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0.1 Norms $\|\cdot\|$

A norm on a linear space X is a function $\|\cdot\| : X \to \mathbb{R}$ with the properties:

- Positive Definite:
  - $\|x\| \geq 0 \; \forall \, x \in X$ (nonnegative)
  - $\|x\| = 0 \iff x = 0$ (strictly positive)
- $\|\lambda x\| = |\lambda| \|x\| \; \forall \, x \in X, \lambda \in \mathbb{C}$(homogeneous)
- $\|x + y\| \leq \|x\| + \|y\| \; \forall \, x, y \in X$ (triangle inequality)

1 Continuity Classes $C^m(\Omega)$

- $C^m(\Omega) = \{f : D^\alpha$ is continuous on $\Omega \, \forall \; |\alpha| \leq m\}$
- $C^\infty(\Omega) = \{f : D^\alpha$ is continuous on $\Omega \, \forall \; |\alpha|\}$ **(Smooth Functions)**
- Analytic Functions: $C^\omega(\Omega) = \{f : f \in C^\infty\}$ and f equals the Taylor series expansion in a neighborhood around every point in Ω
- Continuous and Bounded: $CB(\Omega) = \{f : f$ is continuous and bounded on $\Omega\}$
- Absolutely Continuous: $AC\,[a, b] = \left\{f(x) : f(y) - f(x) = \int_x^y g(s)ds\right\}$ for some $g \in L^1(a, b)$ such that $f \in AC\,[a, b]$ implies $f' = g$ a.e. where f' is the pointwise derivative.

Facts.

- $C^m(\Omega) \subset C^\omega(\Omega) \subset C^\infty(\Omega)$

2 Vector Spaces

2.1 Norms

A norm on a linear space X is a function $\|\cdot\| : X \to \mathbb{R}$ with the properties:

- $\|x\| \geq 0 \; \forall \, x \in X$ (nonnegative)
- $\|\lambda x\| = |\lambda| \|x\| \; \forall \, x \in X, \lambda \in \mathbb{C}$(homogeneous)
- $\|x + y\| \leq \|x\| + \|y\| \; \forall \, x, y \in X$ (triangle inequality)
- $\|x\| = 0 \iff x = 0$ (stricly positive)

2.2 Vector Norms $\|\cdot\|_p$

$$\|x\|_p = \left(\sum |x_i|^p\right)^{\frac{1}{p}}$$

Sum Norm, $p = 1$

$$\|x\|_{sum} = \|x\|_1 = \sum |x_i|$$

Euclidean Norm, $p = 2$

$$\|x\|_{Euclidean} = \|x\|_2 = \sqrt{\sum |x_i^2|}$$

Maximum Norm, $p = \infty$

$$\|x\|_{max} = \|x\|_\infty = max\{|x_i|\}$$

Given a vector x in $\mathbb{C}^N$,

$$\|x\|_{p+a} \leq \|x\|_p \;\forall\; p \geq 1 \;\forall\; a \geq 0$$

$$\|x\|_p \leq \|x\|_r \leq n^{\left(\frac{1}{r}-\frac{1}{p}\right)}\|x\|_p \;\forall\; p \geq 1 \;\forall\; a \geq 0$$

Examples: $\|x\|_2 \leq \|x\|_1$ and $\|x\|_1 \leq \sqrt{n}\|x\|_2$.

3 L^p Spaces

The space of Lebesgue Integrable functions is defined for a given p:

$$L^p(\Omega) = \left\{f : \|f\|_{L^p(\Omega)} < \infty\right\}$$

Where the norm on the space is

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f|^p \, dx\right)^{\frac{1}{p}}$$

- $L^\infty(\Omega)$ is the space of essentially bounded functions. That is, f is bounded on a subset of Ω whose complement has measure zero. The norm on $L^\infty(\Omega)$ is the essential suprenum

$$\|f\|_\infty = \inf\{M \text{ s.t. } |f(x)| \leq M \text{ a.e. } \in \Omega\}$$

- $L^2(\Omega)$ is the space of square-integrable functions. This is a Hilbert Space.
- $L^1(\Omega)$ is the space of Lebesgue integrable functions
- $L^0(\Omega)$ is the space of measurable functions
- $L^1_{\text{loc}}(\Omega)$ is the space of locally integrable functions. That is, functions integrable on any compact subset of an open set $\Omega \subset \mathbb{R}^N$

$$L^1_{\text{loc}}(\Omega) = \{f : f \in L_1(K) \;\forall\; K \subset\subset \Omega, K \text{ compact}\}$$

  Alternative definition

$$L^1_{\text{loc}}(\Omega) = \left\{f : \|f\phi\|_{L^1(\Omega)} < \infty \;\forall\; \phi \in C_0^\infty(\Omega)\right\}$$

  This definition generalizes to $L^p_{\text{loc}}(\Omega)$.

