Random Points, Broken Sticks, and Triangles

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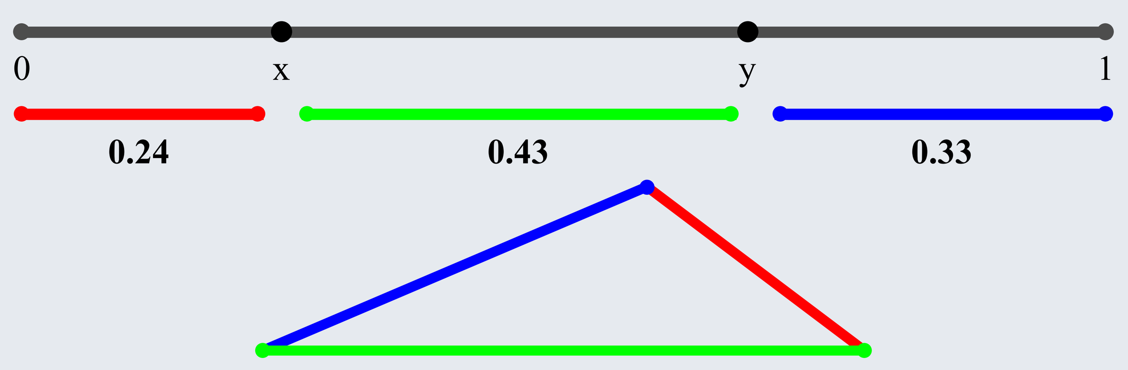
The Broken Stick Problem

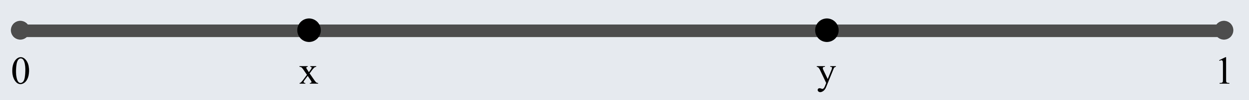
A stick is broken up at two random points along its length. What is the probability that the three pieces form a triangle?



