Assignment 1

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August 25, 2020

Problem 1

Given that $\langle x'|x\rangle = \delta(x'-x)$, we can write the operator X in the position basis (its own eigenbasis) as:

$$\langle x'|X|x\rangle = x\delta\left(x'-x\right) = x'\delta\left(x'-x\right)$$

We also take the canonical commutation relation as one of the postulates:

$$[X,P] = \imath\hbar$$

Now we can write:

$$\langle x'|\left[X,P\right]|x\rangle = \imath\hbar\delta\left(x'-x\right)$$
$$\implies \langle x'|XP|x\rangle - \langle x'|PX|x\rangle = \imath\hbar\delta\left(x'-x\right)$$
$$\implies x'\langle x'|P|x\rangle - x\langle x'|P|x\rangle = \imath\hbar\delta\left(x'-x\right)$$

Therefore,

$$\langle x'|P|x\rangle = \imath\hbar\frac{\delta\left(x'-x\right)}{x'-x} = -\imath\hbar\frac{d}{dx}\left(\delta\left(x-x'\right)\right)$$

Problem 2

In the ket representation,

$$\partial_t|\psi(t)\rangle = -\frac{\imath H(t)}{\hbar}|\psi(t)\rangle$$

The wave function $\psi(x,t)$ is defined as $\langle x|\psi(t)\rangle$. Multiplying the equation by $\langle x|$ on both sides we get:

$$\partial_t\langle x|\psi(t)\rangle = -\imath\langle x|\frac{H(t)}{\hbar}|\psi(t)\rangle$$

$$\implies \partial_t\psi(x,t) = -\imath\langle x|\frac{H(t)}{\hbar}|\psi(t)\rangle$$

We now insert the identity $\int dx'|x'\rangle\langle x'| = 1$, to get

$$\partial_t\psi(x,t) = -\frac{i}{\hbar}\int dx'\langle x|H(t)|x'\rangle\langle x'|\psi(t)\rangle$$

$$\implies \partial_t\psi(x,t) = -\frac{i}{\hbar}\int dx'\psi(x',t)\langle x|H(t)|x'\rangle$$

Now, we assume that $H(t)$ is given by $H(t) = \frac{P^2}{2m} + V$.

$$\langle x|H(t)|x'\rangle = \langle x|\frac{P^2}{2m} + V|x'\rangle = \langle x|\frac{P^2}{2m}|x'\rangle + \langle x|V|x'\rangle$$

We use the position space representation of the momentum and potential operators to get:

$$\langle x|H(t)|x'\rangle = [-\hbar^2 \frac{\partial^2}{\partial x^2} + V(x)]\delta(x - x')$$

Using this, therefore

$$\partial_t \psi(x,t) = -\frac{i}{\hbar}\int dx' \psi(x',t)[-\hbar^2 \frac{\partial^2}{\partial x^2} + V(x)]\delta(x - x')$$

Therefore, we finally get:

$$-\frac{\hbar}{\imath}\frac{\partial}{\partial t}\psi(x,t) = (-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x))\psi(x,t)$$

(Schrodinger's Equation)

Problem 3

For a free non-relativistic particle,

$$H(t) = \frac{P^2}{2m}$$

Using the operator identity in the momentum space (taking the normalisation of plane wave states to be $\frac{1}{\sqrt{(2\pi\hbar)^D}}$ in D dimensions), we have

$$\langle x'|e^{-\frac{\imath}{\hbar}Ht}|x\rangle = \int dp \langle x|p\rangle\langle p|e^{-\frac{\imath}{2\hbar m}P^2 t}|x'\rangle$$

Simplifying,

$$\langle x'|e^{-\frac{\imath}{\hbar}Ht}|x\rangle = \int dp \frac{1}{2\pi\hbar} e^{\frac{i}{\hbar}p\left(x-x'\right) - \frac{\imath}{2\hbar m}p^2 t}$$

(Since, $\langle x|p\rangle = \sqrt{\frac{1}{2\pi\hbar}} e^{\frac{\imath}{\hbar}p\cdot x}$)
Resolving the integral, (by completing the square in the exponent):

$$\langle x'|e^{-\frac{\imath}{\hbar}Ht}|x\rangle = \sqrt{\frac{m}{2\imath t\pi\hbar}} e^{\frac{im(x-x')^2}{2\hbar t}}$$

Working

The power of e is:

$$-\frac{i}{\hbar}\left[\frac{p^2 t}{2m} - p\left(x - x'\right)\right]$$

Putting $\alpha^2 = \frac{t}{2m}$, we get

$$\frac{-i}{\hbar}\left[\alpha^2 p^2 - 2(\alpha p)\frac{(x - x')}{2\alpha} + \frac{(x - x')^2}{4\alpha^2}\right] + \frac{i}{\hbar}\left[\frac{(x - x')^2}{4\alpha^2}\right]$$

which can be written as:

$$-\frac{i}{\hbar} \quad \left(\alpha p - \frac{(x - x')}{2\alpha}\right)^2 + \frac{i}{\hbar}\frac{(x - x')^2}{4\alpha^2}$$

Therefore we can rewrite the integral as: $\left\langle x' \left| e^{-\frac{\imath}{\hbar}Ht} \right| x \right\rangle = \int dp \frac{1}{2\pi\hbar} e^{-\frac{i}{\hbar} \quad \left(\alpha p - \frac{(x-x')}{2\alpha}\right)^2 + \frac{i}{\hbar}\frac{(x-x')^2}{4\alpha^2}}$
Therefore, we can write our integral as,

$\left\langle x' \left| e^{-\frac{\imath}{\hbar}Ht} \right| x \right\rangle = \int dp \frac{1}{2\pi\hbar} e^{-\frac{i}{\hbar} \left(\alpha p - \frac{(x-x')}{2\alpha} \right)^2} e^{\frac{i}{\hbar}\frac{(x-x')^2}{4\alpha^2}}$
$\left\langle x' \left| e^{-\frac{\imath}{\hbar}Ht} \right| x \right\rangle = e^{\frac{i}{\hbar}\frac{(x-x')^2}{4\alpha^2}} \frac{1}{2\pi\hbar} \int dp e^{-\frac{i}{\hbar} \left(\alpha p - \frac{(x-x')}{2\alpha} \right)^2}$
The following is a standard integral,

$$\int_{-\infty}^{\infty} dx e^{-ix^2} = \sqrt{-\imath\pi}$$

Therefore we get,

$$\left\langle x' \left| e^{-\frac{\imath}{\hbar}Ht} \right| x \right\rangle = e^{\frac{i}{\hbar}\frac{(x-x')^2}{4\alpha^2}} \frac{1}{2\pi\hbar} \frac{\sqrt{-\imath\pi}}{\alpha}$$

Putting the value of $\alpha^2 = \frac{t}{2m}$, we get the result stated above.