Angular Momentum Eigenfunctions

We obtained the angular momentum eigenkets in the abstract form and studied some of their properties which were derived purely algebraically

We will now construct explicit eigenfunctions (or wavefunctions) in the position basis.

It is natural to use spherical coordinates to study spherically symmetric systems, which conserve angular momentum

We need to find $\langle \theta, \phi | \ell, m \rangle = \psi_{\ell m}(\theta, \phi)$

Since $\mathbf{L} = \mathbf{r} \times \mathbf{p} = -i\hbar\, \mathbf{r} \times \boldsymbol{\nabla}$, we have to express the gradient in spherical coordinates, which you (should) recall from the E&M course:

$$\nabla = \hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}$$

Since $\mathbf{r} = r\,\hat{\mathbf{r}}$,

$$\mathbf{L} = -i\hbar\left[r(\hat{\mathbf{r}}\times\hat{\mathbf{r}})\frac{\partial}{\partial r} + (\hat{\mathbf{r}}\times\hat{\boldsymbol{\theta}})\frac{\partial}{\partial \theta} + (\hat{\mathbf{r}}\times\hat{\boldsymbol{\phi}})\frac{1}{\sin\theta}\frac{\partial}{\partial \phi}\right]$$

Since $\hat{\mathbf{r}}\times\hat{\mathbf{r}} = 0, \hat{\mathbf{r}}\times\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}$ and $\hat{\mathbf{r}}\times\hat{\boldsymbol{\phi}} = -\hat{\boldsymbol{\theta}}$,

$$\mathbf{L} = -i\hbar\left[\hat{\boldsymbol{\phi}}\frac{\partial}{\partial \theta} - \hat{\boldsymbol{\theta}}\frac{1}{\sin\theta}\frac{\partial}{\partial \phi}\right]$$

Expressing $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\phi}}$ in terms of Cartesian unit vectors:

$$\hat{\boldsymbol{\theta}} = (\cos\theta\cos\phi)\hat{\mathbf{i}} + (\cos\theta\sin\phi)\hat{\mathbf{j}} - \sin\theta\,\hat{\mathbf{k}}$$

$$\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{i}} + \cos\phi\,\hat{\mathbf{j}}$$

$$\mathbf{L} = -i\hbar\left[(-\sin\phi\,\hat{\mathbf{i}} + \cos\phi\ \hat{\mathbf{j}})\ \frac{\partial}{\partial\theta}\right.$$

The raising and lowering operators can be worked out to be

$$L_{\pm} = L_x \pm iL_y = \pm\hbar e^{\pm i\phi}\left(\frac{\partial}{\partial\theta} \pm i\cot\theta\,\frac{\partial}{\partial\phi}\right)$$

And thus

$$L_+L_- = -\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\,\frac{\partial}{\partial\theta} + \cot^2\theta\,\frac{\partial^2}{\partial\phi^2} + i\,\frac{\partial}{\partial\phi}\right)$$

Since $\mathbf{L}^2 = L_+L_- + L_z^2 - \hbar L_z$, we get $\mathbf{L}^2$ in terms of spherical coordinates

$$\mathbf{L}^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

This is exactly the <u>angular part</u> of Laplace's eqn. $\nabla^2 V = 0$
in spherical coordinates encountered in electrostatics !
Recall that one can solve this PDE by using separation of variables technique:
Writing the solution as $Y(\theta, \phi) = \Theta(\theta)\Phi(\phi)$,

$$\Phi(\phi) \text{ satisfies } \frac{d^2\Phi(\phi)}{d\phi^2} = -m^2\Phi(\phi)$$

giving $\Phi(\phi) = \frac{1}{\sqrt{2\pi}} e^{\pm im\phi}$ (normalized to <u>Kronecker</u>-δ function).
Single valuedness of $\Phi(\phi + 2\pi) = \Phi(\phi)$ forces $m = 0, \pm 1, \pm 2, \cdots$
The DE satisfied by $\Theta(\theta)$ is

$$\sin\theta \frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) + (\ell(\ell+1)\sin^2\theta - m^2)\Theta = 0$$

The solution is $\Theta(\theta) = A\, P_\ell^m(\cos\theta)$, where P_ℓ^m is the associated Legendre Polynomials defined by

$$P_\ell^m(x) = (1 - x^2)^{\frac{|m|}{2}} \left(\frac{d}{dx}\right)^{|m|} P_\ell(x)$$

and $P_\ell(x)$ is the ℓ-th Legendre Polynomial defined by

$$P_\ell(x) = \frac{1}{2^\ell} \left(\frac{d}{dx}\right)^{\ell} (x^2 - 1)^\ell$$

The normalized angular wavefunctions are the spherical harmonics:

$$\langle \theta, \phi | \ell, m \rangle = Y_\ell^m(\theta, \phi) = N\, e^{im\phi} P_\ell^m(\cos\theta)$$

where N is the normalization constant

N is determined by the orthonormality condition

$$\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \; |N|^2 \left[Y_\ell^m(\theta,\phi)\right]^* \left[Y_{\ell'}^{m'}\right] = \delta_{\ell\ell'}\delta_{mm'}$$

