Angular Momentum

We review how to obtain eigenvalues of the angular momentum operator

By starting from the classical expression for angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$**,** we get

$$L_x = (\mathbf{r} \times \mathbf{p})_x = YP_z - ZP_y$$
$$L_y = (\mathbf{r} \times \mathbf{p})_y = ZP_x - XP_z$$
$$L_z = (\mathbf{r} \times \mathbf{p})_z = XP_y - YP_x$$

The various commutators involving these operators are

$$[L_x, X] = \left[YP_z - ZP_y, X\right] = 0$$
$$\left[L_y, X\right] = [ZP_x - XP_z, X] = Z[P_x, X] = -i\hbar Z$$
$$[L_z, X] = \left[XP_y - YP_x, X\right] = -Y[P_x, X] = i\hbar Y$$

And similarly commutators with $P_{x,y,z}$ can be calculated

We can now calculate the commutators of the components of the angular momentum operator

$$\begin{aligned}
[L_x, L_y] &= [YP_z - ZP_y, ZP_x - XP_z] \\
&= [YP_z, ZP_x] - [YP_z, XP_z] - [ZP_y, ZP_x] + [ZP_y, XP_z] \\
&= YP_x[P_z, Z] + P_yX\,[Z, P_z] \\
&= i\hbar(XP_y - YP_x) \\
&= i\hbar L_z
\end{aligned}$$

Very similarly one can show that

$$[L_y, L_z] = i\hbar L_x$$
$$[L_z, L_x] = i\hbar L_y$$

<u>Because the three components of **L** do not commute with one another, we cannot measure/determine them simultaneously</u>

In the next step, we calculate the commutator of any one component of $\mathbf{L}$, say L_x, with $\mathbf{L}^2 = \mathbf{L} \cdot \mathbf{L} = L_x^2 + L_y^2 + L_z^2$. Consider,

$$[L_x, L_x^2 + L_y^2 + L_z^2] = [L_x, L_y^2 + L_z^2]$$

(Since obviously $[L_x, L_x^2] = 0$)

$$\text{Now } [L_x, L_y^2] = [L_x, L_y]L_y + L_y[L_x, L_y]$$

$$= i\hbar(L_z L_y + L_y L_z)$$

$$\text{And similarly } [L_x, L_z^2] = [L_x, L_z]L_z + L_z[L_x, L_z]$$

$$= -i\hbar(L_z L_y + L_y L_z)$$

Adding the two parts we get $[L_x, \mathbf{L}^2] = 0$

Very similarly, we can show that $[L_y, \mathbf{L}^2] = [L_z, \mathbf{L}^2] = 0$

Hence, only the magnitude of $\mathbf{L}$ and any one component of $\mathbf{L}$ can be determined simultaneously

We introduce the compact and useful index notation to simplify calculations

Denote

$$X = X_1, Y = X_2,\ Z = X_3\ ;$$
$$P_x = P_1,\ P_y = P_2,\ P_z = P_3$$
$$L_x = L_1, L_y = L_2, L_z = L_3$$

Then in this compact notation $L_i = \epsilon_{ijk} X_j P_k,\ X_i P_j - X_j P_i = \epsilon_{ijk} L_k,$ $i, j, k = 1,2,3$ and of course $[X_i, P_j] = i\hbar\delta_{ij}, [X_i, X_j] = [P_i, P_j] = 0$, with <u>repeated indices summed over all values</u>

The 3 commutators $[L_i, X_j]$ calculated in the previous slide can now be compactly written and generalized as

$$[L_i, X_j] = [\epsilon_{ilm} X_l P_m, X_j] = \epsilon_{ilm} X_l [P_m, X_j]$$
$$= \epsilon_{ilm} X_l (-i\hbar\delta_{mj}) = -i\hbar\epsilon_{ilj} X_l = i\hbar\epsilon_{ijl} X_l$$

Here ϵ_{ijk} is the totally antisymmetric Levi-Civita tensor with $\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = +1$;and $\epsilon_{132} = \epsilon_{321} = \epsilon_{213} = -1$ (Odd permutation of $ijk = 123$)

