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Calculation of tree-level helicity amplitudes using the massless spinor-helicity formalism

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Abstract. Using the massless spinor-helicity formalism the tree-level helicity amplitudes of massless Møller and Compton scattering are calculated in detail. Further, some implications of the fact, that the spinor-helicity expressions for photon polarization vectors manifestly realize the residual gauge freedom of Lorentz gauge, are examined in two additional calculations. The main finding is that the poles of the full amplitude are not only determined by the propagators of the different channels but also by the polarization vectors themselves.

Contents

1 Introduction

Helicity amplitudes are one of the main building blocks of elementary particle physics from which measurable predictions for cross sections and decay rates are computed. The typical mathematical framework in which these calculation are performed is, of course, the Standard Model of particle physics, a perturbative quantum field theory one may argue is the most successful theory of nature humankind has put forth. Hence every simplification in the calculation of helicity amplitudes is very much appreciated by theoretical physicists. One possible method for this has become known as the massless spinor-helicity formalism. This paper aims to put this method to the test by explicitly applying it to two simple tree-level processes.

To accomplish this goal, the note is structured as follows

- Section 2 provides a short overview of the definitions, methods and relations collectively called the massless spinor-helicity formalism. In particular, we will provide identities relating the typical polarization vectors for massless spin-1/2 and spin-1 particles to their spinor-helicity version. Additionally, we will state all the relevant mathematical relations to be used in the subsequent calculations. We will, however, not prove them.

- Section 3 shows the explicit calculation of the tree-level helicity amplitudes for massless, i.e. ultrarelativistic, Møller scattering. We will begin by calculating the individual helicity amplitudes, demonstrating the massless spinor-helicity formalism at work. To verify our result, we will perform the explicit calculation of the unpolarized amplitude squared, i.e. we are going to simply square each of our amplitudes and them sum then up.

- Section 4 follows the same strategy as in Section 3 but applies it to the case of tree-level, massless Compton scattering. In addition, we will study two interesting facets of the manifestly residual-gauge-redundant expressions for the photon polarization vectors in the spinor-helicity formalism.

- Section 5 brings the thesis to a close by summarizing the material covered and considering possible directions for further research.

2 Brief review of the massless spinor-helicity formalism

In this section, we briefly review, without proof, the definition of massless spinor-helicity variables and their mathematical relations relevant for Sections 3 and 4. This is based on Chapter 2 of [1] but we'll use the conventions in Appendix A. So let's begin.

2.1 Massless spin-$1/2$ particles

Given the Dirac polarization vectors $u_{L,R}(p)$ and $\bar{v}_{L,R}(p)$ for a massless spin-1/2 particle of 4-momentum p^μ, we define spinor-helicity variables $\langle p|^\alpha$, $|p\rangle_\alpha$, $[p|_{\dot\alpha}$ and $|p]^{\dot\alpha}$ by

$$\begin{aligned} \bar{v}_L(p) &= \begin{pmatrix} \langle p| & 0 \end{pmatrix} & \qquad u_L(p) &= \begin{pmatrix} |p\rangle \\ 0 \end{pmatrix} \\ \bar{v}_R(p) &= \begin{pmatrix} 0 & [p| \end{pmatrix} & \qquad u_R(p) &= \begin{pmatrix} 0 \\ |p] \end{pmatrix} \end{aligned} \tag{1}$$

and call them *helicity-spinors*. Mathematically speaking these are the 2-component Weyl spinors[1] where their index position and dottedness has an underlying representation theoretic justification. Details can be found in Chapter 13 of [2]. For us the first important implication is that these indices can be raised an lowered by the so-called *spinor metric*

$$\varepsilon = (\!(\varepsilon^{\alpha\beta})\!) = (\!(\varepsilon^{\dot\alpha\dot\beta})\!) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = (\!(-\varepsilon_{\alpha\beta})\!) = (\!(-\varepsilon_{\dot\alpha\dot\beta})\!) = -\varepsilon^{-1} \tag{2}$$

Concretely this means

$$\langle p|^\alpha = \varepsilon^{\alpha\beta}|p\rangle_\beta \qquad\qquad |p\rangle_\alpha = \varepsilon_{\alpha\beta}\langle p|^\beta \tag{3}$$

$$[p|_{\dot\alpha} = \varepsilon_{\dot\alpha\dot\beta}|p]^{\dot\beta} \qquad\qquad |p]^{\dot\alpha} = \varepsilon^{\dot\alpha\dot\beta}[p|_{\dot\beta} \tag{4}$$

and implies that we can build so-called *helicity-spinor products*

$$\langle pq\rangle = \langle p|^\alpha\, |q\rangle_\alpha = \varepsilon^{\alpha\beta}|p\rangle_\beta|q\rangle_\alpha \tag{5}$$

$$[pq] = [p|_{\dot\alpha}\, |q]^{\dot\alpha} \;\; = \varepsilon_{\dot\alpha\dot\beta}|p]^{\dot\beta}|q]^{\dot\beta} \tag{6}$$

which are scalars under $\mathrm{SL}(2,\mathbb{C})$ and anti-symmetric due to Equation (2)

$$\langle pq\rangle = -\langle qp\rangle \tag{7}$$

[1]Roughly speaking one could just think of them as elements of $\mathbb{C}^2$ which transform in a special way under Lorentz transformations.

$$[pq] = -[qp] \tag{8}$$

Thus

$$\langle pp \rangle = 0 = [pp] \tag{9}$$

The second main consequence is the connection between undotted and dotted indices through complex conjugation

$$(\langle p|^{\alpha})^* = |p]^{\dot{\alpha}} \qquad\qquad (|p\rangle_{\alpha})^* = [p|_{\dot{\alpha}} \tag{10}$$

which in turn implies

$$\left(\langle pq \rangle\right)^* = [qp] \tag{11}$$

Moving on, two widely used notational conventions for helicity-spinors are:

1. Given the 4-momentum p_i^{μ} of the i-th particle in a scattering process we write $|i\rangle$ instead of $|p_i\rangle$.
2. Whenever there is an expression which mixes angle with square helicity-spinors or 4-component objects with helicity-spinors, these are interpreted by translating the helicity-spinors back to Dirac spinors using Equation (1).

