Study notes on electromagnetism

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1 Conservation laws

1.1 charge

The charge in a volumn $\mathcal{V}$ is

$$Q(t) = \int_{\mathcal{V}} \rho(r,t)\ d\tau \tag{1}$$

When there is a decrease of Q, there might be current flowing out

$$\frac{dQ}{dt} = -\oint_{\mathcal{S}} \mathbf{J} \cdot d\mathbf{a} \tag{2}$$

Invoking the divergence theorem, we get the continuity equation for charge

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{J} \tag{3}$$

1.2 energy

Consider the work done on charges by some electromagnetic field

$$\begin{aligned} dW_0 &= \mathbf{F} \cdot d\mathbf{l} && (4)\\ &= \rho(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \mathbf{v} dt && (5)\\ &= \mathbf{E} \cdot \mathbf{J}\ dt && (6) \end{aligned}$$

Using Maxwell's equations to express W in fields

$$\begin{aligned} \frac{dW_0}{dt} &= \mathbf{E} \cdot \mathbf{J} && (7)\\ &= \frac{1}{\mu_0}\mathbf{E} \cdot (\nabla \times \mathbf{B}) - \epsilon_0 \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} && (8)\\ &= \frac{1}{\mu_0}\mathbf{B} \cdot (\nabla \times \mathbf{E}) - \frac{1}{\mu_0}\nabla \cdot (\mathbf{E} \times \mathbf{B}) - \epsilon_0 \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} && (9)\\ &= -\frac{1}{\mu_0}\mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} - \epsilon_0 \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} - \frac{1}{\mu_0}\nabla \cdot (\mathbf{E} \times \mathbf{B}) && (10)\\ &= -\frac{\partial}{\partial t}\left(\frac{1}{2}\epsilon_0 \mathbf{E}^2 + \frac{\mathbf{B}^2}{2\mu_0}\right) - \nabla \cdot \mathbf{S} && (11) \end{aligned}$$

Integrate over a volume we get the work done on all charges

$$\frac{dW}{dt} = -\frac{d}{dt}\int_{\mathcal{V}} \frac{1}{2}\epsilon_0 \mathbf{E}^2 + \frac{\mathbf{B}^2}{2\mu_0}\ d\tau - \oint_{\mathcal{S}} \mathbf{S} \cdot d\mathbf{a} \tag{12}$$

Let W=0, we get the continuity equation for energy

$$\frac{\partial u}{\partial t} = -\nabla \cdot \mathbf{S} \tag{13}$$

1.3 momentum

There is a violation of Newton's third law in electrodynamics. However, if the field carries momentum, it will still be conserved. The force on a unit volume of charge is

$$\begin{aligned} \mathbf{f} &= \rho(\mathbf{E} + \mathbf{v} \times \mathbf{B}) && (14)\\ &= \rho \mathbf{E} + \mathbf{J} \times \mathbf{B} && (15) \end{aligned}$$

Expressing $\mathbf{f}$ in fields only we get

$$\begin{aligned}
\mathbf{f} &= \epsilon_0(\nabla\cdot\mathbf{E})\mathbf{E} - \epsilon_0\frac{\partial\mathbf{E}}{\partial t}\times\mathbf{B} + \frac{1}{\mu_0}(\nabla\times\mathbf{B})\times\mathbf{B} && (16)\\
&= \epsilon_0(\nabla\cdot\mathbf{E})\mathbf{E} + \epsilon_0\mathbf{E}\times\frac{\partial\mathbf{B}}{\partial t} - \epsilon_0\frac{\partial}{\partial t}(\mathbf{E}\times\mathbf{B}) - \frac{1}{2\mu_0}\nabla\mathbf{B}^2 + \frac{1}{\mu_0}(\mathbf{B}\cdot\nabla)\mathbf{B} && (17)\\
&= \epsilon_0(\nabla\cdot\mathbf{E})\mathbf{E} + \epsilon_0(\nabla\times\mathbf{E})\times\mathbf{E} - \epsilon_0\frac{\partial}{\partial t}(\mathbf{E}\times\mathbf{B}) - \frac{1}{2\mu_0}\nabla\mathbf{B}^2 + \frac{1}{\mu_0}(\mathbf{B}\cdot\nabla)\mathbf{B} && (18)\\
&= \epsilon_0(\nabla\cdot\mathbf{E})\mathbf{E} - \frac{1}{2}\epsilon_0\nabla\mathbf{E}^2 + \epsilon_0(\mathbf{E}\cdot\nabla)\mathbf{E} - \epsilon_0\frac{\partial}{\partial t}(\mathbf{E}\times\mathbf{B}) - \frac{1}{2\mu_0}\nabla\mathbf{B}^2 + \frac{1}{\mu_0}(\mathbf{B}\cdot\nabla)\mathbf{B} && (19)\\
&= \epsilon_0[(\nabla\cdot\mathbf{E})\mathbf{E} + (\mathbf{E}\cdot\nabla)\mathbf{E}] + \frac{1}{\mu_0}[(\nabla\cdot\mathbf{B})\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{B}] - \nabla(\frac{1}{2}\epsilon_0\mathbf{E}^2 + \frac{1}{2\mu_0}\mathbf{B}^2) - \epsilon_0\frac{\partial}{\partial t}(\mathbf{E}\times\mathbf{B}) && (20)
\end{aligned}$$