Thus, the spherical harmonics $Y_\ell^m(\theta,\phi)$ are the <u>coordinate space</u> eigenfunctions of $\mathbf{L}^2$ and L_z,

$$\mathbf{L}^2 Y_\ell^m(\theta,\phi) = \hbar^2 \ell(\ell+1) Y_\ell^m(\theta,\phi)$$
$$L_z Y_\ell^m(\theta,\phi) = \hbar m Y_\ell^m(\theta,\phi)$$

If you work through the solution for solving the $\Theta(\theta)$ part of the DE (say from Griffith's or any other standard textbook which has a discussion of the Legendre polynomials), it turns out that ℓ is forced to be an integer: $\ell = 0,1,2,\cdots$

For non-integer ℓ, there is no solution for the DE satisfied by $\Theta(\theta)$.
Rodrigues formula only holds for integer ℓ.

However, the algebraic theory of angular momentum, using only the commutation relations of the L_i gives both integer and non-integer values of ℓ.

The non-integer values of ℓ, especially $\ell = \frac{1}{2}$ (describing the electron) is of profound importance for atomic and molecular structure, condensed matter, stability of matter, particle physics, $\cdots$

The non-integer values of ℓ do <u>not</u> have a <u>coordinate representation</u>, i.e., for example, for a spin ½ particle, $\langle\theta, \phi\left|\frac{1}{2}, \pm\frac{1}{2}\right\rangle$ does <u>not</u> exist

Addition of Angular Momenta

The addition of angular momenta of two different particles or systems is an important problem occurring in many areas of physics: Atomic, molecular, nuclear, particle and condensed matter physics….

In many cases the dynamics and properties of a system (i.e., the Hamiltonian) are governed by the rules of angular momentum addition

Classically to add two angular momentum vectors we just add the components

However, in quantum mechanics, we do <u>not</u> have access to all the 3 components of $\mathbf{L}$ because of the uncertainty principle

One can only work with the magnitude $\mathbf{L^2}$ and one component $L_i = (L_x, L_y, L_z)$

Consider two particles with angular momenta $\mathbf{J}_1 = \{J_{1i}\}$ and $\mathbf{J}_2 = \{J_{2i}\}$

These two operator act on two different Hilbert spaces of dimension $2j_1 + 1$ and $2j_2 + 1$ respectively and hence commute with one another

The commutation relations of the two operators can be compactly written as

$$\mathbf{J}_1 \times \mathbf{J}_1 = i\hbar\mathbf{J}_1$$

$$\mathbf{J}_2 \times \mathbf{J}_2 = i\hbar\mathbf{J}_2$$

$$[\mathbf{J}_1, \mathbf{J}_2] = 0$$

We have two Casimir operators $\mathbf{J}_1^2$ and $\mathbf{J}_2^2$ which commute with all components of both the angular momentum, i.e., with all the generators of the algebra

The values of the magnitude of $\mathbf{J}_1^2$ and $\mathbf{J}_2^2$ are $\hbar^2 j_1(j_1 + 1)$ and $\hbar^2 j_2(j_2 + 1)$ respectively

The z-components of the angular momenta can take the range of values

$$J_{1z}: \quad \hbar m_1, \qquad -j_1 \leq m_1 \leq j_1$$
$$J_{2z}: \quad \hbar m_2, \qquad -j_2 \leq m_2 \leq j_2$$

Denote by $\mathcal{E}^{(j_1)}$ the Hilbert space on which $\mathbf{J}_1$ acts and similarly by $\mathcal{E}^{(j_2)}$ the space on which $\mathbf{J}_2$ acts

Since $\mathbf{J}_1$ and $\mathbf{J}_2$ commute, the total Hilbert space is a direct (tensor) product of the two spaces: $$\mathcal{E} = \mathcal{E}^{(j_1)} \otimes \mathcal{E}^{(j_2)}$$

with dimensionality $(2j_1+1)(2j_2+1)$ and basis states a direct product of the two basis states $|j_1, m_1; j_2, m_2\rangle = |j_1, m_1\rangle \otimes |j_2, m_2\rangle$

The definition of the total angular momentum is the same in classical and quantum mechanics, i.e., $$\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2$$

(More precisely this should be written as $\mathbf{J} = \mathbf{J}_1 \otimes \mathbf{I}_2 + \mathbf{I}_1 \otimes \mathbf{J}_2$) where $\mathbf{I}_{1,2}$ is the identity operator of appropriate dimension)