Sometimes a concrete parametrization of the helicity-spinors is needed. Given a 4-momentum p_i^{μ} in spherical coordinates

$$p_i^{\mu} = \Big(E_i,\ E_i \sin(\theta_i)\cos(\phi_i),\ E_i \sin(\theta_i)\sin(\phi_i),\ E_i \cos(\theta_i)\Big)^{\mathsf{T}}$$

a parametrization is given by

$$\begin{aligned}
\langle i|^{\alpha} &= \sqrt{2E_i}\begin{pmatrix} \cos(\frac{\theta_i}{2}) \\ \sin(\frac{\theta_i}{2})\mathrm{e}^{-\mathrm{i}\phi_i} \end{pmatrix} & |i\rangle_{\alpha} &= \sqrt{2E_i}\begin{pmatrix} -\sin(\frac{\theta_i}{2})\mathrm{e}^{-\mathrm{i}\phi_i} \\ \cos(\frac{\theta_i}{2}) \end{pmatrix} \\
[i|_{\dot{\alpha}} &= \sqrt{2E_i}\begin{pmatrix} -\sin(\frac{\theta_i}{2})\mathrm{e}^{+\mathrm{i}\phi_i} \\ \cos(\frac{\theta_i}{2}) \end{pmatrix} & |i]^{\dot{\alpha}} &= \sqrt{2E_i}\begin{pmatrix} \cos(\frac{\theta_i}{2}) \\ \sin(\frac{\theta_i}{2})\mathrm{e}^{+\mathrm{i}\phi_i} \end{pmatrix}
\end{aligned} \tag{12}$$

Let's now turn our attention to their mathematical relations. We have

$$\langle ij \rangle = \sqrt{s_{ij}} \cdot \mathrm{e}^{-\mathrm{i}\phi_{ij}} \tag{13}$$

$$[ji] = \sqrt{s_{ij}} \cdot \mathrm{e}^{+\mathrm{i}\phi_{ij}} \tag{14}$$

$$\langle ij] = 0 = [ij\rangle \tag{15}$$

where $s_{ij} := (p_i + p_j)^2$ and $\phi_{ij} \in \mathbb{R}$. The underlying relation is

$$s_{ij} = \langle ij\rangle[ji] \tag{16}$$

The phase ϕ_{ij} is a frame-dependent quantity which captures analytic properties of helicity amplitudes. Its concrete value is not important, but its relation to other phases is essential for interference effects. One way to ensure correct relative phases is to specialize to a reference frame right at the beginning of a calculation and evaluate them in that frame like in Appendix B. Alternatively, one could use the following frame-independent method: apply Equation (8) to Equation (14) and conclude

$$\phi_{ji} = \phi_{ij} + \pi \tag{17}$$

Now define

$$\hat{\phi}_{ij} = -\phi_{ij} \qquad\qquad \bar{\phi}_{ij} = \phi_{ji} \tag{18}$$

These in combination with the simple relation $s_{ij} = s_{ji}$ allow us to write

$$\langle ij\rangle = \sqrt{s_{ij}}\,\mathrm{e}^{\mathrm{i}\hat{\phi}_{ij}} \tag{19}$$

$$[ij] = \sqrt{s_{ij}}\,\mathrm{e}^{\mathrm{i}\bar{\phi}_{ij}} \tag{20}$$

where we'll refer to the phases ϕ_{ij}, $\hat{\phi}_{ij}$ and $\bar{\phi}_{ij}$ as *massless phases*. Another useful relation is 4-momentum conservation expressed in terms of helicity-spinors

$$\sum_{i=1}^{n} \langle pi\rangle[iq] = 0 \tag{21}$$

where p, q are arbitrary massless 4-momenta.

Before we turn our attention to massless spin-1 particles, we have to investigate a very common object in the spinor-helicity formalism: γ-sandwiches. These are defined as follows

$$\langle i|\gamma^\mu|j\rangle = 0 = [i|\gamma^\mu|j] \tag{22}$$

$$\langle i|\gamma^\mu|j] = \langle i|\sigma^\mu|j] \tag{23}$$

$$[i|\gamma^\mu|j\rangle = [i|\bar{\sigma}^\mu|j\rangle \tag{24}$$

where our conventions for the γ- and Pauli-matrices are collected in Appendix A. We are going to manipulate them in the following way

$$\langle i|\gamma^\mu|j] = [j|\gamma^\mu|i\rangle \tag{25}$$

$$\Big(\langle i|\gamma^\mu|j]\Big)^* = \langle j|\gamma^\mu|i] \tag{26}$$

$$\langle i|\not{p}|j] = \langle ip\rangle[pj] \tag{27}$$

A very useful identity is the Fierz identity

$$\langle 1|\gamma_\mu|2]\langle 3|\gamma^\mu|4] = 2\langle 13\rangle[42] \tag{28}$$

Lastly, there is a generalization to sandwiches of n γ-matrices called n-γ-sandwiches. We'll only make use of the generalization of Equation (22): Let $n \in \mathbb{N}$, then

$$\langle i|\gamma^{\mu_1}\cdots\gamma^{\mu_{2n+1}}|j\rangle = 0 = [i|\gamma^{\mu_1}\cdots\gamma^{\mu_{2n+1}}|j] \tag{29}$$

2.2 Massless spin-1 particles

Given the polarization vectors $\varepsilon^\mu_\pm(p)$ for a massless spin-1 particle, we can express the entire residual gauge orbit[2] of $\varepsilon^\mu_\pm(p)$ in terms of spinor-helicity variables:

$$\varepsilon^\mu_-(p;q) = -\frac{1}{\sqrt{2}}\frac{[q|\gamma^\mu|p\rangle}{[qp]} \qquad\qquad \varepsilon^\mu_+(p;q) = +\frac{1}{\sqrt{2}}\frac{\langle q|\gamma^\mu|p]}{\langle qp\rangle} \tag{30}$$

where q^μ is an arbitrary 4-momentum that is massless, not proportional to p^μ and which parametrizes the residual gauge orbit. In order to choose the gauge spinor correctly one has to keep the following in mind

> Given one Feynman diagram one can choose the gauge spinor freely for each external massless spin-1 particle (except for $|q\rangle$ not proportional to $|p\rangle$) and independently of the other external massless spin-1 particles. But given a perturbative expansion of an amplitude one has to stick to this choice throughout all Feynman diagrams. Only then will the full amplitude be independent of this choice.

[2]The term *residual gauge* refers to the residual gauge symmetry that is left over when imposing Lorentz gauge on free electrodynamics. The term *residual gauge orbit* refers to a set of massless spin-1 polarization vectors which are residually gauge equivalent.

Further, we'll make use of the following mathematical identity

$$\not{\epsilon}_-(p;q) = -\sqrt{2}\,\frac{|p\rangle[q| + |q]\langle p|}{[qp]} \qquad\qquad \not{\epsilon}_+(p;q) = +\sqrt{2}\,\frac{|p]\langle q| + |q\rangle[p|}{\langle qp\rangle} \tag{31}$$

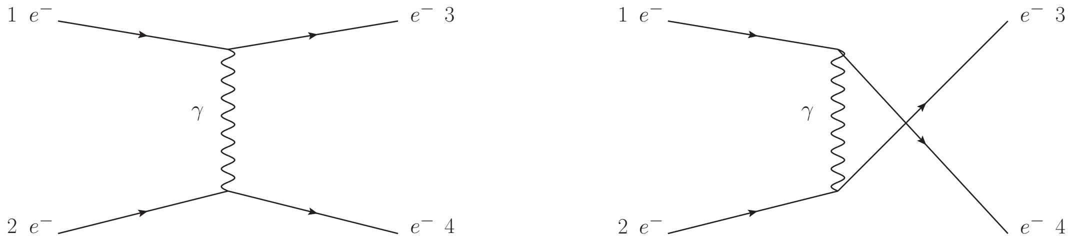
Last but not least, note that we'll sometimes employ variations of the presented relations like $[i|\not{p}|j\rangle = [ip]\langle pj\rangle$ which is a variation of Equation (27). These can easily be derived using other equations presented in this section, e.g. Equations (7) and (8).

