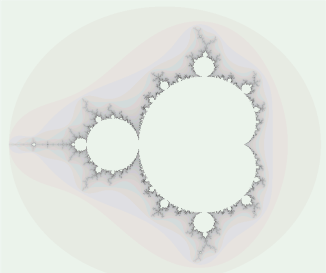
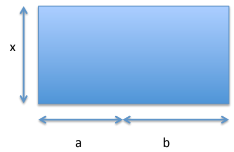
The Mandelbrot set



Complex numbers

1 × 1 = 1
-1 × 1 = -1
1 × -1 = -1
-1 × -1 = 1



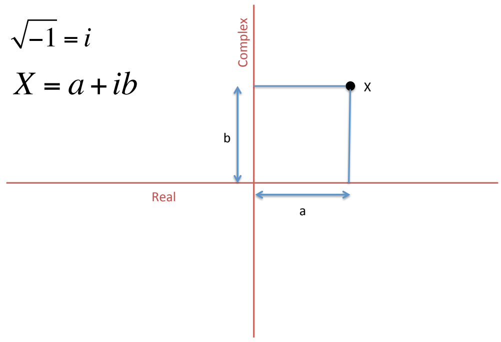
x×(a+b) = x×a + x×b
-1×(3-2) = -1×3 + -1×-2 = -3 + -1×-2
-1×(3-2) = -1×1 = -1
-1 = -3 + -1×-2
So 2 = -1×-2 ■

