Q1:

(a) There are three defaultable loans and a riskless one with the following expected return and risks:

Loans	Expected Returns	Risk
1	0.14	0.06
2	0.08	0.03
3	0.2	0.15
4	0.05	0

In addition, the assets correlate with each other; the correlation coefficients of the returns of the assets are as follow:

$$\rho_{12} = \rho_{21} = 0.5 \qquad \rho_{13} = \rho_{31} = 0.2 \qquad \rho_{23} = \rho_{32} = 0.4.$$

Obtain CML, the optimal portfolio of risky loans on the CML, the market price of risk, and the risk of the optimal portfolio.

[12 marks]

Notation:

- Construct a risky loans portfolio $\Pi = \{w_i \,\text{Loan}_i\}_{i=1}^{3}$ with

$$R_\Pi = w^T R$$
$$\sigma_\Pi^2 = w^T \Sigma w$$

where:

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 & \rho_{13}\sigma_1\sigma_3 \\ \rho_{21}\sigma_1\sigma_2 & \sigma_2^2 & \rho_{23}\sigma_2\sigma_3 \\ \rho_{31}\sigma_1\sigma_3 & \rho_{32}\sigma_2\sigma_3 & \sigma_3^2 \end{bmatrix}$$

- Considering the risk-free asset, construct a risky loans portfolio Π^* with

$$R_{\Pi*} = h^T R + (1 - h^T 1_n) R_f = R_f + h^T (R - R_f 1_n)$$
$$\sigma_{\Pi*}^2 = h^T \Sigma h$$

- Denote

$$1_n = \begin{bmatrix} 1 \\ 1 \\ \dots \\ 1 \end{bmatrix}$$

- $$\text{the market price of risk } = \frac{R_{\Pi*} - R_f}{\sigma_{\Pi*}}$$
$$\text{CML: } R = \frac{R_{\Pi*} - R_f}{\sigma_{\Pi*}} \sigma + R_f$$

Part 0: Optimization Problem ——find the maximum slope

(1) Solve the minimization problem:

$$\max_w \frac{w^T R - R_f}{(w^T \Sigma w)^{1/2}}$$
$$s.t. \{ w^T 1_n = 1, \text{where } 1_n \text{ is a identity matrix}$$

$$L = \frac{w^T R - R_f}{(w^T \Sigma w)^{1/2}} - \lambda (w^T 1_n - 1)$$
$$\begin{cases} \frac{\partial L}{\partial w} = \frac{R(w^T\Sigma w)^{1/2} - (w^TR - R_f)\frac{1}{2}(w^T\Sigma w)^{-1/2} 2\Sigma w}{w^T \Sigma w} - \lambda 1_n = 0 \ldots\ldots (1) \\ \frac{\partial L}{\partial \lambda} = w^T 1_n - 1 = 0 \ldots\ldots\ldots\ldots\ldots (2) \end{cases}$$

Solve λ

$$\begin{aligned} R(w^T\Sigma w) - (w^TR - R_f)\Sigma w =& \lambda 1_n (w^T\Sigma w)^{\frac{3}{2}} \\ (Rw^T\Sigma - w^TR\Sigma - R_f\Sigma)w =& \lambda 1_n (w^T\Sigma w)^{\frac{3}{2}} \\ w^T(Rw^T\Sigma - w^TR\Sigma - R_f\Sigma)w =& \lambda w^T 1_n (w^T\Sigma w)^{\frac{3}{2}} \\ w^T(Rw^T\Sigma - w^TR\Sigma - R_f\Sigma)w =& \lambda (w^T\Sigma w)^{\frac{3}{2}} \\ \frac{w^T(Rw^T\Sigma - w^TR\Sigma - R_f\Sigma)w}{(w^T\Sigma w)^{\frac{3}{2}}} =& \lambda \end{aligned}$$

