Assignment 3

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September 1, 2020

Addition of spins $j = \frac{1}{2}$ and $j = 1$

$|j_1, j_2; j, m\rangle$ basis kets are the eigen-kets of J^2, J_z, J_1^2, J_2^2. The defining equation for the Clebsch-Gordan coefficients is:

$$\begin{aligned}|j_1, j_2; j, m\rangle &= \sum_{m_1 m_2} \langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m\rangle |j_1, j_2; m_1, m_2\rangle \\ &= \sum_{m_1 m_2} C^{jm}_{m_1 m_2} |j_1, j_2; m_1, m_2\rangle\end{aligned}$$

Consider the action of J_+ on the state $|j1, j2; j, m\rangle$,

$$\begin{aligned}\sqrt{(j-m)(j+m+1)}|j_1, j_2; j, m+1\rangle = &\sum_{m_1 m_2} \sqrt{(j_1 - m_1)(j_1 + m_1 + 1)} C^{jm}_{m_1 m_2} |j_1, j_2; m_1 + 1, m_2\rangle + \\ &\sum_{m_1 m_2} \sqrt{(j_2 - m_2)(j_2 + m_2 + 1)} C^{jm}_{m_1 m_2} |j_1, j_2; m_1, m_2 + 1\rangle\end{aligned}$$

Using the orthonormality of the basis kets and the definition of the Clebsch-Gordan coefficients,

$$\sqrt{(j-m)(j+m+1)} C^{j\,m+1}_{m_1'\,m_2'} = \sqrt{(j_1 - m_1' + 1)(j_1 + m_1')} C^{j\,m}_{m_1'-1\,m_2'} + \sqrt{(j_2 - m_2' + 1)(j_2 + m_2')} C^{j\,m}_{m_1'\,m_2'-1}$$

Similarly,

$$\sqrt{(j+m)(j-m+1)} C^{j\,m-1}_{m_1'\,m_2'} = \sqrt{(j_1 + m_1' + 1)(j_1 - m_1')} C^{j\,m}_{m_1'+1\,m_2'} + \sqrt{(j_2 + m_2' + 1)(j_2 - m_2')} C^{j\,m}_{m_1'\,m_2'+1}$$

Calculation for $j = \frac{3}{2}$

$m = \pm\frac{3}{2}$

This state can only be obtained when $(m_1, m_2) = (\pm 1, \pm\frac{1}{2})$. Therefore,

$$C^{\frac{3}{2}\,\frac{3}{2}}_{m_1 m_2} = \delta_{m_1,1}\delta_{m_2,\frac{1}{2}}$$

$$C^{\frac{3}{2}\,-\frac{3}{2}}_{m_1 m_2} = \delta_{m_1,-1}\delta_{m_2,-\frac{1}{2}}$$

$m = \frac{1}{2}$

Since, the Clebsch-gordan coefficients are known for $m = \frac{3}{2}$, using the already obtained relation:

$$\sqrt{(\frac{3}{2}+\frac{3}{2})(\frac{3}{2}-\frac{3}{2}+1)} C^{\frac{3}{2}\,\frac{3}{2}-1}_{m_1'\,m_2'} = \sqrt{(1 + m_1' + 1)(1 - m_1')} C^{\frac{3}{2}\,\frac{3}{2}}_{m_1'+1\,m_2'} + \sqrt{(\frac{1}{2} + m_2' + 1)(\frac{1}{2} - m_2')} C^{\frac{3}{2}\,\frac{3}{2}}_{m_1'\,m_2'+1}$$

Simplifying,

$$\sqrt{3} C^{\frac{3}{2}\,\frac{1}{2}}_{m_1'\,m_2'} = \sqrt{2}\delta_{m_1',0}\delta_{m_2',\frac{1}{2}} + \delta_{m_1',1}\delta_{m_2',-\frac{1}{2}}$$

$m = -\frac{1}{2}$

Since, the Clebsch-gordan coefficients are known for $m = -\frac{3}{2}$, using the already obtained relation:

$$\sqrt{(\frac{3}{2} - -\frac{3}{2})(\frac{3}{2} + -\frac{3}{2} + 1)}C^{\frac{3}{2}\,(-\frac{3}{2}+1)}_{m_1'\,m_2'} = \sqrt{(1 - m_1' + 1)(1 + m_1')}C^{\frac{3}{2}\,-\frac{3}{2}}_{m_1'-1\,m_2'} + \sqrt{(\frac{1}{2} - m_2' + 1)(\frac{1}{2} + m_2')}C^{\frac{3}{2}\,-\frac{3}{2}}_{m_1'\,m_2'-1}$$

Simplifying,

$$\sqrt{3}C^{\frac{3}{2}\,-\frac{1}{2}}_{m_1'\,m_2'} = \sqrt{2}\delta_{m_1',0}\delta_{m_2',-\frac{1}{2}} + \delta_{m_1',-1}\delta_{m_2',\frac{1}{2}}$$