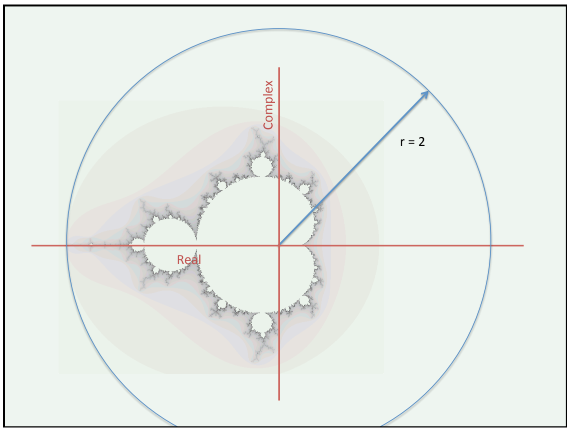
$$\sqrt{-1} = i$$

The complex plane

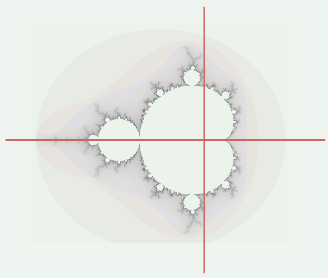
$$\sqrt{-1} = i$$

$$X = a + ib$$





The Mandelbrot set

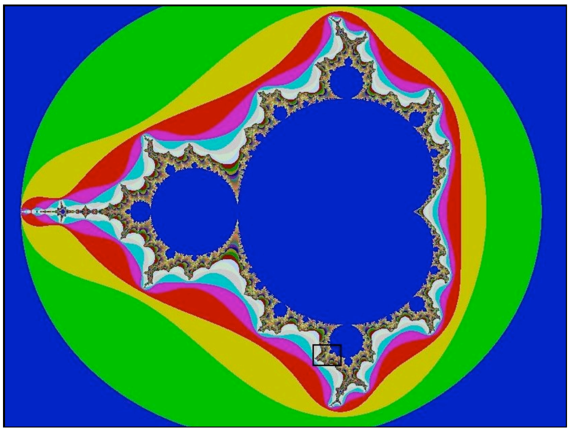


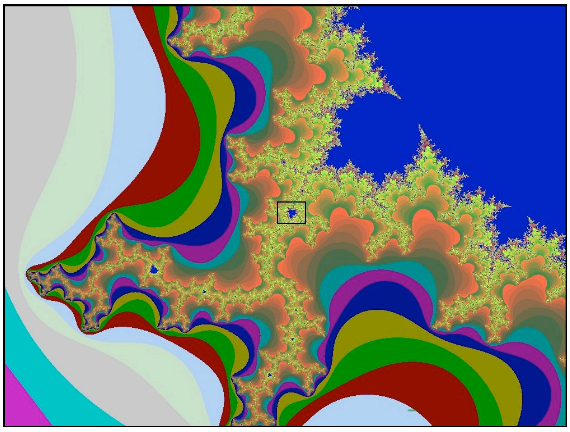
$Z \mapsto Z^2 + C$

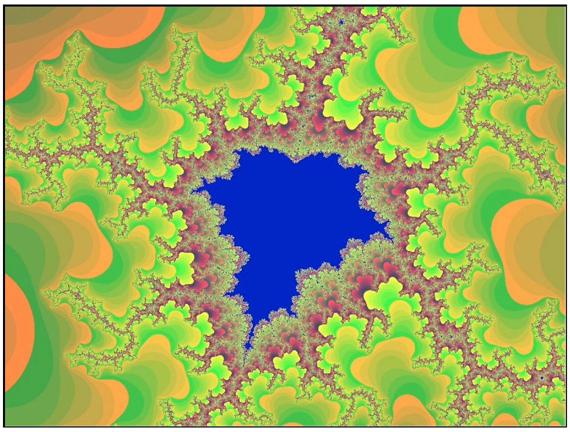
C is given by your position on the complex plane

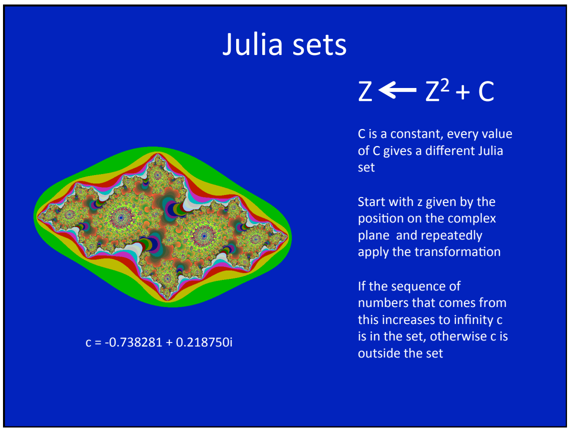
Start with z = 0 and repeatedly apply the transformation

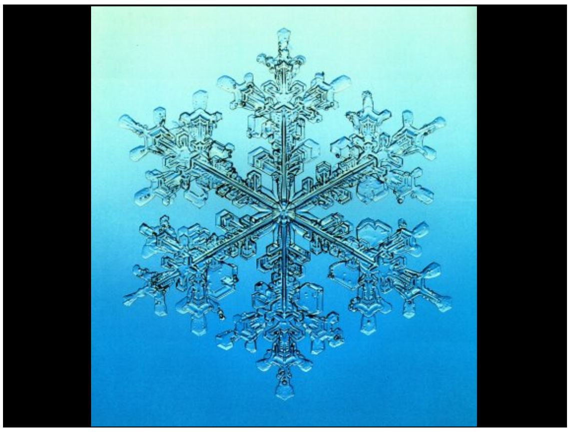
If the sequence of numbers that comes from this increases to infinity c is in the set, otherwise c is outside the set

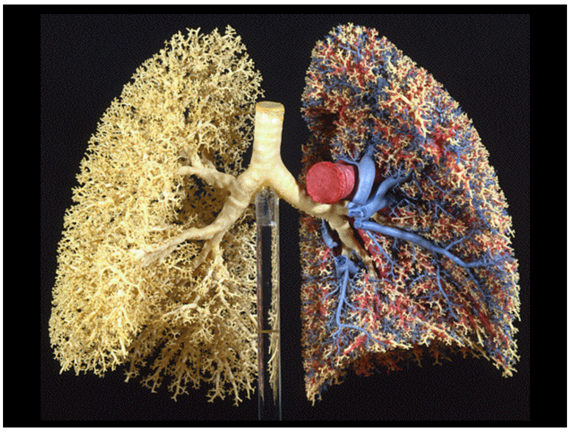






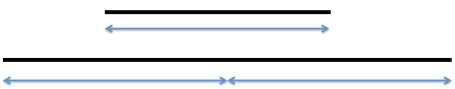

Julia sets
Z ← Z² + C
C is a constant, every value of C gives a different Julia set
Start with z given by the position on the complex plane and repeatedly apply the transformation
If the sequence of numbers that comes from this increases to infinity c is in the set, otherwise c is outside the set
c = -0.738281 + 0.218750i