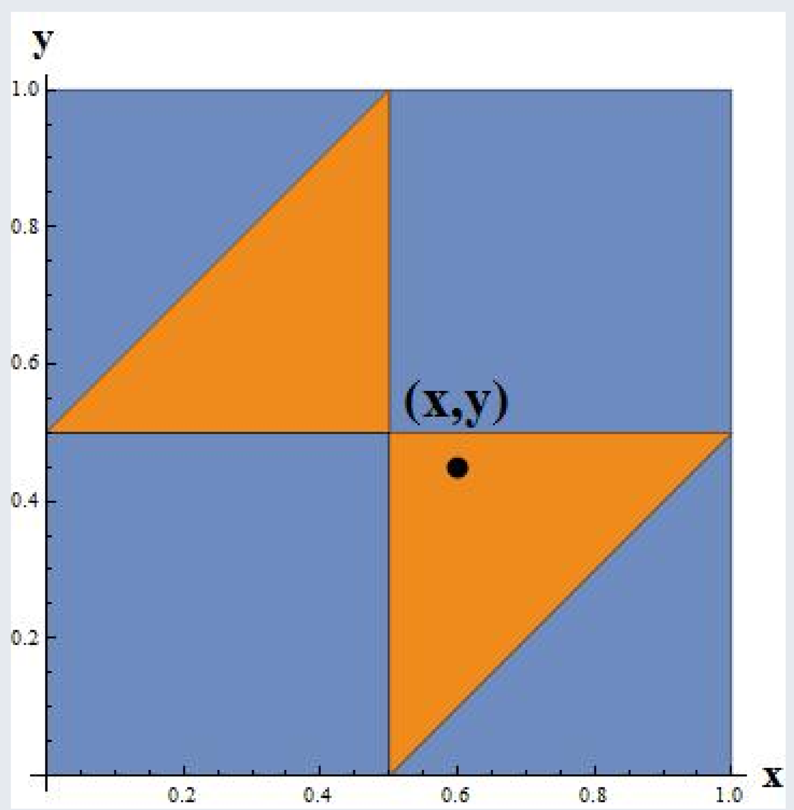
Background

- **Cambridge University Exam (1854).** The Broken Stick Problem first appeared as a problem in the *1854 Senate-House Examination* at Cambridge University, the precursor of the *Mathematical Tripos* examinations.
- **Emile Lemoine (1857):** Published a paper with a solution to the original Broken Stick Problem.
- **Paul Lévy (1939):** Investigated the "broken stick distribution" arising in this problem.
- **Martin Gardner (2001):** Popularized the Broken Stick Problem in his column and books.

Solution

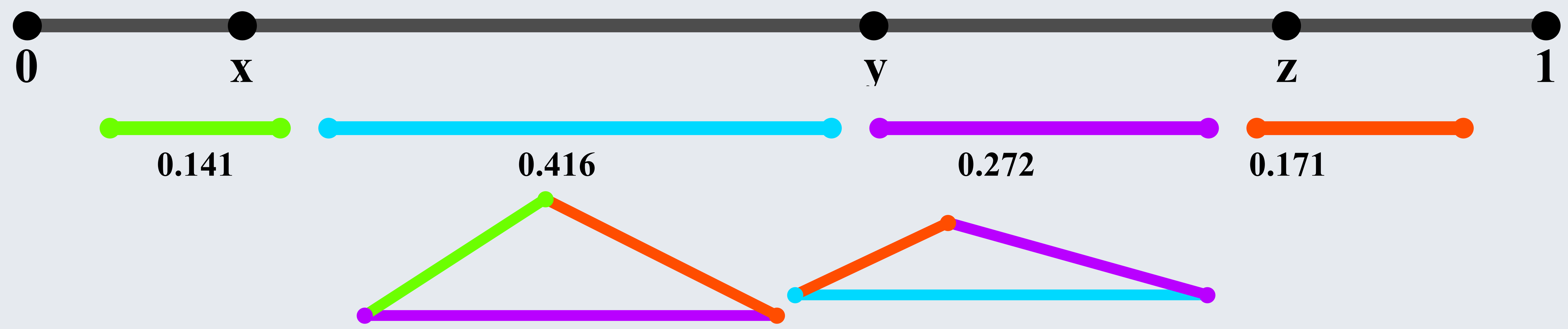


- Represent stick by unit interval $[0, 1]$ and Denote the breaking points by x and y.
- Interpret the pair (x, y) as a point in the unit square.
- Blue region: values (x, y) for which a triangle **can not** be formed.
- Orange region: values (x, y) for which a triangle **can** be formed.
- Orange region occupies $1/4$ of the entire square.
- Hence the probability sought is $1/4$.



The Broken Stick Problem with n Pieces

A stick is broken up at $n-1$ random points, resulting in n pieces. We try to form all possible triangles with these pieces:



Problem 1: At least one triangle

*What is the probability that **at least one** triple of pieces forms a triangle?*

Problem 2: All triangles

*What is the probability that **any** triple of pieces forms a triangle?*

The Broken Stick Problem with n Pieces: Main Result

(1) The probability that **at least one** triple of pieces forms a triangle is given by

$$\boxed{1-\prod_{k=2}^{n}\frac{k}{F_{k+2}-1}}$$

where F_n is the n-th Fibonacci number. (For $n=3$ and $n=4$ this gives $1/4$ and $4/7$, respectively.)

(2) The probability that **any** triple of pieces forms a triangle is given by

$$\boxed{\binom{2n-2}{n}^{-1}}$$

(For $n=3$ and $n=4$ this reduces to $1/4$ and $1/15$, respectively.)