3.1 Useful Facts

- Embeddings:

$$\|u\|_{L^p(\Omega)} \leq meas(\Omega)^{\left(\frac{1}{p}-\frac{1}{q}\right)}\|u\|_{L^q(\Omega)}$$

  Thus if Ω is bounded, then $L^q(\Omega) \subset L^p(\Omega) \subset L^p_{\text{loc}}(\Omega)$ for all $q > p$

$$L^\infty(\Omega) \subset L^2(\Omega) \subset L^1(\Omega) \subset L^1_{\text{loc}}(\Omega)$$

- Continuous functions of compact support are dense in $L^p(\Omega)$
- One can discuss weighted L^p spaces with

$$\|f\|_{L^p(\Omega,wd\mu)} = \left(\int_\Omega w(x)\,|f|^p\,d\mu(x)\right)^{\frac{1}{p}}$$

4 $\ell^p(\mathbb{N})$ Space

This is the space of bi-infinite sequences $\{x_n\}_{-\infty}^{\infty}$ such that $\sum\limits_{n=-\infty}^{\infty} |x_n|^p < \infty$ for $0 < p \leq \infty$. $\ell^p(\mathbb{N})$ is complete with respect to the norm

$$\|x\|_p = \left(\sum_{n=-\infty}^{\infty} |x_n|^p\right)^{\frac{1}{p}}, \qquad p \geq 1$$

and as well for in the case of $p = \infty$

$$\|x\|_\infty = \sup_n |x_n|$$

There is no norm for $0 < p < 1$, but there is a metric

$$d\left(\{x_n\}, \{y_n\}\right) = \sum_{n=-\infty}^{\infty} |x_n - y_n|^p$$

- ℓ^∞ is the space of bounded sequences
- ℓ^2 is the space of square-summable sequences. This is a Hilbert Space.
- ℓ^1 is the space of summable sequences

5 Banach Spaces

6 Hilbert Spaces

A **Hilbert Space** is a vector space with a defined inner product $\langle u, v\rangle$ which has the following properties

- Conjugate symmetry: $\langle u, v\rangle = \overline{\langle v, u\rangle}$
- Linear in the first argument: $\langle au + bw, v\rangle = a\langle u, v\rangle + b\langle w, v\rangle$
- Antilinear in the second argument: $\langle u, av + bw\rangle = \bar{a}\langle u, v\rangle + \bar{b}\langle w, v\rangle$
- Positive definite when $u = v$:
  - $\langle u, u\rangle \geq 0$
  - $\langle u, u\rangle = 0$ if and only if $u = 0$

A Hilbert Space is complete with respect to its norm:

$$\|x\|_{\mathcal{H}} = \sqrt{\langle x, x\rangle_{\mathcal{H}}}$$

6.1 Convergence

- $x_n \in \mathcal{H}$ converges strongly to $x \in \mathcal{H}$ if $\|x_n - x\|_{\mathcal{H}} \to 0$.

- $x_n \in \mathcal{H}$ converges weakly to $x \in \mathcal{H}$ if
$$\langle x_n, y\rangle \to \langle x, y\rangle \ \forall \ y \in \mathcal{H}$$
Thus $\|x\| \ \leq \liminf\limits_{n\to\infty} \|x_n\|$ which is strict inequality when convergence is not strong.

6.2 Important Hilbert Spaces

- ℓ^2 is the space of square-summable sequences with
$$\langle x, y\rangle = \sum_{i=-\infty}^{\infty} x_i y_i$$
- $L^2(\Omega)$ is the space of square integrable fuctions with
$$\langle u, v\rangle = \int_\Omega u\bar{v}dx$$
- $W^{s,2} = \mathcal{H}^s$ are the Sobolev Spaces
$$\langle u, v\rangle = \int_\Omega u\bar{g}dx + \int_\Omega Du \cdot D\bar{g}dx + ... + \int_\Omega D^s u \cdot D^s\bar{g}dx$$
- $\mathbb{C}^{n\times m}$ is a Hilbert space which is the space of all $m \times n$ matrices with the inner product
$$\langle A, B\rangle = tr(A^*B) = \sum_{i=1}^{m}\sum_{j=1}^{n} \bar{a}_{ij}b_{ij}$$
with the Hilbert-Schmidt norm (aka Frobenius norm)
$$\|A\|_{\mathbb{C}^{n\times m}} = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|^2}$$
- $\mathcal{H}^*$, the dual of $\mathcal{H}$, is the Hilbert space of continuous linear functionals from $\mathcal{H}$ into the base field. For every element $u \in \mathcal{H}$, there exists a unique element $\phi_u \in \mathcal{H}^*$ defined by
$$\phi_u(x) = \langle x, u\rangle$$
The norm is
$$\|\phi\|_{\mathcal{H}^*} = \sup_{\|x\| \ =1, x\in\mathcal{H}} |\phi(x)|$$