3 Massless Møller scattering

We are looking at the process

$$e^-(p_1, \lambda_1) + e^-(p_2, \lambda_2) \quad \to \quad e^-(p_3, \lambda_3) + e^-(p_4, \lambda_4)$$

in QED and in the ultrarelativistic limit, i.e. electron mass $m \ll |\vec{p}|$ or practically speaking $m = 0$. The Feynman diagrams at tree level are shown in Figure 1 where these and all following diagrams were created with [3].



(a) t-channel

(b) u-channel

Figure 1: Tree-level diagrams for Møller scattering.

3.1 Helicity amplitudes

Let's first look at the t-channel diagram in Figure 1a. The traditional Feynman rules in our conventions of Appendix A yield[3]

$$\mathcal{M}_t(1^{\pm}_{e^-}, 2^{\pm}_{e^-}, 3^{\pm}_{e^-}, 4^{\pm}_{e^-}) = \frac{e^2}{t} \, \bar{v}_3 \gamma^\mu u_1 \, \bar{v}_4 \gamma_\mu u_2$$

where $t := (p_1 + p_3)^2$ as in Equation (35) and $u_i := u(p_i, \lambda_i)$.

Now we'd like to transition to the spinor-helicity formalism using the spinor-helicity Feynman rules which are conveniently collected in Appendix C. For massless particles we directly have to specialize to a specific helicity. Let us go naively about this and look through some specific choices of polarizations to get a feel for working with the spinor-helicity formalism.

Choosing $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$. Taking only e.g. positive helicity electrons, we have

$$\mathcal{M}_t^{++++} = \frac{e^2}{t} \, [3|\gamma^\mu|1] \, [4|\gamma_\mu|2]$$

[3] Being very conscientious about terminology it'd be wrong to call the t-channel diagram an amplitude. Nonetheless we use our amplitude notation of Appendix A for the diagrams themselves.

According to Equation (22) this expression vanishes

$$= 0$$

The same thing happens if we choose all electrons to have negative helicity, i.e.

$$\mathcal{M}_t^{----} = \frac{e^2}{t} \langle 3|\gamma^\mu|1\rangle \langle 4|\gamma_\mu|2\rangle = 0$$

Choosing $\lambda_1 = \lambda_3$ *or* $\lambda_2 = \lambda_4$. Generalizing above result, we see that whenever $\lambda_1 = \lambda_3$ or $\lambda_2 = \lambda_4$ we get a γ-sandwich involving only angle or square helicity-spinors. These kill the amplitude, i.e.

$$\mathcal{M}_t^{+++-} = \frac{e^2}{t} [3|\gamma^\mu|1] \langle 4|\gamma_\mu|2] = 0 \quad \ldots$$

So we note

The t-channel diagram for this process is non-vanishing only if particles 1 and 3, and likewise particles 2 and 4, have opposite helicities.

which, when using the terminology of Appendix A and taking particles 3 and 4 as outgoing, means that particles 1 and 3, as well as particles 2 and 4, have to have the same physical helicity.

Choosing $\lambda_1 = \lambda_2 \neq \lambda_3 = \lambda_4$. Consider the non-vanishing diagram

$$\begin{aligned}\mathcal{M}_t^{++--} &= \frac{e^2}{t} \langle 3|\gamma^\mu|1] \langle 4|\gamma_\mu|2] \\ &= \frac{2e^2}{t} \langle 34\rangle [21] && \text{by Equation (28)}\end{aligned}$$

and we finally arrived at an expression purely in terms for helicity-spinor products. Basically our work would be done if we'd accept the helicity-spinor products as fundamental objects. We will, however, perform the evaluation in terms of Mandelstam invariants explicitly to see how it works. This will require introducing a reference frame and dealing with the massless phases we already talked about in Section 2. In order to ensure correct relative phases and to postpone the specialization to a concrete frame, we are going to use Equations (19) and (20) instead of Equations (13) and (14) and arrange all helicity-spinor products such that the numbers inside are increasing. This way they are taken care of automatically and kept general. So we have

$$\begin{aligned}
\mathcal{M}_t^{++--} &= -\frac{2e^2}{t}\,\langle 34\rangle\,[12] && \text{by Equation (7)} \\
&= -\frac{2e^2}{t}\,\sqrt{s_{34}}\,\mathrm{e}^{\mathrm{i}\hat{\phi}_{34}}\,\sqrt{s_{12}}\,\mathrm{e}^{\mathrm{i}\bar{\phi}_{12}} && \text{by Equations (19) and (20)} \\
&= -\frac{2e^2}{t}\,\sqrt{s}\,\sqrt{s}\,\mathrm{e}^{\mathrm{i}(\hat{\phi}_{34}+\bar{\phi}_{12})} && \text{by Equation (34)} \\
&= -2e^2\,\frac{s}{t}\,\mathrm{e}^{\mathrm{i}(\hat{\phi}_{34}+\bar{\phi}_{12})}
\end{aligned}$$

So we're essentially left with a ratio of Mandelstam variables and a combination of massless phases. All other polarized diagrams are calculated in an analogous way. Consider for example the diagram with all helicities flipped

$$\begin{aligned}
\mathcal{M}_t^{--++} &= \frac{e^2}{t}\,[3|\gamma^\mu|1\rangle\,[4|\gamma_\mu|2\rangle \\
&= \frac{e^2}{t}\,\langle 1|\gamma^\mu|3]\,\langle 2|\gamma_\mu|4] && \text{by Equation (25)}
\end{aligned}$$

and an analogous calculation to above shows

$$= -2e^2\,\frac{s}{t}\,\mathrm{e}^{\mathrm{i}(\hat{\phi}_{12}+\bar{\phi}_{34})}$$

This is just the complex conjugate[4] of $\mathcal{M}_t^{++--}$ which we could've seen even from the get-go with the help of Equation (26).

Choosing $\lambda_1 = \lambda_4 \neq \lambda_2 = \lambda_3$. In much the same way we can conclude

$$\begin{aligned}
\mathcal{M}_t^{+--+} &= \frac{e^2}{t}\,\langle 3|\gamma^\mu|1]\,[4|\gamma_\mu|2\rangle \\
&= \frac{e^2}{t}\,\langle 3|\gamma^\mu|1]\,\langle 2|\gamma_\mu|4] && \text{by Equation (25)} \\
&= \frac{2e^2}{t}\,\langle 23\rangle\,[14] && \text{by Equation (28) and Equation (7)} \\
&= \frac{2e^2}{t}\,\sqrt{s_{23}}\sqrt{s_{14}}\,\mathrm{e}^{\mathrm{i}(\hat{\phi}_{23}+\bar{\phi}_{14})} && \text{by Equations (19) and (20)} \\
&= 2e^2\,\frac{u}{t}\,\mathrm{e}^{\mathrm{i}(\hat{\phi}_{23}+\bar{\phi}_{14})} && \text{by Equation (36)}
\end{aligned}$$

[4] There is a tedious intricacy due to the abbreviated notation for massless phases of Section 2. Using Equations (17) and (18) we conclude $\hat{\phi}_{ij} + \bar{\phi}_{kl} = -(\hat{\phi}_{kl} + \bar{\phi}_{ij})$ which ensures that our results are indeed complex conjugates of each other.

$$\mathcal{M}_t^{-++-} = ... = 2e^2\,\frac{u}{t}\,\mathrm{e}^{\mathrm{i}(\hat{\phi}_{14}+\bar{\phi}_{23})}$$

Having worked through the calculation of the t-channel diagram, we move on to the u-channel in Figure 1b. The traditional Feynman rules yield

$$\mathcal{M}_u(1^{\pm}_{e^-},\, 2^{\pm}_{e^-},\, 3^{\pm}_{e^-},\, 4^{\pm}_{e^-}) = \frac{e^2}{u}\,\bar{v}_4\gamma^{\mu}u_1\,\bar{v}_3\gamma_{\mu}u_2$$

where $u := (p_1+p_4)^2$ as in Equation (36) and the only difference to $\mathcal{M}_t$ is the exchange of particle 3 and 4.

