Project: Shallow-water modeling of a 2-D vortex past mesoscale mountains

(2022 Spring)

a. Derive the governing equations

The governing equations for a shallow-water flow system with terrain are given by

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\mathrm{g}\frac{\partial h}{\partial x} + fv \qquad (1)$$

$$\frac{Dv}{Dt} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\mathrm{g}\frac{\partial h}{\partial y} - fu \qquad (2)$$

$$\frac{Dh}{Dt} = \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y} = -h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \qquad (3)$$

where u and v are zonal and meridional components of the flow, h is the height of the upper flow surface.

The bottom terrain is given a 2-D function $h_M(x, y)$ and the undisturbed upper flow surface has a constant height H and the Coriolis parameter f. Since the flow is incompressible following

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad (4)$$

Integrating this equation vertically from h_M to h gives

$$\int_{h_M}^{h} \frac{\partial w}{\partial z}\ dz = -\int_{h_M}^{h} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dz$$

i.e., if assuming the motion is 2-D and independent on height,

$$w(h) - w(h_M) = -(h - h_M)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

and with $w(h) = \frac{dh}{dt}$ and $w(h_M) = \frac{dh_M}{dt}$

$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y} - \left(\frac{\partial h_M}{\partial t} + u\frac{\partial h_M}{\partial x} + v\frac{\partial h_M}{\partial y}\right) = -(h - h_M)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$, leading to

or in advection form for $h - h_M$,

$$\frac{\partial (h-h_M)}{\partial t} + u\frac{\partial}{\partial x}(h - h_M) + v\frac{\partial}{\partial y}(h - h_M) = -(h - h_M)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \qquad (5a)$$

or in advection form for h,

$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y} = u\frac{\partial h_M}{\partial x} + v\frac{\partial h_M}{\partial y} - (h - h_M)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \qquad (5b)$$

In flux form,

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}u(h - h_M) + \frac{\partial}{\partial y}v(h - h_M) = 0 \qquad (6)$$

Equation systems (1), (2) and (5) or (1), (2) and (6) can be used to model the 2-D cyclonic vortex past mountain terrain on a constant *f*-plane where $f = 2\Omega\sin\theta$ (θ latitude).

b. Specify the vortex and mountain

1. The tangential velocity V_t for an initial idealized circular vortex is given by

$$V_t = V_{max}\left(\frac{r}{R}\right) exp\left\{\frac{1}{\gamma}\left[1-\left(\frac{r}{R}\right)^{\gamma}\right]\right\} \qquad (7)$$

where V_{max} is the maximum wind speed at the radius of R from the vortex center located at (x_c, y_c), and γ (in range of 0.5-2) is usually set to 2.

2. The 2-D function $h_M(x,y)$ is given by a Gaussian-type mountain as

$$h_M = h_{max}\exp\left[-\frac{(x-x_m)^2}{\sigma_x^2}-\frac{(y-y_m)^2}{\sigma_y^2}\right] \qquad (8)$$

where h_{max} is the height of the mountain peak at location (x_m, y_m) and σ_x and σ_y determine the e-folding decrease of the mountain height in the *x*- and *y*-directions, respectively. Assume $h_M(x,y) \geq 1$.

c. Calculate the streamfunction, vorticity, potential vorticity

Define streamfunction ψ, and $\mathbf{V}_\psi \equiv \mathbf{k}\times\nabla\psi$ is the horizontal velocity defined by the streamfunction as

$$u = -\frac{\partial\psi}{\partial y}, \quad v = \frac{\partial\psi}{\partial x} \qquad (9)$$

and

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} \equiv \nabla^2\psi. \qquad (10)$$

which is called **Possion equation**.

The potential vorticity of the SW flow can be defined as $q = (\zeta + f)/h$.

d. Model domain setup and initial conditions

1. Specify constant grid intervals Δx= Δy = 10 km for a mesh of 501x501 grids.
2. Assume a constant-*f* plane centered at the latitude of 30°N.
3. Assume constant steering zonal flow speed *U* and undisturbed flow height *H*.
4. Specify the mountain $h_M(x,y)$ with a peak heght of h_{max} at (x_m, y_m).
5. Obtain the vortex circulation given the vortex center at (x_c, y_c) and pure tangential wind V_t with maximum intensity of V_{max}.
6. Obtain the pertubation flow height $h_1(x,y)$ using the geostrophic-wind balance.