What is a fractal

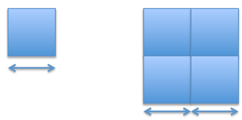
What is not a fractal



A line twice as wide is twice as big   Dimension 1

Width	Size
1	$1 = 1^1$
2	$2 = 2^1$
3	$3 = 3^1$
4	$4 = 4^1$

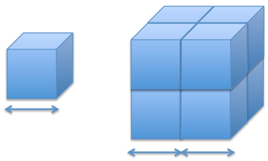
What is not a fractal



A square twice as wide is four times as big   Dimension 2

Width	Size
1	$1 = 1^2$
2	$4 = 2^2$
3	$9 = 3^2$
4	$16 = 4^2$

What is not a fractal



A cube twice as wide is eight times as big

Dimension 3

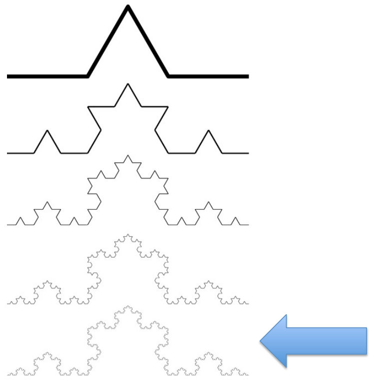
Width	Size
1	1 = 1^3
2	8 = 2^3
3	27 = 3^3
4	64 = 4^3

What is a fractal

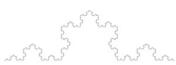
What is a fractal

- Dimension that is not a whole number
- Self similar

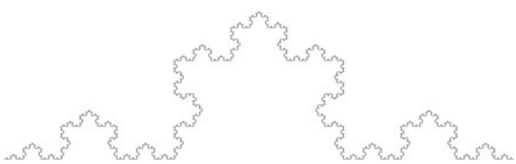
Koch Curve



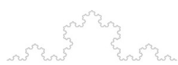
Koch Curve



Dimension	Width	Size
1	3	3 = 3^1
2	3	9 = 3^2
x	3	4 = 3^x



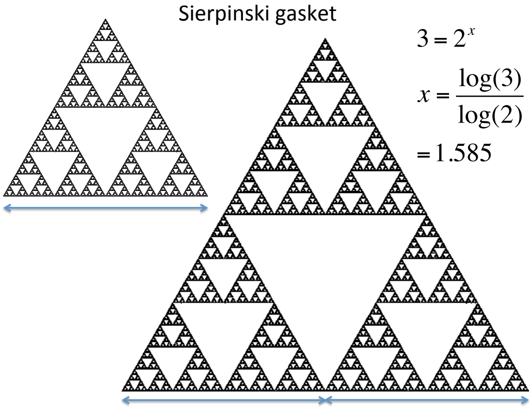
Koch Curve



Dimension	Width	Size
1	3	3 = 3^1
2	3	9 = 3^2
x	3	4 = 3^x

$$4 = 3^x$$
$$\log(4) = \log(3^x)$$
$$\log(4) = x \cdot \log(3)$$
$$x = \frac{\log(4)}{\log(3)} = 1.262$$

