General observations:

Let us have $i \geq j => ai \geq aj\ and\ let\ us\ make\ the\ exchange\ (ai, aj) \to (ai+1, aj-1)$.

Then the sum is preserved and the product decreases. We will call this observation "exchange argument".

Consider

$$a1 * a2 * \ldots * an = a1 + a2 + \cdots + an$$

Let us consider max in the case of an equality. Clearly $a1 \geq 2$. Let $i \geq 2\ and\ ai \geq 2$. Repeatedly using "exchange argument" we will have $(\mathrm{a1}, 2, 1 \ldots 1)$ at the end, so $a1 * 2 \leq a1 + 2 + n - 2 => a1 \leq n\ and\ a1 * a2 * \ldots * an = a1 + a2 + \cdots + an \leq 2 * n$

Due to "exchange argument", if $(\mathrm{a1}, a2 \ldots an)$ is a solution, then for $(a1', a2' \ldots . an') = (\mathrm{a1}, a2 \ldots ai, 1, 1 \ldots ,1)$ is true that $a1' * a2' * \ldots * an' \leq a1' + a2' + \cdots + an'$

We will run a backtrack with parameters $MX - constraint\ for\ a[pos],\ pos - current\ position,\ sum - sum\ of\ already\ fixed\ a,\ prod - product\ of\ already\ fixed\ a$

Optimization:

If at some point we can limit $a1 + a2 + \cdots + an$ in some interval $[low, high]$ with no multiples of the product of the already fixed a, then we can stop generating in this branch.

Idea 1:

We generate all $a1 * a2 * \ldots * ai \leq 2 * n\ and\ ai \geq 2$ and fill the sequence with 1.

Scores 15 points.

Optimization of idea 1:

In idea 1 we have $low = sum + n - pos + 1\ and\ high = sum + n - pos + 1 + (greedy\ multiplication\ to\ 2 * n\ using\ only\ MX)$ due to the "exchange argument".

Scores 15 points.

Idea 2:

$a1$ is uniquely defined by the others, we will generate sequences with $a2 * a2 * a3 * \ldots * ai \leq 2 * n\ and\ ai \geq 2$ and by using the already fixed a we can find $a1$.

Scores 25 points.

Optimization of idea 2:

In idea 2 we have $low = a2 + sum + n - pos + 1 \; and \; high = a2 + sum + n - pos + 1 + (sum + n - pos + 1)/(prod - 1)$ due to the "exchange argument".

Additionally, due to the “exchange argument” we have $ai^i \leq n - i + ai * i$, i.e. $ai \leq (n + 1)^{\frac{1}{i}} + 1$

Scores 55 points.

Idea 3:

We will generate sequences $a3 * a3 * a3 * a4 \ldots * ai \leq 2 * n \; and \; ai \geq 2$

We reach an equation of the type $prod * a1 * a2 = sum\_new + a1 + a2$

$$a2 \leq a1 = \frac{sum\_new + a2}{prod * a2 - 1} => a2 \leq \frac{1 + (prod * sum\_new + 1)^{\frac{1}{2}}}{prod} => \sim O(\left(\frac{n}{prod}\right)^{\frac{1}{2}} - a3)$$

Scores 25 points.

Optimization 1 of idea 3:

In idea 3 we have $low = a3 + a3 + sum + n - pos + 1 \; and \; high = a3 + sum + n - pos + 1 + (sum + n - pos + 1 + a3)/(prod * a3 - 1)$ due to the "exchange argument".

Scores 35 points.

Optimization 2 of idea 3:

$low \leq prod * a1 * a2 = sum\_new + a1 + a2 \leq high$, i.e. we can can go through all t: $prod \; divides \; t$ and get a system of equations:

$a1 * a2 = \frac{t}{Prod} = P \; and \; a1 + a2 = t - sum\_new = S \;\; => a1 = S - a2; (S - a2) * a2 = P \;\; => -a2 * a2 + S * a2 - P = 0 => a2 * a2 + (-S) * a2 + P = 0; D = S * S - 4 * P => a1, a2 = (S \pm (D)^{1/2})/2$

$$O\left(\frac{high - low}{prod}\right) = O\left(\frac{\frac{sum + n - pos + 1 + a3}{prod * a3 - 1} - a3}{prod}\right) \sim O(\frac{n}{prod * prod * a3})$$

Each time we will apply the approach that has smaller O.

Why $S * S$ can be stored in $long \; long$? Using estimates:

$S \leq a3 + a1MX \leq 2 * a1MX \; by \; definition \; for \max a1 + a2$

$$a1MX = \frac{sum + n - pos + 1 + a3}{prod * a3 - 1} \sim \frac{n}{prod * a3}$$

$$O1 \geq O2 => \left(\frac{n}{prod}\right)^{\frac{1}{2}} \geq \left(\frac{n}{prod * prod * a3}\right) => n \leq (prod)^3 * a3 * a3 \leq (prod * a3)^3 =>$$

$$a1MX \leq n^{\frac{2}{3}} \ll 10^9$$

Scores 100 points.

It is possible to score more points with more criteria for pruning in the respective ideas.