7. Obtain the pertubation flow height $h_2(x,y)$ using the gradient-wind balance.
8. Obtain the total flow height $h(x,y) = H + h_1(x,y) + h_2(x,y)$.
9. Obtain the streamfunction from the Poisson equation (10) with proper boundary conditions by Eq. (9). A SOR or ADI scheme may be used to solve the equation.
10. Obtain the vorticity from (10).

e. Numerical methods

1. It may update the advecting field of a scalar ϕ (= *u, v, h*) at grid piont (*i, j*) and time step *n* by the suggested quadratic upstream advection scheme from splitting integration:

$$\phi^*_{i,j} = \begin{cases} \phi^n_{i,j} - \Delta t * u^n_{i,j} * \left(\frac{\phi^n_{i+1,j} - \phi^n_{i-1,j}}{2\Delta x} - \frac{\phi^n_{i+1,j} - 3\phi^n_{i,j} + 3\phi^n_{i-1,j} - \phi^n_{i-2,j}}{6\Delta x}\right) & \text{for } u^n_{i,j} \geq 0 \\ \phi^n_{i,j} - \Delta t * u^n_{i,j} * \left(\frac{\phi^n_{i+1,j} - \phi^n_{i-1,j}}{2\Delta x} - \frac{\phi^n_{i+2,j} - 3\phi^n_{i+1,j} + 3\phi^n_{i,j} - \phi^n_{i-1,j}}{6\Delta x}\right) & \text{for } u^n_{i,j} < 0 \end{cases}$$

for *u*-advection with the zero gradient boundary condition $\frac{\partial \phi^*_b}{\partial x} = 0$ and $\frac{\partial \phi^*_b}{\partial y} = 0$ (*b* represents the grid at lateral boundaries), and

$$\phi^{**}_{i,j} = \begin{cases} \phi^*_{i,j} - \Delta t * v^n_{i,j} * \left(\frac{\phi^*_{i,j+1} - \phi^*_{i,j-1}}{2\Delta y} - \frac{\phi^*_{i,j+1} - 3\phi^*_{i,j} + 3\phi^*_{i,j-1} - \phi^*_{i,j-2}}{6\Delta y}\right) & \text{for } v^n_{i,j} \geq 0 \\ \phi^*_{i,j} - \Delta t * v^n_{i,j} * \left(\frac{\phi^*_{i,j+1} - \phi^*_{i,j-1}}{2\Delta y} - \frac{\phi^*_{i,j+2} - 3\phi^*_{i,j+1} + 3\phi^*_{i,j} - \phi^*_{i,j-1}}{6\Delta y}\right) & \text{for } v^n_{i,j} < 0 \end{cases}$$

for *v*-advection, with zero-gradient boundary condition $\frac{\partial \phi^{**}_b}{\partial x} = 0$ and $\frac{\partial \phi^{**}_b}{\partial y} = 0$.

Note that the first-order upstream advection scheme must be used instead for the grids adjacent to the lateral boundaries:

$$\phi^*_{i,j} = \begin{cases} \phi^n_{i,j} - \Delta t * u^n_{i,j} * \frac{\phi^n_{i,j} - \phi^n_{i-1,j}}{\Delta x} & \text{for } u^n_{i,j} \geq 0 \\ \phi^n_{i,j} - \Delta t * u^n_{i,j} * \frac{\phi^n_{i+1,j} - \phi^n_{i,j}}{6\Delta x} & \text{for } u^n_{i,j} < 0 \end{cases}$$

$$\phi^{**}_{i,j} = \begin{cases} \phi^*_{i,j} - \Delta t * v^n_{i,j} * \frac{\phi^*_{i,j} - \phi^*_{i,j-1}}{\Delta y} & \text{for } v^n_{i,j} \geq 0 \\ \phi^*_{i,j} - \Delta t * v^n_{i,j} * \frac{\phi^*_{i,j+1} - \phi^*_{i,j}}{\Delta y} & \text{for } v^n_{i,j} < 0 \end{cases}$$

(Note that you must determine a proper time step size for stable integration.)

