

Digital Signal Processing

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Gif sur Yvette, November 15, 2012

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I. STATISTICAL DESCRIPTION OF SIGNALS

I.1. Notion of random variable

I.1.1. Summary of random variable

I.1.1.1. Notations

Ω : sample space

$A : \{\omega, \text{ fulfill a given property} \} - A$ forms a family of events of F .

$\forall A \in F, P(A) \in [0, 1]$ (Probability of the event)

The triplet $\{\omega, F, P\}$ forms a probability space.

$(\Omega, F, P) \rightarrow (\mathbb{R}, B, P_X)$

$X : P[X(\omega) \leq x] = F_X(x) \leftarrow$ Cumulative distribution function of the variable X.

I.1.1.2. Discret Case

$X(\omega) \in \{x_1, \dots, x_n\}$

Let us call $p_i = P[X(\omega) = x_i]$

$$\begin{cases} \sum_i p_i &= 1 \\ F_X(x) &= \sum_{i, x_i \leq x} p_i \end{cases}$$

I.1.1.3. Probability density

$$P_X(x) = \sum_i p_i \delta(x - x_i)$$

I.1.1.4. Continuous case

$X(\omega)$ continuous ensemble of values and $F_X(x)$ is continuous and derivable.

$\frac{dF_X(x)}{dx} = p_X(x)$: probability density of X .



$$\begin{aligned} \int_{\mathbb{R}} p_X(x) dx &= 1 \\ \int_{-\infty}^x p_X(u) du &= F_x(x) \\ dF_X(x) &= \underbrace{p_X(x) dx}_{\text{elementary probability}} = P[x \leq X \leq x + dx] \\ F_X(-\infty) &= 0 \text{ and } F_X(+\infty) = 1 \end{aligned}$$

I.1.1.5. Moments of X

$$m_n = E[x^n] = \int_{\mathbb{R}} x^n dF_X(x)$$

$$m_1 = E[X] : \text{statistical mean}$$

$$m_2 = E[X^2] : \text{moment of second order}$$

$$Var(X) = \sigma_X^2 = E[|X - E[X]|^2]$$

$$E[f(X)] = \int_{\mathbb{R}} f(x) dF_X(x)$$

$$\text{If } f(X) = e^{iuX}, E[f(X)] = \int_{\mathbb{R}} e^{iuX} dF_X(x) = \Phi_X(u)$$

Φ_X is the characteristic function of first order.

I.1.1.6. X is continuous

$$\Phi_X(u) = \int_{\mathbb{R}} e^{iuX} p_X(x) dx$$

$$p_X(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-iuX} \Phi_X(u) du$$

I.1.1.7. Characteristic function of second order

$$\Psi_X(u) = \ln \Phi_X(u)$$

$$\Psi_X(u) = \sum_{k=0}^{+\infty} \frac{(iu)^k}{k!} C_k, C_k \text{ is a cumulant.}$$

$$C_1 = m_1$$

$$C_2 = m_2 - m_1^2 \text{ (variance)}$$

I.1.1.8. Couple of random real vectors (X, Y)

* In \mathbb{R}^2 :

$$- E[X, Y^T] = 0, \text{ if } X \perp Y$$

$$- \text{Let us call } X_c = X - E[X] :$$

$$E[X_c, Y_c^T] = Cov(X, Y) = E[(X - E[X])(Y - E[Y])] = E[XY^T] - E[X]E[Y^T]$$

If $Cov(X, Y) = 0$ then X and Y are uncorrelated.