Other than that the translation and calculation in the spinor-helicity formalism is again analogous. Hence we'll just state the result while referring to above calculations for details.

$$\mathcal{M}_u^{++++} = \mathcal{M}_u^{----} = \mathcal{M}_u^{+++-} = \ldots = 0$$

with the conclusion

The u-channel diagram for this process is non-vanishing only if particles 1 and 4, and likewise particles 2 and 3, have opposite helicities.

The non-vanishing u-channel diagrams then are

$$\mathcal{M}_u^{++--} = 2e^2\,\frac{s}{u}\,\mathrm{e}^{\mathrm{i}(\hat{\phi}_{34}+\bar{\phi}_{12})} = \left(\mathcal{M}_u^{--++}\right)^*$$
$$\mathcal{M}_u^{+-+-} = 2e^2\,\frac{t}{u}\,\mathrm{e}^{\mathrm{i}(\hat{\phi}_{24}+\bar{\phi}_{13})} = \left(\mathcal{M}_u^{-+-+}\right)^*$$

Now that we have both channels we can calculate all helicity amplitudes to tree level by:

$$\mathcal{M} = \mathcal{M}_t - \mathcal{M}_u$$

Note that for $\mathcal{M}^{++--}$ and $\mathcal{M}^{--++}$ the phases of the t- and u-channel match up nicely, whereas all other helicity amplitudes either vanish or only contain one non-vanishing channel. To avoid a convoluted final expression littered with combinations of massless phases we state the final result in the frame of Appendix B, i.e. using the massless phases of Equation (46). So we have[5]

$$\mathcal{M}^{\pm\pm\pm\pm} = 2e^2 \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \frac{s}{t}+\frac{s}{u} \end{pmatrix} & \begin{pmatrix} 0 & -\frac{t}{u} \\ -\frac{u}{t} & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & -\frac{u}{t} \\ -\frac{t}{u} & 0 \end{pmatrix} & \begin{pmatrix} \frac{s}{t}+\frac{s}{u} & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \tag{32}$$

3.2 Verification

The unpolarized amplitude squared is given by

$$\frac{1}{4}\sum_{\text{pol}} |\mathcal{M}|^2$$

Explicitly performing the helicity sum using our result in Equation (32) we get

$$\begin{aligned} \frac{1}{4}\sum_{\text{pol}} |\mathcal{M}|^2 &= 2e^4 \left(s^2 \left(\frac{1}{t^2} + \frac{2}{tu} + \frac{1}{u^2} \right) + \frac{u^2}{t^2} + \frac{t^2}{u^2} \right) \\ &= 2e^4 \left(\frac{s^2+u^2}{t^2} + \frac{s^2+t^2}{u^2} + \frac{2s^2}{tu} \right) \end{aligned}$$

which matches the literature result in Equation (10.20) of [4] when setting $m = 0$.

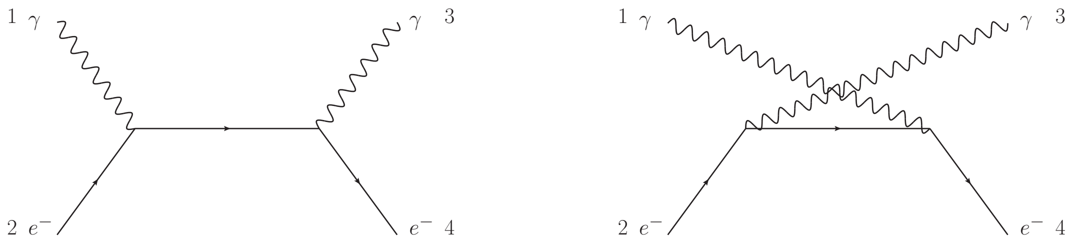
[5]Nested matrices are read as follows: The first two $\pm$'s determine which inner matrix to look at and the second two $\pm$'s the entry in that inner matrix where, when thinking in terms of typical matrix indices, $+$ corresponds to 1 and $-$ to 2.

4 Massless Compton scattering

We are considering the process

$$\gamma(p_1, \lambda_1) + e^-(p_2, \lambda_2) \quad \rightarrow \quad \gamma(p_3, \lambda_3) + e^-(p_4, \lambda_4)$$

in QED and in the ultrarelativistic limit, i.e. electron mass $m \ll |\vec{p}|$ or practically speaking $m = 0$. The Feynman diagrams at tree-level are shown in Figure 2



(a) s-channel (b) u-channel

Figure 2: Tree-level diagrams for Compton scattering.

4.1 Helicity amplitudes

As we are working with massless spin-1 particles we have to keep the remark on choosing gauge spinors of Section 2.2 in mind. The modest number of diagrams in this calculation allows us to ensure this by simply considering both channels simultaneously. Using the traditional Feynman rules for Figures 2a and 2b we get

$$\begin{aligned}\mathcal{M}(1^{\pm}_{\gamma},\, 2^{\pm}_{e^-},\, 3^{\pm}_{\gamma},\, 4^{\pm}_{e^-}) &= \mathcal{M}_s(1^{\pm}_{\gamma},\, 2^{\pm}_{e^-},\, 3^{\pm}_{\gamma},\, 4^{\pm}_{e^-}) + \mathcal{M}_u(1^{\pm}_{\gamma},\, 2^{\pm}_{e^-},\, 3^{\pm}_{\gamma},\, 4^{\pm}_{e^-}) \\ &= -\frac{e^2}{s}\, \bar{v}_4 \not{\varepsilon}_3 (\not{p}_1 + \not{p}_2) \not{\varepsilon}_1 u_2 - \frac{e^2}{u}\, \bar{v}_4 \not{\varepsilon}_1 (\not{p}_2 + \not{p}_3) \not{\varepsilon}_3 u_2\end{aligned}$$

Just as in Section 3 we are going to use the spinor-helicity version of the Feynman rules in Appendix C and – due to the particles being massless – we are going to have to specialize to a specific helicity right away. Without further ado:

Choosing $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$. Taking all particles as having e.g. positive helicity, let's only insert spinor-helicity variables for the Dirac spinors while writing out the Feynman slash

$$\mathcal{M}_s^{++++} = -\frac{e^2}{s}\, \varepsilon_3^{\mu}(p_1^{\nu} + p_2^{\nu})\varepsilon_1^{\rho}\, [4|\gamma_\mu \gamma_\nu \gamma_\rho|2]$$

$$\mathcal{M}_u^{++++} = -\frac{e^2}{u}\, \varepsilon_1^{\mu}(p_2^{\nu} + p_3^{\nu})\varepsilon_3^{\rho}\, [4|\gamma_\mu \gamma_\nu \gamma_\rho|2]$$

This way we see odd n-γ-sandwiches which according to Equation (29) vanish! So we have

$$\mathcal{M}^{++++} = 0$$

Choosing $\lambda_2 = \lambda_4$. This choice always leads to the above odd n-γ-sandwich which allows us to generalize

The helicity amplitude for this process is non-vanishing only if particles 2 and 4 have opposite helicities.

which, when using the terminology of Appendix A and taking particles 3 and 4 as outgoing, means that particles 2 and 4 have to have the same physical helicity.