Sierpinski gasket

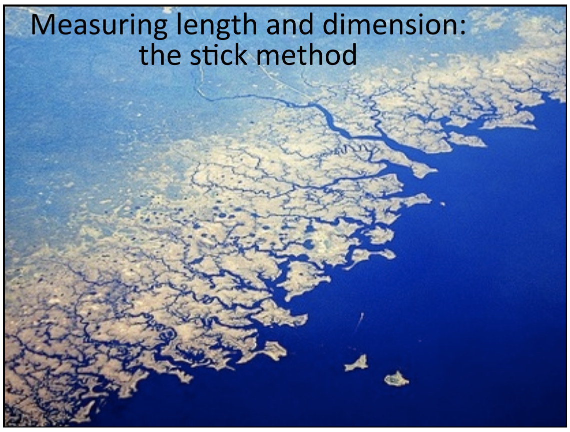


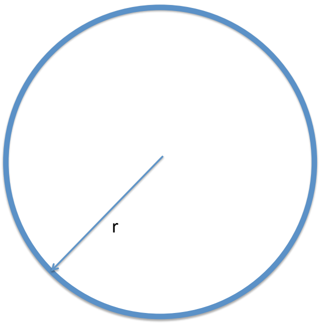
$$3 = 2^x$$

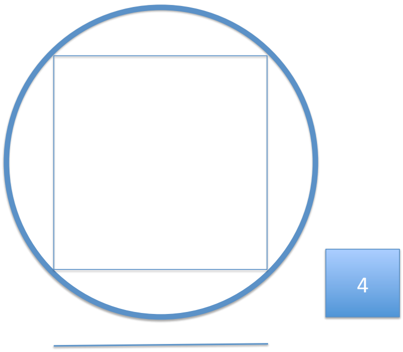
$$x = \frac{\log(3)}{\log(2)}$$

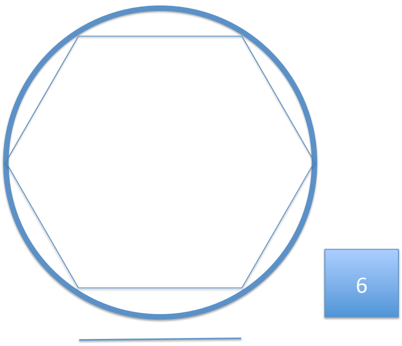
$$= 1.585$$

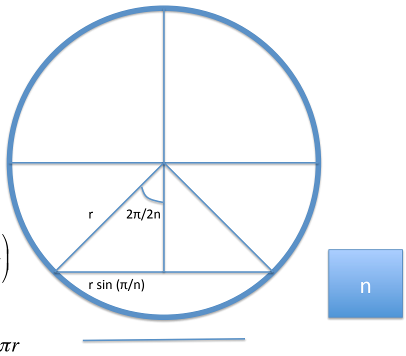
Measuring length and dimension: the stick method







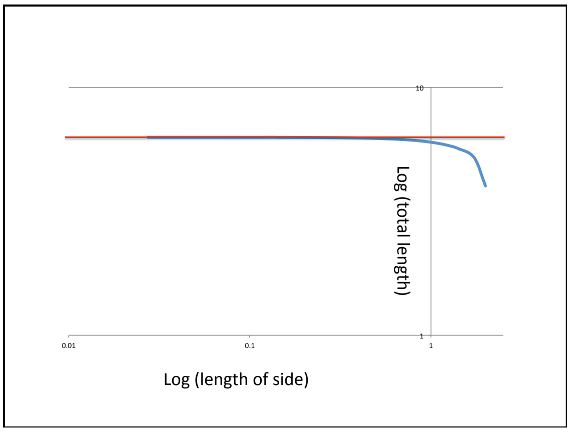


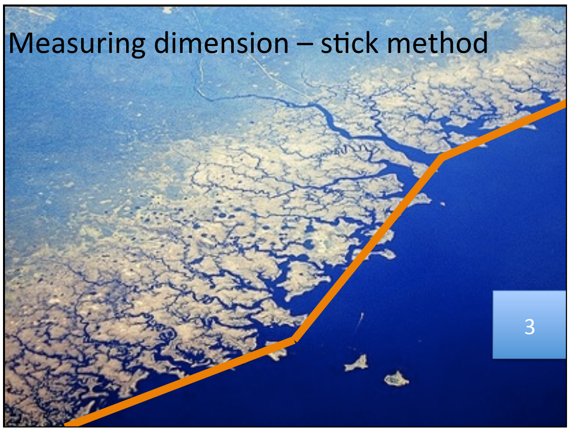


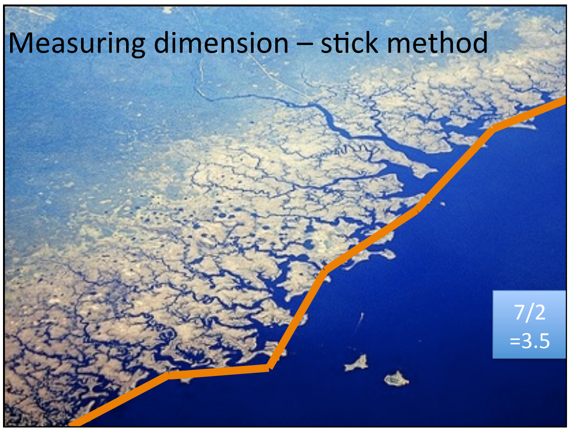
$$c = 2nr\sin\left(\frac{\pi}{n}\right)$$