The algebra satisfied by **J** is

$$[J_i, J_j] = i\epsilon_{ijk}J_k$$

$$[\mathbf{J}^2, J_i] = 0$$

$$[\mathbf{J}^2, \mathbf{J}_1^2] = [\mathbf{J}^2, \mathbf{J}_2^2] = 0$$

since $\mathbf{J}^2 = \mathbf{J}_1^2 + \mathbf{J}_2^2 + 2\,\mathbf{J}_1 \cdot \mathbf{J}_2 = \mathbf{J}_1^2 + \mathbf{J}_2^2 + 2\,\mathbf{J}_2 \cdot \mathbf{J}_1$

However, one can check from the above expression for $\mathbf{J}^2$ that

$$[\mathbf{J}^2, J_{1i}] \neq 0$$

$$[\mathbf{J}^2, J_{2i}] \neq 0$$

This is because $\mathbf{J}^2$ also involves the term $\mathbf{J}_1 \cdot \mathbf{J}_2 = J_{1k}\, J_{2k}$, which does not commute with J_{1i} :

$$[J_{1k}\, J_{2k}, J_{1i}] = i\hbar\epsilon_{kil}J_{1l}\, J_{2k} = i\hbar\epsilon_{ilk}J_{1l}\, J_{2k} = i\,(\mathbf{J}_1 \times \mathbf{J}_2)_i$$

Similar argument for J_{2i}

Thus instead of working in the basis in which $\mathbf{J}_1^2, \mathbf{J}_2^2, J_{1z}$ and J_{2z} are diagonal (i.e., the direct product basis) we can work in the basis in which $\mathbf{J}^2, J_z, \mathbf{J}_1^2$ and $\mathbf{J}_2^2$ are diagonal (i.e., the total angular momentum basis)

Denote the total angular momentum basis states as $|j, m; j_1, j_2\rangle$

The question is: What are the possible value(s) of j and m given j_1 and j_2?

Consider $J_z\, |j_1, m_1; j_2, m_2\rangle = (J_{1z} \otimes I + I \otimes J_{2z})|j_1, m_1\rangle \otimes |j_2, m_2\rangle$

$$= J_{1z}|j_1, m_1\rangle \otimes |j_2, m_2\rangle + |j_1, m_1\rangle \otimes J_{2z}|j_2, m_2\rangle$$

$$= \hbar(m_1 + m_2)|j_1, m_1; j_2, m_2\rangle$$

So the eigenvalues of J_z are of the form $\hbar m$, where $m = m_1 + m_2$

Hence the z-components of the two angular momentum operators simply add

Since $-j_1 \leq m_1 \leq j_1$ and $-j_2 \leq m_2 \leq j_2$, it follows that

$$-(j_1 + j_2) \leq (m_1 + m_2) = m \leq j_1 + j_2$$

But a particular m value may be obtained from different values of m_1 and m_2 so we expect 'degeneracy' of states in this space

Since the total angular momentum $\mathbf{J}$ is an angular momentum operator (i.e, obeys the angular momentum algebra) then the eigenvalues of $\mathbf{J}^2$ are of the form $\hbar^2 j(j+1)$, where j is the total angular momentum quantum number

$$\text{Thus, } -j \leq m \leq j$$

Also $m_{\max} = j_1 + j_2$. Thus it also follows that $j_{\max} = j_1 + j_2$

But $j_{\max}$ cannot be the only value of j !

Then the dimensionality of the space would be $2j_{\max} + 1 = 2j_1 + 2j_2 + 1$

But, in general, $2j_1 + 2j_2 + 1 \neq (2j_1 + 1)(2j_2 + 1)$

The dimensionality of the space should not change as we make a (unitary) change of basis from the direct product basis $|j_1, m_1; j_2, m_2\rangle$ to the total angular momentum basis $|j, m; j_1, j_2\rangle$

Hence j must take on other values as well

To determine the other j values consider $|j_1, j_1\rangle \otimes |j_2, j_2\rangle$

This is the 'maximally stretched' state corresponding to the maximum possible value of m which is $m_1 + m_2$ which is unique:

$$|j_1 + j_2, j_1 + j_2; j_1, j_2\rangle = |j_1, j_1\rangle \otimes |j_2, j_2\rangle$$

(In principle, a m value of $j_1 + j_2$ can be obtained from a j value of $j_1 + j_2$ or $j_1 + j_2 + 1$, $j_1 + j_2 + 2$ and all larger values of $j > j_1 + j_2$)

However, the m values can then exceed $j_1 + j_2$, which cannot be since $m_{\max} = j_1 + j_2$

