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Part I
Partial Differential Equations

1 Introductory Theory and Notation

1.1 Multiindex notation

A multiindex $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n) \in \mathbb{Z}_+^n$ denotes

$$D^\alpha = \frac{\partial^{|\alpha|}}{(\partial x_1)^{\alpha_1}(\partial x_2)^{\alpha_2}...(\partial x_n)^{\alpha_n}}$$

$$x^\alpha = \prod_{i=1}^{n} x_i^{\alpha_i}$$

$$|x| = \sqrt{x_1^2 + ... + x_n^2} = \sqrt{\sum_{i=1}^{n} x_i^2}$$

$$x^\alpha f = \{y : y = x^\alpha f(x)\}$$

For multiindices α and β,

- $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n$
- $\alpha! \prod_{i=1}^{n} \alpha_i!$
- $\alpha + \beta = (\alpha_1 + \beta_1, ..., \alpha_n + \beta_n)$
- $\alpha \geq \beta$ if and only if $\alpha_i \geq \beta_i$ for $i = 1, ..., n$

1.2 Senses of Solution

Let $Lu = \sum_{|\alpha| \leq M} a_\alpha(x) D^\alpha u$ were $a_\alpha(x) \in C^\infty(\Omega)$ so $LT \in \mathcal{D}'(\Omega)$ for $T \in \mathcal{D}'(\Omega)$
Example: If $Lu = u_{tt} - u_{xx}$ then the general solution is $u(x,t) = F(x+t) + G(x-t)$

1.2.0.1 Classical Solution

$Lu = f$ in a **classical sense** if $u \in C^M(\Omega)$ and $u'(x) = f(x)\ \forall\ x \in \Omega$ Example: We require $F, G \in C^2(\Omega)$

1.2.0.2 Weak Solution

$Lu = f$ in a **weak sense** if $u \in L^1_{\text{loc}}$ and $Lu = f$ in $\mathcal{D}'$ sense.
Classical solutons are always also weak solutions
Example: We require $F, G \in L^1(\Omega)$

1.2.0.3 Distributional Solution

$Lu = f$ in a **distributional sense** if $u \in \mathcal{D}'$ and $Lu = f$ in $\mathcal{D}'$ sense.
Classical solutions and weak solutions are always also distributional solutions
Example: We require $F, G = \delta \in \mathcal{D}'(\Omega)$

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1.3 Distributional Solutions

1.3.1 Fundamental Solutions

We say $E \in \mathcal{D}'(\mathbb{R}^N)$ is a **fundamental solution** of

$$L = \sum_{|\alpha| \le M} a_\alpha(x) D^\alpha \qquad a_\alpha(x) \in C^\infty$$

if $LE = \delta$.

- Usually fundamental solutions can be found by taking the FT of $LE = \delta$ which is

$$\left(\sum_{|\alpha| \le M} D^\alpha E \right)^\wedge = (\delta)^\wedge \to \sum_{|\alpha| \le M} (ik)^\alpha \hat{E}(k) = \frac{1}{(2\pi)^{\frac{N}{2}}}$$

- Fundamental solutons are never unique since you can add solutions to the homogenous equation H and the equation $L(E + H) = f$ will still be valid

1.3.1.1 Translational Invariance If E is a fundamental solution to and $a_\alpha(x) = a_\alpha$ (constant coefficients) and the domain of E is all of $\mathbb{R}^N$, then $u = E * f$ is a solution formula for $Lu = f$, and E is **translationally invariant** which means it commutes with translation. That is, given $L : C_0^\infty(\mathbb{R}^N) \to C^\infty(\mathbb{R}^N)$,

$$\tau_h L\phi = L\tau_h\phi \qquad \forall\ \phi \in C_0^\infty,\ \forall\ h \in \mathbb{R}^N$$

Reason:

$$L\tau_h\phi = T(\tau_x(\check{\tau_h \phi})) = T(\tau_{x-h}\check{\phi}) = (T * \phi)(x-h) = L\phi(x-h) = \tau_h L\phi$$

1.3.1.1.1 Convergence If $L : C_0^\infty(\mathbb{R}^N) \to C^\infty(\mathbb{R}^N)$ such that L is translation invariant and continuous then there exists a unique $T \in \mathcal{D}'(\mathbb{R}^N)$ such that $L\phi = T * \phi$

1.3.2 Green's Function

Given a PDE, a fundamental solution that satisfies the boundary conditions is known as a **Green's Function** for that BVP. Given

$$L = \sum_{|\alpha| \le M} a_\alpha(x) D^\alpha \qquad a_\alpha(x) \in C^\infty$$

Then $E = E(x, y)$ is a fundamental solution of $L_x E(x, y) = \delta(x - y)$ if

$$Lu(x) = \int_{\mathbb{R}^N} L_x E(x, y) f(y) dy = \int_{\mathbb{R}^N} \delta(x - y) f(y) dy = f(x)$$

$E(x, y)$ can be of the form $E(x - y)$ but need not be in general.

1.3.2.1 Example

$$K(x,y) = \begin{cases} y(x-1) & 0 \le y \le x \le 1 \\ x(y-1) & 0 \le x \le y \le 1 \end{cases}$$

IF $Lu = u''$ with $u(0) = u(1) = 0$ then $L_x K = \delta(x-y)$. If $E(x) = \frac{|x|}{2}$, then

$$LE = \delta(x) \qquad \text{and } LE(x-y) = \delta(x-y)$$

so they should differ by a solution to the homogenous equation. If $H(x,y) = K(x,y) - E(x-y)$ then

$$H(x,y) = \begin{cases} y(x-1) - \frac{1}{2}(x-y) & 0 \le y \le x \le 1 \\ x(y-1) - \frac{1}{2}(y-x) & 0 \le x \le y \le 1 \end{cases} = \begin{cases} \left(y - \frac{1}{2}\right) x - \frac{1}{2} y & 0 \le x, y \le 1 \end{cases}$$

so $L_x H = 0 \; \forall \; y$. If $u(x) = \int_0^1 K(x,y) f(y) dy$ then $u'' = f$ but also

$$u(0) = \int_0^1 K(0,y) f(y) dy = 0 \text{ and } u(1) = \int_0^1 K(1,y) f(y) dy = 0$$

so we have solved the BVP using the Green's function $K(x,y)$

1.3.2.2 Example: For $N = 3$, $\Delta\left(\frac{-1}{4\pi|x|}\right) = \delta$.

Further, if $f \in C_0^\infty(\mathbb{R}^N)$ then let $u = E * f$ and so

$$Lu = L(E * f) = LE * f = \delta * f = f$$

Example: Let $E(x) = \frac{-1}{4\pi|x|}$ so $\Delta E(x) = \delta$ and so if $\Delta u = f$ then

$$u = (E * f)(x) = \frac{-1}{4\pi} \int_{\mathbb{R}^3} \frac{f(y)}{|x-y|} dy$$

which can be check to also be a classical solution since $E * f \in C^\infty$.