Choosing $\lambda_1 = \lambda_2 = \lambda_3 \neq \lambda_4$. Inserting all spinor-helicity Feynman rules and using Equation (31) we get

$$\begin{aligned}
\mathcal{M}_s^{+++-} &= -\frac{e^2}{s} \langle 4| \sqrt{2}\frac{|3]\langle q_3| + |q_3\rangle[3|}{\langle q_3 3\rangle} (\not{p}_1 + \not{p}_2) \sqrt{2}\frac{|1]\langle q_1| + |q_1\rangle[1|}{\langle q_1 1\rangle} |2] \\
&= -\frac{2e^2}{s} \frac{1}{\langle q_1 1\rangle\langle q_3 3\rangle} \langle 4q_3\rangle [3|\not{p}_1 + \not{p}_2|q_1\rangle [12]
\end{aligned}$$

$$\begin{aligned}
\mathcal{M}_u^{+++-} &= -\frac{e^2}{s} \langle 4| \sqrt{2}\frac{|1]\langle q_1| + |q_1\rangle[1|}{\langle q_1 1\rangle} (\not{p}_2 + \not{p}_3) \sqrt{2}\frac{|3]\langle q_3| + |q_3\rangle[3|}{\langle q_3 3\rangle} |2] \\
&= -\frac{2e^2}{u} \frac{1}{\langle q_1 1\rangle\langle q_3 3\rangle} \langle 4q_1\rangle [1|\not{p}_2 + \not{p}_3|q_3\rangle [32]
\end{aligned}$$

where we've used Equation (15) to simplify. Seeing the two helicity-spinor products $\langle 4q_3\rangle$ and $\langle 4q_1\rangle$ we make the valid choice for the gauge momenta $q_1 = q_3 = p_4$. This kills both channels and yields

$$\mathcal{M}^{+++-} = 0$$

You'd be right to feel uneasy about the gauge spinors and the freedom in choosing them at this point. In fact, one can show that $\mathcal{M}^{+++-}$ vanishes *independent* of the gauge spinor choice, though the calculation is more involved. Since this lies somewhat outside the main discussion, the calculation is shown in Section 4.3.1.

Choosing $\lambda_1 = \lambda_3$. This choice yields a similar result to the choice $\lambda_2 = \lambda_4$ but is a bit more tricky to see as it's connected to the structure of an $\not\epsilon$ and our freedom in picking the gauge spinors. The argument goes as follows: Comparing $\mathcal{M}_s$ and $\mathcal{M}_u$ we see that fermion 2 once shares a vertex with photon 1 and once with photon 3. The same goes for fermion 4. Picking the same helicities for the photons, i.e. $\lambda_1 = \lambda_3$, then implies that in both channels one fermion will only be involved in helicity-spinor products with the photon's gauge spinors as the fermions have to have opposite helicities. This then allows us to pick the gauge momenta to be equal to this fermion's momentum which kills both channels and thus the amplitude. In the previous choice this was realized by fermion 4 and the helicity-spinor products $\langle 4q_3\rangle$ and $\langle 4q_1\rangle$.

So we again conclude

> The helicity amplitude for this process is non-vanishing only if particles 1 and 3 have opposite helicities.

which, when using the terminology of Appendix A and taking particles 3 and 4 as outgoing, means that particles 1 and 3 have to have the same physical helicity.

Choosing $\lambda_1 = \lambda_2 \neq \lambda_3 = \lambda_4$. This is one of only two possible non-vanishing choices left after our previous considerations. In much the same we arrive at

$$\mathcal{M}_s^{++--} = +\frac{2e^2}{s}\frac{1}{\langle q_1 1\rangle[q_3 3]}\langle 43\rangle\,[q_3|\not p_1 + \not p_2|q_1\rangle\,[12]$$

$$\mathcal{M}_u^{++--} = +\frac{2e^2}{u}\frac{1}{\langle q_1 1\rangle[q_3 3]}\langle 4q_1\rangle\,[1|\not p_2 + \not p_3|3\rangle\,[q_3 2]$$

Using a variation of Equation (27)

$$\mathcal{M}_s^{++--} = \frac{2e^2}{s}\frac{1}{\langle q_1 1\rangle[q_3 3]}\langle 43\rangle\left([q_3 1]\langle 1q_1\rangle + [q_3 2]\langle 2q_1\rangle\right)[12]$$

$$\begin{aligned}\mathcal{M}_u^{++--} &= \frac{2e^2}{u}\frac{1}{\langle q_1 1\rangle[q_3 3]}\langle 4q_1\rangle\left([12]\langle 23\rangle + [13]\langle 33\rangle\right)[q_3 2]\\ &= \frac{2e^2}{u}\frac{1}{\langle q_1 1\rangle[q_3 3]}\langle 4q_1\rangle\,[12]\langle 23\rangle\,[q_3 2]\end{aligned}$$

where the last line is due to Equation (9). Choosing $q_1 = p_2$ and $q_3 = p_1$ yields

$$\mathcal{M}_s^{++--} = 0$$

$$\mathcal{M}_u^{++--} = \frac{2e^2}{u} \frac{1}{\langle 21\rangle[13]} \langle 42\rangle[12]\langle 23\rangle[12]$$

So $\mathcal{M}^{++--} = \mathcal{M}_u^{++--}$ in this gauge. We could have of course chosen the gauge spinors differently. An alternative choice and its neat implication that the poles of the full amplitude are not only determined by the propagators of the different channels but also by the polarization vectors are further explained in Section 4.3.2. Continuing the calculation at hand we use Equations (16) and (36) to conclude

$$\mathcal{M}^{++--} = \frac{2e^2}{\langle 23\rangle[32]} \frac{\langle 42\rangle[12]\langle 23\rangle[12]}{\langle 21\rangle[13]}$$

Using Equations (9) and (21) we infer $\langle 21\rangle[13] = -\langle 24\rangle[43] = \langle 42\rangle[43]$. This implies

$$\begin{aligned} &= 2e^2 \frac{[12]^2}{[43][32]} \\ &= 2e^2 \frac{[12]^2}{[34][23]} \end{aligned}$$

As in Section 3 we are happy to have derived a pure helicity-spinor product expression and object to those who'd consider our job done. We power through until we have an expression in terms of Mandelstam variables. Using Equations (19) and (20) we get

$$\begin{aligned} \mathcal{M}^{++--} &= 2e^2 \frac{\left(\sqrt{s_{12}}\mathrm{e}^{\mathrm{i}\bar{\phi}_{12}}\right)^2}{\sqrt{s_{34}}\mathrm{e}^{\mathrm{i}\bar{\phi}_{34}}\sqrt{s_{23}}\mathrm{e}^{\mathrm{i}\bar{\phi}_{23}}} \\ &= 2e^2 \frac{s}{\sqrt{su}} \mathrm{e}^{\mathrm{i}(2\bar{\phi}_{12}-\bar{\phi}_{34}-\bar{\phi}_{23})} && \text{by Equations (34) and (36)} \\ &= 2e^2 \sqrt{\frac{s}{u}} \mathrm{e}^{\mathrm{i}(2\bar{\phi}_{12}-\bar{\phi}_{34}-\bar{\phi}_{23})} \end{aligned}$$