Plug λ back into (1)

$$\begin{aligned} \frac{R(w^T\Sigma w)^{1/2} - (w^TR - R_f)\frac{1}{2}(w^T\Sigma w)^{-1/2} 2\Sigma w}{w^T \Sigma w} =& \frac{w^T(Rw^T\Sigma - w^TR\Sigma - R_f\Sigma)w}{(w^T\Sigma w)^{\frac{3}{2}}} 1_n \\ R(w^T\Sigma w) - (w^TR - R_f)\Sigma w =& w^T(Rw^T\Sigma - w^TR\Sigma - R_f\Sigma)w 1_n \\ (Rw^T\Sigma - w^TR\Sigma - R_f\Sigma)w =& w^T(Rw^T\Sigma - w^TR\Sigma - R_f\Sigma)w 1_n \\ Aw =& \operatorname{tr}(ww^TA) 1_n \\ (A \times 1_n w^T) 1_n =& \operatorname{tr}(ww^TA) 1_n \end{aligned}$$

(2) Now we find it could very hard to solve w from the equation. So we think about alternative ways to get the close solution of w.

- We know that the CML is the tangent of the efficient frontier, and CML passes through the risk-free point. Actually, the original optimization problem equals to find the tantency point. Thus, We can find two methods to replace the original optimization problem:

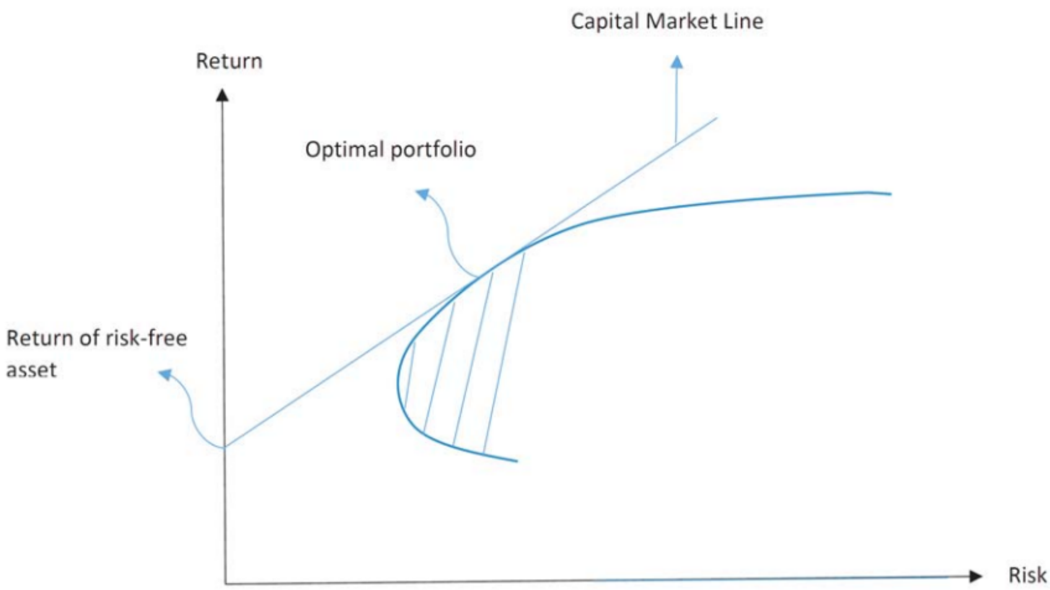

Figure 2.17: Capital Market Line

- **Method 1:**

  (1) Firstly, we try to get the CML. In order to do that, we introduce a constant R_{Π} to create an optimization problem:

$$\min_h h^T \Sigma h$$
$$s.t\{h^T R + (1 - h^T 1_n) R_f = R_{\Pi *}$$

Solve the optimization problem: we can get the CML: $R_{\Pi *} = f(\mathrm{Risk})$, and $h = h(R_{\Pi *})$

(2) Secondly, we bring in some special conditions to solve the constant $R_{\Pi *}$:

At the tangency point, we know:

$$h^T 1_n = 1$$

Thus,

$$h(R_{\Pi *}) 1_n = 1$$

And we can solve the constant $R_{\Pi *}$, and then get h

- **Method 2:**

  (1) Firstly, we try to get the effective frontier. In order to do that, we introduce a constant R_Π to create an optimization problem:

  $$\min_w w^T \Sigma w$$

  $$s.t.\{ \begin{matrix} w^T R = R_\Pi \\ w^T 1_n = 1 \end{matrix}$$

  Solve the optimization problem: we can get the effective frontier: $\mathrm{Risk} = g(R_\Pi)$, and $w = w(R_{\Pi,0})$

  (2) Secondly, we bring in some special conditions to solve the constant R_Π :

  Using the effective frontier, we can easily find the tangency point:

  $$\max_{R_\Pi} \mathrm{Slope} = \frac{R_\Pi - R_f}{\sigma_\Pi} = \frac{R_\Pi - R_f}{g(R_\Pi)}$$

  And we can solve the constant R_Π, and then get w

We show the derivation process with more details in Part 1 & Part 2

Part 1: derivation process about Method 1

(1) Firstly, we find the CML:

$$\min_h h^T \Sigma h$$
$$s.t\{h^T R + (1 - h^T 1_n) R_f = R_{\Pi *}$$

The constraints can be written as

$$\begin{aligned} h^T R + (1 - h^T 1_n) R_f =& R_{\Pi *} \\ R_f + h^T (R - R_f 1_n) =& R_{\Pi *} \\ h_T r =& r_0 \end{aligned}$$

denote $r = R - R_f 1_n$ and $r_0 = R_{\Pi *} - R_f$

- we solve the problem:

$$L = h^T \Sigma h - \lambda(h^T r - r_0)$$

$$\begin{cases} \frac{\partial L}{\partial h} = 2\Sigma h - \lambda r = 0.. \dots\dots\dots\dots\dots (1) \\ \frac{\partial L}{\partial \lambda} = h^T r - r_0 = 0. \dots\dots\dots\dots\dots (2) \end{cases}$$

Using the equation (1):

$$\begin{aligned} 2\Sigma h - \lambda r &= 0 \\ h &= \frac{\lambda}{2}\Sigma^{-1} r \end{aligned}$$

Plug h into the equation (2)

$$\begin{aligned} h &= \frac{\lambda}{2}\Sigma^{-1} r \\ r_0 = r^T h &= r^T \frac{\lambda}{2}\Sigma^{-1} r \\ \lambda &= \frac{2r_0}{r^T \Sigma^{-1} r} \end{aligned}$$

Now plug λ back into $h = \frac{\lambda}{2}\Sigma^{-1} r$:

$$\begin{aligned} h &= \frac{\lambda}{2}\Sigma^{-1} r \\ &= \frac{1}{2}\Sigma^{-1} r \frac{2r_0}{r^T \Sigma^{-1} r} \\ &= r_0 \frac{\Sigma^{-1} r}{r^T \Sigma^{-1} r} \end{aligned}$$

(2) Secondly, we bring in some special conditions to solve the constant r_0. At the tangency point, we have the condition $1_n^T h = 1$, we plug the condition into the solution of h:

$$\begin{aligned} h &= r_0 \frac{\Sigma^{-1} r}{r^T \Sigma^{-1} r} \\ 1 = 1_n^T h &= r_0 \frac{1_n^T \Sigma^{-1} r}{r^T \Sigma^{-1} r} \\ r_0 &= \frac{r^T \Sigma^{-1} r}{1_n^T \Sigma^{-1} r} \end{aligned}$$

Now plug $r_0 = \frac{r^T \Sigma^{-1} r}{1_n^T \Sigma^{-1} r}$ back into the solution of h:

$$\begin{aligned} h &= r_0 \frac{\Sigma^{-1} r}{r^T \Sigma^{-1} r} \\ &= \frac{r^T \Sigma^{-1} r}{1_n^T \Sigma^{-1} r} \frac{\Sigma^{-1} r}{r^T \Sigma^{-1} r} \\ &= \frac{\Sigma^{-1} r}{1_n^T \Sigma^{-1} r} \end{aligned}$$