2. Update the mometum field due to the horiozontal height graident term and the Coriolis term:

$$u^{n+1}_{i,j} = u^{**}_{i,j} - \Delta t * \mathrm{g}\left(\frac{h^n_{i+1,j} - h^n_{i-1,j}}{2\Delta x}\right) + \Delta t * f * v^{**}_{i,j}$$

$$v^{n+1}_{i,j} = v^{**}_{i,j} - \Delta t * \mathrm{g}\left(\frac{h^n_{i,j+1} - h^n_{i,j-1}}{2\Delta y}\right) - \Delta t * f * u^{**}_{i,j}$$

3. Update the flow height field with the updated $u_{i,j}^{n+1}$ and $v_{i,j}^{n+1}$ using the splitting advection and the forward-backward integration. The updated height after including the horizontal advection and the terrain term $u\frac{\partial h_M}{\partial x} + v\frac{\partial h_M}{\partial y}$ is denoted as $h_{i,j}^{**}$.
4. Now further update the height field $h_{i,j}^{**}$ from the horizontal divergence term using the forward-backward integration:

$$h_{i,j}^{n+1} = h_{i,j}^{**} - \Delta t * \left(h_{i,j}^{n} - h_{Mi,j}\right)\left[\left(\frac{u_{i+1,j}^{n+1} - u_{i-1,j}^{n+1}}{2\Delta x}\right) + \left(\frac{v_{i,j+1}^{n+1} - v_{i,j-1}^{n+1}}{2\Delta y}\right)\right]$$

5. Apply the lateral boundary conditions to update the prognostic variables u, v and h. Zero gradient at the lateral boundaries (Neumann b.c.) is simple to use. Radiation b.c. may also be applied for reducing the reflection near the boundaries. You may try different lateral boundary conditions, like geostrophic balance.

f. Problems

A: Westbound cyclones past a single mountain range

1. Set
   (x_{m1}, y_{m1}) = (231, 251), (x_c, y_c) = (341, 251)
2. Repeat the previous steps until you reach the 96-h simulation end of a vortex for the following cases:
   W: U = 4, 8 m s^{-1}, $V_{max} = 40$ m s^{-1}, R =150 km, $h_{max} = 3$ km, $\sigma_x = 50$ km, $\sigma_y = 200$ km, H = 5 km (wide-terrain experiment)
   N: U = 4, 8 m s^{-1}, $V_{max} = 40$ m s^{-1}, R =150 km, $h_{max} = 3$ km, $\sigma_x = 200$ km, $\sigma_y = 50$ km, H = 5 km (narrow-terrain experiments)

B: Westbound cyclones past a set of three mountain ranges

1. Set
   (x_{m1}, y_{m1}) = (231, 201), (x_c, y_c) = (341, 251)
   (x_{m2}, y_{m2}) = (231, 301), (x_c, y_c) = (341, 251)
   (x_{m3}, y_{m3}) = (181, 251), (x_c, y_c) = (341, 251)
2. Repeat the previous steps until you reach the 96-h simulation end of a vortex for the following cases:
   S: U = 4, 8 m s^{-1}, $V_{max} = 40$ ms^{-1}, R =150 km, $h_{max} = 3$ km, $\sigma_x = 150$ km, $\sigma_y = 150$ km, H = 5 km (small-cyclone experiment)
   L: U = 4, 8 m s^{-1}, $V_{max} = 40$ ms^{-1}, R =300 km, $h_{max} = 3$ km, $\sigma_x = 150$ km, $\sigma_y = 150$ km, H = 5 km (large-cyclone experiment)

g. Linear instability

1. For the SW system, the time step Δt used for a stable integration of advection depends on the Courant number defined as

$C = \max(u_{i,j}^n)\Delta t \le 1$ and $C = \max(v_{i,j}^n)\Delta t \le 1$ (CFL condition)

for most Eulerian schems. However, the time-split quadratic upstream scheme has a more stringent constrant for C that must be far less than 1. It usually chooses a safe time step so that $C \le 1/4$.

2. In addition to the advection, gravity waves will be genertaed and will also affect the size of the time step. When a forward-backward integration is adpoted in 2-D SW, a maximum stable time step may be allowed for $C = \max(c_g)\Delta t \le \frac{1}{\sqrt{2}}$.

You then need to try a a time step by estimating the maximum gravity wave speed.

3. A maximum time step for the SW system is allowed when both criteria in 1 and 2 are met.

h. Nonlinear instability (due to aliasing errors caused by nonlinear advection)

Instability that is induced by nonlinear aliasing of shorher waves must be well controlled by application of a 2-Δx filter or additional numerical diffusion to the prognostic variables, *u*, *v*, and *h* after each integration. The Shapiro filter (smoother and desmoother) often is suggested as

$$\phi_{i,j}^{n+1*} = (1-\mu)\phi_{i,j}^{n+1} + \mu(\phi_{i+1,j}^{n+1} + \phi_{i-1,j}^{n+1} + \phi_{i,j-1}^{n+1} + \phi_{i,j+1}^{n+1})/4 \ , \ \mu = 0.125$$

$$\phi_{i,j}^{n+1**} = (1-\mu)\phi_{i,j}^{n+1*} + \mu(\phi_{i+1,j}^{n+1*} + \phi_{i-1,j}^{n+1*} + \phi_{i,j-1}^{n+1*} + \phi_{i,j+1}^{n+1*})/4 \ , \ \mu = -0.125$$