For the total angular momentum state corresponding to a m value of $j_1 + j_2 - 1$, there are <u>two</u> possible direct product states with this quantum number:

$$|j_1, j_1\rangle \otimes |j_2, j_2 - 1\rangle \quad \text{and} \quad |j_1, j_1 - 1\rangle \otimes |j_2, j_2\rangle$$

Both give the same value for $m = j_1 + j_2 - 1$

In the total angular momentum basis $|j, m; j_1, j_2\rangle$ on the other hand the state $|j_1 + j_2, j_1 + j_2 - 1; j_1, j_2\rangle$ gives one state with $m = j_1 + j_2 - 1$

The second state has to be $|j_1 + j_2 - 1, j_1 + j_2 - 1; j_1, j_2\rangle$, which also has $m = j_1 + j_2 - 1$

And hence $j = j_1 + j_2 - 1$ is another allowed j value

Similarly a m value of $j_1 + j_2 - 2$ can be obtained by

$$m_1 = j_1, \qquad m_2 = j_2 - 2$$
$$m_1 = j_1 - 1, \quad m_2 = j_2 - 1$$
$$m_1 = j_1 - 2, \qquad m_2 = j_2$$

These 3 states can be obtained by $j = j_1 + j_2$, $j = j_1 + j_2 - 1$ or $j = j_1 + j_2 - 2$

Thus $j = j_1 + j_2 - 2$ is another allowed j value

Proceeding this way we can show that the total quantum number j can take values $j_1 + j_2, j_1 + j_2 - 1, j_1 + j_2 - 2, \cdots\cdots, j_{\min}$

The value of $j_{\min}$ is fixed by the requirement that the dimensionality of the total space be $(2j_1 + 1)(2j_2 + 1)$

For each j value the dimensionality of the space is $2j + 1$

This constraint leads to

$$\sum_{j_{\min}}^{j_1+j_2} (2j+1) = (2j_1+1)(2j_2+1)$$

$$\text{i.e.}, (j_1+j_2)(j_1+j_2-1) - j_{\min}(j_{\min}-1) + j_1 + j_2 - j_{\min} + 1 = (2j_1+1)(2j_2+1)$$

$$j_{\min}^2 = j_1^2 + j_2^2 + 2j_1j_2 + j_1 + j_2 + j_1 + j_2 + 1 - (4j_1j_2 + 2j_1 + 2j_2 + 1) = j_1^2 + j_2^2 - 2j_1j_2 = (j_1 - j_2)^2$$

Hence $j_{\min} = |j_1 - j_2|$ (Modulus sign since j does not take negative values)

Thus the total angular momentum value takes values from $j_1 + j_2$ to $|j_1 - j_2|$ is steps of one and

$$\mathcal{E} = \mathcal{E}^{(j_1)} \otimes \mathcal{E}^{(j_2)} = \sum_{j=|j_1-j_2|}^{j_1+j_2} \oplus \, \mathcal{E}^{(j)}$$

Clebsch-Gordon coefficients

A composite system with angular momentum eigenkets $|j_1, m_1\rangle$ and $|j_2, m_2\rangle$ can be equivalently described in terms of two alternative basis: the total angular momentum basis $|j, m\rangle$ and the direct product basis $|j_1, m_1\rangle \otimes |j_2, m_2\rangle$

Two equivalent basis can be related to each other through a unitary transformation.

Since each set is a complete basis, one can express one basis as a linear combination of the other.

The expansion coefficients are called the Clebsch-Gordon-Wigner coefficients

Thus we can write

$$|j, m; j_1, j_2\rangle = \sum_{m_1, m_2} C(j, j_1, j_2; m, m_1, m_2)\, |j_1, m_1; j_2, m_2\rangle$$

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Where the expansions coefficients $C(j, j_1, j_2; m, m_1, m_2)$ are given by the overlap of the two different sets of basis vectors

$$C(j, j_1, j_2; m, m_1, m_2) = \langle j_1, m_1; j_2, m_2 | j, m; j_1, j_2 \rangle$$

So that

$$|j, m; j_1, j_2\rangle = \sum_{m_1, m_2} C(j, j_1, j_2; m, m_1, m_2)\, |j_1, m_1; j_2, m_2\rangle$$

$$= \sum_{m_1, m_2} \langle j_1, m_1; j_2, m_2 | j, m; j_1, j_2 \rangle \;\; |j_1, m_1; j_2, m_2\rangle$$

$$= \sum_{m_1, m_2} \underbrace{|j_1, m_1; j_2, m_2\rangle\langle j_1, m_1; j_2, m_2|}\, j, m; j_1, j_2 \rangle$$

The quantity in the under-brackets is the identity operator in the direct product space