Now, we get the solution of risky portfolio (tangency point) weights:

$$h = \frac{\Sigma^{-1} r}{1_n^T \Sigma^{-1} r} = \frac{\Sigma^{-1}(R - R_f 1_n)}{1_n^T \Sigma^{-1}(R - R_f 1_n)}$$

Part 2: derivation process about Method 2

(1) Firstly, find the effective front:

$$\min_w w^T \Sigma w$$

$$s.t. \begin{cases} w^T R = R_\Pi \\ w^T 1_n = 1 \end{cases}$$

Solve the problem:

$$L = w^T\Sigma w - \lambda_1(w^T R - R_\Pi) - \lambda_2(w^T 1_n - 1)$$

$$\begin{cases} \frac{\partial L}{\partial w} = 2\Sigma w - \lambda_1 R - \lambda_2 1_n = 0 \ldots\ldots (1) \\ \frac{\partial L}{\partial \lambda_1} = w^T R - R_\Pi = 0 \ldots\ldots\ldots\ldots (2) \\ \frac{\partial L}{\partial \lambda_2} = w^T 1_n - 1 = 0 \ldots\ldots\ldots\ldots\ldots (3) \end{cases}$$

Now substitute equation (1) into (2) (3)

$$w = \frac{1}{2}\Sigma^{-1}(\lambda_1 R + \lambda_2 1_n)$$

$$\begin{cases} R_\Pi = \frac{1}{2}\left(\lambda_1 R^T\Sigma^{-1}R + \lambda_2 1_n^T\Sigma^{-1}R\right) \\ 1 = \frac{1}{2}\left(\lambda_1 R^T\Sigma^{-1}1_n + \lambda_2 1_n^T\Sigma^{-1}1_n\right) \end{cases}$$

which can be rewrited as following:

$$\begin{bmatrix} R_\Pi \\ 1 \end{bmatrix} = \frac{1}{2} \cdot \begin{bmatrix} R^T\Sigma^{-1}R & 1_n^T\Sigma^{-1}R \\ R^T\Sigma^{-1}1_n & 1_n^T\Sigma^{-1}1_n \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \ldots . (4)$$

Now substitute equation (1) into $w^T\Sigma w$

$$\begin{aligned} w^T\Sigma w &= \frac{1}{4}\left(\lambda_1 R^T\Sigma^{-1} + \lambda_2 1_n^T\Sigma^{-1}\right) \cdot \Sigma\Sigma^{-1}\left(\lambda_1 R + \lambda_2 1_n\right) \\ &= \frac{1}{4}\left({\lambda_1}^2 R^T\Sigma^{-1}R + 2\lambda_1\lambda_2 R^T\Sigma^{-1}1_n + {\lambda_2}^2 1_n^T\Sigma^{-1}1_n\right) \\ &= \frac{1}{4}\begin{bmatrix} \lambda_1 & \lambda_2 \end{bmatrix} \begin{bmatrix} R^T\Sigma^{-1}R & R^T\Sigma^{-1}1_n \\ R^T\Sigma^{-1}1_n & 1_n^T\Sigma^{-1}1_n \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \ldots\ldots (5) \end{aligned}$$

solve λ_1, λ_2 from (4), and then substitute the solution into (5):

$$\begin{aligned} \sigma_\Pi^2 =& \begin{bmatrix} R_\Pi & 1 \end{bmatrix} \begin{bmatrix} R^T\Sigma^{-1}R & 1_n^T\Sigma^{-1}R \\ R^T\Sigma^{-1}1_n & 1_n^T\Sigma^{-1}1_n \end{bmatrix}^{-1} \begin{bmatrix} R^T\Sigma^{-1}R & R^T\Sigma^{-1}1_n \\ R^T\Sigma^{-1}1_n & 1_n^T\Sigma^{-1}1_n \end{bmatrix} \begin{bmatrix} R^T\Sigma^{-1}R & 1_n^T\Sigma^{-1}R \\ R^T\Sigma^{-1}1_n & 1_n^T\Sigma^{-1}1_n \end{bmatrix}^{-1} \begin{bmatrix} R_\Pi \\ 1 \end{bmatrix} \\ =& \begin{bmatrix} R_\Pi & 1 \end{bmatrix} \begin{bmatrix} R^T\Sigma^{-1}R & 1_n^T\Sigma^{-1}R \\ R^T\Sigma^{-1}1_n & 1_n^T\Sigma^{-1}1_n \end{bmatrix}^{-1} \begin{bmatrix} R_\Pi \\ 1 \end{bmatrix} \\ =& \begin{bmatrix} R_\Pi & 1 \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix}^{-1} \begin{bmatrix} R_\Pi \\ 1 \end{bmatrix} \\ =& \frac{cR_\Pi^2 - 2bR_\Pi + a}{ac - b^2} \end{aligned}$$