You may try to apply a heavier smoothing with larger μ.

i. Modeling, Plots and Reports*

1. Plot the flow vectors, streamfunctions, and potential vorticity shaped contours for the model output every 6 h (the terrain height contours should be indicated as well)
2. Find the vortex center by searching the location of circulation center.
3. Superimpose the track at an interval of 3-h on the plot of potential vorticity and vector.
4. First check the initial conditions and then the results for simplified conditions (is the vortex moving straight westward without decaying in the absence of the terrain?)
5. Can you plot the asymmetric wind of the vortex?
6. Do any sensitivity tests (e.g., *R*, γ, g, μ, time steps, latertal boundary conditions, advection schemes, numerical smoothing, etc.) as you think worthy to test.
7. Write your report (including plots) about the modeling results on all the experiments and any conducted sensitivityt ests, and summarize your conclusions.

*You may use Fortran F90 with NCL (or RIP), or alternatively, Matlab. Be careful that the Matlab code uses a default convention for the matrix A(i,j) where i is the row number (northward grid index) and j is the column number (eastward grid index). For consistency with meteorological definitions, use A(j,i) in SW modeling with j for the northward grid and i for the eastward grid.

Solving the Poisson equation with SOR (successive over-relaxation):

Given the Poisson equation,

$\nabla^2\psi = \zeta$

which can be expressed by

$$\left[\frac{\psi_{i+1,j}+\psi_{i-1,j}-2\psi_{i,j}}{(\Delta x)^2}+\frac{\psi_{i,j+1}+\psi_{i,j-1}-2\psi_{i,j}}{(\Delta y)^2}\right] = \zeta_{i,j}$$

Let the residual defined as

$$R_{i,j} = \left[\frac{\psi_{i+1,j}+\psi_{i-1,j}-2\psi_{i,j}}{(\Delta x)^2}+\frac{\psi_{i,j+1}+\psi_{i,j-1}-2\psi_{i,j}}{(\Delta y)^2}\right] - \zeta_{i,j}$$

Thus,

$$(\Delta y)^2(\psi_{i+1,j}+\psi_{i-1,j}-2\psi_{i,j}) + (\Delta x)^2(\psi_{i,j+1}+\psi_{i,j-1}-2\psi_{i,j}) = (\Delta x)^2(\Delta y)^2\zeta_{i,j}$$

$$\psi_{i,j}\{-2[(\Delta x)^2+(\Delta y)^2]\} + (\Delta x)^2\left(\psi_{i,j+1}+\psi_{i,j-1}\right) + (\Delta y)^2(\psi_{i+1,j}+\psi_{i-1,j}) = (\Delta x)^2(\Delta y)^2\zeta_{i,j}$$

$$\psi_{i,j} + \frac{(\Delta x)^2\left(\psi_{i,j+1}+\psi_{i,j-1}\right)+(\Delta y)^2(\psi_{i+1,j}+\psi_{i-1,j})}{\{-2[(\Delta x)^2+(\Delta y)^2]\}} = \frac{(\Delta x)^2(\Delta y)^2\zeta_{i,j}}{\{-2[(\Delta x)^2+(\Delta y)^2]\}}$$

The solution can be obtained by the iteration method with the updated vorticity $\zeta_{i,j}^{n+1}$ by rewriting it as an iterative form

$$\psi_{i,j}^{(m+1)} - \psi_{i,j}^{(m)} + \psi_{i,j}^{(m)} + \frac{(\Delta x)^2\left(\psi_{i,j+1}^{(m)}+\psi_{i,j-1}^{(m)}\right)+(\Delta y)^2(\psi_{i+1,j}^{(m)}+\psi_{i-1,j}^{(m)})}{\{-2[(\Delta x)^2+(\Delta y)^2]\}} = \frac{(\Delta x)^2(\Delta y)^2\zeta_{i,j}^{n+1}}{\{-2[(\Delta x)^2+(\Delta y)^2]\}}$$