1.3.2.3 Example: In $\Omega = \mathbb{R}^2$,

$$Lu = u_{tt} - u_{xx} \qquad u(x,0) = h(x) \qquad u_t(x,0) = g(x)$$

Let $E(x,t) = \frac{1}{2} H\left(()\, t - |x|\right)$, where H is the Heaviside function. So

$$E(x,t) = \begin{cases} 0 & t < |x| \\ \frac{1}{2} & t > |x| \end{cases} \qquad \text{(Indicator function for the forward light cone)}$$

If $f \in C_0^\infty$, $u = E * f$ is a soluion of $Lu = f$.

$$u(x,t) = E * f = \int_{\mathbb{R}^2} E(x-y, t-s) dy ds = \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H(t - s - |x-y|) f(y,s) dy ds$$

So

$$u(x,t) = \frac{1}{2} \int_{-\infty}^{t} \int_{x+t-s}^{x+s-t} f(y,s) dy ds \qquad \text{(Indicator function for backward light cone)}$$

If we assume $f(x,t) = 0$ for $t < 0$, this becomes

$$u(x,t) = \frac{1}{2}\int_0^t \int_{x+t-s}^{x+s-t} f(y,s)dyds \qquad \text{(Integral of cone with vertex at } x,y \text{)}$$

Note $u(x,0) = u_t(x,0) = 0$

Also $(E \underset{(x)}{*} g) = \int_{-\infty}^{\infty} E(x-y,t)g(y)dy = \frac{1}{2}\int_{x-t}^{x+t} g(y)dy$ (Part of D'Alembert's Solution Formula)
and $\frac{\partial}{\partial t}(E \underset{(x)}{*} h)(x,t) = \frac{1}{2}(h(x+t) + h(x-t))$
so solution of PDE is

$$u(x,t) = (E * f)(x,y) + (E \underset{(x)}{*} g) + \frac{\partial}{\partial t}(E \underset{(x)}{*} h)(x,t)$$

1.3.3 Symbol of Operator L

Define the **Symbol of** L to be

$$P(k) = (2\pi)^{\frac{N}{2}} \sum_{|\alpha| \leq M} (ik)^\alpha$$

with $P(k)\hat{E}(k) = 1$.

- The coefficients of the operator L yield a unique symbol and vice versa
- $P_m(k) = (2\pi)^{\frac{N}{2}} \sum\limits_{|\alpha| = M} (ik)^\alpha$ is the **principal symbol of** L and excludes lower order terms
  L is elliptic if $P_m(k) = 0$ if and only if $k = 0$ for $k \in \mathbb{R}^N$: that is $P_m(k)$ has no real roots except 0
- A fundamental solution should satisfy (assuming $P(k) \neq 0$)

$$\hat{E}(k) = \frac{1}{P(k)} \implies E(x) = \frac{1}{(2\pi)^{\frac{N}{2}}} \int_{\mathbb{R}^N} \frac{e^{ikx}}{P(k)} dk$$

  But this is not clear if $\frac{1}{p} \notin \mathcal{S}$
- **Malgrange Ehrenpreis Theorem** If $L \neq 0$ everywere then a fundamental solution to $Lu = f$ exists.
- **Theorem** If L is elliptic, $f \in C^\infty(\mathbb{R}^N)$, $u \in \mathcal{D}'(\mathbb{R}^N)$ and $Lu = f$, then $u \in C^\infty(\mathbb{R}^N)$

1.3.3.1 Example: $L = \Delta$ then $P(k) = -(2\pi)^{\frac{N}{2}} \sum\limits_{j=1}^{N} k_j^2$ so $P(k) = 0$ when $k_1 = k_2 = 0$ so is therefore elliptic

1.3.3.2 Example: $Lu = u_{tt} - u_{xx}$ then $P(k) = 2\pi(k_x^2 - k_t^2)$ so $P(k) = 0$ when $k_t = \pm k_x$ and is therefore not elliptic.

1.3.4 Duhamel's Principle

Duhamel's principle is a general method for obtaining solutions to inhomogeneous linear evolution equations (Ex: see heat equation) by convolution of a fundamental solution with the inhomogenous term.

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1.3.4.1 Regularity of Solutions For $\Delta u = 0$ all solutions are classical, weak, and distributional solutions.
Thus, the regularity of the solution depends on the PDE.

1.4 Characteristic Curves

of a PDE are the lines on which the solution is constant.

1.5 Types of PDEs

1.5.1 Linear PDEs

A PDE of order m is **linear** if it can be expressed as

$$Lu(x) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha u(x) = f(x)$$

An order k linear PDE over $\mathbb{R}^n$ will have $\begin{pmatrix} n+k-1 \\ k-1 \end{pmatrix}$ distinct terms

1.5.2 Semilinear PDEs

A PDE of order m is **semilinear** if it can be expressed as

$$\sum_{|\alpha|=m} a_\alpha(x) D^\alpha u(x) + a_0\left(D^{k-1}u, ..., Du, u, x\right) = f(x)$$

1.5.3 Quasilinear PDEs

A PDE of order m is **quasilinear** if it can be expressed as

$$\sum_{|\alpha|=m} a_\alpha\left(D^\alpha u(x), ..., Du, u, x\right) + a_0\left(D^{k-1}u, ..., Du, u, x\right) = f(x)$$

1.5.4 Nonlinear PDEs

A PDE of order m is **nonlinear** if it depends nonlinearly upon the highest order derivatives.
PDEs are classified by order and type (elliptic, hyperbolic, parabolic)

1.6 First Order Equations $(m = 1)$

1.6.1 Linear Equations

1.6.1.1 Constant Coefficients

$$\vec{\nabla} u \cdot \vec{\theta} = \frac{\partial u}{\partial \vec{\theta}} = a u_x + b u_y = 0$$

A directional derivative vanishes for all points (x, y).
A characteristic curve though point $(x_0, 0)$ obeys $ay = b(x - x_0)$, so $x_0 = \frac{bx - ay}{b}$
If $u(x, 0) = f(x)$, then $u(x, y) = u(x_0, 0) = f(x_0) = f\left(\frac{bx-ay}{b}\right)$

1.6.1.2 Nonconstant Coefficients

$$a(x, y)u_x + b(x, y)u_y = 0$$

A directional derivative at each point vanishes.
A characteristic curve parameterized by $(x(t), y(t))$ satisfying the ODE system

$$\frac{dx}{dt} = a(x, y) \qquad \frac{dy}{dt} = b(x, y)$$

$$\frac{d}{dt}u(x(t), y(t)) = u_x\frac{dx}{dt} + u_y\frac{dy}{dt} = au_x + bu_y = 0$$

u is a solution of the PDE $\implies$ the curves are characteristics $\implies$ u is constant along these curves
Let $f(x) = u(x, 0)$, if (x, y) and $(x_0, 0)$ lie on the same characteristic then $u(x, y) = u(x_0, 0) = f(x_0)$

1.6.1.3 Cauchy Problem

A solution u of the PDE is specified by $f(s)$ on a curve γ

- γ can be defined parametrically by $x = \phi(s)$ and $y = \psi(s)$ then $u(\phi(s), \psi(s)) = f(s)$
- γ can be nowhere characteristic nor can it be tangent to the characteristic direction $(x', y') = (a(x, y), b(x, y))$ because if $f = 0$ anywhere, then f touches a characteritic, which means it is a characteristic
- Solution: Existence isn't guaranteed. If γ is nowhere tangent to (a, b) and $(a, b) \neq (0, 0)$ then a unique solution exists locally (near γ)

Solve the ODE for $(x(t, s), y(t, s))$, a fixed s which is a characteristic through $(\phi(s), \psi(s))$

$$\frac{dx}{dt} = a(x, y) \qquad x(0, s) = \phi(s)$$
$$\frac{dy}{dt} = b(x, y) \qquad y(0, s) = \psi(s)$$

Perform a transformation $(x(t, s), y(t, s)) \to (t(x, y), s(x, y))$
Which means $u(0, s) = f(s)$ becomes $u(x, y) = f(s(x, y))$
Solution is valid at a point (x, y) if it can be conncted to γ by a characteristic curve

1.6.2 Semilinear Equations

$$a(x, y)u_x + b(x, y)u_y = c(x, y, u)$$

So we have an ODE system

$$\frac{dx}{dt} = a(x, y) \qquad \frac{dy}{dt} = b(x, y)$$

$$\frac{d}{dt}\left(u(x(t), y(t))\right) = a(x, y)u_x + b(x, y)u_y = c(x, y, u) = c(x(t), y(t), u(x(t), y(t)))$$

Parameterize γ (see Cauchy Problem), solve system to get $t(x, y)$ and $s(x, y)$ and solve

$$u' = c(x, y, u) \qquad u(s, 0) = f(s)$$

using previously obtained $x(t, s)$ and $s(t, s)$ to set $u(t, s)$ and substitute to get $u = u(x, y)$

