

Odds, Risks, and Bootstraps

- ▶ Introduce some terms from Epidemiology. (required material)
 - ▶ Risk difference
 - ▶ Risk ratio
 - ▶ Odds ratio
- ▶ Go over another “recipe” for creating confidence intervals (bootstrap)
Bootstrapping will be included only as extra credit – this is not required material for the course

****note**, bootstrapping is not in your textbook, and the way your book describes the epidemiologic terms is not consistent with the way we teach them at Berkeley. For these topics lecture materials supersede the textbook.

An Interlude for Epidemiology

What is a risk?

$P(\text{event})$ in some defined time period such as:

- ▶ probability of acquiring malaria in a transmission season
- ▶ probability of death in the five years following a diagnosis
- ▶ probability of developing type 2 diabetes in a lifetime

We are very often interested in comparing the risk in two groups based on some exposure of interest:

$P(\text{Disease} \mid \text{exposure})$ vs. $P(\text{Disease} \mid \text{no exposure})$

remember that we could also write the probability of disease in the unexposed group as:

$P(\text{Disease} \mid \text{exposure}^C)$ or $P(\text{Disease} \mid \overline{\text{exposure}})$

These are conditional risks, because we are “conditioning on” membership/exposure

When we make comparisons we usually do this in one of two ways:

- ▶ absolute difference
- ▶ relative difference

Which one of these comparisons have we covered in hypothesis testing?

Absolute: Example Lung cancer among men in the U.S.

These numbers represent the probability of ever dying from lung cancer in men over the age of 35

Group	Lung Cancer	No Lung Cancer	Total
Smoker	13	4987	5000
non-smoker	1	4999	5000
Total	14	9986	10000

What is the absolute difference in risk?

$$\text{Risk Difference (RD)} = P(\text{LC} \mid \text{Smoker}) - P(\text{LC} \mid \text{non-smoker}) = ?$$

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What is the absolute difference in risk?

$$\text{Risk Difference (RD)} = P(\text{LC} \mid \text{Smoker}) - P(\text{LC} \mid \text{non-smoker}) = (13/5000) - (1/5000) = 0.0024$$

Another common way you will see differences presented is in relative terms: -
Risk ratio

$$RR = \frac{P(D|E)}{P(D|\bar{E})}$$

- Odds ratio

$$OR = \frac{\frac{P(D|E)}{1-P(D|E)}}{\frac{P(D|\bar{E})}{1-P(D|\bar{E})}}$$

Absolute: Example Lung cancer among men in the U.S.

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What are the RR and OR for these data?

► Risk ratio

$$RR = \frac{\frac{13}{5000}}{\frac{1}{5000}} = 13$$

► Odds ratio

$$OR = \frac{\frac{\frac{13}{5000}}{1 - \frac{13}{5000}}}{\frac{\frac{1}{5000}}{1 - \frac{1}{5000}}} = \frac{0.002606778}{0.00020004} = 13.03$$

- ▶ So far we've (mostly) been using formulas of the form estimate \pm margin of error to generate confidence intervals.
- ▶ We've applied these formulas to the mean \bar{x} and the proportion \hat{p} . These "large sample" procedures use the Central Limit Theorem to calculate the standard error of the sampling distribution for \bar{x} and \hat{p}

- ▶ We did this when the data met certain assumptions, one of which being that the data were drawn from a population that is Normally distributed.
- ▶ Though there was some leeway around this assumption, what would happen if you were sure the assumption was violated? How would you calculate a 95% confidence interval in that case?
- ▶ We need a procedure to calculate confidence intervals that do not rely on the assumption of Normality.

- ▶ What if we wanted a confidence interval for the median? Or a confidence interval for the first quartile, Q_1 , or some other parameter?
- ▶ We couldn't use the same procedure because the CLT does not apply to the sampling distribution of the median or Q_1 , or to many other statistics.
- ▶ We need a procedure to calculate confidence intervals for other parameters, such as the median or any other quartile or percentile.

