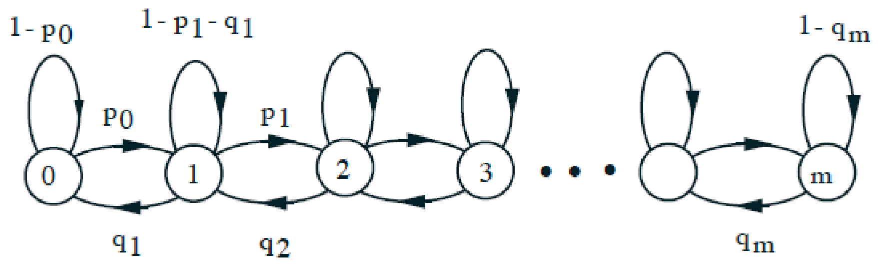
IC566c: Random Variables and Random Processes (Spring 2023)

Assignment 6 (Due: May 31, 2023)

1. A birth-death process is shown below with m+1 states with transition probability p_i from state i to i+1, and q_i from i to i-1.



(a) Explain that this process has steady-state probabilities π_i. (i.e., the n-step transition probabilities converge to π_i.**)**

(b) Assume $p_i = p$ and $q_i = q$ (> p) for every i. Also assume that m goes to infinity. Find π_i.

2. Consider a Markov process with two states and transition probability matrix

$$P = \begin{bmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{bmatrix}$$

(a) Draw a Markov chain showing two states and the transition probabilities.

(b) Find the stationary distribution π_1 and π_2 of the chain.

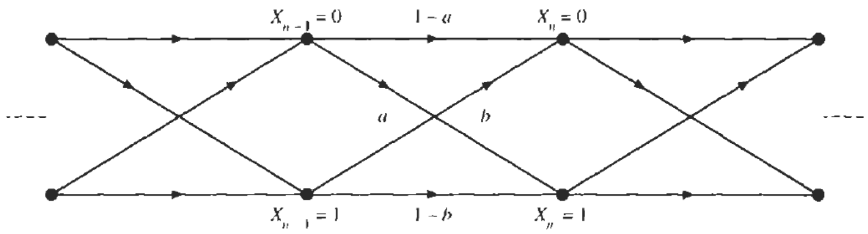
3. An example of a two-state Markov chain is provided by a communication network consisting of the sequence (or cascade) of stages of binary communication channels shown in the figure below. Here X_n denotes the digit leaving the nth stage of the channel and X_0 denotes the digit entering the first stage. The transition probability matrix of this communication network is often called the channel matrix, which is

$$P = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix} \qquad 0 < a < 1,\ 0 < b < 1$$

Assume that a = 0.1 and b = 0.2, and the initial distribution is $P(X_0 = 0) = P(X_0 = 1) = 0.5$.

(a) Find the distribution of X_n.

(b) Find the distribution of X_n when n goes to infinity.



4. There are two types of calls to the Dalseong chicken delivery house. Menu A calls arrive as a Poisson process with rate $\lambda_A = 1$. Menu B calls arrive as an independent Poisson process with rate $\lambda_B = 2$. Let us fix t to be the time that you started to solve this homework.

(a) What is the expected length of the interval that t belongs to? That is, the interval from the last call before t until the first call after t.

(b) What is the probability that t belongs to an AA interval? That is, the first event before, as well as the first event after time t are both of menu A.

(c) What is the probability that between t and t+1, we have exactly two events, one of menu A, followed by one of menu B?

(d) Suppose that a menu B call just arrived. Find the expected time until the end of the first future AA interval? (Hint: Construct a Markov chain with states of A, B and END. The END state is for the end of the first future AA interval. Then, find a pair of equations for the expected time to the END state starting from the A state and the B state.)

5. A weather condition over the satellite link is modeled to be a three-state Markov chain with Sunny, Cloudy, and Rainy states of the transition probability matrix

$$P = \begin{bmatrix} 3/4 & q & 0 \\ 1/2 & r & 1/4 \\ 0 & 3/4 & s \end{bmatrix}$$

(a) Draw a Markov chain showing three states and the transition probabilities.

(b) Find the values of q, r, and s.

(c) Explain that this process has steady-state probabilities π_i. (i.e., the n-step transition probabilities converge to π_i.)

6. You flip a fair coin 36 times. Let S_{36} denote the number of heads resulting from the 36 tosses. Now you want to calculate $P(S_{36} = 19)$ by using the following two methods.

(a) If you use the binomial coefficient $\binom{n}{k}$ with some integers n and k, what is the exact expression (or number) of the probability?

(b) If you use the central limit theorem and approximate the probability as $P(18.5 < S_{36} < 19.5)$, what is the expression (or number) of the approximate probability? You may use $\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$ with some real number z.

7. You want to design a poll for one Yes-or-No question, to which the fraction f of the total population say Yes.

(a) Let X_i be equal to 1 if the ith person polls Yes, and 0 if No. Denote M_n as the sample mean of X_i's with sample size n. Assuming that all X_i's are i.i.d., what are the mean and variance of M_n?

(b) Given that $P(|Z|>1.96) = 0.05$ for a standard normal random variable Z, what is the minimum number of the poll size to guarantee the 95% confidence of less than 1% error?

(c) Now you do not have access to the standard normal distribution table, and should instead use the Chebyshev's inequality

$$P(|X - \mu| \geq c) \leq \frac{\sigma^2}{c^2}$$

What is the minimum number of the poll size to guarantee the 95% confidence of less than 1% error?