where *m* indicates the *m*th iteration and $\psi_{i,j}^{(0)} = \psi_{i,j}^{n}$

$$\psi_{i,j}^{(m+1)} = \psi_{i,j}^{(m)} + \frac{(\Delta x)^2\left(\psi_{i,j+1}^{(m)}+\psi_{i,j-1}^{(m)}\right)+(\Delta y)^2(\psi_{i+1,j}^{(m)}+\psi_{i-1,j}^{(m)})}{2[(\Delta x)^2+(\Delta y)^2]} - \psi_{i,j}^{(m)} + \frac{(\Delta x)^2(\Delta y)^2\zeta_{i,j}^{n+1}}{\{-2[(\Delta x)^2+(\Delta y)^2]\}}$$

$$\psi_{i,j}^{(m+1)} = \psi_{i,j}^{(m)} + \frac{(\Delta x)^2\left(\psi_{i,j+1}^{(m)}+\psi_{i,j-1}^{(m)}\right)+(\Delta y)^2(\psi_{i+1,j}^{(m)}+\psi_{i-1,j}^{(m)})-2[(\Delta x)^2+(\Delta y)^2]\psi_{i,j}^{(m)}-(\Delta x)^2(\Delta y)^2\zeta_{i,j}^{n+1}}{2[(\Delta x)^2+(\Delta y)^2]}$$

With the ω factor to accelerate the convergence rate

$$\psi_{i,j}^{(m+1)} = \psi_{i,j}^{(m)} + \omega\frac{(\Delta x)^2\left(\psi_{i,j+1}^{(m)}+\psi_{i,j-1}^{(m)}\right)+(\Delta y)^2(\psi_{i+1,j}^{(m)}+\psi_{i-1,j}^{(m)})-2[(\Delta x)^2+(\Delta y)^2]\psi_{i,j}^{(m)}-(\Delta x)^2(\Delta y)^2\zeta_{i,j}^{n+1}}{2[(\Delta x)^2+(\Delta y)^2]}$$

i.e.,

$$\psi_{i,j}^{(m+1)} = \psi_{i,j}^{(m)} + \omega\frac{(\Delta x)^2\left(\psi_{i,j+1}^{(m)}+\psi_{i,j-1}^{(m)}-2\psi_{i,j}^{(m)}\right)+(\Delta y)^2(\psi_{i+1,j}^{(m)}+\psi_{i-1,j}^{(m)}-2\psi_{i,j}^{(m)})-(\Delta x)^2(\Delta y)^2\zeta_{i,j}^{n+1}}{2[(\Delta x)^2+(\Delta y)^2]}$$

$$\psi_{i,j}^{(m+1)} = \psi_{i,j}^{(m)} + \omega \frac{\left(\psi_{i,j+1}^{(m)}+\psi_{i,j-1}^{(m)}-2\psi_{i,j}^{(m)}\right)/(\Delta y)^2+(\psi_{i+1,j}^{(m)}+\psi_{i-1,j}^{(m)}-2\psi_{i,j}^{(m)})/(\Delta x)^2-\zeta_{i,j}^{n+1}}{2[(\Delta x)^2+(\Delta y)^2]/[(\Delta x)^2(\Delta y)^2]}$$

$$\psi_{i,j}^{(m+1)} = \psi_{i,j}^{(m)} + \omega \frac{(\psi_{i+1,j}^{(m)}+\psi_{i-1,j}^{(m)}-2\psi_{i,j}^{(m)})/(\Delta x)^2+\left(\psi_{i,j+1}^{(m)}+\psi_{i,j-1}^{(m)}-2\psi_{i,j}^{(m)}\right)/(\Delta y)^2-\zeta_{i,j}^{n+1}}{2[(\Delta x)^2+(\Delta y)^2]/[(\Delta x)^2(\Delta y)^2]}$$

$$\psi_{i,j}^{(m+1)} = \psi_{i,j}^{(m)} + \omega \frac{[(\Delta x)^2(\Delta y)^2]}{2[(\Delta x)^2+(\Delta y)^2]} \mathrm{R}_{i,j}^{(m)}$$

where

$$\mathrm{R}_{i,j}^{(m)} = \frac{(\psi_{i+1,j}^{(m)} + \psi_{i-1,j}^{(m)} - 2\psi_{i,j}^{(m)})}{(\Delta x)^2} + \frac{\left(\psi_{i,j+1}^{(m)} + \psi_{i,j-1}^{(m)} - 2\psi_{i,j}^{(m)}\right)}{(\Delta y)^2} - \zeta_{i,j}^{n+1}$$