# of pieces	P_{all}: Experimental	P_{all}: Theoretical	$P_{\geq 1}$: Experimental	$P_{\geq 1}$: Theoretical
3	0.24765	$\frac{1}{4}=0.25000\ldots$	0.24916	$\frac{1}{4}=0.25000\ldots$
4	0.0664	$\frac{1}{15}=0.06666\ldots$	0.56983	$\frac{4}{7}=0.571429\ldots$
5	0.01782	$\frac{1}{56}=0.01786\ldots$	0.82128	$\frac{23}{28}=0.821429\ldots$
6	0.00471	$\frac{1}{210}=0.00476\ldots$	0.94645	$\frac{53}{56}=0.946429\ldots$
7	0.00124	$\frac{1}{792}=0.00126\ldots$	0.98818	$\frac{87}{88}=0.988636\ldots$
8	0.00028	$\frac{1}{3003}=0.00033\ldots$	0.99835	$\frac{593}{594}=0.998316\ldots$
n		$\binom{2n-2}{n}^{-1}$		$1-\prod_{k=2}^{n}\frac{k}{F_{k+2}-1}$

Probabilities for forming triangles from the pieces of a Broken Stick: P_{all} is the probability that **all** triples of pieces form a triangle. $P_{\geq 1}$ is the probability that **at least one** triple of pieces forms a triangle. The experimental values were obtained by running $100,000$ random simulations. Note that the formula for $P_{\geq 1}$ involves F_k, the k-th Fibonacci number.

Proof of Main Result: Key Ideas

- Denote the lengths of the pieces by $y_1, \ldots, y_n$ and note that their sum must equal the length of the stick, 1.

  (1) $\quad y_1 + y_2 + \cdots + y_n = 1.$
- Use symmetry. Assume that the y_i's are in increasing order:

  (2) $\quad 0 \leq y_1 \leq y_2 \leq \cdots \leq y_n \leq 1$
- Observe that, under the additional assumption (3), **no** triangle can be formed if and only if

  (3) $\quad y_{i+2} \geq y_i + y_{i+1}$ for all indices i
- Compute the integrals in $\mathbb{R}^n$ representing the volumes of the $(n-1)$-dimensional regions given by conditions (1)–(2), resp. (1)–(3).
- The probability that **at least one** triple of pieces forms a triangle is 1 minus the ratio between these two volumes. A similar argument gives the probability that **any** triple of pieces forms a triangle.

Related Work

- **Polygons:** The probability of getting a polygon from an n-piece broken stick is $1-n2^{-n+1}$. (D'Andrea and Gómez, 2006)
- **Broken Stick Model:** Probabilistic model based on a randomly broken stick. Found to be a good match for a wide range of real-world phenomena, such as the distribution of species and resources (MacArthur, 1957)), the allocation of assets (Tashman and Frey, 2009)), and intervals between twin births in Champaign-Urbana (Ghent and Hanna, 1968).

References

- C. D'Andrea and E. Gómez, *The broken spaghetti noodle*, Amer. Math. Monthly 113 (2006), 555-557.
- M. Gardner, *The colossal book of mathematics*, W. W. Norton, New York, 2001. (Chapter 21)
- A. W. Ghent and B. P. Hanna, *Application of the "Broken Stick" formula to the prediction of random time intervals*, American Midland Naturalist 79(2) (April 1968), 273–288.
- Goodman, *The problem of the broken stick reconsidered*, Math. Intelligencer 30 (2008), 43–49.
- I. Lemoine, *Sur une question de probabilités*, Bull. Soc. Math, de France 1 (1875), 39–40.
- P. Lévy, *Sur la division d'un segment par des points choisis au hasard*, C. R. Acad. Sci. Paris 208 (1939), 147–149.
- R. H. MacArthur, *One the relative abundance of bird species*, Proc. Nat. Acad. Sci. USA 43 (1957), 293–295
- A. Tashman, R. J. Frey, *Modeling risk in arbitrage strategies using finite mixtures*, Quant. Finance 9 (2009), 495–503.

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