6.3 Hilbert Bases

To check if a given sequence If $\{x_n\}$ is a basis for $\mathcal{H}$, we check the following

1. $\{x_n\}$ is closed if the set of all finite linear combinations of them is dense in $\mathcal{H}$
2. $\{x_n\}$ is complete if and only if there is no nonzero vector orthogonal to all x_n, that is
$$\langle x, x_n\rangle = 0 \ \forall \ n \iff x = 0$$
3. If $\{x_n\}$ are orthonormal they are **maximal orthonormal** if $\{x_n\}$ is not contianed in any strictly larger orthonormal set.

6.3.1 Orthonormal Basis

Let $\{e_n\}_{n=1}^{\infty}$ be orthonormal, the following are equivalent

1. $\{e_n\}$ is maximal orthonormal
2. $\{e_n\}$ is closed
3. $\{e_n\}$ is complete
4. $\{e_n\}$ is a basis of $\mathcal{H}$
5. Bessel's Inequality holds for all $x \in \mathcal{H}$
6. $\sum\limits_{n=1}^{\infty} \langle x, e_n \rangle \langle e_n, y \rangle = \langle x, y \rangle \ \forall\, x, y \in \mathcal{H}$

So we know a complete orthonormal set forms a basis, with $\langle x, e_n \rangle$ known as the n^{th} **Generalized Fourier Coefficient** of x with respect to $\{e_n\}$

6.3.2 Gram Schmidt

If $M = \mathcal{L}\{e_1, e_2, ..., e_n\}$ is linearly independent but not orthonormal, then create a new orthonormal basis $M = \mathcal{L}\{f_1, f_2, ..., f_n\}$ with

$$f_n = \frac{f_n'}{\|f_n'\|} \qquad f_n' = e_n - \sum_{i=1}^{n} \langle e_n, f_i \rangle f_i$$

6.3.3 Reisz Fischer Critertion

For an infinite dimensional basis,
$\sum\limits_{n=1}^{\infty} c_n e_n$ converges in $\mathcal{H}$ if and only if $\sum\limits_{n=1}^{\infty} |c_n|^2 < \infty$
This implies $\sum\limits_{n=1}^{\infty} c_n e_n$ converges if and only if $\{c_n\}_{i=1}^{\infty} \in \ell^2$

6.3.4 Riemann Lebesgue Lemma

Version 1
If given $\{e_n\}_{n=1}^{\infty}$ is ON and $x \in \mathcal{H}$ then $\sum\limits_{n=1}^{\infty} |\langle x, e_n \rangle|^2 < \infty$ implies $\lim\limits_{n \to \infty} \langle x, e_n \rangle = 0$

6.3.5 Bessel's Inequality

For an infinite dimensional basis, $S_N = \sum \langle x, e_n \rangle e_n = P_{M_N} x$ where $M_N = \mathcal{L}\{e_1, e_2, ..., e_n\}$, which implies

$$\sum_{n=1}^{\infty} |\langle x, e_n \rangle|^2 \leq \|x\|^2$$

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7 Sobolev Spaces

Intuitively, a Sobolev space is a space of functions with sufficiently many derivatives for some application domain, such as partial differential equations, and equipped with a norm that measures both the size and regularity of a function. It is a vector space of functions equipped with a norm that is a combination of L^p norms and the function itself as well as derivatives up to a given order. The derivatives are understood in a suitable weak sense to make the space complete, thus a Banach space. Their importance comes from the fact that solutions of partial differential equations are naturally found in Sobolev spaces, rather than in spaces of continuous functions and with the derivatives understood in the classical sense.

7.1 Weak Derivatives

Given $f \in L^p(\Omega)$, $D^\alpha f$ is defined in the $\mathcal{D}'$ sense for any α if $\exists\, g \in L^q(\Omega)$ s.t. $D^\alpha f = g$ in the $\mathcal{D}'$ sense then we say g is the **weak** $\alpha-$**derivative** of f in $L^q(\Omega)$. That is,

$$\int_\Omega f(x) D^\alpha \phi(x) dx = (-1)^{|\alpha|} \int g(x)\phi(x) dx \qquad \forall\, \phi \in C_0^\infty(\Omega)$$

- Weak derivatives need not be uniformly convergent
- Weak derivatives are defined almost everywhere
- Weak and classical derivatives coincide when $u \in C^k(\Omega)$ and $|\alpha| = k$