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1.7 Second Order Equations $(m = 2)$

Typically made easier using a linear transformation of coordinates

Let $\eta = \phi(x, y)$, $\xi = \psi(x, y)$ be an invertible transformation $(x, y) \to (\eta, \xi)$. We need to find u_x and u_{xx} in terms of the new system. Using chain rule,

$$\begin{aligned}
u_x &= u_\eta \phi_x + u_\xi \psi_x \\
u_{xx} &= u_{\eta\eta}\phi_x^2 + u_\eta + u_{\xi\eta}\psi_x\phi_x + u_\xi + u_{\eta\xi}\phi_x\psi_x + u_\eta + u_{\xi\xi}\psi_x^2 + u_\xi \\
u_{xx} &= u_{\eta\eta}\phi_x^2 + 2u_{\xi\eta}\psi_x\phi_x + u_{\xi\xi}\psi_x^2 + 2u_\eta + 2u_\xi
\end{aligned}$$

Now we must find u_y and u_{yy} in the new system.

$$\begin{aligned}
u_y &= u_\eta \phi_y + u_\xi \psi_y \\
u_{yy} &= u_{\eta\eta}\phi_y^2 + u_\eta + u_{\xi\eta}\psi_y\phi_y + u_\xi + u_{\eta\xi}\phi_y\psi_y + u_\eta + u_{\xi\xi}\psi_y^2 + u_\xi \\
u_{yy} &= u_{\eta\eta}\phi_y^2 + 2u_{\xi\eta}\psi_y\phi_y + u_{\xi\xi}\psi_y^2 + 2u_\eta + 2u_\xi
\end{aligned}$$

Now we must compute the mixed derivative u_{xy} based on u_x

$$\begin{aligned}
u_{xy} &= u_{\eta\eta}\phi_y\phi_x + u_\eta + u_{\xi\eta}\phi_y\psi_x + u\eta + u_{\eta\xi}\psi_y\phi_x + u_\xi + u_{\xi\xi}\psi_y\psi_x + u_\xi \\
u_{xy} &= u_{\eta\eta}\phi_y\phi_x + u_{\xi\eta}\left(\phi_y\psi_x + \psi_y\phi_x\right) + u_{\xi\xi}\psi_y\psi_x + 2u_\eta + 2u_\xi
\end{aligned}$$

Adding everything together, a second order equation in the original coordinates

$$au_{xx} + 2bu_{xy} + cu_{yy} + du_x + fu_y + hu = g(x, y)$$

becomes in the new coordinate system:

$$Au_{\eta\eta} + Bu_{\eta\xi} + Cu_{\xi\xi} + Du_\eta + Fu_\xi + Hu = g(\eta, \xi)$$

where

$$\begin{aligned}
A(\eta, \xi) =& a\phi_x^2 + b\phi_x\phi_y + c\phi_y^2 \\
B(\eta, \xi) =& 2a\phi_x\psi_x + b(\phi_y\psi_x + \phi_x\psi_y) + 2c\psi_y\phi_y \\
C(\eta, \xi) =& a\psi_x^2 + b\psi_x\psi_y + c\psi_y^2 \\
D(\eta, \xi) =& d\phi_x + f\phi_y + 2a + 2b + 2c \\
F(\eta, \xi) =& d\psi_x + f\psi_y + 2a + 2b + 2c \\
H(\eta, \xi) =& h
\end{aligned}$$

If the transformation $\eta = \phi(x, y)$, $\xi = \psi(x, y)$ is linear, it can be expressed in the form

$$\begin{pmatrix} \eta \\ \xi \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

with $\alpha\delta - \beta\gamma \neq 0$ to ensure invertibility. This means $\phi_x = \alpha$, $\phi_y = \beta$, $\psi_x = \gamma$, $\psi_y = \delta$, so

$$\begin{aligned}
A(\eta, \xi) =& a\alpha^2 + b\alpha\beta + c\beta^2 \\
B(\eta, \xi) =& 2a\alpha\gamma + b(\beta\gamma + \alpha\delta) + 2c\beta\delta \\
C(\eta, \xi) =& a\gamma^2 + b\gamma\delta + c\delta^2 \\
D(\eta, \xi) =& d\alpha + f\beta + 2a + 2b + 2c \\
F(\eta, \xi) =& d\gamma + f\delta + 2a + 2b + 2c \\
H(\eta, \xi) =& h
\end{aligned}$$

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1.7.1 Types of Equations

1.7.1.1 Parabolic Equations $b^2 - ac = 0$ With equations of this type, one can choose $b(\eta, \xi) = c(\eta, \xi) = 0$ to obtain

$$u_{\eta\eta} = 0$$

or equivalently $b(\eta, \xi) = a(\eta, \xi) = 0$ to get

$$u_{\xi\xi} = 0$$

1.7.1.2 Hyperbolic Equations $b^2 - ac > 0$ With equations of this type, one can choose $a(\eta, \xi) = c(\eta, \xi) = 0$ to obtain

$$u_{\xi\eta} + D^{|1|}u = 0$$

or equivalently $b(\eta, \xi) = 0$ to get

$$u_{\xi\xi} - u_{\eta\eta} + D^{|1|}u = 0$$

Where $D^{|1|}u$ is the lower order terms
If $D^{|1|}u = 0$, the solutions to these equations are of the form

$$u(\eta, \xi) = \Phi(\eta) + \Psi(\xi)$$

for some functions $\Phi(\eta), \Psi(\xi)$

1.7.1.3 Elliptic Equations $b^2 - ac < 0$ With equations of this type, one can choose $a(\eta, \xi) = c(\eta, \xi) = 0$ to obtain

$$u_{\xi\eta} + D^{|1|}u = 0$$

or equivalently $b(\eta, \xi) = 0$ to get

$$u_{\xi\xi} - u_{\eta\eta} + D^{|1|}u = 0$$

Where $D^{|1|}u$ is the lower order terms

1.7.2 Separation of Variables

Seeking separable solutions $u(x_1, ..., x_n) = X_1(x_1) \times ... \times X_n(x_n)$ works for many kinds of PDEs in multiple dimensions. $(0 \le \theta < 2\pi, r \ge 0)$

1.7.3 Boundary Conditions

For a well-posed mth order PDE, you will have up to m side conditions, usually in the form of

$$F(u) = 0 \text{ on } \partial\Omega$$

Boundary conditions are said to be **homogenous** if they are closed under linear combinations, such as

$$u = 0 \text{ on } \partial\Omega$$

The most common boundary conditions are:

1.7.3.1 Dirichlet Conditions (First Type)

$$u = g \text{ on } \partial\Omega$$

1.7.3.2 Neumann (Second Type)

$$\frac{\partial u}{\partial n} = \nabla u \cdot \vec{\mathbf{n}} = g \text{ on } \partial\Omega$$

where $\vec{\mathbf{n}}$ is the unit outward normal

1.7.3.3 Robin (Third Type), aka Mixed

$$\frac{\partial u}{\partial n} + \sigma u = \nabla u \cdot \vec{\mathbf{n}} + \sigma u = g \text{ on } \partial\Omega$$

1.7.4 Common Well-Posed Problems

The following equations are well-posed and have classic solutions

1.7.4.1 Poisson's Equation

$$\Delta u = f \qquad \text{(Poisson's Equation)}$$

$$\Delta u = 0 \qquad \text{(Laplace's Equation)}$$

Solutions are harmonic functions, typically with Dirichlet, Neumann, or Robin conditions

$$u_{xx} + u_{yy} + u_{zz} = f \qquad \text{(rectangular coordinates)}$$

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = f \qquad \text{(polar coordinates)}$$

Requires product solution families to be 2π periodic in θ:

$$u_n(\theta, r) = \begin{cases} c_1 r^n \cos(n\theta) + c_2 r^n \sin(n\theta) + c_3 r^{-n}\cos(n\theta) + c_4 r^{-n}\sin(n\theta) & n = 1, 2, 3... \\ c_1 + c_2 log(r) & n = 0 \end{cases}$$