Similarly $[L_i, P_j] = [\epsilon_{ilm} X_l P_m, P_j] = i\hbar \epsilon_{ijm} P_m$

In fact, we can *define* a vector operator to be any set of 3 operators V_i which obey $[L_i, V_j] = i\hbar \epsilon_{ijk} V_k$

In index notation how do we calculate the commutator $[L_i, L_j]$?

$$[L_i, L_j] = [L_i, \epsilon_{jmn} X_m P_n]$$

$$= \epsilon_{jmn}([L_i, X_m]P_n + X_m[L_i, P_n])$$

$$= \epsilon_{jmn}(i\hbar \epsilon_{imk} X_k P_n + i\hbar \epsilon_{ink} X_m P_k)$$

$$= i\hbar[(\delta_{ji}\delta_{nk} - \delta_{jk}\delta_{ni})X_k P_n - (\delta_{ji}\delta_{mk} - \delta_{jk}\delta_{mi})X_m P_k]$$

$$= i\hbar[(\delta_{ij} X_k P_k - X_j P_i - \delta_{ji} X_k P_k + X_i P_j]$$

$$= i\hbar[X_i P_j - X_j P_k] = i\hbar \epsilon_{ijk} L_k$$

So finally we get the familiar relation $[L_i, L_j] = i\hbar\epsilon_{ijk}L_k$

Here we have used a very useful property involving the contraction of two ϵ tensors:

$$\epsilon_{ijk}\epsilon_{klm} = \begin{vmatrix} \delta_{il} & \delta_{im} \\ \delta_{jl} & \delta_{jm} \end{vmatrix} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

which can be verified by 'brute force'

The square of the angular momentum vector is a scalar operator $\mathbf{L}^2 = \mathbf{L} \cdot \mathbf{L} = L_i L_i = \sum_{i=1}^{3} L_i L_i$

This operator commutes with all the components of the angular momentum:

$$[L_i, \mathbf{L}^2] = [L_i, L_j L_j] = L_j[L_i, L_j] + [L_i, L_j]L_j$$
$$= L_j(i\hbar\epsilon_{ijk}L_k) + (i\hbar\epsilon_{ijk}L_k)L_j$$
$$= i\hbar\epsilon_{ijk}(L_j L_k + L_k L_j) = 0$$

In group, theory an operator which commutes with all the generators of the group is called a Casimir operator

A theory is invariant under rotations if the generators commute with the Hamiltonian; i.e, for all $i = 1,2,3$, i.e., $[L_i, H] = 0$

For such rotationally invariant systems it also follows that

$$[\mathbf{L}^2, H\,] = [L_i\, L_i, H] = L_i[L_i, H] + [L_i, H]L_i = 0$$

Since the different components of $\mathbf{L}$ do <u>not</u> commute with one another, we can simultaneously diagonalize $H, \mathbf{L}^2$ and <u>only one component</u> of $\mathbf{L}$ (conventionally chosen to be $L_3 = L_z$)

An example of a rotationally invariant theory is $H = \frac{\mathbf{P}^2}{2m} + V(R) = \frac{\mathbf{P}^2}{2m} + V(\mathbf{X}^2)$, so that the potential depends only on the radial coordinate and independent of direction.

Exs. are Coulomb and gravitation potential, 3 dimensional isotropic harmonic oscillator with potential $V(x, y, z) = \frac{1}{2}m\omega^2(x^2 + y^2 + z^2) = V(R^2)$ etc.