Example: If $f(x) = |x|$ on $(-1,1)$, then $f' = [f'] + \Delta f_0 \delta = \operatorname{sign}(x)$. So since $f; \in L^q(\Omega)$ for $1 \leq q \leq \infty$, f has a weak first derivative in $L^q(\Omega)$
Example: If $f(x) = H(x)$ then $f' = \delta \notin L^q(\Omega)$ for any q. So f has does not have a weak first derivative in $L^q(\Omega)$ for any q

7.2 Definition

The Sobolev space $W^{k,p}(\Omega)$ of **order** k is defined to be the set of all functions $u \in L^p(\Omega)$ such that for every multi-index α with $|\alpha| \leq k$, the weak partial derivative $D^\alpha u$ belongs to $L^p(\Omega)$, i.e. for an open set Ω and for $1 \leq p \leq \infty$

$$W^{k,p}(\Omega) = \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega)\ \forall\ |\alpha| \leq k\}$$

- The first (default) of norm on $W^{k,p}(\Omega)$ is

$$\|u\|_{W^{k,p}(\Omega)} = \begin{cases} \left(\sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^p(\Omega)}^p \right)^{\frac{1}{p}} & 1 \leq p < \infty \\ \max_{|\alpha| \leq k} \|D^\alpha u\|_{L^\infty(\Omega)} & p = \infty \end{cases}$$

- An equivalent norm is

$$\|u\|_{W^{k,p}(\Omega)} = \begin{cases} \left(\|u\|_{L^p(\Omega)}^p + \left\|D^k u\right\|_{L^p(\Omega)}^p \right)^{\frac{1}{p}} & 1 \leq p < \infty \\ \max_{|\alpha| \leq k} \|D^\alpha u\|_{L^\infty(\Omega)} & p = \infty \end{cases}$$

- Another equivalent norm on $W^{k,p}(\Omega)$ is

$$\|u\|'_{W^{k,p}(\Omega)} = \sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^p(\Omega)}$$

- $W^{k,p}(\Omega)$ is a Banach space
- Alternative definiton: We say f has a strong $\alpha-$derivative in $L^p(\Omega)$ if $\exists\, f_n, g \in C^\infty(\Omega)$ s.t. $f_n \to f$ in $L^p(\Omega)$ and $D^\alpha f_n \to g$ in $L^p(\Omega)$
  It turns out that the space of $f \in L^p(\Omega)$ that have a strong $\alpha-$derivative for $|\alpha| \leq k$ is the same space as $W^{k,p}(\Omega)$. That is for $|\alpha| \leq k$ f has a strong $\alpha-$derivative if and only if it has a weak $\alpha-$derivative

7.3 Embeddings

- $W^{k,2}(\Omega)$ is the Hilbert space $\mathcal{H}^k(\Omega)$ with norm $\|\cdot\|_{W^{k,2}(\Omega)}$ that can be equivalently defined as

$$\mathcal{H}^k(\Omega) = \left\{ f \in L^2(\Omega) : \sum_{n=-\infty}^{\infty} \left(1 + n^2 + n^4 + ... + n^{2k}\right) \left|\hat{f}(n)\right|^2 < \infty \right\}$$

  with norm

$$\|f\|^2_{\mathcal{H}^k(\Omega)} = \sum_{n=-\infty}^{\infty} (1 + |n|^2)^k \left|\hat{f}(n)\right|^2$$

  and inner product

$$\langle f, g\rangle_{\mathcal{H}^k(\Omega)} = \sum_{i=0}^{k} \left\langle D^i f, D^i g\right\rangle_{L^2(\Omega)}$$

  - $\mathcal{H}^1_0(\Omega)$ is subspace of $\mathcal{H}^1(\Omega)$ where functions vanish on the bondary $\partial\Omega$, complete with respect to its norm

$$\|f\|_{\mathcal{H}^1(\Omega)} = \left(\int_\Omega \left(|f|^2 + |\nabla f|^2\right)\right)^{\frac{1}{2}}$$

  - $f \in \mathcal{H}^1(\mathbb{T})$ if $c_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}dx$ satisfies $\sum\limits_{n=-\infty}^{\infty} |nc_n|^2 < \infty$
  - $\mathcal{H}^1(\mathbb{T}) \subset L^2(\mathbb{T})$

