k, t) = \frac{1}{(2\pi)^{\frac{N}{2}}} \int_{\mathbb{R}^N} u(x, t) e^{-ikx} dx \qquad \text{and } (u_t)^\wedge = (\hat{u})_t$$

So $(-\Delta u)^\wedge = |k|^2 \hat{u}$ so $\hat{u}_t + |k|^2 \hat{u} = 0$, which is an ODE for $\hat{u}$ for fixed k. Solving,

$$\hat{u}(k, t) = \hat{f} e^{-|k|^2 t} \implies u(x, t) = \frac{1}{(2\pi)^{\frac{N}{2}}} \mathcal{F}^{-1}\left(\left((f * g)\right)^\wedge\right) = \frac{1}{(4\pi t)^{\frac{N}{2}}} \int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{4t}} f(y) dy$$

where $\hat{g}(k) = e^{-|k|^2 t}$ so $g(x) = \frac{e^{-\frac{|x|^2}{4t}}}{(2t)^{\frac{N}{2}}}$. This is valid for $f \in L^p(\mathbb{R}^N)$ for some $1 \leq p \leq \infty$, $t > 0$, and $\lim_{t \to 0^+} u(x, t) = f(x)$ a.e.

13.7.4.7.2 Using FT and Duhamel's Principle By Duhamel's Principle, we can do the same for $Lu = h$. Let $v(x,t,s)$ satisfy $v_t - \Delta v = 0$ $x \in \mathbb{R}^N$, $t > 0$, $v(x,0,s) = h(x,s)$ and let $u(x,t) = \int_0^t v(x,t-s,s)ds$.

$$u_t = v(x,0,t) + \int_0^t v_t(x,t-s,s)ds = h(x,t) + \int_0^t \Delta v(x,t-s,s)ds = h(x,t) + \Delta u$$

So for $u_t - \Delta u = h(x,t)$ for $t > 0$ with $u(x,0) = 0$, we have

$$v(x,t,s) = \frac{1}{(2\pi t)^{\frac{N}{2}}} \int_{\mathbb{R}^N} e^{\frac{-|x-y|^2}{4t}} h(y,s)dy \implies u(x,t) = \int_0^t \int_{\mathbb{R}^N} \frac{1}{(2\pi(t-s))^{\frac{N}{2}}} e^{\frac{-|x-y|^2}{4(t-s)}} h(y,s)dyds = E * h$$

where $E = \frac{H(t)e^{\frac{-|x|^2}{4t}}}{(2\pi t)^{\frac{N}{2}}}$ and $h(x,t) = 0$ for $t < 0$. So our solution is

$$u(x,t) = (E * h)(x,t) + (E \underset{(x)}{*} f)(x,t)$$

So E is the fundamental solution to the Heat Equation. Note: for $t > 0$, $E(x,t) = \left(\frac{1}{\sqrt{t}}\right)^N F\left(\frac{x}{\sqrt{t}}\right)$ if $F(x) = \frac{e^{-\frac{|x|^2}{4}}}{(2\pi)^{\frac{N}{2}}}$. Here $F \geq 0$, $F \in L^1$, and $\int_{\mathbb{R}^N} F(x)dx = 1$. By previous discussion of approximate identities let $k = \frac{1}{\sqrt{t}}$

$$k^N F(kx) \to \delta \text{ as } k \to \infty \implies E(x,t) \to \delta \in \mathcal{D}' \text{ as } t \to 0^+$$

so $(E \underset{(x)}{*} f)(x,t) = E(\cdot,t) * f \to f$ as $t \to 0^+$ (similar to approximate identities)

13.7.4.8 Maximum Principle The solution u is bounded by the extremes of the initial condition and the Dirichlet boundary conditions.