How do we find the simultaneous eigenstates of $H, \mathbf{L}^2$ and L_3? It turns out we can solve for the eigenvalue spectrum only using the underlying commutation relations $[L_i, L_j] = i\hbar\epsilon_{ijk}L_k$ and without appealing to any properties of the orbital angular momentum operator $\mathbf{L} = \mathbf{r} \times \mathbf{p}$

Define two new operators

$$L_+ = L_1 + iL_2$$
$$L_- = L_1 - iL_2$$

with $L_- = L_+^\dagger$. Since $\mathbf{L}^2$ commutes with all components L_i, it follows that

$$[L_+, \mathbf{L}^2] = [L_-, \mathbf{L}^2] = 0$$

On the other hand, $L_\pm$ do not commute with L_3:

$$[L_+, L_3] = [L_1 + iL_2, L_3] = -i\hbar L_2 + i(i\hbar L_1)$$
$$= -\hbar(L_1 + iL_2) = -\hbar L_+$$

$$[L_-, L_3] = [L_1 - iL_2, L_3] = -i\hbar L_2 - i(i\hbar L_1)$$
$$= \hbar(L_1 - iL_2) = \hbar L_-$$

Furthermore

$$\begin{aligned}[L_+, L_-] &= [L_1 + iL_2, L_1 - iL_2] \\ &= [L_1, -iL_2] + [iL_2, L_1] \\ &= (-i)(i\hbar L_3) + i(-i\hbar L_3) \\ &= 2\hbar L_3\end{aligned}$$

Since in a rotationally invariant theory $[H, L_i] = 0$, it also follows that

$$[L_+, H] = [L_-, H] = 0$$

Let $|j, m\rangle$ represent the simultaneous eigenstates of the operators $\mathbf{L}^2$ and L_3 such that

$$L_3|j, m\rangle = \hbar m|j, m\rangle$$
$$\mathbf{L}^2|j, m\rangle = \hbar^2 j|j, m\rangle$$

Next to check the effect of L_+ on a given eigenstate $|j,m\rangle$ consider:

$$\begin{aligned} L_3 L_+|j,m\rangle &= ([L_3, L_+] + L_+L_3)|j,m\rangle \\ &= (\hbar L_+ + L_+L_3)|j,m\rangle \\ &= (\hbar + \hbar m)L_+|j,m\rangle \\ &= \hbar(m+1)L_+|j,m\rangle \end{aligned}$$

Similarly for $\mathbf{L}^2$,

$$\begin{aligned} \mathbf{L}^2 L_+|j,m\rangle &= ([\mathbf{L}^2, L_+] + L_+L^2)|j,m\rangle \\ &= L_+\mathbf{L}^2|j,m\rangle \\ &= \hbar^2 j L_+|j,m\rangle \end{aligned}$$

This shows that the effect of L_+ acting on $|j,m\rangle$ is to increase it's m value by 1, to take it to a state where the eigenvalue of L_3 is raised by one unit of $\hbar$

The eigenvalue of $\mathbf{L}^2$ is unchanged when acting on $L_+|j,m\rangle$

Since $L_+|j,m\rangle \propto |j,m+1\rangle$, we can write $L_+|j,m\rangle = C_+|j,m+1\rangle$, where C_+ is a normalization constant depending on j and m

Similarly we can easily show that

$$L_3 L_-|j,m\rangle = \hbar(m-1)L_-|j,m\rangle$$

$$\mathbf{L}^2 L_-|j,m\rangle = L_-\mathbf{L}^2|j,m\rangle = \hbar^2 j L_-|j,m\rangle$$

This shows that the operator L_- acting on $|j,m\rangle$ lowers the eigenvalue of L_3 by one unit of $\hbar$, while not effecting the eigenvalue of $\mathbf{L}^2$.

Since $L_-|j,m\rangle \propto |j,m-1\rangle$, we can write $L_-|j,m\rangle = C_-|j,m-1\rangle$

Since L_+ and L_- raise and lower the eigenvalue of L_3 (i.e., $m\hbar$) they are called raising and lowering operators

Further, by repeatedly applying L_+ and L_- respectively on $|j,m\rangle$ we can generate a sequence of states $|j,m+1\rangle, |j,m+2\rangle, \cdots$ and $|j,m-1\rangle, |j,m-$

However this sequence of states generated by repeated application of $L_{\pm}$ cannot go on infinitely. It has to terminate at some state. Since

$$\mathbf{L}^2 = L_1^2 + L_2^2 + L_3^2, \qquad \mathbf{L}^2 - L_3^2 = L_1^2 + L_2^2 \geq 0$$