As in this process $s > 0$ and $u \leq 0$ we write

$$= 2e^2 \sqrt{\frac{s}{-u}} \mathrm{e}^{\mathrm{i}(2\bar{\phi}_{12}-\bar{\phi}_{34}-\bar{\phi}_{23}-\frac{\pi}{2})}$$

So finally a ratio of Mandelstam variables and a combination of massless phases. Is the case of all helicities flipped again just the complex conjugate of this result? Let's see:

after applying Equation (15) we get

$$\mathcal{M}_s^{--++} = \frac{2e^2}{s} \frac{1}{\langle q_3 3\rangle [q_1 1]} [43] \langle q_3 | \not{p}_1 + \not{p}_3 | q_1] \langle 12\rangle$$

$$\mathcal{M}_u^{--++} = \frac{2e^2}{u} \frac{1}{\langle q_3 3\rangle [q_1 1]} [4q_1] \langle 1 | \not{p}_2 + \not{p}_3 | 3] \langle q_3 2\rangle$$

which are indeed just the complex conjugates of our initial expressions for $\mathcal{M}_s^{++--}$ and $\mathcal{M}_u^{++--}$ when thinking back to Equations (11) and (26). So we have

$$\mathcal{M}^{--++} = \left(\mathcal{M}^{++--}\right)^*$$

Choosing $\lambda_1 = \lambda_4 \neq \lambda_2 = \lambda_3$. This is the second possible non-vanishing choice we can make. There is nothing new to show here so we'll just state the result in the gauge $q_1 = q_3 = p_2$.

$$\begin{aligned}
\mathcal{M}^{+--+} &= 2e^2 \sqrt{\frac{-u}{s}}\, \mathrm{e}^{\mathrm{i}(\hat{\phi}_{23} + \bar{\phi}_{14} - \hat{\phi}_{12} - \bar{\phi}_{23} - \frac{\pi}{2})} \\
\mathcal{M}^{-++-} &= \left(\mathcal{M}^{+--+}\right)^*
\end{aligned}$$

Finally, let us collect our results for the helicity amplitudes

$$\mathcal{M}^{\pm\pm\pm\pm} = 2e^2\mathrm{i} \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{\frac{s}{-u}} \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ \sqrt{\frac{-u}{s}} & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & -\sqrt{\frac{-u}{s}} \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} -\sqrt{\frac{s}{-u}} & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \tag{33}$$

where we've again specialized to the CM-frame of Appendix B using the massless phases of Equation (46).

4.2 Verification

The unpolarized amplitude squared is again given by

$$\frac{1}{4} \sum_{\text{pol}} |\mathcal{M}|^2$$

Inserting our results from Equation (33) into an explicitly helicity sum yields

$$\frac{1}{4}\sum_{\text{pol}}|\mathcal{M}|^2 = 2e^4\left(\frac{s}{-u}+\frac{-u}{s}\right)$$

which matches the literature result in Equation (13.141) of [5].[6]

4.3 Additional calculations

Let us now come to some additional calculations that are somewhat of an aside to the main goal of determining the tree-level helicity amplitudes for Compton scattering.

4.3.1 Gauge-independent calculation

Picking up right where we left of in Section 4.1 on massless Compton scattering our tree-level result for $\mathcal{M}^{+++-}$ before choosing a gauge spinor was

$$\mathcal{M}_s^{+++-} = -\frac{2e^2}{s}\frac{1}{\langle q_1 1\rangle\langle q_3 3\rangle}\langle 4q_3\rangle\,[3|\not{p}_1+\not{p}_2|q_1\rangle\,[12]$$

$$\mathcal{M}_u^{+++-} = -\frac{2e^2}{u}\frac{1}{\langle q_1 1\rangle\langle q_3 3\rangle}\langle 4q_1\rangle\,[1|\not{p}_2+\not{p}_3|q_3\rangle\,[32]$$

and upon picking the gauge spinors in a smart way, namely $q_1 = q_3 = p_4$, we saw

$$\mathcal{M}^{+++-} = \mathcal{M}_s^{+++-} + \mathcal{M}_u^{+++-} = 0$$

Now we want to show this for general q_1 and q_3. So let's get started. We first apply a variation of Equation (27)

$$\mathcal{M}_s^{+++-} = -\frac{2e^2}{s}\frac{1}{\langle q_1 1\rangle\langle q_3 3\rangle}\langle 4q_3\rangle\Big([31]\langle 1q_1\rangle+[32]\langle 2q_1\rangle\Big)[12]$$

$$\mathcal{M}_u^{+++-} = -\frac{2e^2}{u}\frac{1}{\langle q_1 1\rangle\langle q_3 3\rangle}\langle 4q_1\rangle\Big([12]\langle 2q_3\rangle+[13]\langle 3q_3\rangle\Big)[32]$$

Using a variation of Equation (21) we can easily conclude

$$[31]\langle 1q_1\rangle+[32]\langle 2q_1\rangle = -[33]\langle 3q_1\rangle-[34]\langle 4q_1\rangle = -[34]\langle 4q_1\rangle$$

$$[12]\langle 2q_3\rangle+[13]\langle 3q_3\rangle = -[11]\langle 1q_3\rangle-[14]\langle 4q_3\rangle = -[14]\langle 4q_3\rangle$$

[6]Note: In [5] the momenta are labelled differently, hence the t's and not u's.

which simplifies our channel expressions

$$\begin{aligned}
\mathcal{M}_s^{+++-} &= -\frac{2e^2}{s} \frac{1}{\langle q_1 1\rangle\langle q_3 3\rangle} \langle 4q_3\rangle \Big(-[34]\langle 4q_1\rangle\Big) [12] \\
&= +2e^2 \frac{\langle q_1 4\rangle\langle q_3 4\rangle}{\langle q_1 1\rangle\langle q_3 3\rangle} \frac{[34][12]}{s}
\end{aligned}$$

$$\begin{aligned}
\mathcal{M}_u^{+++-} &= -\frac{2e^2}{u} \frac{1}{\langle q_1 1\rangle\langle q_3 3\rangle} \langle 4q_1\rangle \Big(-[14]\langle 4q_3\rangle\Big) [32] \\
&= +2e^2 \frac{\langle q_1 4\rangle\langle q_3 4\rangle}{\langle q_1 1\rangle\langle q_3 3\rangle} \frac{[14][32]}{u}
\end{aligned}$$