Coefficients are found using boundary conditions, initial conditions, and Fourier theory

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^2 u}{\partial z^2} = 0 \qquad \text{(cylindrical coordinates)}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2 sin(\theta)}\frac{\partial^2}{\partial \theta^2}\left(\sin(\theta)\frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2\sin^2(\theta)}\frac{\partial^2 u}{\partial \phi^2} = 0 \qquad \text{(spherical coordinates)}$$

1.7.4.1.1 Using Distributions

$u(x) = \frac{-1}{4\pi|x|}$, note $u \in L^1_{\text{loc}}(\mathbb{R}^3)$ since

$$\int_{\mathbb{R}} |u(x)|\, dx = \int_0^R \int_0^{2\pi} \int_{-\pi}^{\pi} \frac{1}{4\pi r} r^2 \sin(\theta) d\phi dr d\theta = \frac{C}{2}R^2$$

Claim: $\Delta u = \delta$. Reason:

$$u(\Delta\phi) = \int u\Delta\phi = \lim_{\epsilon \to 0}\int_{\epsilon < |x| < R} u\Delta\phi dx$$

Since ϕ has compact support and so for some R $\phi(|x| > R) = 0$. By Green's Identity this becomes

$$\lim_{\epsilon\to 0}\int_{\epsilon<|x|<R} u\Delta\phi dx = \lim_{\epsilon \to 0}\left(\int_{\epsilon<|x|<R}\phi(x)\Delta u dx + \int_{|x|=\epsilon}\phi\frac{\partial u}{\partial n} - u\frac{\partial \phi}{\partial n}ds\right)$$

Note that $\Delta u \propto \Delta \frac{1}{r} = 0$, and

$$\left| \int_{|x|=\epsilon} u \frac{\partial \phi}{\partial n} ds \right| \leq \frac{1}{4\pi\epsilon} \int_{|x|=\epsilon} \left| \frac{\partial \phi}{\partial n} \right| ds = \frac{1}{4\pi\epsilon} \max \left(\left| \frac{\partial \phi}{\partial n} \right| \right) 4\pi\epsilon^2 \leq C\epsilon \to 0$$

since the $4\pi\epsilon^2$ is the surface area of a sphere. We also see that since $\frac{\partial u}{\partial n} = -\frac{\partial u}{\partial r}$,

$$\int_{|x|=\epsilon} \phi \frac{\partial u}{\partial n} ds = -\int_{|x|=\epsilon} \phi \frac{\partial u}{\partial r} ds = -\int_{|x|=\epsilon} \phi \frac{1}{4\pi\epsilon^2} ds = \frac{1}{4\pi\epsilon^2} \int_{|x|=\epsilon} \phi ds$$

which is the average of ϕ over the sphere $|x| = \epsilon$ which must tend to the value at the center. That is, as $\epsilon \to 0$, $\frac{1}{4\pi\epsilon^2} \int_{|x|=\epsilon} \phi ds \to \phi(0)$, so

$$\Delta u(\phi) = u(\Delta \phi) = \phi(0) = \delta(\phi)$$

In higher dimensions,

$$u(x) = \begin{cases} \frac{-1}{2\pi} \log(x) & N = 2 \\ \frac{1}{A_N |x|^{N-2}} & N \geq 3 \end{cases}$$

Where A_N is the surface area of the N-Sphere.

1.7.4.2 Fundamental Solution

$$E(x) = \begin{cases} \frac{|x|}{2} & N = 1 \\ \frac{1}{2\pi} \log |x| & N = 2 \\ \frac{C_N}{|x|^{N-2}} & N \geq 3 \end{cases}$$

Where $C_N = \frac{???}{N(N-1)a_n}$ and a_n is the volume of $B_1(0)$ in $\mathbb{R}^N$ If Δ is replaced by the positive operator $-\Delta$, the the FS is $E_-(x) = -E(x)$

1.7.4.3 Wave Equation

$$u_{tt} - c^2 u_{xx} = 0$$

Typically with Dirichlet, Neumann, or Robin conditions in space and some

$$\begin{aligned} u(\vec{\mathbf{x}}, 0) =& f(\vec{\mathbf{x}}) \\ u_t(\vec{\mathbf{x}}, 0) =& g(\vec{\mathbf{x}}) \qquad x \in \Omega \end{aligned}$$

$\eta = x + ct$ and $\xi = x - ct$, then

$$u(x,t) = \Phi(x + ct) + \Psi(x - ct)$$

If initial conditions $u(x, 0) = f(x)$ and $u_t(x, 0) = g(x)$ are given,
then $f(x) = \Phi(x) + \Psi(x)$ and $g(x) = \Phi'(x) + \Psi'(x)$ and so $\int_0^x g(y)dy = \Phi(x) + \Psi(x) + C$
Combining, $F(x) = \frac{1}{2} \left(f(x) + \int_0^x g(y)dy + C \right)$ giving us
D'Alembert's Solution Formula

$$u(x,t) = \frac{1}{2} \left(f(x + ct) + f(x - ct) \right) + \frac{1}{2c} \int_{x-ct}^{x+ct} g(y)dy$$

1.7.4.4 Fundamental Solution

$$E(x,t) = \begin{cases} \frac{1}{2}H(t-|x|) & N=1 \\ \frac{1}{2}\frac{H(t-|x|)}{\sqrt{t^2-|x|^2}} & N=2 \\ \frac{\delta(t-|x|)}{4\pi|x|} & N=3 \end{cases}$$

Example: For $N = 3$

$$(E * f) = \int_{\mathbb{R}^4} \frac{\delta(s-|y|)}{4\pi|y|} f(x-y,t-s)dsdy = \int_{\mathbb{R}^3, s=|y|} \frac{1}{4\pi|y|} f(x-y,t-|y|)dy$$

Assume $f(x,t) < 0$ for $t < 0$, so

$$(E * f) = \int_{\mathbb{R}^3, |y|<t} \frac{1}{4\pi|y|} f(x-y,t-|y|)dy = \int_{B(x,t)} \frac{1}{4\pi|x-y|} f(y,t-|x-y|)dy$$

So the solution of $u_{tt} - \Delta u = f$, $u(x,0) = h(x)$, $u_t(x,0) = g(x)$ can be shown to be

$$u(x,t) = (E \underset{(x)}{*} g) + \frac{\partial}{\partial t}(E \underset{(x)}{*} h) + (E * f)$$

1.7.4.4.1 Uniqueness Showing the uniqueness of solutions to this problem is equivalent to showing that the homogenous problem

$$u_{tt} - c^2 u_{xx} = 0 \qquad u_t(\vec{\mathbf{x}},0) = u(\vec{\mathbf{x}},0) = 0$$

with homogenous boundary conditions has only a trivial solution.