The operator $\mathbf{L}^2 - L_3^2$ is a positive semi-definite operator, hence it's eigenvalues are ≥ 0 and so $\hbar^2 j - \hbar^2 m^2 \geq 0$, or $j \geq m^2$

This implies that there must exist states with a minimum and maximum value of m such that

$$L_+|j, m_{\max}\rangle = 0$$

Taking the inner product of the above ket with itself and noting that $L_+^\dagger = L_-$

$$\text{i.e,} \qquad \langle j, m_{\max}|L_- L_+|j, m_{\max}\rangle = 0$$

$$\text{i.e,} \qquad \langle j, m_{\max}|(\mathbf{L}^2 - L_3^2 - \hbar L_3)|j, m_{\max}\rangle = 0$$

$$\text{i.e,} \qquad (\hbar^2 j - \hbar^2 m_{\max}^2 - \hbar^2 m_{\max})\langle j, m_{\max}|j, m_{\max}\rangle = 0$$

$$\text{Hence } j - m_{\max}(m_{\max} + 1) = 0$$

Similarly there must exist a state with a minimum value of m such that

$$L_{-}|j, m_{\min}\rangle = 0$$

$$\text{i.e,} \qquad \langle j, m_{\min}|L_{+}L_{-}|j, m_{\min}\rangle = 0$$

$$\text{i.e,} \qquad \langle j, m_{\min}|(\mathbf{L}^2 - L_3^2 + \hbar L_3)|j, m_{\min}\rangle = 0$$

$$\text{i.e,} \qquad \left(\hbar^2 j - \hbar^2 m_{\min}^2 + \hbar^2 m_{\min}\right)\langle j, m_{\min}|j, m_{\min}\rangle = 0$$

Hence $j - m_{\min}(m_{\min} - 1) = 0$

Comparing the two equations involving $j, m_{\max}$ and $m_{\min}$ we get

$$m_{\min} = -m_{\max}$$

(The other solution $m_{\max} = m_{\min} - 1$ violates our assumption that $m_{\max} > m_{\min}$ and that $m_{\min}$ denotes the minimum value of m)

Let us say one can go from the state $|j, m_{\min}\rangle$ to $|j, m_{\max}\rangle$ by applying the raising operator k number of times. So $k = 0,1,2,\cdots$ must be an integer Since every time L_+ is applied to increases the m value by one $(m \to m+1)$, it follows that

$$m_{\max} - m_{\min} = k$$

$$\text{i.e,} \qquad 2m_{\max} = k$$

$$\text{i.e,} \qquad m_{\max} = -m_{\min} = \frac{k}{2}$$

where k is an integer. In terms of k then,

$$j = m_{\max}(m_{\max}+1) = \frac{k}{2}\left(\frac{k}{2}+1\right)$$

Define $\ell = \frac{k}{2}$ so that ℓ takes only multiples of half integral values, i.e., $0, \frac{1}{2}, 1, \frac{3}{2}, \cdots$

So finally, the eigenvalues of $\mathbf{L}^2$ and L_3 are of the form

$$\mathbf{L}^2|\ell,m\rangle = \ell(\ell+1)\hbar^2\,|\ell,m\rangle$$

$$L_3|\ell,m\rangle = m\hbar\,|\ell,m\rangle$$

Where $\ell = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots$ and m takes values $-\ell \leq m \leq \ell$

Since m can go from $-\ell$ to ℓ in steps of one, there are $2\ell+1$ states

Since $|\ell,m\rangle$ are constructed as eigenstates of $\mathbf{L}^2$ and L_3, these operators, viewed as matrices in the $|\ell,m\rangle$ basis, are already diagonal:

$$\langle\ell',m'|\mathbf{L}^2|\ell,m\rangle = \ell(\ell+1)\hbar^2\,\delta_{\ell'\ell}\delta_{m'm}$$

i.e., more simply $\mathbf{L}^2$ is simply proportional to the identity matrix:

$$\mathbf{L}^2 = \ell(\ell+1)\hbar^2\,\mathbf{1}$$

Where $\mathbf{1}$ is the identity matrix of dimension $(2\ell+1)\times(2\ell+1)$