Just as $|\psi\rangle = \sum |n\rangle\langle n|\psi\rangle = \sum c_n\, |n\rangle$, where $c_n = \langle n|\psi\rangle$ are the expansion coefficients

The CG coefficients $C(j, j_1, j_2; m, m_1, m_2) = \langle j_1, m_1; j_2, m_2 | j, m; j_1, j_2\rangle$
Are <u>non-zero</u> only if $|j_1 - j_2| \leq j \leq j_1 + j_2$

Applying the $J_z = J_{1z} + J_{2z}$ operator on both sides

$$J_z|j, m; j_1, j_2\rangle = \sum_{m_1, m_2} C(j, j_1, j_2; m, m_1, m_2)\,(J_{1z} + J_{2z})|j_1, m_1; j_2, m_2\rangle$$

$$\text{i.e.,} \sum_{m_1, m_2} (m - m_1 - m_2) C(j, j_1, j_2; m, m_1, m_2)|j_1, m_1; j_2, m_2\rangle = 0$$

This implies that $C(j, j_1, j_2; m, m_1, m_2) = 0, \quad \text{if} \quad m \neq m_1 + m_2$

In other words $C(j, j_1, j_2; m, m_1, m_2) = \langle j_1, m_1; j_2, m_2 | j, m; j_1, j_2 \rangle = 0$
if $m = m_1 + m_2$ and $|j_1 - j_2| \leq j \leq j_1 + j_2$ does <u>not</u> hold

So effectively we can write

$$|j, m; j_1, j_2\rangle = \sum_{m_1} C(j, j_1, j_2; m_1, m - m_1) \, |j_1, m_1; j_2, m - m_1\rangle$$

The normalization condition for the (total angular momentum) basis states

$$\langle j', m'; j_1, j_2 | j, m; j_1, j_2 \rangle = \delta_{j'j} \delta_{m'm}$$

Leads to the relation

$$C^*(j', j_1, j_2; m_1', m' - m_1') C(j, j_1, j_2; m_1, m - m_1) \delta_{m_1' m_1} \delta_{m'm} = \delta_{j'j} \delta_{m'm}$$

Or $C^*(j',j_1,j_2;m_1,m'-m_1)C(j,j_1,j_2;m_1,m-m_1)\delta_{m'm} = \delta_{j'j}\delta_{m'm}$

This is the orthogonality relation for CG coefficients

The phases of the CG coefficients are fixed by demanding that

$$\langle j_1,j_1;j_2,j-j_1|j,j;j_1,j_2\rangle = C(j,j_1,j_2;j_1,j-j_1)$$

is <u>real</u> and <u>positive</u>

Even though we discussed about the addition of angular momenta for two distinct particles, the same analysis holds even if we are adding the orbital angular momentum **L** and spin **S** of the same particle

One can also invert the expansion and write the direct product states as linear combinations of the total angular momentum states:

$$|j_1, m_1; j_2, m_2\rangle = \sum_{m,j} |j, m; j_1, j_2\rangle\langle j, m; j_1, j_2|j_1, m_1; j_2, m_2\rangle$$

$$= \sum_{m,j} C^*\,(j, j_1, j_2; m, m_1, m_2)|j, m; j_1, j_2\rangle$$

Let us consider the sum of two angular momenta with eigenvalues ½ each and analyze the resulting eigenvalues and eigenstates. In this case, we have

$$j_1 = \frac{1}{2}, \qquad j_2 = \frac{1}{2}$$

and, therefore,

$$m_1 = \pm\frac{1}{2}, \qquad m_2 = \pm\frac{1}{2}$$

The basis states for each of the angular momentum operators, is

$$|j_1, m_1\rangle = \left|\frac{1}{2}, \frac{1}{2}\right\rangle \text{ and } \left|\frac{1}{2}, -\frac{1}{2}\right\rangle$$

$$|j_2, m_2\rangle = \left|\frac{1}{2}, \frac{1}{2}\right\rangle \text{ and } \left|\frac{1}{2}, -\frac{1}{2}\right\rangle$$

Since we are adding two angular momenta with fixed j values of ½ each we can label the angular momentum states by their m vales and drop the j

So define

$$\left|\frac{1}{2},\frac{1}{2}\right\rangle = \left|\frac{1}{2}\right\rangle = |+\rangle \quad \text{and} \left|\frac{1}{2},-\frac{1}{2}\right\rangle = \left|-\frac{1}{2}\right\rangle = |-\rangle$$

So the direct product states $|j_1, m_1; j_2, m_2\rangle \equiv |j_1, m_1\rangle \otimes |j_2, m_2\rangle$ can be more conveniently labelled by avoiding clutter as

$$|j_1, m_1\rangle \otimes |j_2, m_2\rangle \equiv |m_1\rangle \otimes |m_2\rangle = |m_1, m_2\rangle$$