13.7.4.9 Shrodinger Equation

$$u_t - i\Delta u = 0 \qquad\qquad (\vec{\mathbf{x}},t) \in \Omega \times [0,\infty)$$

Typically with Dirichlet, Neumann, or Robin conditions in space and some $u(\vec{\mathbf{x}},0) = f(\vec{\mathbf{x}}), x \in \Omega$

13.7.4.10 Fundamental Solution

$$E(x,t) = \frac{H(t)}{(2\pi t)^{\frac{N}{2}}} e^{i(N-2)\frac{\pi}{4}} e^{\frac{-|x|^2}{4it}}$$

13.7.4.11 Helmholtz Equation

$$\Delta u - k^2 u = 0 \qquad\qquad (\vec{\mathbf{x}},t) \in \Omega \times [0,\infty)$$

13.7.4.12 Fundamental Solution

$$E(x,t) = \begin{cases} \frac{1}{2\pi} K_0(k\,|x|) & N = 2 \\ \frac{-e^{-k|x|}}{4\pi|x|} & N = 3 \end{cases}$$

Where K_i is the ith order Modified Bessel Function.
Note as $k \to 0$, $E(x)$ tends toward the $E(x)$ corresponding to $L = \Delta$

13.7.4.13 Euler-Tricomi Equation

$$u_{xx} - xu_{yy} = 0$$

Hyperbolic in the half plane $x > 0$, parabolic at $x = 0$, elliptic in $x < 0$
Characteristics are along $y \pm \frac{2}{3}x^{\frac{2}{3}} = C$
Particular solutions are

$$\begin{aligned} u =& c_1 xy + c_2 x + c_3 y + c_4 \\ u =& c_1 \left(3y^2 + x^3\right) + c_2 \left(y^3 + x^3 y\right) + c_3 \left(6xy^2 + x^4\right) \end{aligned}$$

13.7.4.14 Biharmonic Oscillator

$$\Delta^2 u = 0 \qquad\qquad (\vec{\mathbf{x}}, t) \in \Omega \times [0, \infty)$$

13.7.4.15 Fundamental Solution ??

13.7.4.16 Klein Gordon Equation

$$Lu = u_{tt} - u_{xx} + u \qquad\qquad (\vec{\mathbf{x}}, t) \in \Omega \times [0, \infty)$$

13.7.4.17 Fundamental Solution

$$E(x,t) = \frac{1}{2} H(t - |x|) J_0(\sqrt{t^2 - x^2})$$

14 Distribution Theory in DE

If $f'(x) = 0$ for $a \leq x \leq b$ then $f(x) = c$ classically If $T' = 0 \in \mathcal{D}'(a,b)$ then $T = c$
Reason: Choose $\phi_0 \in C_0^\infty$ with $\int_a^b \phi_0(x)dx = 1$. If $\phi \in C_0^\infty(a,b)$ let $\psi(x) = \phi(x) - \int_a^b \phi(y)dy\phi_0(x)$. Note $\int_a^b \psi(x)dx = 0$. Let $\zeta = \int_a^x \psi(s)ds$ so $\zeta' = \psi(x)$ since $\zeta \in C^\infty(a,b)$ and further $\zeta \in C_0^\infty$ since $\zeta(a) = \zeta(b) = 0$ and $\zeta' = 0$ for $x < a$ or $x > b$. Then $0 = T'(\zeta) = -T(\zeta') = -T(\psi) = -T(\phi) + \left(\int_a^b \phi y dy\right) T(\phi_0)$ so $T(\phi) = \int_a^b T(\phi_0)\phi(y)dy = \int_a^b c\phi(y)dy = c$ in $\mathcal{D}'$ sense.

14.1 ODEs in $\mathcal{D}'$ sense

$$T' = f \qquad f \in L^1_{\text{loc}}(\mathbb{R})$$

Let $g(x) = \int_a^x f(s)ds$ (antiderivative of f). Claim $g' = f$ in $\mathcal{D}'(a,b)$. Reason:

$$T_g'(\phi) = -T_g(\phi') = -\int_a^b g(x)\phi'(x)dx = -\int_a^b \int_a^x f(s)ds\phi'(x)dx = -\int_a^b f(s) \int_s^b \phi'(x)dxds$$

Using FTC and compact support of ϕ,

$$T_g'(\phi) = -\int_a^b f(s)(\phi(b) - \phi(s))ds = \int_a^b f(s)\phi(s)ds = T_f(\phi)$$

So the general solution of $T' = f$ in $\mathcal{D}'(a,b)$ is

$$T = \int_a^x f(s)ds + C$$