Similarly, in the $|\ell, m\rangle$ basis L_3 is also diagonal with eigenvalues

$$-\ell\hbar, (-\ell+1)\hbar, \cdots\cdots (\ell-1)\hbar,\ \ \ell\hbar$$

$$L_3 = \hbar \begin{pmatrix} \ell & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & -\ell \end{pmatrix}$$

So for a spin half particle, $\ell = \frac{1}{2}$, the eigenvalue of $\mathbf{L}^2$ is $\frac{1}{2}\left(\frac{1}{2}+1\right)\hbar^2 = \frac{3}{4}\hbar^2$

The L_3 eigenvalues are $-\frac{1}{2}\hbar$ and $\frac{1}{2}\hbar$. Hence the $\mathbf{L}^2$ and L_3 matrices are

$$\mathbf{L}^2 = \frac{3}{4}\hbar^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$L_3 = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

How do we determine the matrices for L_x and L_y in the $|\ell, m\rangle$ eigenbasis?

Note that $|\ell, m\rangle$ are not eigenstates of L_x and L_y, since they do not commute with L_z

We can do this by going back to the definitions of $L_\pm$:

$$L_+ = L_1 + iL_2$$

$$L_- = L_1 - iL_2$$

Adding and subtracting them we get

$$L_1 = L_x = \frac{1}{2}(L_+ + L_-)$$

$$L_2 = L_y = \frac{1}{2i}(L_+ - L_-)$$

Hece if we determine the matrix representations of $L_\pm$ we can figure out the L_x and L_y matrices

Now we know that

$$L_+|\ell, m\rangle = C_+|\ell, m+1\rangle$$

Taking the inner product of this ket with itself,

$$\langle \ell, m|L_- L_+|\ell, m\rangle = |C_+|^2 \text{ since } L_+^\dagger = L_-$$

Now

$$L_- L_+ = (L_1 - iL_2)(L_1 + iL_2) = L_1^2 + L_2^2 + i[L_1, L_2] = \mathbf{L}^2 - L_3^2 - \hbar L_3$$

Hence the above inner product becomes

$$\text{or } \langle \ell, m|(\mathbf{L}^2 - L_3^2 - \hbar L_3)|\ell, m\rangle = |C_+|^2$$

$$\text{or } \hbar^2[\ell(\ell+1) - m(m+1)] = |C_+|^2$$

This determines C_+ upto a phase. Taking the phase factor to be one:

$$C_+ = C_+^* = \hbar\sqrt{\ell(\ell+1) - m(m+1)}$$

This determines the action of L_+ on $|\ell, m\rangle$:

$$L_+|\ell, m\rangle = \hbar\sqrt{\ell(\ell+1) - m(m+1)}\,|\ell, m+1\rangle$$

Very similarly, considering $L_-|j, m\rangle = C_-|j, m-1\rangle$, we can show

$$L_-|\ell, m\rangle = \hbar\sqrt{\ell(\ell+1) - m(m-1)}\,|\ell, m-1\rangle$$

So the only non-zero matrix elements of $L_{\pm}$ are

$$\langle \ell, m+1|L_+|\ell, m\rangle = C_+$$
$$\langle \ell, m-1|L_-|\ell, m\rangle = C_-$$

for each m which takes values from $-\ell$ to $+\ell$ in steps of one

Note that the $L_{\pm}$ matrices cannot connect states with different ℓ, they only change the value of m for a given ℓ

All this can be easily calculated for the case $\ell = \frac{1}{2}$ and $m = \pm\frac{1}{2}$

In his case the only non-zero matrix element of $L_{\pm}$ is

$$\langle \frac{1}{2}, \frac{1}{2}|L_+|\frac{1}{2}, -\frac{1}{2}\rangle = \hbar\sqrt{\frac{1}{2}\left(\frac{1}{2}+1\right) - \left(-\frac{1}{2}\right)\left(-\frac{1}{2}+1\right)}$$
$$= \hbar$$