So there are four direct product states labelled by the z-component of the angular momentum

$$|+,+\rangle, |+,-\rangle, |-,+\rangle, |-,-\rangle$$

So the total z-component of the angular momentum acting on the product state $|+,+\rangle$, for example, is

$$
\begin{aligned}
J_z|+,+\rangle &= (J_{1z} \otimes I + I \otimes J_{2z})|+,+\rangle \\
&= J_{1z}|+\rangle \otimes I\,|+\rangle + I\,|+\rangle \otimes J_{2z}|+\rangle \\
&= \left(\frac{1}{2}\hbar + \frac{1}{2}\hbar\right)|+,+\rangle \\
&= \hbar|+,+\rangle
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
J_z|-,+\rangle &= 0 \\
J_z|+,-\rangle &= 0 \\
J_z|-,-\rangle &= -\hbar|-,-\rangle
\end{aligned}
$$

This can be represented as a matrix

$$J_z = \begin{array}{c} (+,+) \\ (+,-) \\ (-,+) \\ (-,-) \end{array} \begin{pmatrix} +\hbar & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\hbar \end{pmatrix}$$

So J_z is diagonal with the allowed values of m being $1,0,-1$

However, the product states are <u>not</u> eigenstates of the total angular momentum. Using,

$$\mathbf{J}^2 = \mathbf{J}_1^2 + \mathbf{J}_2^2 + 2\,\mathbf{J}_1 \cdot \mathbf{J}_2$$

$$= \mathbf{J}_1^2 + \mathbf{J}_2^2 + 2\left(J_{1x}J_{2x} + J_{1y}J_{2y} + J_{1z}J_{2z}\right)$$

We do not know how the operator $J_{1x}J_{2x} + J_{1y}J_{2y}$ acts on the product kets

So Consider $J_{1+}J_{2-} = (J_{1x} + i\,J_{1y})(J_{2x} - i\,J_{2y})$

$$= J_{1x}J_{2x} + J_{1y}J_{2y} + i(J_{1y}J_{2x} - J_{1x}J_{2y})$$

And similarly,

$$J_{1-}J_{2+} = (J_{1x} - i\,J_{1y})(J_{2x} + i\,J_{2y})$$
$$= J_{1x}J_{2x} + J_{1y}J_{2y} + i(J_{1x}J_{2y} - J_{1y}J_{2x})$$

Thus adding $J_{1+}J_{2-}$ and $J_{1-}J_{2+}$ we get

$$J_{1+}J_{2-} + J_{1-}J_{2+} = 2(J_{1x}J_{2x} + J_{1y}J_{2y})$$

Using this relation, we can express $\mathbf{J}^2$ in terms of operators whose action on the product kets is known

$$\mathbf{J}^2 = \mathbf{J}_1^2 + \mathbf{J}_2^2 + 2\,J_{1z}J_{2z} + J_{1+}J_{2-} + J_{1-}J_{2+}$$

Verify that the action of $\mathbf{J}^2$ on the product states gives,

$$\mathbf{J}^2|+,+\rangle = 2\hbar^2|+,+\rangle$$
$$\mathbf{J}^2|+,-\rangle = \hbar^2(|+,-\rangle + |-,+\rangle$$
$$\mathbf{J}^2|-,+\rangle = \hbar^2(|-,+\rangle + |+,-\rangle$$
$$\mathbf{J}^2|-,-\rangle = 2\hbar^2|-,-\rangle$$

Written as a matrix in the basis of the product ket states

$$\mathbf{J}^2 = \hbar^2 \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

So even though the states $|+,+\rangle$ and $|-,-\rangle$ are eigenstates of $\mathbf{J}^2$ with eigenvalues $1(1+1)\hbar^2 = 2\hbar^2$ (and thus $\ell = 1$), the states $|+,-\rangle$ and $|-,+\rangle$ are <u>not</u> states with definite value of total angular momentum, as reflected in the off diagonal elements in $\mathbf{J}^2$

We can now easily form states of definite value of total angular momentum by forming <u>appropriate linear combinations</u> of the direct product states, i.e., by diagonalizing the $\mathbf{J}^2$ matrix

One can show that the states

$$|+,+\rangle$$

$$\frac{1}{\sqrt{2}}(|+,-\rangle + |-,+\rangle)$$

$$|-,-\rangle$$

Are eigenstates of total angular momentum $|j = 1, m = \pm 1{,}0\rangle$

And the single state

$$\frac{1}{\sqrt{2}}(|+,-\rangle - |-,+\rangle)$$

is a state with total angular momentum zero $|j = 0, m = 0\rangle$

We can identify

$$|j = 1, m = 1\rangle = |+,+\rangle$$

$$|j = 1, m = 0\rangle = \frac{1}{\sqrt{2}}(|+,-\rangle + |-,+\rangle)$$

$$|j = 1, m = -1\rangle = |-,-\rangle$$

As the "triplet states" with $|j = 1, m = \pm 1{,}0\rangle$

And the singlet state

$$|j = 0, m = 0\rangle = \frac{1}{\sqrt{2}}(|+,-\rangle - |-,+\rangle)$$

with $j = 0$ (and of course $m = 0$ trivially)