1.7.4.4.2 Four Point Property for solutions to the wave equation $u_{xx} - u_{tt} = 0$ on $\Omega \subset \mathbb{R}^2$ containing the tilted rectangle with vertices $(x,t), (x+h-k, t+h+k), (x+h, t+h), u(x-k, t+k)$,

$$u(x,t) + u(x+h-k, t+h+k) = u(x+h, t+h) + u(x-k, t+k)$$

1.7.4.4.3 Using Distributions

Let $F \in L^1_{\text{loc}}(\mathbb{R})$, then $u(x,t) = F(x+t)$ is one solution to the wave equation in $\mathcal{D}'(\mathbb{R}^2)$ since

$$T(\phi) = (u_{tt} - u_{xx})(\phi) = u(\phi_{tt} - \phi_{xx}) = \int\int_{\mathbb{R}^2} F(x+t)(\phi_{tt}(x,t) - \phi_{xx}(x,t))dxdt$$

By change of coordinates, $\xi = x+t$, $\eta = x-t$, $\phi_{tt} - \phi_{xx} = -4\phi_{\xi\eta}$, $dxdt = \frac{\partial(x,t)}{\partial(\xi,\eta)} = -\frac{1}{2}d\xi d\eta$

$$= 2\int_{-\infty}^{\infty} F(\eta) \int_{-\infty}^{\infty} \phi_{\xi\eta} d\xi d\eta = 0$$

since ϕ has compact support

1.7.4.5 Heat Equation

$$u_t - \Delta u = 0 \qquad\qquad (\vec{\mathbf{x}}, t) \in \Omega \times [0, \infty)$$

Typically with Dirichlet, Neumann, or Robin conditions in space and some

$$u(\vec{\mathbf{x}}, 0) = f(\vec{\mathbf{x}}), x \in \Omega$$

Look for separable solutions of the form $u(t, x) = \Phi(t)\Psi(x)$ where

$$\Phi(t) = c_1 e^{kt} + c_2 e^{-kt} \qquad \Psi(x) = c_1 sin(kx) + c_2 cos(kx)$$

Solutions will usually involve $k = k_n$ so the set of product solutions is

$$\{\Phi_n(t)\Psi_n(x)\}_0^\infty = \left\{c_1 e^{k_n^2 t} sin(k_n x) + c_2 e^{k_n^2 t} cos(k_n x) + c_3 e^{-k_n^2 t} sin(k_n x) + c_4 e^{-k_n^2 t} sin(k_n x)\right\}_0^\infty$$

Where the $c_1, ..., c_4$ is chosen to satify the boundary and initial conditions using Fourier theory.

1.7.4.5.1 Uniqueness Showing the uniqueness of solutions to this problem is equivalent to showing that the homogenous problem

$$u_t - \Delta u = 0 \qquad u_t(\vec{\mathbf{x}}, 0) = u(\vec{\mathbf{x}}, 0) = 0$$

with homogenous boundary conditions has only a trivial solution.

1.7.4.6 Fundamental Solution

$$E(x, t) = \frac{H(t)}{(2\pi t)^{\frac{N}{2}}} e^{\frac{-|x|^2}{4t}}$$

1.7.4.7 Reason:

1.7.4.7.1 Using FT For $Lu = u_t - u_{xx}$ with $u(x, 0) = f(x)$ taking the partial FT in x yields

$$\hat{u}(k, t) = \frac{1}{(2\pi)^{\frac{N}{2}}} \int_{\mathbb{R}^N} u(x, t) e^{-ikx} dx \qquad \text{and } (u_t)^\wedge = (\hat{u})_t$$

So $(-\Delta u)^\wedge = |k|^2 \hat{u}$ so $\hat{u}_t + |k|^2 \hat{u} = 0$, which is an ODE for $\hat{u}$ for fixed k. Solving,

$$\hat{u}(k, t) = \hat{f} e^{-|k|^2 t} \implies u(x, t) = \frac{1}{(2\pi)^{\frac{N}{2}}} \mathcal{F}^{-1}\left(\left((f * g)\right)^\wedge\right) = \frac{1}{(4\pi t)^{\frac{N}{2}}} \int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{4t}} f(y) dy$$

where $\hat{g}(k) = e^{-|k|^2 t}$ so $g(x) = \frac{e^{-\frac{|x|^2}{4t}}}{(2t)^{\frac{N}{2}}}$. This is valid for $f \in L^p(\mathbb{R}^N)$ for some $1 \le p \le \infty$, $t > 0$, and $\lim_{t \to 0^+} u(x, t) = f(x)$ a.e.

1.7.4.7.2 Using FT and Duhamel's Principle By Duhamel's Principle, we can do the same for $Lu = h$. Let $v(x,t,s)$ satisfy $v_t - \Delta v = 0$ $x \in \mathbb{R}^N$, $t > 0$, $v(x,0,s) = h(x,s)$ and let $u(x,t) = \int_0^t v(x,t-s,s)ds$.

$$u_t = v(x,0,t) + \int_0^t v_t(x,t-s,s)ds = h(x,t) + \int_0^t \Delta v(x,t-s,s)ds = h(x,t) + \Delta u$$

So for $u_t - \Delta u = h(x,t)$ for $t > 0$ with $u(x,0) = 0$, we have

$$v(x,t,s) = \frac{1}{(2\pi t)^{\frac{N}{2}}} \int_{\mathbb{R}^N} e^{\frac{-|x-y|^2}{4t}} h(y,s)dy \implies u(x,t) = \int_0^t \int_{\mathbb{R}^N} \frac{1}{(2\pi(t-s))^{\frac{N}{2}}} e^{\frac{-|x-y|^2}{4(t-s)}} h(y,s)dyds = E * h$$

where $E = \frac{H(t)e^{\frac{-|x|^2}{4t}}}{(2\pi t)^{\frac{N}{2}}}$ and $h(x,t) = 0$ for $t < 0$. So our solution is

$$u(x,t) = (E * h)(x,t) + (E \underset{(x)}{*} f)(x,t)$$

So E is the fundamental solution to the Heat Equation. Note: for $t > 0$, $E(x,t) = \left(\frac{1}{\sqrt{t}}\right)^N F\left(\frac{x}{\sqrt{t}}\right)$ if $F(x) = \frac{e^{-\frac{|x|^2}{4}}}{(2\pi)^{\frac{N}{2}}}$. Here $F \geq 0$, $F \in L^1$, and $\int_{\mathbb{R}^N} F(x)dx = 1$. By previous discussion of approximate identities let $k = \frac{1}{\sqrt{t}}$

$$k^N F(kx) \to \delta \text{ as } k \to \infty \implies E(x,t) \to \delta \in \mathcal{D}' \text{ as } t \to 0^+$$

so $(E \underset{(x)}{*} f)(x,t) = E(\cdot,t) * f \to f$ as $t \to 0^+$ (similar to approximate identities)

1.7.4.8 Maximum Principle The solution u is bounded by the extremes of the initial condition and the Dirichlet boundary conditions.

1.7.4.9 Shrodinger Equation

$$u_t - i\Delta u = 0 \qquad (\vec{\mathbf{x}}, t) \in \Omega \times [0,\infty)$$

Typically with Dirichlet, Neumann, or Robin conditions in space and some $u(\vec{\mathbf{x}},0) = f(\vec{\mathbf{x}}), x \in \Omega$

1.7.4.10 Fundamental Solution

$$E(x,t) = \frac{H(t)}{(2\pi t)^{\frac{N}{2}}} e^{i(N-2)\frac{\pi}{4}} e^{\frac{-|x|^2}{4it}}$$

1.7.4.11 Helmholtz Equation

$$\Delta u - k^2 u = 0 \qquad (\vec{\mathbf{x}}, t) \in \Omega \times [0,\infty)$$

1.7.4.12 Fundamental Solution

$$E(x,t) = \begin{cases} \frac{1}{2\pi} K_0(k|x|) & N = 2 \\ \frac{-e^{-k|x|}}{4\pi|x|} & N = 3 \end{cases}$$

Where K_i is the ith order Modified Bessel Function.
Note as $k \to 0$, $E(x)$ tends toward the $E(x)$ corresponding to $L = \Delta$

1.7.4.13 Euler-Tricomi Equation

$$u_{xx} - xu_{yy} = 0$$

Hyperbolic in the half plane $x > 0$, parabolic at $x = 0$, elliptic in $x < 0$
Characteristics are along $y \pm \frac{2}{3}x^{\frac{2}{3}} = C$
Particular solutions are

$$\begin{aligned} u =& c_1 xy + c_2 x + c_3 y + c_4 \\ u =& c_1 \left(3y^2 + x^3\right) + c_2 \left(y^3 + x^3 y\right) + c_3 \left(6xy^2 + x^4\right) \end{aligned}$$

1.7.4.14 Biharmonic Oscillator

$$\Delta^2 u = 0 \qquad\qquad (\vec{\mathbf{x}}, t) \in \Omega \times [0, \infty)$$