7.3.1 Sobolev Embedding Theorem

  - If $k > l$, $1 \leq p, q \leq \infty$, $(k-l)p < n$, and $\frac{1}{p} - \frac{1}{q} = \frac{k-l}{n}$ then

$$W^{k,p}(\mathbb{R}^N) \subset W^{l,q}(\mathbb{R}^N)$$

    When $k = 1$ and $l = 0$, let p^* be the **Sobolev conjugate** of p: $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$

$$W^{1,p}(\mathbb{R}^N) \subset L^{p^*}(\mathbb{R}^N)$$

  - Existence of sufficiently many weak derivatives implies some continuity of the classical derivatives.

$$\mathcal{H}^s(\mathbb{R}^N) \subset C^k_0(\mathbb{R}^N)\ \forall\ s > k + \frac{N}{2}$$

    Notably,

$$\mathcal{H}^1(\mathbb{R}) \subset C_0(\mathbb{R})$$

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7.4 Facts

- For finite p, $W^{k,p}(\Omega)$ is a separable space (contains a countable dense subset).

$$\text{If } E \subset W^{k,p}(\Omega), E \neq \emptyset, \{u_n\}_{n\in\mathbb{N}} \in W^{k,p}(\Omega), \text{ then } \exists\, n \text{ s.t. } u_n \in E$$

- $W^{0,p}(\Omega)$ is simply $L^p(\Omega)$
- $W^{1,1}(\Omega)$ is the space of **absolutely continuous** functions $AC(\Omega)$
- $W^{1,\infty}(\Omega)$ is the space of **Lipschitz continous** functions on Ω

- Formally if $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$ then the weak derivative of f is $f'(x) = \sum_{n=-\infty}^{\infty} c_n i n e^{inx}$
Bessel's equality says $\|f'\|_{L^2(\mathbb{R}^N)} = \sum_{n=-\infty}^{\infty} n {c_n}^2 < \infty$

8 Linear Operators

The space of Linear Operators $\mathcal{L}X, Y$ The norm on this space is the **Operator norm**

$$\|T\| = \sup_{x\in D(T), x\neq 0} \frac{\|Tx\|}{\|x\|} = \sup_{x\in D(T), \|x\| =1} \|Tx\| = \sup_{x\in D(T), 0<\|x\| \leq 1} \|Tx\|$$

which has the following properties

- $\|Tu\| \leq \|T\| \, \|u\|$
- $\|T+S\| \leq \|T\| + \|S\|$
- $\|TS\| \leq \|T\| \, \|S\|$
- $\|T\| = 0$ if and only if it is the **Zero map** $0u = 0$

8.1 Convergence

- The space of Bounded linear operators $\mathcal{B}(\mathcal{H})$ is a Banach space complete with respect to the operator norm with the property that $\|T\| < \infty$
- If $\|T_n - T\| \to 0$, we say $T_n \to T$ in $\mathcal{B}(\mathcal{H})$ uniformly
- If $T_n \to T$ **pointwise** or **strongly** if $T_n x \to Tx \; \forall \, x \in \mathcal{H}$ Uniform convergence implies strong convergence. Converse is false.
- The **Identity Map** $I \; X \to X$ is bounded on any normed space and has $\|I\| = 1$.

A series $\sum_{n=1}^{\infty} T_n$ converges uniformly or strongly if the corresponding sequence of partial sums does. We know $\sum_{n=0}^{\infty} T_n$ converges uniformly if $\sum_{n=0}^{\infty} \|T_n\| < \infty$

Example $T \in \mathcal{B}$, $S = \sum\limits_{n=0}^{\infty} \frac{T^n}{n!}$ converges uniformly since

$$\sum_{n=0}^{\infty} \left\| \frac{T^n}{n!} \right\| \leq \sum_{n=0}^{\infty} \frac{\|T\|^n}{n!} < \infty$$

Limit is deifned to be e^T
Let $E \in \mathcal{B}(\mathcal{H})$, $\|E\| < 1$. Then $S = \sum\limits_{n=0}^{\infty} E^n$ is convergent since

$$\sum_{j=0}^{\infty} \left\| E^j \right\| \leq \sum_{j=0}^{\infty} \|E\|^j = \frac{1}{1 - \|E\|} < \infty$$

If $S_N = \sum\limits_{n=0}^{n} E^n$,

$$S_N(I - E) = (I + E + E^2 + ... + E^N)(I - E) = I - E^{N+1}$$

as $N \to \infty$, $\left\| E^{N+1} \right\| \leq \|E\|^{N+1} \to 0$ so

$$S(I - E) = I \implies S = (I - E)^{-1}$$

or

$$(I - E)^{-1} = \sum_{n=0}^{\infty} E^n$$

In particular, $\left\| (I - E)^{-1} \right\| \leq \sum\limits_{j=0}^{\infty} \|E\|^j = \frac{1}{1 - \|E\|}$ This implies $1 \in \rho(E)$

8.2 Compact Operators

The space of **compact operators** is the linear operators $\mathcal{L}(X, Y)$ such that Y is precompact (the closure of Y is compact).

