Quarto

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1 Welcome

This is a Quarto website in "book" style.

To learn more about Quarto websites visit https://quarto.org/docs/websites.

This is an example showing how I use Quarto for my lectures notes. I wrote a blog post about it here.

Part I

Course notes

2 Geodesics

Here is an excerpt from the GR lecture notes.

2.1 Parallel-transport and geodesics

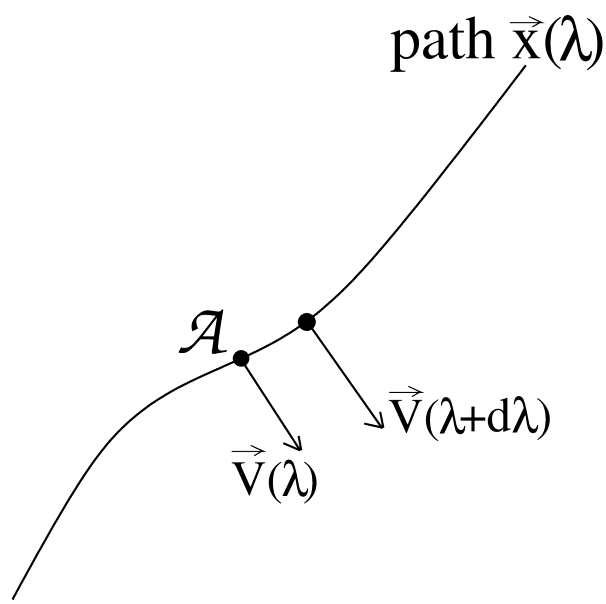

Figure 2.1: Parallel transport of a vector $\vec{v}(\lambda)$ where λ is an affine parameter. For geodesics the vector $\vec{v}$ is a tangent vector, $\vec{u}(\lambda)$.

We can use the idea of parallel transport to construct *geodesics*, defined as curves that parallel-transport their own tangent vectors. That is, for a geodesic

$$\nabla_{\vec{u}}\vec{u} = 0$$

$$u^\beta u^\alpha{}_{;\beta} = 0$$

$$u^\beta u^\alpha{}_{,\beta} + \Gamma^\alpha{}_{\beta\gamma} u^\beta u^\gamma = 0$$

$$\frac{d}{d\lambda}\left(\frac{dx^\alpha}{d\lambda}\right) + \Gamma^\alpha{}_{\beta\gamma}\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda} = 0 \tag{2.1}$$

where in the last of these expressions (often called the *geodesic equation*, though the first is also the geodesic equation), λ is a continuous parameter along the curve.

We have some freedom to choose λ – if we choose it to be the proper time of a particle with the curve as its world line, then $\vec{u}$ is the velocity of the particle. However, λ is a more general quantity and can be used also for light rays with no proper time. Any linear transformation of λ, such as $\phi = a\lambda + b$ with a, b constants, has $\vec{x}(\phi)$ a valid solution of the geodesic equation (try the transformation $\lambda \to \phi$) – we refer to λ (or ϕ) as an **affine parameter**.

In a locally-flat region, where the Christoffel symbols vanish, clearly the geodesic equation (Equation 2.1) reduces to

$$\frac{d^2x^\alpha}{d\lambda^2} = 0$$

which solves to the straight-line solution

$$x^\alpha = A^\alpha\lambda + B^\alpha.$$

In fact we can say, in a very real sense, that all geodesics are **straight**. This definition about "parallel transport of the tangent vector" is the only sensible definition of a straight line – it means that the curve at each point keeps moving in the direction of its local tangent vector. No other frame-independent definition of "straight" makes sense.

Part II

Problems

3 Problems 1

An example question from the first problems class.

3.1 Gravitational potential

A uniform spherical mass, M, has radius R.

(a) Calculate the Newtonian gravitational potential function $\Phi(r)$, from the Poisson equation in the form

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi}{dr}\right) = 4\pi G\rho$$

appropriate for a spherically-symmetric object, where ρ is the (constant) density of the mass.

Answer to part (a)

The Poisson equation

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi}{dr}\right) = 4\pi G\rho$$

for a Newtonian potential can be integrated once as

$$r^2\frac{d\Phi}{dr} = \int 4\pi G\rho r^2\, dr + C_1$$

where C_1 is a first integration constant (this is an indefinite integral). For a constant-density object of radius R there are two cases to consider — with $r \leq R$ and with $r \geq R$.
For $r \leq R$,

$$r^2\frac{d\Phi}{dr} = C_1 + \frac{4}{3}\pi G\rho r^3$$

where C_1 is the first constant of integration. Then

$$\Phi(r) = \int \frac{C_1}{r^2}\,dr + C_2 + \frac{4}{3}\pi G\rho \int r\,dr = -\frac{C_1}{r} + C_2 + \frac{2}{3}\pi G\rho r^2\,.$$

The C_1 term represents the effect of a point mass at the centre of the body, but we don't have one here, so $C_1 = 0$. C_2 will be found by matching solutions with the exterior solution: being a constant added to the potential, it doesn't represent a gravitational force.
At $r \geq R$,

$$r^2\frac{d\Phi}{dr} = D_1 + \frac{4}{3}\pi G\rho R^3$$

where we apparently need D_1 as a constant of integration. Rearrange and integrate again,

$$\Phi(r) = -\frac{D_1}{r} - \frac{4}{3}\frac{\pi G\rho R^3}{r} + D_2\,.$$

Here we can set $D_2 = 0$, since by convention we put $\Phi \to 0$ as $r \to \infty$. D_1 can also be taken as zero since it also represents a gravitational potential from a point mass at $r = 0$, and none is present.
It remains to match the solutions at $r = R$, so that there isn't an infinite force felt there. This means that

$$C_2 + \frac{2}{3}\pi G\rho R^2 = -\frac{4}{3}\pi G\rho R^2$$

so

$$C_2 = -2\pi G\rho R^2$$

and the overall solution is

$$\Phi(r) = \begin{cases} -2\pi G\rho R^2\left(1 - \frac{1}{3}\left(\frac{r^2}{R^2}\right)\right) & r \leq R \\ -\frac{4}{3}\frac{\pi G\rho R^3}{r} & r \geq R \end{cases}$$

which can be written in terms of mass as

$$\Phi(r) = -\frac{GM}{R}\begin{cases} \frac{3}{2} - \frac{1}{2}\left(\frac{r^2}{R^2}\right) & r \leq R \\ \frac{R}{r} & r \geq R \end{cases}$$

The maximum (negative) potential is at $r = 0$, and has value

$$\Phi_0 = -\frac{3}{2}\frac{GM}{R}\,.$$

4 About

This is an example Quarto website created to demonstrate how to use Quarto for lecture notes. If you have any questions, feel free to email me at Andy.Young@bristol.ac.uk.