We introduce the Maxwell stress tensor

$$T_{ij} = \epsilon_0(E_iE_j - \frac{1}{2}\delta_{ij}E^2) + \frac{1}{\mu_0}(B_iB_j - \frac{1}{2}\delta_{ij}B^2) \tag{21}$$

Where T_{ij} is the force in the i direction acting on the surface element in j direction.

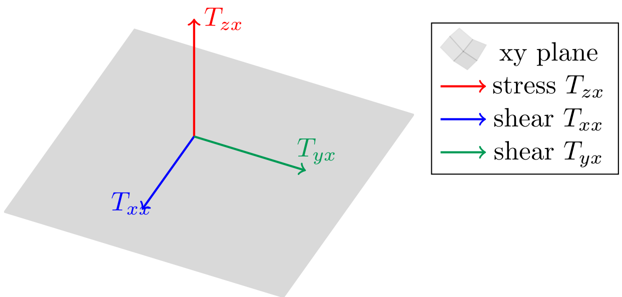

Figure 1: force on a surface element

One can form the product of tensor with vector

$$(\mathbf{a}\cdot\overleftrightarrow{\mathbf{T}})_j = \sum_i a_iT_{ij} \tag{22}$$

$$(\overleftrightarrow{\mathbf{T}}\cdot\mathbf{a})_j = \sum_i T_{ji}a_i \tag{23}$$

The divergence of Maxwell stress tensor is

$$(\nabla\cdot\overleftrightarrow{\mathbf{T}})_i = \epsilon_0[(\nabla\cdot\mathbf{E})E_i + (\mathbf{E}\cdot\nabla)E_i] + \frac{1}{\mu_0}[(\nabla\cdot\mathbf{B})B_i + (\mathbf{B}\cdot\nabla)B_i] - \nabla_i(\frac{1}{2}\epsilon_0\mathbf{E}^2 + \frac{1}{2\mu_0}\mathbf{B}^2) \tag{24}$$

The force per unit volume can be written in terms of the Maxwell stress tensor as

$$\mathbf{f} = \nabla\cdot\overleftrightarrow{\mathbf{T}} - \mu_0\epsilon_0\frac{\partial\mathbf{S}}{\partial t} \tag{25}$$

Invoking the divergence theorem we get

$$F = \frac{d\mathbf{p}_{\text{mech}}}{dt} = -\frac{d}{dt}\int\mathbf{g}\,d\tau + \oint_S\overleftrightarrow{\mathbf{T}}\cdot d\mathbf{a} \tag{26}$$

Where $\mathbf{g}$ is the momentum density. The first term is the momentum stored in the field and the second term is the momentum flows through the surface per unit time, and

$$\mathbf{g} = \mu_0\epsilon_0\mathbf{S} = \epsilon_0(\mathbf{E}\times\mathbf{B}) \tag{27}$$

If $d\mathbf{p}_{\text{mech}}/dt = 0$ or the space is empty, we get the continuity equation for momentum

$$\frac{\partial\mathbf{g}}{\partial t} = -\nabla\cdot\overleftrightarrow{\mathbf{T}} \tag{28}$$

1.4 angular momentum

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial t} &= \mathbf{r} \times \mathbf{f} &(29)\\
&= \mathbf{r} \times (\nabla \cdot \overleftrightarrow{\mathbf{T}} - \frac{\partial \mathbf{g}}{\partial t}) &(30)\\
&= -\nabla \cdot (\overleftrightarrow{\mathbf{T}} \times \mathbf{r}) - \mathbf{r} \times \frac{\partial \mathbf{g}}{\partial t} &(31)
\end{aligned}
$$

$$
\frac{d\mathbf{L}}{dt} = -\frac{d}{dt}\int_{\mathcal{V}} \mathbf{r} \times \mathbf{g}\ d\tau - \oint_{\mathcal{S}} (\overleftrightarrow{\mathbf{T}} \times \mathbf{r}) \cdot d\mathbf{a}
$$