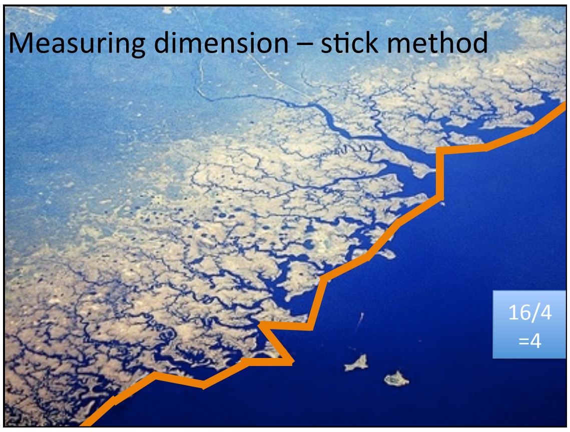
$$\sin(\theta) \approx \theta$$

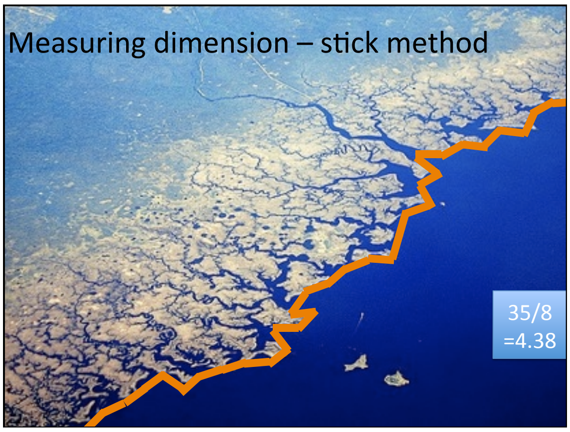
$$c \approx 2nr\frac{\pi}{n} = 2\pi r$$

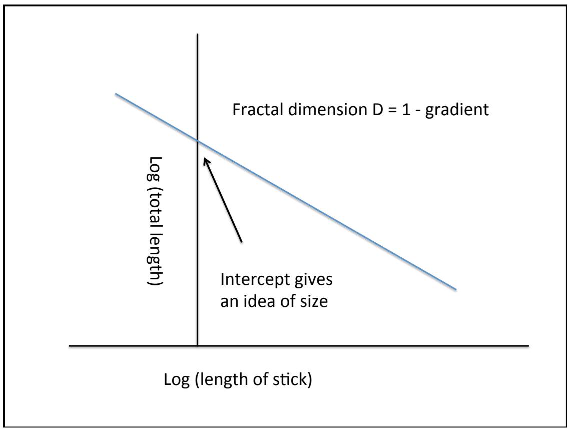

10
Log (total length)
0.01
0.1
1
1
Log (length of side)


Measuring dimension – stick method
3

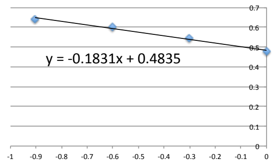

Measuring dimension – stick method
7/2
=3.5


Measuring dimension – stick method
16/4
=4


Measuring dimension – stick method
35/8
=4.38


Fractal dimension D = 1 - gradient
Log (total length)
Intercept gives
an idea of size
Log (length of stick)

The dimension of various coastlines



D = 1.18 for this example

D = 1.25 for the west coast of Britain,
D = 1.15 for the land frontier of Germany,
D = 1.14 for the land frontier of Portugal,
D = 1.13 for the Australian coast, and
D = 1.02 for the South African coast,