The Bootstrap approach

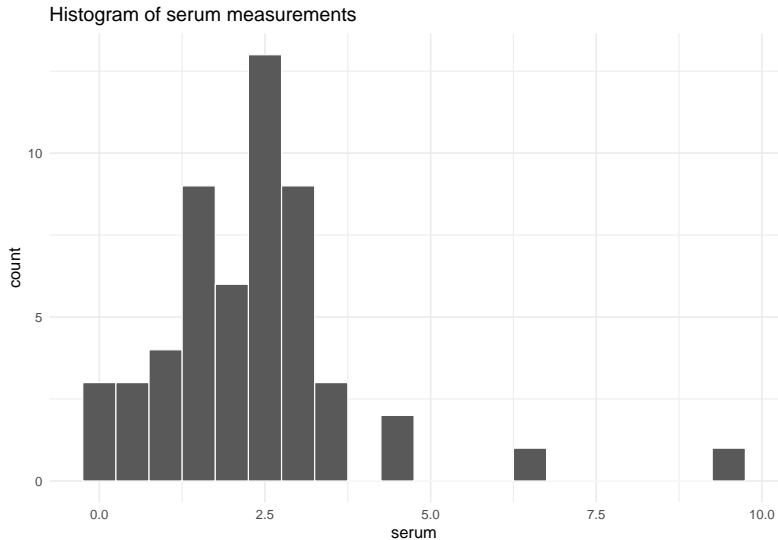
Enter: The Bootstrap Confidence Interval

- ▶ The Bootstrap CI does not require the underlying distribution to be Normally distributed.
- ▶ The Bootstrap CI can be made for any parameter (based on its sample statistic).
- ▶ It was so-named because it uses only the information from the sample you have to estimate the CI, such that the sample is “pulling itself up by its bootstraps”.
- ▶ This method gained popularity in 1979 when popularized by Bradley Efron, and harnesses current computing power.
- ▶ It is a very popular method used to compute confidence intervals today

Bootstrap example

This example came from bootstrap's advocate, Bradley Efron. Suppose we have measures of serum from 54 patients:

```
serum_data <- data.frame(serum = c(0.1, 0.1, 0.2, 0.4, 0.4, 0.6, 0.8, 0.8,  
                                0.9, 0.9, 1.3, 1.3, 1.4, 1.5, 1.6, 1.6, 1.  
                                1.7, 1.7, 1.8, 2.0, 2.0, 2.2, 2.2, 2.2, 2.  
                                2.3, 2.4, 2.4, 2.4, 2.4, 2.4, 2.4, 2.5, 2.  
                                2.5, 2.7, 2.7, 2.8, 2.9, 2.9, 2.9, 3.0, 3.  
                                3.1, 3.2, 3.2, 3.3, 3.3, 3.5, 4.4, 4.5, 6.  
                                9.4))
```



First, calculate the theory-based 95% CI: dplyr()

```
t_star <- qt(0.975, df = 53) # find t* to use to calculate the 95% CI
```

```
serum_data %>% summarise(mean_serum = mean(serum),  
                           se_serum = sd(serum)/sqrt(n()),  
                           lower_CI = mean_serum - t_star * se_serum,  
                           upper_CI = mean_serum + t_star * se_serum)
```

```
##   mean_serum se_serum lower_CI upper_CI  
## 1    2.318519 0.2078991 1.901526 2.735511
```

First, calculate the theory-based 95% CI: `t.test()`

```
t.test(serum_data %>% pull(serum))

##
## One Sample t-test
##
## data:  serum_data %>% pull(serum)
## t = 11.152, df = 53, p-value = 1.62e-15
## alternative hypothesis: true mean is not equal to 0
## 95 percent confidence interval:
##  1.901526 2.735511
## sample estimates:
## mean of x
##  2.318519
```

The 95% CI provides our best guess of where the true proportion lies. The 95% CI for μ is 1.90 to 2.74. We found this interval using a method that gives an interval that captures μ 19 times out of 20.

95% CI for the median

- ▶ We can't use the above method to compute a 95% CI for the median because the CLT does not apply to the median.
- ▶ However, we can use a method that relies on the assumption that the sample is a SRS from the underlying population

How the Bootstrap CI is made

- ▶ If we truly have a SRS from the underlying population, this means that the distribution of serum in the sample should *approximate* the distribution of serum in the population
- ▶ Graphically, this means the shape of the histogram for the sample data should approximate the shape of the density plot for the entire population
- ▶ The key: If we take repeated samples (with replacement) from our sample, we can approximate the sampling distribution for any statistic we'd like
- ▶ This is the process of bootstrapping
- ▶ Let's apply this method to calculate the 95% CI for the median using the serum data