(2) Secondly, using the effective frontier, we can easily find the tangency point:

$$\max_{R_\Pi} \text{Slope} = \frac{R_\Pi - R_f}{\sigma_\Pi}$$

solve the problem:

$$\begin{aligned}
\frac{\partial \text{Slope}}{\partial R_\Pi} =& \frac{\sigma_\Pi - \frac{\partial \sigma_\Pi}{\partial R_\Pi}(R_\Pi - R_f)}{\sigma_\Pi^2} = 0 \\
\sigma_\Pi =& \frac{\partial \sigma_\Pi}{\partial R_\Pi}(R_\Pi - R_f)
\end{aligned}$$

$$\begin{aligned}
\frac{1}{R_\Pi - R_f} ==& \frac{\partial \sigma_\Pi}{\sigma_\Pi \partial R_\Pi} \\
=& \frac{\partial (\sigma_\Pi^2)^{1/2}}{\sigma_\Pi \partial R_\Pi} \\
=& \frac{1}{\sigma_\Pi} \frac{1}{2} (\sigma_\Pi^2)^{-\frac{1}{2}} \frac{\partial \sigma_\Pi^2}{\partial R_\Pi} \\
=& (\frac{cR_\Pi^2 - 2bR_\Pi + a}{ac - b^2})^{-1} \frac{1}{2} \frac{2cR_\Pi - 2b}{ac - b^2} \\
=& \frac{ac - b^2}{cR_\Pi^2 - 2bR_\Pi + a} \frac{cR_\Pi - b}{ac - b^2} \\
=& \frac{cR_\Pi - b}{cR_\Pi^2 - 2bR_\Pi + a}
\end{aligned}$$

$$\begin{aligned}
cR_\Pi^2 - 2bR_\Pi + a =& cR_\Pi^2 - bR_\Pi - cR_f R_\Pi + bR_f \\
(-b + cR_f) R_\Pi =& bR_f - a \\
R_\Pi =& \frac{a - bR_f}{b - cR_f}
\end{aligned}$$

And then, we get the solution of risky portfolio (tangency point) weights:

$$\begin{aligned}
w =& \frac{1}{2} \Sigma^{-1} (\lambda_1 R + \lambda_2 1_n) \\
=& \frac{1}{2} \Sigma^{-1} \begin{bmatrix} R & 1_n \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \\
=& \frac{1}{2} \Sigma^{-1} \begin{bmatrix} R & 1_n \end{bmatrix} 2 \begin{bmatrix} R^T \Sigma^{-1} R & 1_n^T \Sigma^{-1} R \\ R^T \Sigma^{-1} 1_n & 1_n^T \Sigma^{-1} 1_n \end{bmatrix}^{-1} \begin{bmatrix} R_\Pi \\ 1 \end{bmatrix} \\
=& \Sigma^{-1} \begin{bmatrix} R & 1_n \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix}^{-1} \begin{bmatrix} R_\Pi \\ 1 \end{bmatrix} \\
=& \Sigma^{-1} \begin{bmatrix} R & 1_n \end{bmatrix} (\begin{bmatrix} R^T \\ 1_n^T \end{bmatrix} \Sigma^{-1} \begin{bmatrix} R & 1_n \end{bmatrix})^{-1} \begin{bmatrix} R_\Pi \\ 1 \end{bmatrix}
\end{aligned}$$