We see that the triplet states with $j = 1$ is symmetric under the interchange of the two particles whereas the singlet state with $j = 0$ is antisymmetric (the singlet and triplet terminology comes from atomic spectroscopy)
However, note that according to the Pauli principle for identical particles, it is the total wavefunction, i.e., the product of the space and spin parts, which has to be antisymmetric under the exchange of the particles

So, say for identical electrons in an atom, $\Psi_{\text{total}} = \psi_{\text{space}}(\mathbf{x})\psi_{\text{spin}}$

Hence if the co-ordinate part of the wavefunction is symmetric, it has to be combined with the anti-symmetric spin zero singlet spin state to make the overall wavefunction anti-symmetric

And if the co-ordinate part of the wavefunction is anti-symmetric, it has to be combined with the symmetric spin one triplet spin state to make the overall wavefunction anti-symmetric

[<u>Note</u>: If there are other degrees of freedom or quantum numbers describing the state of a fermion (like "color" for spin half quarks, the color part of the wavefunction is also to be included for the symmetrisation/antisymmetrization requirement**]**

The unitary matrix which implements the change of basis from the product state to the total angular momentum states can be written as

$$\begin{pmatrix} |1,1\rangle \\ |1,0\rangle \\ |1,-1\rangle \\ |0,0\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} |+,+\rangle \\ |+,-\rangle \\ |-,+\rangle \\ |-,-\rangle \end{pmatrix}$$

The elements of this unitary matrix connecting the two sets of basis states are the Clebsch-Gordan coefficients

So we can write the composition of angular momenta as

$$\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0$$

Or, in terms of the dimensions of the spin states

$$\mathbf{2} \otimes \mathbf{2} = \mathbf{3} \oplus \mathbf{1}$$

What we have done here is conceptually the same as forming irreducible representations of the direct product of two vectors $\mathbf{A} = (A_x, A_y, A_z)$ and $\mathbf{B} = (B_x, B_y, B_z)$

The direct product of the two vectors consists of 9 objects obtained by multiplying each component of $\mathbf{A}$ by each component of $\mathbf{B}$:

$$A_x B_x, A_x B_y, A_x B_z$$
$$A_y B_x, A_y B_y, A_y B_z$$
$$A_z B_x, A_z B_y, A_z B_z$$

We know that under rotations ordinary vectors transform under rotations as

$$A'^i = R^{ij} A^j$$

$$B'^i = R^{ij} B^j$$

where R^{ij} are the elements of the 3×3 rotation matrix

Hence the product $A^i B^j$ with $i, j = 1,2,3$ transforms under rotations as a second rank tensor, with two factors of the rotation matrix R

$$A'^i B'^j = R^{il} R^{jm} A^l B^m$$

(with summation over l and m understood)

However, this tensor representation is <u>reducible</u>

This is a crucial new concept

By taking appropriate linear combinations of the nine elements $A^i B^j$, we can show that the nine elements reduce to a smaller set of objects with definite transformation properties under rotation

For ex., the single combination $A_x B_x + A_y B_y + A_z B_z = A^i B^i = \mathbf{A} \cdot \mathbf{B}$ is the usual scalar product which is invariant under rotations and symmetric under $A \leftrightarrow B$ interchange

Similarly the components of the cross product

$$(\mathbf{A} \times \mathbf{B})_i = \left(A_y B_z - A_z B_y, A_z B_x - B_z A_x, A_x B_y - A_y B_x\right)$$

are a set of three anti-symmetric objects which transforms as a vector under rotations

The remaining <u>five</u> objects not included in the scalar or vector product form a symmetric 2nd rank <u>tensor</u> $A^i B^j$

These are the irreducible representations

The 3 dimensional vector, transforming with a single factor of the rotation matrix R is the <u>fundamental representation</u> of the rotation group, called $\mathrm{SO}(3)$ in group theory

In general, we can build higher dimensional <u>irreducible</u> representations by taking products of the fundamental representation and forming symmetric and anti-symmetric combinations and removing the trace from the symmetric part (i.e., the dot product, which is a scalar or singlet)

This follows from general principles of group theory for forming higher dimensional representations starting from the fundamental representation

The transformation properties of ordinary 3-component vectors are closely related to the transformation properties of the three $j = 1$ angular momentum states and thus vectors transform like spin 1

For $\mathrm{SO}(3)$ we can write the above as

$$\mathbf{3} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{3} \oplus \mathbf{5}$$

Or in terms of combining angular momenta

$$1 \otimes 1 = 2 \oplus 1 \oplus 0$$

One very non-trivial result from group theory is that $\mathrm{SO}(3)$ has representations other than scalar, vector, tensor etc.