- Compact operators are bounded and contiuous.
- Any operator with finite rank is a compact operator.
- Any compact operator is a limit of finite rank operators.
- If $T_n \in \mathcal{K}(\mathcal{H})$ and $\|T_n - T\|_{\mathcal{L}(X,Y)} \to 0$, then T is compact

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9 Distribution Theory

9.1 Test Functions

For open $\Omega \subset \mathbb{R}^n$, $C_0^\infty(\Omega) = \mathcal{D}(\Omega)$ is the vector space of test functions

$$\mathcal{D}(\Omega) = \{f \in C^\infty(\Omega) : \operatorname{supp} f \subset\subset \mathbb{R}\} \qquad (f \text{ with compact support})$$

- C_0^∞ is not a metric space, but rather a Topological Vector Space (TVS)

9.1.1 Convergence in $C_0^\infty(\mathbb{R}^N)$

Let $\phi_n \in C_0^\infty$
We say $\phi_n \to 0$ in C_0^∞ if

1. $\exists\, K \subset\subset \Omega$ s.t. $\operatorname{supp} \phi_n \subset K \;\forall\, n$
2. $\|D^\alpha \phi_n\|_{C(\overline{\Omega})} \to 0$ as $n \to \infty \;\forall\, \alpha$

And so we say $\phi_n \to \phi$ in C_0^∞ if

1. $\|\phi_n - \phi\|_\infty \to 0$ (uniformly) in C_0^∞

9.1.2 Properties of C_0^∞

$C_0^\infty(\Omega)$ is closed under the following operations:
Given $\phi_1(x) \in C_0^\infty(\Omega)$, C_0^∞ is closed under the following operations:

- Scaling $\phi(x) = \alpha\phi_1(x)$
- Translation $\phi(x) = \phi_1(x - c)$
- Dilation $\phi(x) = \phi_1(\alpha x)$ for any $\alpha \neq 0$
- Differentiation $\phi(x) = D^\alpha \phi_1(x)$ for any multiindex α
- Sums of such terms
- Products of such terms
- If $\phi_1(x) \in C_0^\infty(\Omega)$ then $\phi_1(|x|) \in C_0^\infty(\Omega)$

Example: The Common Mollifier

$$\phi(x) = \begin{cases} e^{\frac{1}{x^2-1}} & |x| < 1 \\ 0 & |x| > 1 \end{cases}$$

9.2 Distributions $T \in \mathcal{D}'(\Omega)$

Distributions T are continuous linear functionals on $C_0^\infty(\Omega)$ that satisfy

1. $T : C_0^\infty(\Omega) \to \mathbb{C}$
2. $T(c_1\phi_1 + c_2\phi_2) = c_1 T(\phi_1) + c_2 T(\phi_2) \;\forall\, c_1, c_2 \in \mathbb{C}$ and $\forall\, \phi_1, \phi_2 \in C_0^\infty(\Omega)$
3. If $\phi_n \to \phi$ in $C_0^\infty(\Omega)$ then $T(\phi_n) \to T(\phi)$

That is, $\mathcal{D}'(\Omega)$ is the set of distributions over Ω , and it is the dual space of C_0^∞

9.2.1 Convergence in $\mathcal{D}'(\Omega)$

$T_n \to T$ in $\mathcal{D}'(\Omega)$ if $T_n(\phi) \to T(\phi)$ for all $\phi \in C_0^\infty(\Omega)$

$$\langle T_n, \phi \rangle \to \langle T, \phi \rangle \;\forall\; \phi \implies T_n \to T$$

9.2.2 Regular Distributions

T is a **regular distribution** if $T = T_f : \phi \to T_f(\phi)$ for some $f \in L^1_{\text{loc}}$

- The set of T_f s.t. $f \in L^1_{\text{loc}}(\Omega)$ may be regarded as a subset of $\mathcal{D}'(\Omega)$ and is defined

$$T_f(\phi) = \int_\Omega f(x)\phi(x)dx$$

- $T_{f_1} = T_{f_2}$ if and only if $f_1 = f_2$ a.e.
- If $f_n \to f$ in the L^1_{loc} sense, then $T_{f_n} \to T_f$ since $|T_{f_n}(\phi) - T_f(\phi)| \leq \|f_n - f\|_{L^1(K)}\|\phi\|_{L^\infty(K)} \to 0$
  We abuse notation and say $f_n \to f$ in the $\mathcal{D}'$ sense
- Every distribution is either **regular** or **singular**: δ is singular