The angular momentum density in an electromagnetic field is

$$
\mathcal{L} = \mathbf{r} \times \mathbf{g} \tag{32}
$$

2 Electromagnetic waves

2.1 description

A wave is a disturbance of continuous medium that propagates with a fixed shape at constant velocity

$$
f(z,t) = f(z - vt, 0) = g(z - vt) \tag{33}
$$

By Newton's second law and small angle approximation

$$
F = \mu(\Delta z)\frac{\partial^2 f}{\partial t^2} = T\sin(\theta - \sin\theta') \approx T\frac{\partial^2 f}{\partial z^2}\Delta z
$$

We can get the wave equation as

$$
\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} \tag{34}
$$

where $v = \sqrt{T/\mu}$ with the general solution

$$
f(z,t) = g(z - vt) + h(z + vt) \tag{35}
$$

For a wave in the real and complex form

$$
\begin{aligned}
f(z,t) &= A\cos[k(z - vt) + \delta]\\
\tilde{f}(z,t) &= \tilde{A}e^{i(kz - \omega t)}
\end{aligned}
$$

where A, $k(z-vt)+\delta$, δ, $z = vt - \delta/k$, δ/k are the amplitude, phase, phase constant, central maximum and its delay. Consider a two-segment string scenario, boundary conditions are the continuity of $\tilde{f}$ and $\partial\tilde{f}/\partial z$.

$$
\tilde{f}(z,t) = \begin{cases} \tilde{A}_i e^{i(k_1 z - wt)} + \tilde{A}_r e^{i(-k_1 z - wt)}, & z > 0\\ \tilde{A}_t e^{i(k_2 z - wt)}, & z < 0 \end{cases}
$$

with the results in complex and real forms given by

$$
\begin{aligned}
\tilde{A}_r &= \frac{v_2 - v_1}{v_1 + v_2}\tilde{A}_i, \quad \tilde{A}_t = \frac{2v_2}{v_1 + v_2}\tilde{A}_i\\
A_r &= \frac{v_2 - v_1}{v_1 + v_2}A_i, \quad A_t = \frac{2v_2}{v_1 + v_2}A_i, \quad (\mu_1 > \mu_2)\\
A_r &= \frac{v_1 - v_2}{v_1 + v_2}A_i, \quad A_t = \frac{2v_2}{v_1 + v_2}A_i, \quad (\mu_1 < \mu_2)
\end{aligned}
$$

The general polarization vector $\mathbf{n} = \cos\theta\mathbf{x} + \sin\theta\mathbf{y}$ will indicate the plane of vibration with $\mathbf{n}\cdot\mathbf{k} = 0$

$$
\begin{aligned}
\mathbf{f}(z,t) &= A\cos(kz - \omega t + \delta)\mathbf{x} \pm A\sin(kz - \omega t + \delta)\mathbf{y}\\
\tilde{\mathbf{f}}(z,t) &= \tilde{A}e^{i(kz - \omega t)}(\mathbf{x} \mp i\mathbf{y})
\end{aligned}
$$

which are real/complex representation of right/left circular polarization

2.2 energy and momentum in vacuum

four Maxwell's equations can be solved by decoupling technique

$$\begin{aligned}
\nabla \times (\nabla \times \mathbf{E}) &= \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} \\
&= \nabla \times (-\frac{\partial \mathbf{B}}{\partial t}) = -\frac{\partial}{\partial t}(\nabla \times \mathbf{B}) = -\mu_0\epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} \\
\nabla \times (\nabla \times \mathbf{B}) &= \nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B} \\
&= \nabla \times (\mu_0\epsilon_0 \frac{\partial \mathbf{E}}{\partial t}) = \mu_0\epsilon_0 \frac{\partial}{\partial t}(\nabla \times \mathbf{E}) = -\mu_0\epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2} \\
\nabla^2 \mathbf{E} = \mu_0\epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}, &\quad \nabla^2 \mathbf{B} = \mu_0\epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}
\end{aligned}$$

notice that they are just 3d wave equation with $c = 1/\sqrt{\mu_0\epsilon_0}$, notice that the equations also restrict solutions which will consist of monochromatic plane em waves (with $\mathbf{E}$ defines their polarization) as

$$\begin{aligned}
\tilde{\mathbf{E}}(\mathbf{r}, t) &= \tilde{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\mathbf{n} \\
&= \frac{c^2}{\omega}\tilde{B}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}(\mathbf{n}_B \times \mathbf{k}) = c\tilde{B}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}(\mathbf{n}_B \times \mathbf{k}/|\mathbf{k}|) \\
&= \frac{c^2}{\omega}\tilde{\mathbf{B}} \times \mathbf{k} = c\tilde{\mathbf{B}} \times \mathbf{k}/|\mathbf{k}| \\
\tilde{\mathbf{B}}(\mathbf{r}, t) &= \tilde{B}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\mathbf{n}_B \\
&= \frac{1}{\omega}\tilde{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}(\mathbf{k} \times \mathbf{n}) = \frac{1}{c}\tilde{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}(\mathbf{k}/|\mathbf{k}| \times \mathbf{n}) \\
&= \frac{1}{\omega}\mathbf{k} \times \tilde{\mathbf{E}} = \frac{1}{c}\mathbf{k}/|\mathbf{k}| \times \tilde{\mathbf{E}}
\end{aligned}$$