For $\Delta x = \Delta y = \Delta$,

$$\psi_{i,j}^{(m+1)} = \psi_{i,j}^{(m)} + \omega \frac{\left(\psi_{i,j+1}^{(m)}+\psi_{i,j-1}^{(m)}+\psi_{i+1,j}^{(m)}+\psi_{i-1,j}^{(m)}-4\psi_{i,j}^{(m)}-\Delta^2\zeta_{i,j}^{n+1}\right)}{4} \qquad (1)$$

1) Solving the Poisson equation requires and the updated vorticity as $\zeta_{i,j}^{n+1}$ and the conditions on the lateral boundaries given (as a choice) by

$\mathrm{u} = -\frac{\partial\psi}{\partial \mathrm{y}}$ on the northern and southern boundaries, and

$\mathrm{v} = \frac{\partial\psi}{\partial \mathrm{x}}$ on the western and eastern boundaries; both can be expressed by

$$\mathrm{u}_{i,b}^n = -\frac{\psi_{i,b+1}^{(m)}-\psi_{i,b-1}^{(m)}}{2\Delta y} \qquad (b = 1, ny) \qquad \text{(2a) and}$$

$$v_{b,j}^n = -\frac{\psi_{b+1,j}^{(m)}-\psi_{b-1}^{(m)}}{2\Delta x} \qquad (b = 1, nx) \qquad \text{(2b).}$$

Substituting (2a) and (2b) for the undefined grid values in (1).

2) The iterations will proceed until the residual reaches the criterion or the maximum relative change in $\psi_{i,j}^{(m)}$ is less than a very small value such that

$$\max[(\psi^{(m+1)} - \psi^{(m)})/\overline{\psi}^{(m)}] < \varepsilon$$

where ε can be 0.1% or smaller.

Successive over-relaxation ($\omega > 1$) under-relaxation ($\omega < 1$) can be applied to faster the convergence rate.

Note that with the given Neumann boundary conditions for the Poisson equation, the solution if existing plus any constant is also a solution. Thus, there is no unique solution! How can we do under such a situation?

3) Finally, the updated streamfunction is given by

$$\psi_{i,j}^{n+1} = \psi_{i,j}^{(m+1)}$$

and the updated velocity is given by

$$u_{i,j}^{n+1} = -\frac{\left(\psi_{i,j+1}^{n+1} - \psi_{i,j-1}^{n+1}\right)}{2\Delta y}$$ and

$$v_{i,j}^{n+1} = \frac{\left(\psi_{i+1,j}^{n+1} - \psi_{i-1,j}^{n+1}\right)}{2\Delta x}$$

Solving the Poisson equation with SOR for the balanced height

From the momentum equations with constant f

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -g\frac{\partial h}{\partial x} + fv \qquad \text{(A1)}$$

$$\frac{Dv}{Dt} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -g\frac{\partial h}{\partial y} - fu \qquad \text{(A2)}$$

Taking the divervence of the momentum equations,

$\frac{\partial(\text{A1})}{\partial x} + \frac{\partial(\text{A2})}{\partial y}$ gives

$$\frac{\partial u_x}{\partial t} + \frac{\partial v_y}{\partial t} = -\left(uu_x + vu_y\right)_x - gh_{xx} + fv_x - \left(uv_x + vv_y\right)_y - gh_{yy} - fu_y - uf_y$$

$$= f(v_x - u_y) - g(h_{xx} + h_{yy}) - \left(uu_x + vu_y\right)_x - \left(uv_x + vv_y\right)_y$$

$$= f(v_x - u_y) - g(h_{xx} + h_{yy}) - (u_x u_x + uu_{xx} + v_x u_y + vu_{yx} + u_y v_x + uv_{xy} + v_y v_y + vv_{yy})$$

$$= f\zeta - g\left(h_{xx} + h_{yy}\right) - u\left(u_{xx} + v_{xy}\right) - v\left(u_{yx} + v_{yy}\right) - \left(u_x u_x + v_x u_y + u_y v_x + v_y v_y\right)$$

$$= f\zeta - g\left(h_{xx} + h_{yy}\right) - u\left(u_x + v_y\right)_x - v\left(u_x + v_y\right)_y - \left(u_x u_x + v_x u_y + u_y v_x + v_y v_y\right) \quad \text{(A3)}$$

With the continuity equation $u_x + v_y = 0$，(A3) reduces to

$$g\left(h_{xx} + h_{yy}\right) = f\zeta - (u_x u_x + v_x u_y + u_y v_x + v_y v_y)$$

$$= f\zeta - u_x(-v_y) - u_y v_x + u_x v_y - v_x u_y$$

$$= f\zeta + 2\left(u_x v_y - u_y v_x\right)$$

$$= f\zeta + 2J(u, v) \qquad \text{(A4)}$$

(A4) is the nonlinear balance equation. Thus, the Poisson equation

$$h_{xx} + h_{yy} = \frac{1}{g}\left(f\zeta + 2J(u, v)\right) = F(x, y)$$

can be solved using the SOR method for the height with the given boundary conditions:

$$-g\frac{\partial h}{\partial x} + fv = 0 \quad \text{and} \quad -g\frac{\partial h}{\partial y} - fu = 0,$$

and the known source term $F(x, y)$.