Using Equations (16), (34) and (36) we get

$$\begin{aligned}
\mathcal{M}_s^{+++-} &= 2e^2 \frac{\langle q_1 4\rangle\langle q_3 4\rangle}{\langle q_1 1\rangle\langle q_3 3\rangle} \frac{[34][12]}{\langle 21\rangle[12]} \\
&= 2e^2 \frac{\langle q_1 4\rangle\langle q_3 4\rangle}{\langle q_1 1\rangle\langle q_3 3\rangle} \frac{[34]}{\langle 21\rangle}
\end{aligned}$$

$$\begin{aligned}
\mathcal{M}_u^{+++-} &= 2e^2 \frac{\langle q_1 4\rangle\langle q_3 4\rangle}{\langle q_1 1\rangle\langle q_3 3\rangle} \frac{[14][32]}{\langle 23\rangle[32]} \\
&= 2e^2 \frac{\langle q_1 4\rangle\langle q_3 4\rangle}{\langle q_1 1\rangle\langle q_3 3\rangle} \frac{[14]}{\langle 23\rangle}
\end{aligned}$$

Having simplified the channels as much as possible we move on to their sum

$$\begin{aligned}
\mathcal{M}^{+++-} &= 2e^2 \frac{\langle q_1 4\rangle\langle q_3 4\rangle}{\langle q_1 1\rangle\langle q_3 3\rangle} \left(\frac{[34]}{\langle 21\rangle} + \frac{[14]}{\langle 23\rangle}\right) \\
&= 2e^2 \frac{\langle q_1 4\rangle\langle q_3 4\rangle}{\langle q_1 1\rangle\langle q_3 3\rangle} \frac{1}{\langle 21\rangle\langle 23\rangle} \Big(\langle 23\rangle[34] + \langle 21\rangle[14]\Big)
\end{aligned}$$

Taking a closer look at the parentheses we conclude using Equation (21)

$$\langle 23\rangle[34] + \langle 21\rangle[14] = -\langle 21\rangle[14] + \langle 21\rangle[14] = 0$$

and thus finally

$$\mathcal{M}^{+++-} = 0$$

without ever having chosen a specific gauge spinor.

4.3.2 A different gauge and its implications

In this section we want to examine a different choice of gauge spinors for the calculation in Section 4.1 of $\mathcal{M}^{++--}$. Before choosing a gauge we arrived at

$$\mathcal{M}_s^{++--} = \frac{2e^2}{s} \frac{1}{\langle q_1 1\rangle [q_3 3]} \langle 43\rangle \Big([q_3 1]\langle 1 q_1\rangle + [q_3 2]\langle 2 q_1\rangle \Big) [12]$$

$$\mathcal{M}_u^{++--} = \frac{2e^2}{u} \frac{1}{\langle q_1 1\rangle [q_3 3]} \langle 4 q_1\rangle [12]\langle 23\rangle [q_3 2]$$

and went on to choose $q_1 = p_2$ and $q_3 = p_1$. Here we are going to go with $q_1 = q_3 = p_2$ which yields by Equation (9)

$$\mathcal{M}_s^{++--} = \frac{2e^2}{s} \frac{1}{\langle 21\rangle [23]} \langle 43\rangle [21]\langle 12\rangle [12]$$

$$\mathcal{M}_u^{++--} = 0$$

So $\mathcal{M}^{++--} = \mathcal{M}_s^{++--}$ in this gauge. Due to gauge invariance we of course expect that this choice of gauge spinors leads to the same result as in Section 4.1 which implies something quite interesting:

> One would naively expect that the poles of the full amplitude for $s \to 0$ or $u \to 0$ are determined by the propagators of the different channels. But in this section we see quite clearly that, depending on the gauge spinor choice, either $\mathcal{M}_s^{++--}$ or $\mathcal{M}_u^{++--}$ vanishes, which means that the poles stem from both the propagators and the external particles.

For example seeing a $1/\sqrt{-u}$ in the result for $\mathcal{M}^{++--}$ one could think that this term can only come from the u-channel diagram. But a suitable gauge spinor choice can kill the u-channel diagram and thus the s-channel diagram produces this $1/\sqrt{-u}$ through the interplay between the denominators of the photon polarization vectors in this gauge and the propagator.

Pleased with ourselves for spotting this, we can now finish the calculation to convince ourselves of gauge invariance once more. Using Equations (16) and (34) we see

$$\mathcal{M}^{++--} = \frac{2e^2}{s} \frac{1}{\langle 21\rangle [23]} \langle 43\rangle \, s \, [12]$$

$$= 2e^2 \frac{\langle 43\rangle[12]}{\langle 21\rangle[23]}$$

$$= 2e^2 \frac{\langle 34\rangle[12]}{\langle 12\rangle[23]}$$

Performing the same steps as in Section 4.1 to arrive at an expression in terms of Mandelstam variables one gets

$$= 2e^2 \sqrt{\frac{s}{-u}}\, \mathrm{e}^{\mathrm{i}(\hat{\phi}_{34}+\bar{\phi}_{12}-\hat{\phi}_{12}-\bar{\phi}_{23}-\frac{\pi}{2})}$$

and upon remembering Equation (18) and doing some simple manipulations one can see that this phase matches the one in Section 4.1.

5 Summary and Outlook

In this note we briefly reviewed the massless spinor-helicity formalism for both massless spin-1/2 and spin-1 particles. We used these techniques to determine the tree-level helicity amplitudes of massless Møller and Compton scattering and confirmed these by verifying that their respective unpolarized amplitude squared match the literature result. In the context of Compton scattering we additionally studied some consequences of the spinor-helicity expressions for the photon polarization vectors (see Equation (30)). These deviate from the traditional expressions in that they make the residual gauge symmetry of Lorentz gauge manifest. To begin with we performed the calculation of one helicity amplitude without specializing to one particular residual gauge. We then determined another helicity amplitude in a different residual gauge. This last calculation implies something quite interesting: The poles of the full amplitude are not only determined by the propagators of the different channels but also by the polarization vectors themselves.

Having discussed the massless spinor-helicity formalism, one might wonder if there is a massive extension. Indeed there is [6–8], although it seems that it is not present in the educational literature of perturbative quantum field theory. Nonetheless, the general spinor-helicity formalism for both massless and massive particles is used in a fundamentally different approach to particle physics called on-shell methods [1, 8]. As resources combining the two views on the spinor-helicity formalism are not widespread, a possible direction for future work could be a comprehensive review of the general spinor-helicity formalism from both the on-shell and the traditional perturbative perspectives. Another natural step would be to apply these methods to related processes.

A Conventions

In this appendix we collect our conventions. These are mainly taken from [9].

Physics

We are only considering ultrarelativistic, i.e. practically speaking massless, processes. Hence all 4-momenta are light-like or massless unless specified otherwise.

Making use of crossing symmetry, we will take all 4-momenta of a process to be *incoming*. This has the following implications

1. The only Dirac polarization vectors we'll use are u-type and $\bar{v}$-type ones.
2. The Mandelstam variables s, t, and u for a 4-particle process are defined as

$$s := (p_1 + p_2)^2 = (p_3 + p_4)^2 \tag{34}$$

$$t := (p_1 + p_3)^2 = (p_2 + p_4)^2 \tag{35}$$

$$u := (p_1 + p_4)^2 = (p_2 + p_3)^2 \tag{36}$$

3. The typical helicity amplitude for a process with n_{in} incoming and n_{out} outgoing particles can be reconstructed from a helicity amplitude given in this convention by flipping the helicities of the n_{out} particles that are supposed to be outgoing.

