

# Assignment 1

Mytraya Gattu, Roshni Singh, Ramakant Pal  
180050032, 18B030020, 180260028

August 25, 2020

The ladder operators are defined as

$$L_+ = L_x + iL_y$$

$$L_- = L_x - iL_y$$

$$\begin{aligned} L_+ L_z &= L_z L_+ + [L_+, L_z] \\ &= L_z L_+ + [L_x, L_z] + i[L_y, L_z] \\ &= L_z L_+ + (-i\hbar L_y) + i(\hbar L_x) \\ &= L_z L_+ - \hbar(L_x + iL_y) \\ &= L_z L_+ - \hbar L_+ \\ [L_i, L_j] &= i\hbar \epsilon_{ijk} L_k \end{aligned}$$

Therefore,

$$\begin{aligned} L_+ L_z |l, m\rangle &= L_z L_+ |l, m\rangle - \hbar L_+ |l, m\rangle \\ \hbar m L_+ |l, m\rangle + \hbar L_+ |l, m\rangle &= L_z L_+ |l, m\rangle \end{aligned}$$

From the orthonormality of the vectors in the set  $\bigcup_i \{|l, m_i\rangle\}$ , we can conclude that

$$L_+ |l, m\rangle \propto |l, m+1\rangle$$

and similarly,

$$L_- |l, m\rangle \propto |l, m-1\rangle$$

Let,

$$L_+ |l, m\rangle = C_{l,m}^+ |l, m+1\rangle$$

and

$$L_- |l, m\rangle = C_{l,m}^- |l, m-1\rangle$$

From this, we conclude that

$$\langle l, m | L_- L_+ |l, m\rangle = C_{l,m+1}^- C_{l,m}^+$$

$$\langle l, m | L_+ L_- |l, m\rangle = C_{l,m-1}^+ C_{l,m}^-$$

Also,

$$\begin{aligned} L_- L_+ &= (L_x - iL_y)(L_x + iL_y) \\ &= L_x^2 - i(L_y L_x - L_x L_y) + L_y^2 \\ &= L_x^2 + L_y^2 + L_z^2 - L_z^2 - \hbar L_z \\ &= L^2 - L_z^2 - \hbar L_z \end{aligned}$$

Similarly,

$$L_+L_- = L^2 - L_z^2 + \hbar L_z$$

This implies,

$$\langle l, m | L_- L_+ | l, m \rangle = \hbar^2 (l^2 + l - m^2 - m)$$

$$\langle l, m | L_+ L_- | l, m \rangle = \hbar^2 (l^2 + l - m^2 + m)$$

Since,  $L_+^\dagger = L_-$ ,

$$L_+ |l, m\rangle \leftrightarrow \langle l, m | L_+^\dagger \\ \langle l, m | L_-$$

Hence,

$$\langle l, m | L_- L_+ | l, m \rangle = C_{l, m+1}^- C_{l, m}^+ = C_{l, m}^{+*} C_{l, m}^+$$

$$\langle l, m | L_+ L_- | l, m \rangle = C_{l, m-1}^+ C_{l, m}^- = C_{l, m}^{-*} C_{l, m}^-$$

Using the above relations,

$$C_{l, m}^+ = e^{i\phi_{l, m}^+} \hbar \sqrt{l^2 + l - m^2 - m}$$

$$C_{l, m}^- = e^{i\phi_{l, m}^-} \hbar \sqrt{l^2 + l - m^2 + m}$$

Since, the phase carries no actual physical significance, we set  $\phi^+ = \phi^- = 0$ .

$$\langle l, m' | L_+ | l, m \rangle = \hbar \delta_{m+1, m'} \sqrt{l^2 + l - m^2 - m}$$

$$\langle l, m' | L_- | l, m \rangle = \hbar \delta_{m-1, m'} \sqrt{l^2 + l - m^2 + m}$$

It is also clear, from the derivation that  $m$  ranges from  $-l$  to  $l$  in steps of 1. Dimensionality of these matrices is therefore:  $(2l+1)^2$  Using the formulae for the matrix elements for  $L_+$  and  $L_-$  in matrix form, for  $l = \frac{1}{2}$ :

$$L_+ = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$L_- = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and for  $l = 1$ :

$$L_+ = \hbar \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}$$

$$L_- = \hbar \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}$$

From the definitions for  $L_+$  and  $L_-$ ,

$$L_x = \frac{1}{2} (L_+ + L_-)$$

$$L_y = \frac{1}{2i} (L_+ - L_-)$$

Therefore, for  $l = 1/2$ :

$$L_x = \hbar \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$$

$$L_y = \hbar \begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix}$$

and for  $l = 1$ :

$$L_x = \hbar \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

$$L_y = \hbar \begin{pmatrix} 0 & -\frac{i}{\sqrt{2}} & 0 \\ \frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ 0 & \frac{i}{\sqrt{2}} & 0 \end{pmatrix}$$

$L_z$  can be found by

$$L_z = -\frac{i}{\hbar} [L_x, L_y]$$

For  $l = 1$ ,

$$L_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$