9.2.3 Properties of $\mathcal{D}'$

- $\mathcal{D}'$ is closed under linear combinations.
- Multiplication for distributions is not defined in a general way.
- For $a(x) \in C^\infty$, we define $(aT)(\phi) = T(a\phi)$ since $a\phi \in C_0^\infty$ and linearity is preserved. Similiarly, $aT_f = T_f(a\phi)$.
  Example: $T = \delta$, $(a\delta)(\phi) = \delta(a\phi) = a(0)\phi(0) = a(0)\delta(\phi)$ so $a\delta = a(0)\delta$
- Distributional Derivatives are defined as $\langle T', \phi \rangle = -\langle T, \phi' \rangle$.
  For $T = T_f$ for $f \in C^1(\mathbb{R})$ we define $T_f'(\phi) = -T_f(\phi')$
- In $\mathbb{R}^N$, for a multiindex α, $(D^\alpha T)(\phi) = (-1)^\alpha T(D^\alpha \phi)$. Rather,

$$\langle D^\alpha T, \phi \rangle = (-1)^\alpha \langle T, D^\alpha \phi \rangle$$

- Any locally integrable function ($f \in L^1_{loc}(\Omega)$) has a distributional derivative
- $T'(\phi) \in \mathcal{D}'$ since $T' : \phi \to \mathbb{C}$ and is linear and if $\phi_n \to 0$ then $-T(\phi_n') \to 0$
- All distributions are infinitely differentiable in the sense of distribution, and you may suffle mixed partial derivatives.
- **Product Rule** For $a \in C^\infty$, $(\frac{\partial}{\partial x_i}(aT))(\phi) = \left(a\left(\frac{\partial}{\partial x_i}T\right)\right)(\phi) + \frac{\partial a}{\partial x_i}T(\phi)$, or

$$\langle (aT)', \phi \rangle = \langle a'T, \phi \rangle + \langle aT', \phi \rangle$$

- $D^\alpha : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$. That is, if $T_n \to T$ in $\mathcal{D}'(\Omega)$ then $D^\alpha T_n \to D^\alpha T$ in $\mathcal{D}'(\Omega)$ for all α
  Example: If $f_n \to \delta$ then $f_n' \to \delta'$
  Classically function convergence does not imply derivative convergence

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9.2.4 Properties of Distributions

- $T(\phi) = \phi(x_0)$ for some $x_0 \in \Omega$ and $\phi \in C_0^\infty$ is a distribution but not a function
- The **Dirac Delta Function** is defined as such: pick $x_0 \in \Omega$, let $T(\phi) = \phi(x_0)$, denoted

$$\delta_{x_0} : \delta_{x_0}(\phi) = \phi(x_0) \qquad \delta : \delta(\phi) = \phi(0)$$

  1. $\delta_{x_0}(x - x_0) = 0$ if $x \neq 0$
  2. $\int_{-\infty}^{\infty} \delta(x) dx = 1$

- The **Dipole Distribution**is defined

$$T(\phi) = \delta' = -\phi'(x_0) \in \mathcal{D}'$$

- $H'(\phi) = \phi(0) = \delta(\phi)$ and $H''(\phi) = \delta'(\phi) = -\delta(\phi') = -\phi'(0)$ (the dipole distributon)
- **Convolutions** in $\mathcal{D}'$ is defined for $f \in C_0^\infty$: $f * T$

$$\langle f * T, \phi \rangle = \left\langle T, \check{f}\phi \right\rangle, \qquad (f * T)(x) = T(\tau_x \check{f})$$

  and for $S * T$ where T is a distribution of compact support:

$$S * (T * \phi) = (S * T) * \phi$$

9.2.5 Aproximations to the Identity

f is said to be an **approximate identity** if $f \in L^1(\mathbb{R}^n)$ such that

1. $\int_{\mathbb{R}^n} f_n(x) dx = 1 \; \forall \; n$
2. $\exists \; c$ s.t. $\|f_n\|_{L^1(\mathbb{R}^n)} \leq c$
3. $\lim\limits_{n \to \infty} \int_{|x| > \epsilon} |f_n(x)| \, dx = 0 \; \forall \; \epsilon > 0$

Then $f_n \to \delta$ in $\mathcal{D}'$ sense

9.2.6 Principle Value Distributions

We see that

$$T(\phi) = \int_{-\infty}^{\infty} \frac{\phi(x)}{x} dx$$

looks like T_f for $f(x) = \frac{1}{x}$ but $f(x) = \frac{1}{x} \notin L^1_{\text{loc}}$, so instead we consider the **Principle Value** of the integral

$$T(\phi) = \lim_{\epsilon \to 0} \int_{|x| > \epsilon} \frac{\phi(x)}{x} dx = \text{pv} \int_{-\infty}^{\infty} \frac{\phi(x)}{x} dx = \text{pv} \left(\frac{1}{x} \right)$$

Note that the principle value distribution is not the same as the improper integral.