This is the complex 2-dimensional spinor representation which describes spin ½ particles

Note that the anti-symmetric cross-product, which transforms as a vector, is special to three space dimensions

For a N-dimensional vector in N-space dimensions, the "vector cross-product" is actually an anti-symmetric 2nd rank tensor. The decomposition is

$$N \otimes N = 1 \oplus \frac{N(N-1)}{2} \oplus \left(\frac{N(N+1)}{2} - 1\right)$$

It is only in three space dimensions, $N = 3$, that the anti-symmetric representation has 3 components, i.e., the same as the fundamental vector representation and has the same transformation properties as a vector
In two dimensions the anti-symmetric "cross product" is actually a single component scalar

Moving on, how do we calculate the CG coefficients for combining higher angular momenta?

Consider adding $j = 1$ with $j = 1/2$. This gives,

$$1 \otimes \frac{1}{2} = \frac{3}{2} \oplus \frac{1}{2}$$

Or in terms of dimensions $\mathbf{3} \otimes \mathbf{2} = \mathbf{4} \oplus \mathbf{2}$

As before we can write $\mathbf{J}^2$ in the direct product basis and diagonalize it finding the eigenvalues (which will be $\frac{3}{2}$ and $\frac{1}{2}$ obviously) and eigenvectors (which are states of total angular momentum)

An easier alternative method is as follows:

Start from the largest value of j and with the maximum value of $m = 3/2$, i.e., the "maximally stretched" state given by

$$|1,1\rangle \otimes \left|\frac{1}{2},\frac{1}{2}\right\rangle = \left|\frac{3}{2},\frac{3}{2}\right\rangle$$

Obviously the LHS will also be maximally stretched since $m_{\max} = m_{1\max} + m_{2\max}$

Apply the lowering operator in the form $J_- = J_{1-} + J_{2-}$ and use

$$J_-|j,m\rangle = \hbar\sqrt{j(j+1) - m(m-1)}\,|j,m-1\rangle$$

where J_- is the lowering operator acting on the total angular momentum kets and J_{1-} and J_{2-} act on the states of particle 1 and 2 respectively

(As always, remember $J_{1-} + J_{2-} = J_{1-} \otimes I + I \otimes J_{2-}$; this pedantic detail is usually omitted)

Now $J_- \left|\frac{3}{2},\frac{3}{2}\right\rangle = \hbar\sqrt{\frac{3}{2}\left(\frac{3}{2}+1\right)-\frac{3}{2}\left(\frac{3}{2}-1\right)}\ \left|\frac{3}{2},\frac{1}{2}\right\rangle = \sqrt{3}\,\hbar\left|\frac{3}{2},\frac{1}{2}\right\rangle$

Similarly for J_{1-} and J_{2-}

$$J_-|1,1\rangle = \hbar\sqrt{1(1+1)-1(1-1)}\,|1,0\rangle = \sqrt{2}\,\hbar|1,0\rangle$$

$$J_-\left|\frac{1}{2},\frac{1}{2}\right\rangle = \hbar\sqrt{\frac{1}{2}\left(\frac{1}{2}+1\right)-\frac{1}{2}\left(\frac{1}{2}-1\right)}\ \left|\frac{1}{2},-\frac{1}{2}\right\rangle = \hbar\left|\frac{1}{2},-\frac{1}{2}\right\rangle$$

Thus, cancelling factors of $\hbar$ on both sides, we get

$$\left|\frac{3}{2},\frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}}\,|1,0\rangle \otimes \left|\frac{1}{2},\frac{1}{2}\right\rangle + \sqrt{\frac{1}{3}}|1,1\rangle \otimes \left|\frac{1}{2},-\frac{1}{2}\right\rangle$$

We keep applying the appropriate lowering operator on both sides till we reach the "lowest state", i.e., the state with the minimum m value on both sides which is

$$|1,-1\rangle \otimes \left|\frac{1}{2},-\frac{1}{2}\right\rangle = \left|\frac{3}{2},-\frac{3}{2}\right\rangle$$

The "topmost" and "lowermost" states (i.e. states with m_{max} and m_{min}) have CG coefficients one

The states with $j = \frac{1}{2}$ are obtained as similar linear combinations of the product states and the coefficients are fixed by demanding that all these states be orthonormal to all the $j = \frac{3}{2}$ states

The calculation of the full set of CG coefficients is left as a (rather standard) exercise