To not get confused about this helicity flip, we'll refer to the spin-projection of a particle's spin state onto the 3-momentum in this convention as *helicity*, whereas we'll say *physical helicity* for the spin-projection onto the 3-momentum specified by a typical process of n_{in} incoming and n_{out} outgoing particles. So for incoming particles the terms coincide and for outgoing particles the only difference is a sign flip.

Moreover, we use the following notational convention for denoting helicity amplitudes. Given a scattering process of n particles of type f_i, momentum p_i and polarization λ_i

$$f_1(p_1, \lambda_1) + f_2(p_2, \lambda_2) + ... \quad \to \quad f_k(p_k, \lambda_k) + ... + f_n(p_n, \lambda_n)$$

we denote the helicity amplitude by

$$\mathcal{M}(1_{f_1}^{\lambda_1}, 2_{f_2}^{\lambda_2}, ..., k_{f_k}^{-\lambda_k}, ..., n_{f_n}^{-\lambda_n})$$

if we want to highlight the particle types and enumeration and by

$$\mathcal{M}^{\{\lambda_1\}\,\{\lambda_2\}\,\dots\,\{-\lambda_k\}\,\dots\,\{-\lambda_n\}}$$

if we want to highlight the polarization structure. We omit the braces when there are no problems with legibility.

Maths

We choose the mostly minus convention for the Minkowski metric

$$g = (\!(g^{\mu\nu})\!) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{37}$$

For the Pauli matrices we use

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{38}$$

So with the usual $\vec{\sigma} := (\sigma^1,\, \sigma^2,\, \sigma^3)$ we get

$$(\!(\sigma^\mu)\!) := (\sigma^0, \vec{\sigma}) \tag{39}$$

$$(\!(\bar{\sigma}^\mu)\!) := (\sigma^0, -\vec{\sigma}) \tag{40}$$

Additionally, we use the Dirac matrices in the chiral basis

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \tag{41}$$

Thus all Dirac spinors are also always in the chiral basis.

B Calculation of massless phases for a 4-particle CMS process

In this appendix we determine the massless phases of helicity-spinor products in the following center-of-mass-system for the type of process of Sections 3 and 4, i.e., an ultra-relativistic 4-particle process.

The calculation will involve specifying the 4-momenta in our chosen frame, determining the expression of the Mandelstam variables in that frame and calculating all angle helicity-spinor products in that frame. Comparing steps two and three will allow us to derive all massless phases as using Equation (18) we conclude

$$\bar{\phi}_{ij} = -\hat{\phi}_{ij} + \pi \tag{42}$$

i.e. it suffices to determine $\hat{\phi}_{ij}$.

Specifying the 4-momenta. In the CM-frame we choose the following kinematical parametrization of the 4-momenta

$$\begin{aligned} p_1 &= +(E,\ 0,\ 0,\ E)^{\mathsf{T}} \\ p_2 &= +(E,\ 0,\ 0,\ -E)^{\mathsf{T}} \\ p_3 &= -(E,\ E\sin\theta,\ 0,\ E\cos\theta)^{\mathsf{T}} \\ p_4 &= -(E,\ -E\sin\theta,\ 0,\ -E\cos\theta)^{\mathsf{T}} \end{aligned} \tag{43}$$

where θ is the scattering angle. As we are working with spherical coordinates, it is important to get the various minuses accounted for in the correct way, i.e., using the following kinematical parametrization in spherical coordinates

particle	energy	|3-momentum|	polar angle	azimuthal angle
1	E	E	0	0
2	E	E	π	0
3	$-E$	$-E$	θ	0
4	$-E$	$-E$	$\pi - \theta$	π

Table 1: Spherical coordinate kinematics of ultra-relativistic 4-particle CMS process

Determining the Mandelstam variables. Using Equation (43) and the definition of the

Mandelstam variables in Equations (34) to (36) we conclude

$$\begin{aligned} s &= 4E^2 \\ t &= -2E^2(1-\cos\theta) = -4E^2\sin^2(\tfrac{\theta}{2}) \\ u &= -2E^2(1+\cos\theta) = -4E^2\cos^2(\tfrac{\theta}{2}) \end{aligned} \tag{44}$$

Calculating the helicity-spinor products. Inserting the kinematics of Table 1 into the helicity-spinor parametrization in Equation (12) and calculating the spinor products one sees

$$\begin{aligned} \langle 12\rangle &= -2E = \sqrt{s}\,\mathrm{e}^{\mathrm{i}\pi} \\ \langle 34\rangle &= -2E = \sqrt{s}\,\mathrm{e}^{\mathrm{i}\pi} \\[1em] \langle 13\rangle &= -\mathrm{i}\cdot 2E\sin(\tfrac{\theta}{2}) = \sqrt{t}\,\mathrm{e}^{\mathrm{i}\pi} \\ \langle 24\rangle &= \mathrm{i}\cdot 2E\sin(\tfrac{\theta}{2}) = \sqrt{t} \\[1em] \langle 14\rangle &= \mathrm{i}\cdot 2E\cos(\tfrac{\theta}{2}) = \sqrt{u} \\ \langle 23\rangle &= \mathrm{i}\cdot 2E\cos(\tfrac{\theta}{2}) = \sqrt{u} \end{aligned} \tag{45}$$

Reading off the massless phases. From the previous step and Equation (42) we conclude

$$\begin{aligned} &\hat{\phi}_{12} = \pi\,, \quad \bar{\phi}_{12} = 0 && \hat{\phi}_{13} = \pi\,, \quad \bar{\phi}_{13} = 0 && \hat{\phi}_{14} = 0\,, \quad \bar{\phi}_{14} = \pi \\ &\hat{\phi}_{34} = \pi\,, \quad \bar{\phi}_{34} = 0 && \hat{\phi}_{24} = 0\,, \quad \bar{\phi}_{24} = \pi && \hat{\phi}_{23} = 0\,, \quad \bar{\phi}_{23} = \pi \end{aligned} \tag{46}$$

C Massless spinor-helicity Feynman rules

This appendix summarizes the translation between traditional Feynman rules for massless external particles and the spinor-helicity version. These are simply the spinor-helicity expressions of the various polarization vectors given in Equations (1) and (30)

Keep in mind that the *take all momenta as incoming* which means that we only have to specify the Feynman rules for incoming particles.

C.1 Massless spin-$\frac{1}{2}$

Fermions

$$u^{-}(p_i) = |i\rangle \tag{47}$$

$$u^{+}(p_i) = |i] \tag{48}$$

Anti-fermions

$$\bar{v}^{-}(p_i) = \langle i| \tag{49}$$

$$\bar{v}^{+}(p_i) = [i| \tag{50}$$

C.2 Massless spin-1

$$\varepsilon_{-}^{\mu}(p_i; q) = -\frac{1}{\sqrt{2}} \frac{[q|\gamma^{\mu}|i\rangle}{[qi]} \tag{51}$$

$$\varepsilon_{+}^{\mu}(p_i; q) = +\frac{1}{\sqrt{2}} \frac{\langle q|\gamma^{\mu}|i]}{\langle qi\rangle} \tag{52}$$

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Eigenständigkeitserklärung

Ich habe die vorliegende Arbeit selbstständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt. Ich versichere, dass die Arbeit in gleicher oder ähnlicher Form bisher noch keiner anderen Prüfungsbehörde vorgelegt wurde.

Regensburg, 29.09.2025 P. Erhard

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