9.2.7 Pseudofunctions

For asymptotic singularities, we cannot use $\Delta f(c)$

Example: $f(x) = \begin{cases} \log(x) & x > 0 \\ 0 & x < 0 \end{cases}$, note that $f \notin L^1_{\text{loc}}$

$$T_f'(\phi) = -\lim_{\epsilon \to 0^+} \int_\epsilon^\infty \log(x)\phi'(x)dx = \lim_{\epsilon \to 0^+} \left(-\left[\log(x)\phi(x)\right]_\epsilon^\infty + \int_\epsilon^\infty \frac{\phi(x)}{x} dx \right)$$

Neither limit exists but their sum does due to cancellation. Using $\phi(\epsilon) = \phi(0) + \phi'(\theta)\epsilon$ by MVT,

$$T_f'(\phi) = \lim_{\epsilon \to 0^+} \left(\log(\epsilon)\phi(0) + \int_\epsilon^\infty \frac{\phi(x)}{x} dx \right) = \text{pf}\, \frac{H(x)}{x}$$

Where pf means 'pseudofunction' or 'finite part', and is itself a distribution

9.3 Tempered Distributions

Norm on the Schwartz Space

$$p_{\alpha,\beta}(\phi) = \|\phi\|_{\alpha,\beta} = \sup_{x \in \mathbb{R}^N} \left| x^\alpha \partial^\beta \phi(x) \right|$$

9.3.1 Definition of the Schwartz Space $\mathcal{S}$

The **Schwartz Space** $\mathcal{S}(\mathbb{R}^N)$ consists of all functions for which all derivatives are bounded and all derivatives are bonded by multiplication by x^α for any α. That is,

$$\mathcal{S} = \left\{ \phi \in C^\infty(\mathbb{R}^N) : p_{\alpha,\beta}(\phi) < \infty \ \forall\ \alpha, \beta \in \mathbb{Z}^n_+ \right\}$$

If $\phi \in \mathcal{S}$, then there exists a constant $C_{d,\alpha}$ such that

$$|\partial^\alpha \phi(x)| \leq \frac{C_{d,\alpha}}{(1 + |x|)^{\frac{d}{2}}} \qquad \forall\ x \in \mathbb{R}^N$$

Thus, and element of $\mathcal{S}$ is a smooth function such that the function and all of its derivatives decay faster than the reciprocal of any polynomial (that is it decays faster than any polynomial grows) as $|x| \to \infty$. Elements of $\mathcal{S}$ are called **Schwartz functions** and known as **rapidly decreasing functions**.

Schwartz Functions Facts

- $C_0^\infty(\mathbb{R}^N) \subset \mathcal{S}$
- Gaussians multiplied by any finite degree polynomial are Schwartz functions

$$q(x)e^{-c|x-x_0|^2} \in \mathcal{S} \ \forall\ c > 0, x_0 \in \mathbb{R}^N, q(x) = \sum_{|\alpha| \leq d} c_\alpha x^\alpha$$

- $\mathcal{S} \subset L^1(\mathbb{R}^N)$
- $\mathcal{S}$ is dense in $L^p(\mathbb{R}^N)$ for $1 \leq p < \infty$ since C_0^∞ is dense in $\mathbb{R}^N$

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9.4 Distributions of Compact Support

E is the subspace of distributions with compact support.
Define $E = C^\infty(\mathbb{R}^N)$ with $\phi_n \to \phi \in E$ if $\|D^\alpha(\phi_n - \phi)\|_{L^\infty(K)} \to 0$ for all $K \subset\subset \mathbb{R}^N$
That is, $T \in E$ if and only if $T \in \mathcal{D}'$ and $\operatorname{supp} T \subset\subset \mathbb{R}^N$
So $C_0^\infty(\mathbb{R}^N) \subset \mathcal{S} \subset E$ are toplogical inclusions and so $E' \subset \mathcal{S}' \subset \mathcal{D}'$

- **Support of Distributions**: $x \in \operatorname{supp} T$ if and only if $\forall\, \epsilon > 0\ \exists\, \phi \in C_0^\infty(B_\epsilon(x))$ s.t. $T(\phi) \neq 0$
- $T = \delta$ If $x \neq 0$ let $\epsilon = \frac{|x|}{2}$ then $\phi \in C_0^\infty(B_\epsilon(x))$ then $\delta(\phi) = \phi(0) = 0$ and so $\operatorname{supp} \delta = \{0\}$
- $\hat{T}$ is defined for all $T \in E'$ since $E' \subset \mathcal{D}'$