The stick method formalized

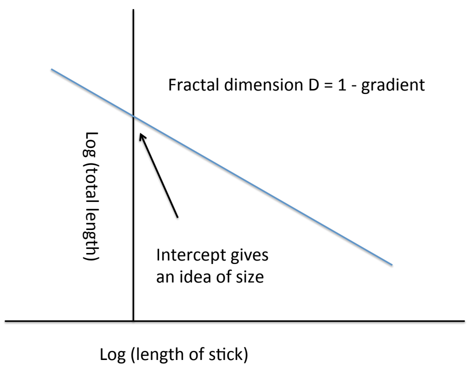
Total length (stick length) Dimension

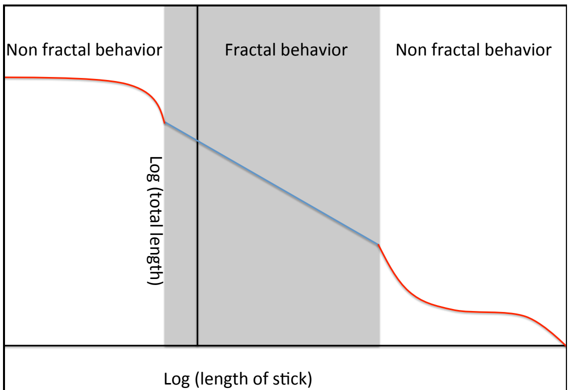
$$C(\delta) = K\delta^{1-D}$$

Constant Stick length

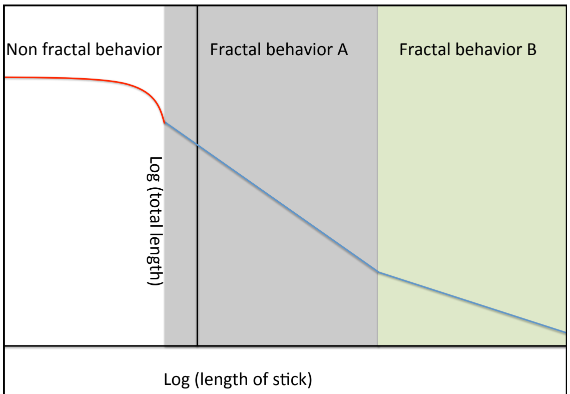
$$\log(C(\delta)) = \log(K) + (1-D)\log(\delta)$$

intercept gradient

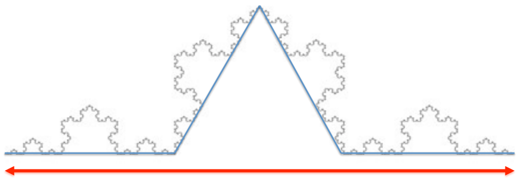




Halley et al.

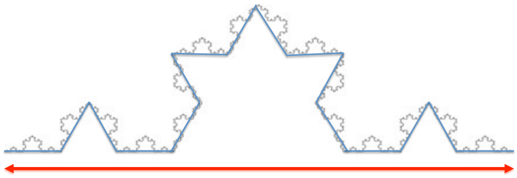


Koch Curve



Stick size	Length
1	1
1/3	4 × 1/3

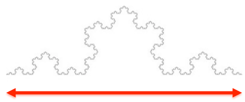
Koch Curve



1

Stick size	Length
1	1
1/3	4 × 1/3
1/9	16 × 1/9

Koch Curve

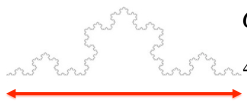


1

Stick size	Length
1	1
1/3	4 × 1/3
1/9	16 × 1/9
1/27	64 × 1/27
3^{-n}	$4^n \times 3^{-n}$

$$C(\delta) = K\delta^{1-D}$$
$$4^n \cdot 3^{-n} = K\left(3^{-n}\right)^{1-D}$$
$$4^n = K(3^{-n})^{-D}$$
$$4^n = (3^{-n})^{-D}$$
$$n\log(4) = Dn\log(3)$$
$$D = \frac{\log(4)}{\log(3)}$$

Koch Curve

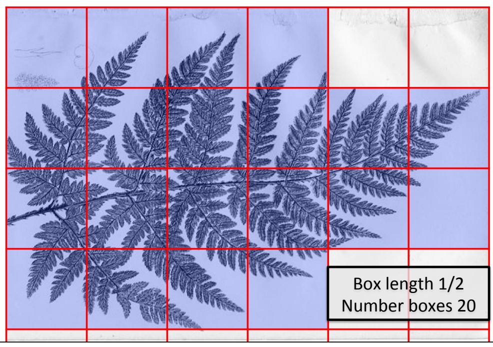


3

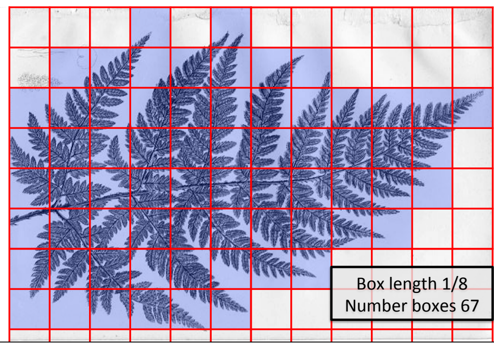
Stick size	Length
3	3
1	4 × 1
1/3	16 × 1/3
1/9	64 × 1/9
$3^{(1-n)}$	$4^n \times 3^{(1-n)}$

$$C(\delta) = K\delta^{1-D}$$
$$4^n \cdot 3^{(1-n)} = K\left(3^{(1-n)}\right)^{1-D}$$
$$4^n = K(3^{(1-n)})^{-D}$$
$$4^n = 4(3^{(1-n)})^{-D}$$
$$(n-1)\log(4) = D(n-1)\log(3)$$
$$D = \frac{\log(4)}{\log(3)}$$