Bootstrap confidence interval for the median

1. Calculate the median for the original sample of size 54. Denote this value by m . This is our estimate of the median for the underlying population. We need to create a 95% CI around m .

```
median_serum <- serum_data %>% summarise(median_serum = median(serum))  
median_serum
```

```
##      median_serum  
## 1                2.35
```

2. Resample with replacement from the original sample a new sample, also of size 54.

Bootstrap confidence interval for the median

3. Calculate the median based on resample #1. Call this median m_1^* .
4. Resample again. Calculate the median based on resample #2. Call this median m_2^* repeat this resampling procedure several thousand times.
5. Make a histogram of $m_1^*, m_2^*, \dots, m_{1000}^*$. This histogram approximates the sampling distribution for the median

6. Calculate the bounds such that the middle 95% of the observations are between the lower and upper bounds. In R, we can do this using `quantile(sample_median, 0.025)` and `quantile(sample_median, 0.975)` to locate the 2.5th and 97.5th percentiles of the variable `sample_median`.

Bootstrap confidence interval for the median

The code embeded in the lecture here resamples the data 1000 times and calculates the median for each resample. You do not need to know how to do this in R. It stores the median in a data frame called `many_sample_medians`

```
head(many_sample_medians)
```

```
##      sample_median sample.id
## 1             2.4         1
## 2             2.4         2
## 3             2.5         3
## 4             2.4         4
## 5             2.4         5
## 6             2.4         6
```

Calculate the lower and upper bounds of the 95% bootstrap CI

*# Understand this code. It takes the data frame containing the bootstrap
sample medians and calculates the lower and upper CI using the quantile function*

```
bounds <- many_sample_medians %>%  
  summarise(lower_CI = quantile(sample_median, 0.025),  
            upper_CI = quantile(sample_median, 0.975))
```

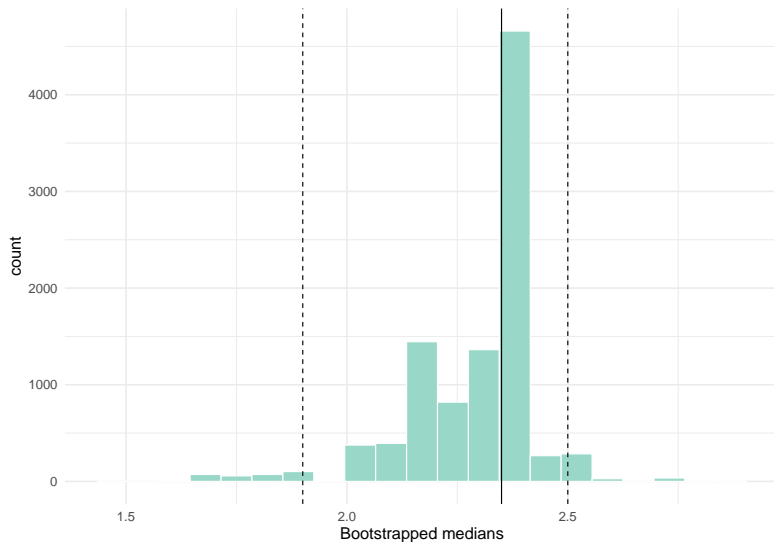
bounds

```
##   lower_CI upper_CI  
## 1      1.9      2.5
```

how would the code change if we were interested in the 99% CI?

Thus, our best estimate of the median is 2.35. The bootstrapped 95% confidence interval for the median is 1.9 to 2.5.

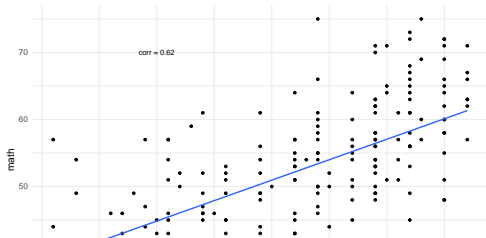
Plot the histogram of the bootstrapped medians and denote the 95% confidence interval



Another example: Calculate the bootstrap CI for the correlation coefficient

You have data on student demographics and scores across multiple subjects. You are interested in the correlation between the scores for writing (write) and mathematics (math)

```
## Warning in geom_text(aes(x = 40, y = 70, label = "corr = 0.62"), check_over  
## i Please consider using 'annotate()' or provide this layer with data conta  
## a single row.  
  
## 'geom_smooth()' using formula = 'y ~ x'
```



Calculate the bootstrapped CI for the correlation

1. Calculate the correlation for the original sample of size 200. Denote this value by m . This is our estimate of the median for the underlying population. We need to create a 95% CI around m .

```
corr_math_write <- hsb2 %>% summarise(correlation = cor(write, math))  
corr_math_write
```

```
## correlation  
## 1 0.6174493
```

2. Resample with replacement from the original sample a new sample, also of size 200.
3. Calculate the correlation based on resample #1. Call this correlation r_1^* .
4. Resample again. Calculate the correlation based on resample #2. Call this median r_2^* repeat this resampling procedure several thousand times.