1.7.4.15 Fundamental Solution ??

1.7.4.16 Klein Gordon Equation

$$Lu = u_{tt} - u_{xx} + u \qquad\qquad (\vec{\mathbf{x}}, t) \in \Omega \times [0, \infty)$$

1.7.4.17 Fundamental Solution

$$E(x,t) = \frac{1}{2} H(t - |x|) J_0(\sqrt{t^2 - x^2})$$

2 Distribution Theory in DE

If $f'(x) = 0$ for $a \leq x \leq b$ then $f(x) = c$ classically If $T' = 0 \in \mathcal{D}'(a,b)$ then $T = c$
Reason: Choose $\phi_0 \in C_0^\infty$ with $\int_a^b \phi_0(x)dx = 1$. If $\phi \in C_0^\infty(a,b)$ let $\psi(x) = \phi(x) - \int_a^b \phi(y)dy \phi_0(x)$. Note $\int_a^b \psi(x)dx = 0$. Let $\zeta = \int_a^x \psi(s)ds$ so $\zeta' = \psi(x)$ since $\zeta \in C^\infty(a,b)$ and further $\zeta \in C_0^\infty$ since $\zeta(a) = \zeta(b) = 0$ and $\zeta' = 0$ for $x < a$ or $x > b$. Then $0 = T'(\zeta) = -T(\zeta') = -T(\psi) = -T(\phi) + \left(\int_a^b \phi y dy\right) T(\phi_0)$ so $T(\phi) = \int_a^b T(\phi_0)\phi(y)dy = \int_a^b c\phi(y)dy = c$ in $\mathcal{D}'$ sense.

2.1 ODEs in $\mathcal{D}'$ sense

$$T' = f \qquad f \in L^1_{\text{loc}}(\mathbb{R})$$

Let $g(x) = \int_a^x f(s)ds$ (antiderivative of f). Claim $g' = f$ in $\mathcal{D}'(a,b)$. Reason:

$$T_g'(\phi) = -T_g(\phi') = -\int_a^b g(x)\phi'(x)dx = -\int_a^b \int_a^x f(s)ds\phi'(x)dx = -\int_a^b f(s)\int_s^b \phi'(x)dxds$$

Using FTC and compact support of ϕ,

$$T_g'(\phi) = -\int_a^b f(s)(\phi(b) - \phi(s))ds = \int_a^b f(s)\phi(s)ds = T_f(\phi)$$

So the general solution of $T' = f$ in $\mathcal{D}'(a,b)$ is

$$T = \int_a^x f(s)ds + C$$

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3 Advanced Theory

4 New Notes

Notation Let $A \in \mathbb{C}^{n\times m}$, $B \in \mathbb{C}^{n\times m}$, then the Hadamar product

$$A : B = \sum_{i=1}^{m}\sum_{i=1}^{n} a_{ij}b_{ij}$$

$\|A\| \ = |A : A|^2$

$\mathbb{S}^n$ are the symmetric $n \times n$ matrices. $\mathbb{S}^n \subset \mathbb{C}^{n\times n}$.

$B(x, r)$ is the unit ball in $\mathbb{R}^n$ centered at x $B(x, 1)$ is the unit ball in $\mathbb{R}^n$ centered at x. $B(0, 1)$ is the unit ball in $\mathbb{R}^n$ centered at the origin.

S^{n-1} is the unit sphere in $\mathbb{R}^n$. $S^{n-1} = \partial B(0, 1)$

e_i is the ith coordinate vector in $\mathbb{R}^n$.

$\mathbb{R}^n_+ = \{y \in \mathbb{R}^n : y_n > 0\}$ is the upper half space of $\mathbb{R}^n$.

Let U, V, W be open subsets of $\mathbb{R}^n$. $V \subset\subset U$ means V is compactly contained in U. That is $\overline{V} \subset U$, $\overline{V}$ is compact. ∂U is the boundary of U. $\overline{U} = \partial U \cup U$ $U_T = U \times (0, T]$

$\Gamma_T = \overline{U}_T - U_T$ (parabolic boundary) $\alpha(n) = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$ is the volume of the $B(0, 1)$ in $\mathbb{R}^N$. $n\alpha(n)$ is the surface area of $\partial B(0, 1)$ in $\mathbb{R}^N$ $\natural_0 = \natural \cup \{0\}$

Du is the Gradient of u D^2u is the Hessian of u

The Laplacian of u is $\Delta u = \sum_{i=1}^{n} u_{x_i x_i} = \mathrm{tr}\,(D^2u)$

Airy eqn -3rd order linear

$$u_t + u_{xxx} = g(x)$$

KdV eqn -3rd order semilinear

$$u_t + uu_x + u_{xxx} = g(u)$$

Monge-Ampere eqn - fully nonlinear, second order

$$\det(D^2u) = f(u)$$

Transport equation solutions

$$u_t + b \cdot Du = f, (\vec{\mathbf{x}}, t) \in \mathbb{R}^n \times (0, \infty)$$

Laplace Poisson eqn

$$\Delta u = f$$

Heat Equation

$$u_t - \Delta u = f$$

Wave Eqn

$$u_{tt} - \Delta u = f$$

Wave Eqn

$$\Delta u = \int_{B(0,\epsilon)} \Phi(y)\Delta_x f(x-y)dy + \int_{\mathbb{R}^N \setminus B(0,\epsilon)} \Phi(y)\Delta_x f(x-y)dy$$

The first part vanishes by FTC, and the second part becomes

$$= \int_{\mathbb{R}^N \setminus B(0,\epsilon)} \Delta_y \Phi(y)\Delta_x f(x-y)dy - \int_{\partial B(0,\epsilon)} \frac{\partial}{\partial n}\Phi(y) f(x-y)dS(y) + \int_{\partial B(0,\epsilon)} \Phi(y)\frac{\partial}{\partial n} f(x-y)dS(y)$$

$\frac{\partial}{\partial n} f(x-y) = 0$ away from the origin. We also note that

$$-D_n\Phi(y) = -\left(-\frac{d}{dr}\right)\left(\frac{-1}{2\pi}\log(r)\right)|_{r=\epsilon} = \frac{1}{2\pi\epsilon}$$

So

$$\Delta u(x) = -\int_{\partial B(0,\epsilon)} \frac{f(x-y)}{2\pi\epsilon} dS(y) = -f(x)\int_{\partial B(x,\epsilon)} \frac{1}{2\pi\epsilon} dS(y) = -f(x)$$

So $\Delta u(x) \to f(x)$ uniformly as $\epsilon \to 0$.

If $u \in C^2(U)$ is Harmonic, then it satisfies the Mean Value Property

$$u(x) = \fint_{\partial B(x,r)} u dS = \frac{\int_{\partial B(x,r)} u dS}{\int_{\partial B(x,r)} dS}$$

For all $B(x,r) \subset U$

Proof. Set

$$\varphi(\nu) = \fint_{\partial B(x,r)} v(\nu) dS(y) = \frac{\int_{\partial B(0,1)} v(x+rz) dS(z) r^{n-1}}{\|\partial B(0,1)\|} = \fint_{\partial B(0,1)} v(x+rz) dS(z)$$

Since $\|\partial B(0,1)\| \ = n\alpha(n) r^{n-1}$

The average value of a Harmonic function over a sphere is the value at the center of the sphere.

$$\varphi'(r) = \fint_{\partial B(0,1)} Du(x+rz) z dS(z) = \fint_{\partial B(x,r)} Du(y) \cdot \frac{y-x}{r} dS(z)$$

Since $\frac{y-x}{r} = \frac{z}{n}$ (?) it is the outward unit normal

$$= \fint_{\partial B(x,r)} \frac{\partial u}{\partial n} dS = \frac{\int_{\partial B(x,r)} \Delta u ds}{|\partial B(x,r)|} = 0$$

This implies $\varphi(r)$ is a constant.