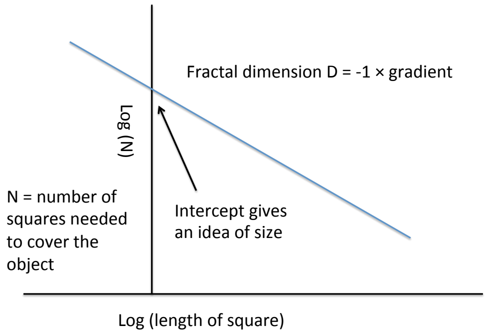
Box counting algorithm



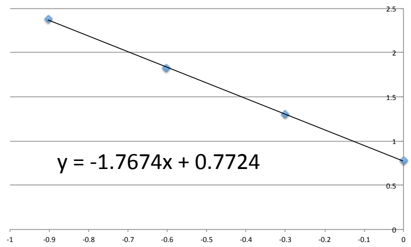
Box counting algorithm



Box counting algorithm

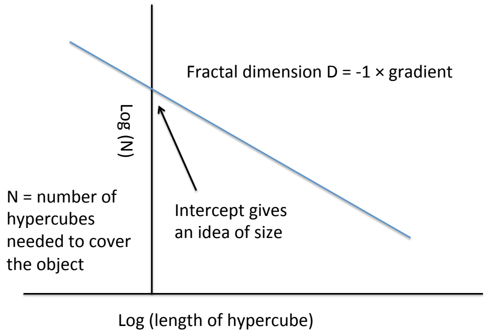


In our example



Fern D = 1.767

Box counting algorithm



The box counting algorithm formalized

number of hypercubes needed to cover the object

Dimension

$$N(\delta) = K\delta^{-D}$$

Constant

Hypercube length

$$\log(N(\delta)) = \log(K) + -D\log(\delta)$$

intercept

gradient

Comparing box and stick methods

number of sticks needed to go round the perimeter

Dimension

$$N(\delta) = K\delta^{-D}$$

Constant

Stick length

Comparing box and stick methods

number of sticks needed to go round the perimeter

Dimension

$$N(\delta) = K\delta^{-D}$$

Constant

Stick length

Total length

Dimension

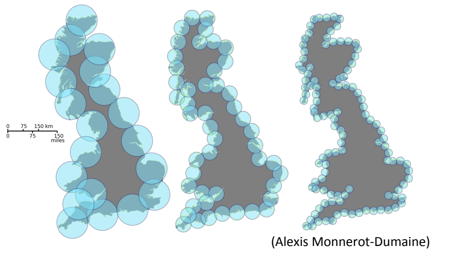
$$C(\delta) = \delta N(\delta) = K\delta^{1-D}$$

Constant

Stick length

Hausdorff dimension

The hausdorff dimension d of an object has the property that the number of balls of radius r needed to cover the object grows proportionally to r^{-d} as r becomes small



(Alexis Monnerot-Dumaine)

Hausdorff dimension

$$\inf\left\{d \geq 0 \middle| C^d = 0\right\}$$

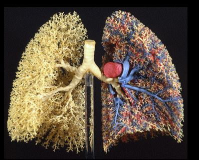
$$C^d = \inf\left\{\sum_i r_i^d \middle| \text{ Balls of radius } r_i > 0 \text{ can cover the object } \right\}$$

(Alexis Monnerot-Dumaine)

Why do fractals appear in nature?

In organisms

- A simple set of rules (DNA)
- Fractal structures are heritable
  (Bailey *et al.* 2004)
- Need to maximize surface area but minimize volume
- Efficiency of transportation



Why do fractals appear in nature?

In geography

- Same processes at multiple scales

In landscape ecology

In animal behavior

Chaos

Wait for the next lecture