Similarly the only non-zero matrix element of L_- is

$$\langle \frac{1}{2}, -\frac{1}{2} | L_- | \frac{1}{2}, \frac{1}{2} \rangle = \hbar \sqrt{\frac{1}{2}\left(\frac{1}{2}+1\right) - \left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)}$$

$$= \hbar$$

Hence the matrices $L_\pm$ in the $|\ell, m\rangle$ basis are

$$L_+ = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$L_- = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\therefore L_1 = L_x = \frac{1}{2}(L_+ + L_-) = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\text{and } L_2 = L_y = \frac{1}{2i}(L_+ - L_-) = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Thus the 3 angular momentum matrices for the case $\ell = \frac{1}{2}$ in the $|\ell, m\rangle$ are

$$L_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, L_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, L_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Apart from the factor of $\frac{\hbar}{2}$, these are the famous Pauli matrices

And of course $\mathbf{L}^2 = \frac{1}{2}\left(\frac{1}{2} + 1\right)\hbar^2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$= \frac{3}{4}\hbar^2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The $2\ell + 1$ orthonormal states $|\ell, m\rangle$ (with $m = -\ell, -\ell + 1, \cdots\cdots, \ell - 1, \ell$) constitute the simultaneous eigenkets of the Hermitian operators $\mathbf{L}^2$ and L_3 . They define a subspace $\mathcal{E}^{(\ell)}$ of the complete Hilbert space consisting of eigenkets $|j, m\rangle$ of all possible (allowed) angular momentum values $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots$

The operators $\mathbf{L}^2, L_3, L_\pm$ take any eigenvector in $\mathcal{E}^{(\ell)}$ to another vector in the same space , i.e., the action of these operators on $\mathcal{E}^{(\ell)}$ leaves this space invariant, since the magnitude of the angular momentum (characterised by ℓ) does not change under $\mathbf{L}^2, L_3, L_\pm$

Exercise: Construct the matrices $\langle \ell, m' | L_+ | \ell, m \rangle$ and $\langle \ell, m' | L_- | \ell, m \rangle$

What is the dimensionality of the matrices? Construct the matrices for L_x and L_y for $\ell = \frac{1}{2}$ and $\ell = 1$. (The matrices for $\mathbf{L}^2$ and L_z are of course diagonal in the $|\ell, m\rangle$ basis)

We have shown how to construct the 2×2 matrices for $\ell = 1/2$

Construct the 3×3 matrices for $\ell = 1$. Show that we get

$$\text{And} \quad L_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad L_y = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & i \\ 0 & i & 0 \end{pmatrix},$$

$$L_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{for } \ell = 1$$

Verify that for the matrices for spin ½ and spin 1 obey the fundamental commutation relations $[L_i, L_j] = i\hbar\epsilon_{ijk}L_k$

In general for an arbitrary ℓ, in the angular momentum basis $|\ell, m\rangle$, the matrix elements of the various relevant operators are

$$\langle \ell', m' | \mathbf{L}^2 | \ell, m \rangle = \hbar^2 \ell(\ell + 1)\, \delta_{\ell',\ell}\, \delta_{m',m}$$

$$\langle \ell', m' | L_3 | \ell, m \rangle = \hbar m\, \delta_{\ell',\ell}\, \delta_{m',m}$$

$$\langle \ell', m' | L_+ | \ell, m \rangle = \hbar \sqrt{\ell(\ell + 1) - m(m + 1)}\, \delta_{\ell',\ell}\, \delta_{m',m+1}$$

$$\langle \ell', m' | L_- | \ell, m \rangle = \hbar \sqrt{\ell(\ell + 1) - m(m - 1)}\, \delta_{\ell',\ell}\, \delta_{m',m-1}$$

where $\ell = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots$ and $m = -\ell, -\ell + 1, \cdots, \ell - 1, \ell$