Calculation for $j = \frac{1}{2}$

$m = \frac{1}{2}$

This state can be constructed by the superposition of $|1, -\frac{1}{2}\rangle$ and $|0, \frac{1}{2}\rangle$. By convention, the Clebsch-Gordan coefficients are taken to be orthonormal, which in turn implies from:

$$|C^{\frac{1}{2}\,\frac{1}{2}}_{1\,-\frac{1}{2}}|^2 + |C^{\frac{1}{2}\,\frac{1}{2}}_{0\,\frac{1}{2}}|^2 = 1\,,$$

that

$$C^{\frac{1}{2}\,\frac{1}{2}}_{1\,-\frac{1}{2}} = \cos\theta$$

$$C^{\frac{1}{2}\,\frac{1}{2}}_{0\,\frac{1}{2}} = \sin\theta$$

Consider the action of J_+ on the state in discussion.

$$\sqrt{(\frac{1}{2} - \frac{1}{2})(\frac{1}{2} + \frac{1}{2} + 1)}C^{\frac{1}{2}\,\frac{1}{2}+1}_{m_1'\,m_2'} = \sqrt{(1 - m_1' + 1)(1 + m_1')}C^{\frac{1}{2}\,\frac{1}{2}}_{m_1'-1\,m_2'} + \sqrt{(\frac{1}{2} - m_2' + 1)(\frac{1}{2} + m_2')}C^{\frac{1}{2}\,\frac{1}{2}}_{m_1'\,m_2'-1}$$

Simplifying,

$$0 = \sqrt{(1 - m_1' + 1)(1 + m_1')}C^{\frac{1}{2}\,\frac{1}{2}}_{m_1'-1\,m_2'} + \sqrt{(\frac{1}{2} - m_2' + 1)(\frac{1}{2} + m_2')}C^{\frac{1}{2}\,\frac{1}{2}}_{m_1'\,m_2'-1}$$

Now, let $m_1' = 1$ and $m_2' = \frac{1}{2}$. Substituting in the above equation,

$$\begin{aligned} 0 &= \sqrt{(1 - 1 + 1)(1 + 1)}C^{\frac{1}{2}\,\frac{1}{2}}_{1-1\,\frac{1}{2}} + \sqrt{(\frac{1}{2} - \frac{1}{2} + 1)(\frac{1}{2} + \frac{1}{2})}C^{\frac{1}{2}\,\frac{1}{2}}_{1\,\frac{1}{2}-1} \\ &= \sqrt{2}\sin\theta + \cos\theta \\ \implies \tan\theta &= -\frac{1}{\sqrt{2}} \\ \implies \sin\theta &= \pm\sqrt{\frac{1}{3}} \\ \implies \cos\theta &= \mp\sqrt{\frac{2}{3}} \end{aligned}$$

Without loss of generality, I choose one sign. Therefore,

$$C^{\frac{1}{2}\,\frac{1}{2}}_{m_1 m_2} = \sqrt{\frac{2}{3}}\delta_{m_1,1}\delta_{m_2,-\frac{1}{2}} - \sqrt{\frac{1}{3}}\delta_{m_1,0}\delta_{m_2,\frac{1}{2}}$$

$m = -\frac{1}{2}$

Here, although an analogous calculation to the previous sub-section is possible, we choose to use the operation of J_- on the $m = \frac{1}{2}$ state, to preserve the sign-convention adopted.

$$\sqrt{(\frac{1}{2}+\frac{1}{2})(\frac{1}{2}-\frac{1}{2}+1)}C^{\frac{1}{2}\,\frac{1}{2}-1}_{m_1'\,m_2'} = \sqrt{(1+m_1'+1)(1-m_1')}C^{\frac{1}{2}\,\frac{1}{2}}_{m_1'+1\,m_2'} + \sqrt{(\frac{1}{2}+m_2'+1)(\frac{1}{2}-m_2')}C^{\frac{1}{2}\,\frac{1}{2}}_{m_1'\,m_2'+1}$$

Only non-zero values occur when either $(m_1', m_2') = (-1, \frac{1}{2})$ or $(m_1', m_2') = (0, -\frac{1}{2})$ Therefore,

$$C^{\frac{1}{2}\,-\frac{1}{2}}_{-1\,\frac{1}{2}} = \sqrt{2}C^{\frac{1}{2}\,\frac{1}{2}}_{0\,\frac{1}{2}} + \sqrt{(\frac{1}{2}+\frac{1}{2}+1)(\frac{1}{2}-\frac{1}{2})}C^{\frac{1}{2}\,\frac{1}{2}}_{-1\,\frac{3}{2}} = -\sqrt{\frac{2}{3}}$$

Similarly,

$$C^{\frac{1}{2}\,-\frac{1}{2}}_{0\,-\frac{1}{2}} = \sqrt{\frac{1}{3}}$$

Hence,

$$C^{\frac{1}{2}\,-\frac{1}{2}}_{m_1 m_2} = -\sqrt{\frac{2}{3}}\delta_{m_1,-1}\delta_{m_2,\frac{1}{2}} + \sqrt{\frac{1}{3}}\delta_{m_1,0}\delta_{m_2,-\frac{1}{2}}$$