Relationship between the wind components (U, V) at Cartesian coordinates and cylindrical coordinates (V_r, V_t)

$V_r = U \cos A + V \sin A$
$V_t = - U \sin A + V \cos A$

and

$U = V_r \cos A - V_t \sin A$
$V = V_r \sin A + V_t \cos A$

where A is the azimuthal angle of the location of the wind defined in the polar coordinates or cylindrical coordinates.

A generalized SOR (successive over-relaxation) and an alternative method for solving second-order pdf

The second-order elliptical pdf is given by

$$a\phi_{xx} + b\phi_{yy} + c\phi_{yy} + d\phi_x + e\phi_y + f\phi_z + g\phi_{xy} + h\phi_{xz} + i\phi_{yz} = s(x,y,z) \quad (1)$$

where *a* to *i* are coefficients that may be functions of *x*, *y* and *z*, and *s* denotes the nonhomogeneous term as a function of *x*, *y* and *z*. SOR can be formulated more easily when *g*, *h* and *i* terms are not existent (i.e., without the cross-derivative terms). The **pdf** can be elliptical without these cross-derivative terms provided that $b^2 - 4ac \geq 0$. However, it will be somewhat lengthy to derive a **generalized** SOR with the standard procedure for the pdf even with constant δx, δy and δz. We leave it out for an exercise.

The above problem can be alternatively tackled by transferring the elliptical pdf to a parabolic pdf as

$$\phi_t + a\phi_{xx} + b\phi_{yy} + c\phi_{yy} + d\phi_x + e\phi_y + f\phi_z + g\phi_{xy} + h\phi_{xz} + i\phi_{yz} = s(x,y,z) \quad (2)$$

with the time-derivative term ϕ_t added. The steady-state solution of this pdf (2) will be the solution of pdf (1), and the solution technique is called *the artificial-compressibility method* resulting from solving the Poisson equation for the balanced pressure with incompressible flow.

The finite-difference scheme for this parabolic pdf (2) can be easily formulated using typical methods for uneven grids as

$$\phi_t + a\phi_{xx} + b\phi_{yy} + c\phi_{yy} + d\phi_x + e\phi_y + f\phi_z + g\phi_{xy} + h\phi_{xz} + i\phi_{yz} = s(x,y,z)$$

With $\delta t = t_{n+1} - t_n$, $\delta x_i = x_{i+1} - x_i$, $\delta y_j = y_{j+1} - y_j$, $\delta z_k = z_{k+1} - z_k$

$$\phi_t = \frac{\phi_{i,j,k}^{n+1} - \phi_{i,j,k}^{n}}{\delta t} ,$$

$$\phi_{xx} = \left(\frac{\phi_{i+1,j,k}^{n} - \phi_{i,j,k}^{n}}{\delta x_i} - \frac{\phi_{i,j,k}^{n} - \phi_{i-1,j,k}^{n}}{\delta x_{i-1}}\right) / [(\delta x_i + \delta x_{i-1})/2] ,$$

$$\phi_{yy} = \left(\frac{\phi_{i,j+1,k}^{n} - \phi_{i,j,k}^{n}}{\delta y_j} - \frac{\phi_{i,j,k}^{n} - \phi_{i,j-1,k}^{n}}{\delta y_{j-1}}\right) / [(\delta y_j + \delta y_{j-1})/2] ,$$

$$\phi_{zz} = \left(\frac{\phi_{i,j,k+1}^{n} - \phi_{i,j,k}^{n}}{\delta z_k} - \frac{\phi_{i,j,k}^{n} - \phi_{i,j,k-1}^{n}}{\delta z_{k-1}}\right) / [(\delta z_k + \delta z_{k-1})/2] ,$$

$\phi_x = \frac{\phi^n_{i+1,j,k} - \phi^n_{i-1,j,k}}{\delta x_i + \delta x_{i-1}}$,

$\phi_y = \frac{\phi^n_{i,j+1,k} - \phi^n_{i,j-1,k}}{\delta y_j + \delta y_{j-1}}$,