Part3: The procedure of Calculation:

we follow Method 1 to calculate the results:

$$R = \frac{2}{100} \begin{bmatrix} 7 \\ 4 \\ 10 \end{bmatrix}$$

$$\Sigma = \frac{9}{10000} \begin{bmatrix} 4 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 25 \end{bmatrix}$$

$$\Sigma^{-1} = \frac{10000}{9} \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} & 0 \\ -\frac{1}{3} & \frac{32}{21} & -\frac{2}{21} \\ 0 & -\frac{2}{21} & \frac{1}{21} \end{bmatrix}$$

$$a = R^T\Sigma^{-1}R = \frac{2}{100} * \frac{2}{100} * \frac{10000}{9}[7 \quad 4 \quad 10]\begin{bmatrix} \frac{1}{3} & -\frac{1}{3} & 0 \\ -\frac{1}{3} & \frac{32}{21} & -\frac{2}{21} \\ 0 & -\frac{2}{21} & \frac{1}{21} \end{bmatrix}\begin{bmatrix} 7 \\ 4 \\ 10 \end{bmatrix} = \frac{4}{9}1_n^T\begin{bmatrix} \frac{49}{3} & -\frac{28}{3} & 0 \\ -\frac{28}{3} & \frac{512}{21} & -\frac{80}{21} \\ 0 & -\frac{80}{21} & \frac{100}{21} \end{bmatrix}1_n = \frac{403}{21}\frac{4}{9}$$

$$b = R^T\Sigma^{-1}1_n = \frac{2}{100} * \frac{10000}{9}[7 \quad 4 \quad 10]\begin{bmatrix} \frac{1}{3} & -\frac{1}{3} & 0 \\ -\frac{1}{3} & \frac{32}{21} & -\frac{2}{21} \\ 0 & -\frac{2}{21} & \frac{1}{21} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{200}{9}1_n^T\begin{bmatrix} \frac{7}{3} & -\frac{7}{3} & 0 \\ -\frac{4}{3} & \frac{128}{21} & -\frac{8}{21} \\ 0 & -\frac{20}{21} & \frac{10}{21} \end{bmatrix}1_n = \frac{82}{21}\frac{200}{9}$$

$$c = 1_n^T\Sigma^{-1}1_n = \frac{10000}{9}[1 \quad 1 \quad 1]\begin{bmatrix} \frac{1}{3} & -\frac{1}{3} & 0 \\ -\frac{1}{3} & \frac{32}{21} & -\frac{2}{21} \\ 0 & -\frac{2}{21} & \frac{1}{21} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{10000}{9}1_n^T\begin{bmatrix} \frac{1}{3} & -\frac{1}{3} & 0 \\ -\frac{1}{3} & \frac{32}{21} & -\frac{2}{21} \\ 0 & -\frac{2}{21} & \frac{1}{21} \end{bmatrix}1_n = \frac{22}{21}\frac{10000}{9}$$

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix}^{-1} = (\frac{4}{9*21}\begin{bmatrix} 403 & 82*50 \\ 82*50 & 22*2500 \end{bmatrix})^{-1} = \frac{9*21}{4} * \frac{1}{5355000}\begin{bmatrix} 22*2500 & -82*50 \\ -82*50 & 403 \end{bmatrix}$$

R_Π:

$$R_\Pi = \frac{a - bR_f}{b - cR_f} = \frac{403 \times 4 - 82 \times 200 \times 0.05}{82 \times 200 - 22 \times 10000 \times 0.05} = \frac{403 \times 4 - 82 \times 10}{82 \times 200 - 22 \times 500} = \frac{1}{100}\frac{1612 - 820}{164 - 110} = \frac{792}{54}\% = \frac{132}{900}$$

weights:

$$\begin{aligned} w =& \Sigma^{-1}[R \quad 1_n]\begin{bmatrix} a & b \\ b & c \end{bmatrix}^{-1}\begin{bmatrix} R_\Pi \\ 1 \end{bmatrix} \\ =& \frac{10000}{9}\begin{bmatrix} \frac{1}{3} & -\frac{1}{3} & 0 \\ -\frac{1}{3} & \frac{32}{21} & -\frac{2}{21} \\ 0 & -\frac{2}{21} & \frac{1}{21} \end{bmatrix}\frac{2}{100}\begin{bmatrix} 7 & 50 \\ 4 & 50 \\ 10 & 50 \end{bmatrix}\frac{9*21}{4} * \frac{1}{5355000}\begin{bmatrix} 22*2500 & -82*50 \\ -82*50 & 403 \end{bmatrix}\frac{1}{900}\begin{bmatrix} 132 \\ 900 \end{bmatrix} \\ =& \frac{1}{18 \times 5355000}\begin{bmatrix} 7 & -7 & 0 \\ -7 & 32 & -2 \\ 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} 7 & 50 \\ 4 & 50 \\ 10 & 50 \end{bmatrix}\begin{bmatrix} 22*2500 & -82*50 \\ -82*50 & 403 \end{bmatrix}\begin{bmatrix} 132 \\ 900 \end{bmatrix} \\ =& \frac{1}{18 \times 5355000}\begin{bmatrix} 21 & 0 \\ 59 & 1150 \\ 2 & -50 \end{bmatrix}\begin{bmatrix} 35700 \\ -1785 \end{bmatrix}100 \\ =& \frac{1}{18 \times 53550}\begin{bmatrix} 749700 \\ 53550 \\ 160650 \end{bmatrix} \\ =& \frac{1}{18}\begin{bmatrix} 14 \\ 1 \\ 3 \end{bmatrix} \end{aligned}$$

σ^2_Π:

$$\begin{aligned} \sigma^2_\Pi =& w^T\Sigma w \\ =& \frac{1}{18}[14 \quad 1 \quad 3]\frac{9}{10000}\begin{bmatrix} 4 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 25 \end{bmatrix}\frac{1}{18}\begin{bmatrix} 14 \\ 1 \\ 3 \end{bmatrix} \\ =& \frac{1}{36 \times 10000}1_n^T\begin{bmatrix} 784 & 14 & 84 \\ 14 & 1 & 6 \\ 84 & 6 & 225 \end{bmatrix}1_n \\ =& \frac{1218}{36 \times 10000} \\ =& \frac{203}{6}\%\% \end{aligned}$$

$$\sigma_\Pi = \sqrt{\frac{203}{6}}\% = 5.817\%$$

Slope:

$$\begin{aligned}\text{Slope} =& \frac{R_\Pi - R_f}{\sigma_\Pi}\\ =& \frac{132/9\% - 0.05}{\sqrt{203/6\%}}\\ =& \frac{132/9 - 5}{\sqrt{203/6}}\\ =& 1.662\end{aligned}$$

CML:

$$R = 1.662\sigma + 0.05$$

The market price of risk = 1.662

(b) Suppose that an investor has the risk tolerance of 0.02. Find the composition of the optimal portfolio on the CML by specifying the percentage of his wealth to be invested in the risk-free asset and the risky loans obtained in part a, i.e. you need to find four proportions.

[3 marks]

[Total marks 15]

$$\begin{aligned}0.02^2 =& t^2\sigma_\Pi^2\\ t^2 =& 4\%\% \frac{6}{203\%\%}\\ t =& \sqrt{\frac{24}{203}}\\ =& 0.3438\end{aligned}$$

w:

$$w_{new} = \begin{bmatrix}\frac{14}{18}t\\ \frac{1}{18}t\\ \frac{3}{18}t\\ 1-t\end{bmatrix} = \begin{bmatrix}0.2674\\ 0.0191\\ 0.0573\\ 0.6562\end{bmatrix}$$

return

$$R_{new} = \begin{bmatrix}\frac{14}{18}t & \frac{1}{18}t & \frac{3}{18}t & \frac{18(1-t)}{18}\end{bmatrix} \frac{1}{100}\begin{bmatrix}14\\ 8\\ 20\\ 5\end{bmatrix} = \frac{1}{100}\frac{(196+8+60-90)t+90}{18} = \frac{174t+90}{18} = 8.324\%$$