$$\lim_{\epsilon\to 0} \varphi(\epsilon) = \lim_{\epsilon \to 0} ?/ \implies \phi(r) =$$

Corollary: Same holds on $B(x,r)$. $u(x) = \fint_{B(x,r)} u(y) dy$. Proof. Use the shell method

$$\int_{B(x,r)} u(y) dy = \int_0^r \left(\int_{\partial B(x,s)} u dS\right) ds = ... = u(x)\alpha(n) r^n = u(x)\left|\partial B(x,r)\right|$$

Theorem. Converse. If $u \in C^2(U)$ satisfies $u(x) = \fint_{\partial B(x,r)} u ds \ \forall \ B(x,r) \subset U$ then u is harmonic. Proof. If not, there exists $r > 0 : \Delta u > 0$ on $B(x_0, r)$. As before,

$$\varphi'(r) = 0 = \frac{\int_{B(x_0,r)} \Delta u dy}{|\partial B(x_0,r)|} > 0 \Rightarrow\Leftarrow$$

Strong Maximum Principle Suppose $u \in C^2(U) \cap C(\overline{U})$ is harmonic on U, and assume U is bounded. Let $M = \max_{\overline{U}} u$.

1. $\max_{\partial U} = M$
2. If U is connected and $\exists \ x_0 \in U : u(x_0) = M$, then $u = M$.

Corollary, Strong Minimum principle also holds. Suppose $u \in C^2(U) \cap C(\overline{U})$ is harmonic on U, and assume U is bounded. Let $m = \min_{\overline{U}} u$.

1. $\min_{\partial U} = m$
2. If U is connected and $\exists\, x_0 \in U : u(x_0) = m$, then $u = m$.

Proof of Strong Maximum Principle If $\exists\, x_0 \in U : u(x_0) = M$ then if $B(x_0, r) \subset U$, $M = \fint_{B(x_0,r)} u(y)dy \leq M$ with equality if and only if $u = M$ on $B(x_0, r)$. This implies $u = M$ on $B(x_0, r)$. This implies if $S = \{x \in U : u(x) = M\} \implies S$ is relatively open in U. u is continuous $u^{-1}(M)$ is closed in $U \implies S$ is relatively open, closed. If U is connected, nonempty then $U = S$. So (2) holds. The continuity implies that (1) holds.

Corollary. Consider

$$\begin{cases} \Delta u = & 0, x \in U \\ v = & g, x \in \partial U \end{cases}$$

Assume U is connected, $u \in C^2(U) \cap C(\overline{U})$.

- If $g \geq 0$, then $u \geq 0$.
- If also $\exists\, s_0 \in \partial U : g(s_0) > 0$, then $u > 0$ in U.

Theorem 5. This problem has at most one solution. If u_1, u_2 are solutions, then $y = u_1 - u_2$ solves this problem with $g = 0$ (the homogenous case). Maximum principle implies $y \geq 0$, minimum principle implies $y \leq 0$, so $y = 0 \implies u_1 = u_2$.

Mollifiers

The standard mollifier

$$\eta_(x) = \begin{cases} ce^{\left(|x|^2-1\right)^{-1}} & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$$

$\eta_(x) \in C_0^\infty(\mathbb{R}^N)$

If $f : U \to \mathbb{R} \in L^1_{\text{loc}}$, define $f^\epsilon = \eta_\epsilon * f = \int_U \eta_\epsilon(x-y)f(y)dy$ in U_ϵ. $\eta_\epsilon = \frac{1}{\epsilon^n}\eta\left(\frac{x}{\epsilon}\right)$

$\eta_\epsilon(x)$ is a delta sequence

- $\int_{\mathbb{R}^N} \eta_\epsilon dx = 1$
- $\operatorname{supp} \eta_\epsilon = \overline{B(0,\epsilon)} \subset \overline{B(0,1)}$
- $\eta_\epsilon \geq 0$.

Theorem p 714

- $f \in C^\infty(U_\epsilon)$, $U_\epsilon = \{x \in U : \operatorname{dist}(x, \partial U) > \epsilon\}$
- $f^\epsilon \to f$ a.e. as $\epsilon \to 0$
- If $f \in C(U)$ then $f^\epsilon \to f$ uniformly on compact subsets of U
- $f \in L^P_{\text{loc}}(U) \implies f^\epsilon \to f$ in $L^p_{\text{loc}}(U)$

Theorem If $U \in C(U)$ satisfies the mean value property on each $B(x, r) \subset U$ then $u \in C^\infty$ by showing

$$U(x) =?^\epsilon(x), x \in U^\epsilon(x) = \int_U \eta_\epsilon(x-y)u(y)dy = \frac{1}{\epsilon^n}\int_{B(x,\epsilon)} \eta\left(\frac{|x-y|}{\epsilon}\right)u(y)dy$$

$$= \frac{1}{\epsilon^n}\int_0^\epsilon \eta\left(\frac{r}{\epsilon}\right)\left(\int_{\partial B(x,r)} U ds\right)dr$$

$$= \frac{1}{\epsilon^n}\int_0^\epsilon \eta\left(\frac{r}{\epsilon}\right)U(x)\left|\partial B(x,r)\right|dr$$

$$= U(x)\int_{B(0,\epsilon)} \eta_\epsilon dy = U(x)$$

If u is harmonic on U then $|D^\alpha u(x_0)| \le \frac{c_k}{r^{n+k}}\|u\|_{L^1(B(,x_0,r))}$ for all $B(x_0,r)\subset U$. $|\alpha| = k$. where $C_0 = \frac{1}{\alpha(n)}$ and $C_k = \frac{(2^{n+1}nk)^k}{\alpha(n)}$

Proof $k=0$

$$|U(x_0)| = \left|\fint_{B(x_0,r)} u(x)dx\right| \le \frac{\int_{B(x_0,r)}|u|\,dx}{\alpha(n)r^n} = \frac{C_0}{r^n}\|U\|_{L^1(B(x_0,r)}$$

Proof $k=1$

$$|U_{x_i}(x_0)| = \left|\fint_{B(x_0,\frac{r}{\epsilon})} u_{x_i}(x)dx\right| \le \frac{\int_{B(x_0,\left(\frac{r}{\epsilon}\right))} u\cdot n_i dx}{\alpha(n)\left(\frac{r}{\epsilon}\right)^n} \le \|u\|_\infty \frac{\int_{B(x_0,\left(\frac{r}{\epsilon}\right)} ds}{\left|B(x_0,\left(\frac{r}{\epsilon}\right))\right|} = \|U\|_{L^1(B(x_0,r))} = \|U\|_{L^1(B(x_0,r))}\frac{2n}{r}$$

...

Liouville's Theorem. Suppose $u:\mathbb{R}^n\to\mathbb{R}$ is harmonic and bounded. Then $u\equiv \text{const.}$.

Proof. Let $B = B(x_0,r)$

$$|D(x_0)| \le \frac{\sqrt{n}C_1}{r^{n+1}}\|U\|_{L^1(B)} \le \frac{C_1}{r^{n+1}}\|U\|_{L^\infty(B)}|B| \le \frac{C}{r}\|U\|_{L^\infty(\mathbb{R})}\ \forall\ r$$

$$\implies |D(x_0)|\to 0 r\to\infty \implies |Du| = 0 \implies u\equiv\text{const.}$$

Theorem 9. Let $f\in C_0^2(\mathbb{R}^n)$, $n\ge 3$. Then any bdd solution of $-\Delta u = f$ in $\mathbb{R}^n$ satisfies

$$u = \int_{\mathbb{R}^n}\Phi(x-y)f(y)dy + C$$

Proof. For $n\ge 3$ Φ is bdd, $\Phi\to 0$ as $|x|\to\infty \implies \int_{\mathbb{R}^n}\Phi(x-y)f(y)dy$ is bounded. Let v be another bdd solution to $\Delta v = f$, then $y = u - v$ satisfies $\Delta y = 0$, y bdd. so $y\equiv\text{const.}$.