$$\begin{aligned}
\mathbf{E}(\mathbf{r}, t) &= E_0 \cos(\mathbf{k}\cdot\mathbf{r}-\omega t+\delta)\mathbf{n} \\
&= \frac{c^2}{\omega}B_0 \cos(\mathbf{k}\cdot\mathbf{r}-\omega t+\delta)(\mathbf{n}_B \times \mathbf{k}) = cB_0 \cos(\mathbf{k}\cdot\mathbf{r}-\omega t+\delta)(\mathbf{n}_B \times \mathbf{k}/|\mathbf{k}|) \\
&= \frac{c^2}{\omega}\mathbf{B} \times \mathbf{k} = c\mathbf{B} \times \mathbf{k}/|\mathbf{k}| \\
\mathbf{B}(\mathbf{r}, t) &= B_0 \cos(\mathbf{k}\cdot\mathbf{r}-\omega t+\delta)\mathbf{n}_B \\
&= \frac{1}{\omega}E_0 \cos(\mathbf{k}\cdot\mathbf{r}-\omega t+\delta)(\mathbf{k} \times \mathbf{n}) = \frac{1}{c}E_0 \cos(\mathbf{k}\cdot\mathbf{r}-\omega t+\delta)(\mathbf{k}/|\mathbf{k}| \times \mathbf{n}) \\
&= \frac{1}{\omega}\mathbf{k} \times \mathbf{E} = \frac{1}{c}\mathbf{k}/|\mathbf{k}| \times \mathbf{E}
\end{aligned}$$

from which we use $|\mathbf{k}| = \omega/c$ and can conclude that $|\mathbf{E}|/|\mathbf{B}| = c$ with their geometry relation as

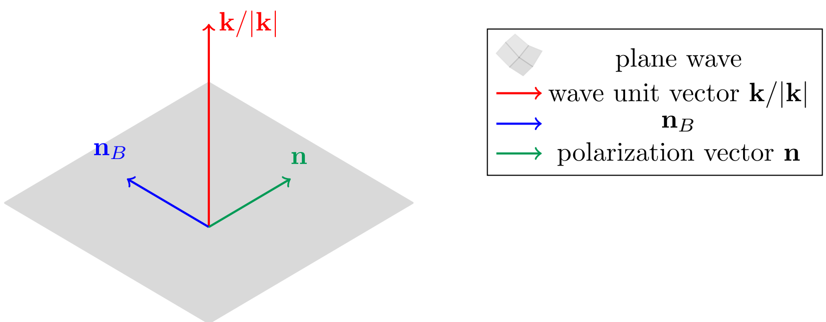

Figure 2: geometry relation in em wave

then we can calculate the energy density, Poynting vector, linear momentum density and their average values

$$u = \frac{1}{2}\epsilon_0 \mathbf{E}^2 + \frac{\mathbf{B}^2}{2\mu_0} = \epsilon_0 \mathbf{E}^2 \tag{36}$$

$$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}) = \frac{1}{c\mu_0}\mathbf{E}^2\mathbf{k}/|\mathbf{k}| = c\epsilon_0\mathbf{E}^2\mathbf{k}/|\mathbf{k}| = cu\mathbf{k}/|\mathbf{k}| \tag{37}$$

$$\mathbf{g} = \mu_0\epsilon_0\mathbf{S} = \frac{1}{c}\epsilon_0\mathbf{E}^2\mathbf{k}/|\mathbf{k}| = \frac{u}{c}\mathbf{k}/|\mathbf{k}| \tag{38}$$

$$\langle u\rangle = \frac{1}{T}\int_0^T u dt \tag{39}$$

$$= \frac{\omega}{2\pi}\epsilon_0 E_0^2 \int_0^{2\pi/\omega} \cos^2(\mathbf{k}\cdot\mathbf{r} - \omega t) dt \tag{40}$$

$$= \frac{1}{2}\epsilon_0 E_0^2 \frac{\omega}{2\pi}\left\{t - \frac{1}{2\omega}\sin[2(\mathbf{k}\cdot\mathbf{r} - \omega t)]\right\}\Big|_0^{2\pi/\omega} \tag{41}$$

$$= \frac{1}{2}\epsilon_0 E_0^2 \tag{42}$$

$$\tag{43}$$

2.3 in linear/conducting medium