$\phi_z = \frac{\phi^n_{i,j,k+1} - \phi^n_{i,j,k-1}}{\delta z_k + \delta z_{k-1}}$, and the cross-derivative terms

$$\phi_{xy} = (\phi_x)_y = \left[\frac{\phi^n_{i+1,j+1/2,k} - \phi^n_{i-1,j+1/2,k}}{(\delta x_i + \delta x_{i-1})/2} - \frac{\phi^n_{i+1,j-1/2,k} - \phi^n_{i-1,j-1/2,k}}{(\delta x_i + \delta x_{i-1})/2}\right] / [(\delta y_j + \delta y_{j-1})/2] =$$
$$[\phi^n_{i+1,j+1,k} + \phi^n_{i+1,j,k} - \phi^n_{i-1,j+1,k} - \phi^n_{i-1,j,k} - \phi^n_{i+1,j,k} - \phi^n_{i+1,j-1,k} + \phi^n_{i-1,j,k} + \phi^n_{i-1,j-1,k}] / [(\delta x_i + \delta x_{i-1})(\delta y_j + \delta y_{j-1})/4]/2 =$$
$$[\phi^n_{i+1,j+1,k} + \phi^n_{i-1,j-1,k} - \phi^n_{i-1,j+1,k} - \phi^n_{i+1,j-1,k}] / [(\delta x_i + \delta x_{i-1})(\delta y_j + \delta y_{j-1})/8]$$
$$\phi_{xz} = (\phi_x)_z = \left[\frac{\phi^n_{i+1,j,k+1/2} - \phi^n_{i-1,j,k+1/2}}{(\delta x_i + \delta x_{i-1})/2} - \frac{\phi^n_{i+1,j,k-1/2} - \phi^n_{i-1,j,k-1/2}}{(\delta x_i + \delta x_{i-1})/2}\right] / [(\delta z_k + \delta z_{k-1})/2] =$$
$$[\phi^n_{i+1,j,k+1} + \phi^n_{i-1,j,k-1} - \phi^n_{i-1,j,k+1} - \phi^n_{i+1,j,k-1}] / [(\delta x_i + \delta x_{i-1})(\delta z_k + \delta z_{k-1})/8]$$
$$\phi_{yz} = (\phi_y)_z = \frac{\phi^n_{i,j+1,k+1/2} - \phi^n_{i,j-1,k+1/2}}{(\delta y_j + \delta y_{j-1})/2} - \frac{\phi^n_{i,j+1,k-1/2} - \phi^n_{i,j-1,k-1/2}}{(\delta y_j + \delta y_{j-1})/2} =$$
$$[\phi^n_{i,j+1,k+1} + \phi^n_{i,j-1,k-1} - \phi^n_{i,j-1,k+1} - \phi^n_{i,j+1,k-1}] / [(\delta y_j + \delta y_{j-1})(\delta z_k + \delta z_{k-1})/8]$$

where a simple forward-in-time and center-in-space explicit scheme is employed. Since solving the pdf is only a one-time job, we may choose a relatively small time step to ensure a stable integration with this simple scheme. Complicated implicit schemes for (2) that allow much larger time steps may also be developed and formulated, but these schemes usually require complex matrix calculations that are much more difficult for coding. A safe time step may be tried by considering

$C_1 = \max(a_{i,j,k})\, \Delta t/(\delta x_i)^2 \le 1/4$
$C_2 = \max(b_{i,j,k})\, \Delta t/(\delta y_j)^2 \le 1/4$
$C_3 = \max(c_{i,j,k})\, \Delta t/(\delta z_k)^2 \le 1/4$
$C_4 = \max(d_{i,j,k})\, \Delta t/(\delta x_i) \le 1$
$C_5 = \max(e_{i,j,k})\, \Delta t/(\delta x_i) \le 1$
$C_6 = \max(f_{i,j,k})\, \Delta t/(\delta x_i) \le 1$
$C_7 = \max(g_{i,j,k})\, \Delta t/(\delta x_i \delta y_j) \le 1/4$
$C_8 = \max(h_{i,j,k})\, \Delta t/(\delta x_i \delta z_k) \le 1/4$
$C_9 = \max(i_{i,j,k})\, \Delta t/(\delta y_j \delta z_k) \le 1/4$

(It is more stringent to choose a stable time step with the forward-in-time and center-in-space explicit scheme. But, it should be OK if considering only one-fourth of the criteria for all $C_k (k = 1, \ldots, 9)$