Calculate the bootstrapped CI for the correlation

5. Make a histogram of $r_1^*, r_2^*, \dots, r_{10000}^*$. This histogram approximates the sampling distribution for the correlation coefficient
6. Calculate the bounds such that the middle 95% of the observations are between the lower and upper bounds. In R, we can do this using `quantile(sample_corr, 0.025)` and `quantile(sample_corr, 0.975)` to locate the 2.5th and 97.5th percentiles of the variable `sample_corr`.

```
##      sample_corr sample.id
## 1      0.6032971         1
## 2      0.6020023         2
## 3      0.6171369         3
## 4      0.6083906         4
## 5      0.7292170         5
## 6      0.6268646         6
```


Calculate the lower and upper bounds of the 95% bootstrap CI for the correlation coefficient

*# Understand this code. It takes the data frame containing the bootstrap
sample medians and calculates the lower and upper CI using the quantile fun*

```
bounds <- many_sample_correlations %>%  
  summarise(lower_CI = quantile(sample_corr, 0.025),  
            upper_CI = quantile(sample_corr, 0.975))
```

```
bounds
```

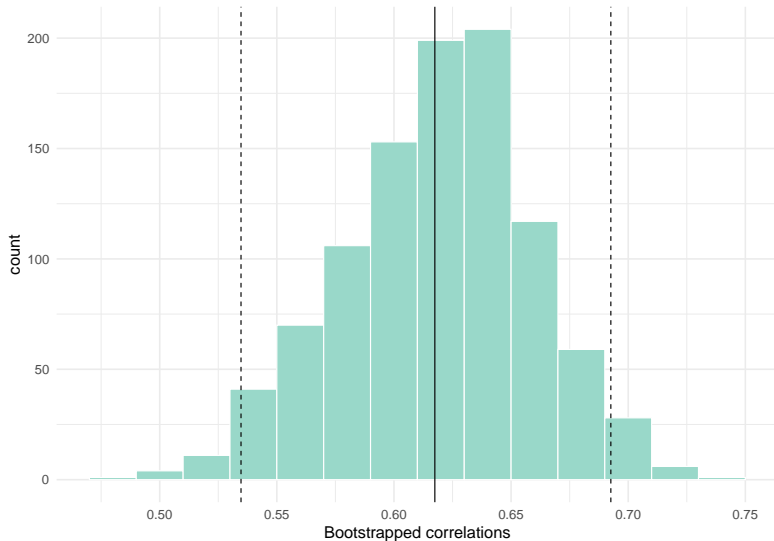
```
##      lower_CI  upper_CI  
## 1 0.5347083 0.6925803
```

how would the code change if we were interested in the 99% CI?

Plot the histogram of the bootstrapped correlations and show the 95% CI

```
plot_boots_cor<-ggplot(many_sample_correlations, aes(x = sample_corr)) +  
  geom_histogram(binwidth = 0.02, col = "white", fill = "#99d8c9") +  
  labs(x = "Bootstrapped correlations") +  
  geom_vline(aes(xintercept = corr_math_write %>% pull())) +  
  theme_minimal(base_size = 15) +  
  geom_vline(aes(xintercept = bounds %>% pull(lower_CI)), lty = 2) +  
  geom_vline(aes(xintercept = bounds %>% pull(upper_CI)), lty = 2)
```

Plot the histogram of the bootstrapped correlations and show the 95% CI



- ▶ The bootstrap is a method that we use to estimate confidence intervals
- ▶ It is particularly useful when:
 - ▶ We don't have a nice formula to calculate the CI, or we don't know what the formula is
 - ▶ The underlying assumptions of using a “large sample” formula are not satisfied
- ▶ We can make bootstrap CIs around many statistics we've learnt about: the median, the quartiles, the correlation coefficient.

Parting humor, courtesy of irregular webcomic

DATA: BY THE NUMBERS