Theorem 10. u harmonic in U implies u analytic on U. u is analytic on U if for all $x_0\in U$, $u(x)$ is given by its Taylor series in a neighborhood of x_0. Ie $u(x) = \sum_\alpha \frac{D^\alpha}{u}(x_0)\alpha!(x-x_0)^\alpha$. Idea of pf, use Cauchy estimates to show that (the Taylor remainder term) $\mathbb{R}_N(x_0,x)\to 0$ as $N\to\infty$.

Harmek's theorem. For all connected open sets $V\subset\subset U$, for all nonnegative Harmonic functions u defined on U, there exists $C>0$ for which $(C = C_V)$

$$\sup_V u \le C\inf_V u$$

In particular,

$$\frac{1}{C}u(y)\le u(x)\le Cu(y)\ \forall\ x,y\in V$$

Proof. Let $r = \frac{\text{dist}(V,\partial U)}{4}$. Let $x,y\in V$, $|x-y|\le r$.

$$u(x) = \fint_{B(x,2r)} \ge \frac{1}{\alpha(n)(2r)^n}\int_{B(y,r)} udz = \frac{1}{2^n}\fint_{B(y,r)} udz = \frac{1}{2^n}u(y)$$

so $\frac{1}{2^n}u(y)\le u(x)\le 2^n u(y)$, where the upper inequality is obtained by symmetry of the first inequality. Then cover $\overline{V}$ with a finite set of balls of radius $\frac{r}{2}$ (since V is compact and connected), then apply this theorem to each ball $\{B_i\}_{i=1}^N$ with $B_i\cap B_{i-1}\ne\emptyset, i = 2,3,...N$.

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5 Green's function for $-\Delta u = f$

$$-\Delta u = f \; \forall \; x \in U, u = g \; \forall \; x \in \partial U$$

for some open bounded U where $\partial U \in C^1$.
Let $V_\epsilon = U - B(x, \epsilon)$, $x \in U$. Apply Green's formula to $\Phi(y-x), u(y)$.

$$\int_{V_\epsilon} u(y)\Delta\Phi(y-x) - \Phi(y-x)\Delta(y) = \int_{\partial V_\epsilon} u(y)\frac{\partial}{\partial n}\Phi(y-x) - \Phi(y-x)\frac{\partial}{\partial n}u(y)dS(y)$$

Which as $\epsilon \to 0$, we obtain

$$u(x) = \int_{\partial U} \Phi(y-x)\frac{\partial}{\partial n}u(y) - g(y)\frac{\partial}{\partial n}\Phi(y-x)dS(y) + \int_U \Phi(y-x)f(x)dy$$

Corrector function. Suppose we can find $\phi^*(y)$ that solves

$$\Delta_y u = 0\Phi(y-x) \in U, \phi^*(y) = \Phi(y-x) \in \partial U$$

Apply Green's function to $u_1\phi^*$:

$$-\int_U \phi^*(y)\Delta u(y)dy = \int_{\partial U} u(y)\frac{\partial}{\partial n}\phi^*(y) - \Phi(y-x)\frac{\partial}{\partial n}u(y)dS(y)$$

Combining this with the above equation,

$$u(x) = \int_{\partial U} u(y)\left(\frac{\partial}{\partial n}\phi^*(y) - \frac{\partial}{\partial n}\Phi(y-x)\right)dS(y) + \int_U \left(\psi^*(y) - \Phi(y-x)\right)\Delta u(y)dS(y)$$

So we have the Green's function $G(x,y) = \Phi(y-x) - \phi^*(y) \; x, y \in U \; x \neq y$. So

$$u(x) = -\int_{\partial U} u(y)\frac{\partial}{\partial n}G(x,y)dS(y) - \int_U G(x,y)\Delta u(y)dy, x \in U$$

Theorem 12. If $u \in C^2(\overline{U})$ solves the Poisson problem, then

$$u(x) = -\int_{\partial U} g(y)\frac{\partial}{\partial n}G(x,y)dS(y) - \int_U G(x,y)f(y)dy, x \in U$$

That is, IF a solution exists and if ϕ^* can be found, the above is the solution.

5.1 Properties of Green's functions

- $G(x,y) = G(y,x)$

Green's functions on a half-space. Let $\mathbb{R}^n_+ = \{(x_1, ..., x_n) : x_n > 0\}$. If $x \in \mathbb{R}^n_+$, define The reflection of x as $\tilde{x} = (x_1, ..., -x_n)$. Define $\phi^*(y) = \Phi(y - \tilde{x})$, $x, y \in \mathbb{R}^n_+$. Then

$$\Delta\phi^* = 0, x \in \mathbb{R}^*_n, \phi^*(y)|_{\partial\mathbb{R}^n_+} = \Phi(y-x) = \Phi(y-\tilde{x})$$

Consequently, $G(x,y) = \Phi(y-x) - \Phi(y-\tilde{x})$ is the Green's function.
So a solution to the Poisson Problem is

$$\frac{\partial}{\partial n}G = -G_{y_n}(x,y) = -\left(y_n\Phi(y-x) - y_n\Phi(y-\tilde{x})\right) = \frac{1}{n\alpha(n)}\left[\frac{y_n - x_n}{|y-x|^n} - \frac{y_n + x_n}{|y-\tilde{x}|^n}\right]$$

so

$$\frac{\partial}{\partial n} G|_{\partial \mathbb{R}^n_+} = \frac{-2x_n}{n\alpha(n)\left|x - y^n\right|}$$

Define the Poisson Kernel

$$K(x,y) = \frac{2x_n}{n\alpha(n)\left|x - y^n\right|}$$

then

$$u(x) = \int_{y_n=0} K(x,y)g(y)dy + \int_U G(x,y)f(y)dy$$

Assume $g \in C(\mathbb{R}^{n-1}) \cap L^\infty(\mathbb{R}^{n-1})$ then $u = \int_{??} K(x,y)g(y)dy$ satisfies

- $u = C^\infty(\mathbb{R}^{n-1}_+) \cap L^\infty(\mathbb{R}^{n-1}_+)$
- $\Delta u = 0$ in $\mathbb{R}^n_+$
- $\lim_{x \to x_0} u(x) = g(x_0) \; \forall \; x_0 \in \partial \mathbb{R}^n_+$

5.1.1 Green's function on the unit ball

If $x \in \mathbb{R}^n \setminus \{0\}$ define $\tilde{x} = \frac{x}{|x|^2}$. $\tilde{x}$ is the dual point to x with respect to $\partial B(0,1)$. Note $x, \tilde{x}$ are parallel vectors, and $|\tilde{x}| = \frac{1}{|x|}$. We say $R : x \to \tilde{x}$ is the **inversion map**.

Find $\phi^*(y)$ such that

$$\Delta \phi^*(y) = 0 \in B^0(0,1), \phi^*(y) = \Phi(y - x) \in \partial B(0,1)$$

Note that $\Phi(y - \tilde{x})$ is harmonic in y in $B^0(0,1)$ for all $x \in B(0,1)$ so $\Phi(|x|\,(y - \tilde{x}))$ is harmonic in y in $B^0(0,1)$. Define the Green's function for the unit Ball

$$G(x,y) = \Phi(y - x) - \Phi(|x|\,(y - \tilde{x})) = \Phi(y - x) - \phi^*(y)$$

where

$$\frac{\partial}{\partial n} G(x,y) = \frac{-1}{n\alpha(n)} \frac{1 - |x|^2}{|x - y|^n}$$

Define Poisson Kernel for $B(0,1)$ by

$$K(x,y) = \frac{1}{n\alpha(n)} \frac{1 - |x|^2}{|x - y|^n}$$

Then $u = \int_{\partial B(0,1)} K(x,y)u(y)dy$ and so $\Delta u = 0 \in B(0,1)$, and $u = g \in \partial B(0,1)$.