3.Tìm tất cả các hàm số f N^*->N^* thỏa mãn:
$f^{f(n)}(m) \mid m+n+1$

Case 1: There is a v such that $f(v)=1$
Case 1.1:There is a u different from v such that $f(u)=1$
We have that f is bounded (plugging $n=u$ and $n=v$). We have $f(n)\mid n+v+1$ for all n, therefore for all large primes p, $f(p-v-1)=1$. Plugging $n=p-v-1$, $f(m)\mid m+p-v$ for all m and large primes p. Using Dirichlet theorem, it is easy to see that $f(m)=1$ or $f(m)=2$ for all m. If $f(1)=1$ then $f(n) \mid n+2$ for all n, therefore $f(2k+1)=1$. If $f(2)=1$ then $f(n)\mid n+3$, therefore $f(2k)=1$. **Thus f(n)=1 is a solution**
If $f(2)=2$ then $f^2(m)\mid m+3$, plugging $m=2$ we reach a contradiction.
If $f(1)=2$ and $f(2)=2$, we have $f^2(m)\mid m+2$ for all m, plugging $m=1$ we reach a contradiction.
The case $f(1)=2$ and $f(2)=1$ will be solved later.
Case 1.2: $f(v)=1$ if and only if $v=1$
We have $f(m) \mid m+v+1$ and $f^{f(m)-1}(1) \mid m+v+1$
For all largre primes we have $f(p-v-1)=p$ and $f^{p-1}(1) \mid p$
Case 1.2.1: f is injective
We have $f^{f(p-2)}(1) \mid p$
If the sequence $1,f(1),f(f(1)),f(f(f(1))),\dots$ not bounded $f^{p-1}(1)= f^{f(p-2)}(1)=p$ for all large primes p, thus $f(p-2)=p-1$. But $p-1 \mid p-2+v+1$, contradiction.
If the sequence $1,f(1),f(f(1)),f(f(f(1))),\dots$ bounded $f^{p-1}(1)=1$ for all large primes p. Thus the sequence has period 1 or 2. Thus $f(f(1))=1$. Therefore $f(1)=1$ or $f(1)=2$ and $f(2)=1$. The case $f(1)=2$ and $f(2)=1$ will be solved later.
Case 1.2.2: f is not injective
Consider $f(a)=f(b)=M$ then $f^M(n) \mid a-b$ for all n.
Consider the sequence, $1,f(1),f(f(1)),f(f(f(1))),\dots$ there must be two equal values therefore the sequence is bounded, therefore $f^{p-1}(1)=1$ for all primes p, therefore $f(1)=1$ or $f(1)=2$ and $f(2)=1$.
We will solve the case f(1)=2,f(2)=1 and f(x) ≠ 1 for all x ≠ 2. We have $f(m)\mid m+3$ and $f(f(m)) \mid m+2$. Plugging $m=3$, we have $f(3)\mid 6$ thus $f(3)=2,3$ or 6. If $(3)=3$, since $f(f(m)) \mid m+2$ we reach a contradiction. If $f(3)=6$ then $f(6)=5$ but $f(6)\mid 9$, contradiction. If $f(3)=2$ then $f(f(m)) \mid m+4$ therefore $f(f(2k+1))=1$ this mean $f(2k+1)=2$ for all k
The case $f(1)=2$, $f(2)=1$ and $f(x) \mid 2$ for all x can be solved similarly.
We conclude that f(2k+1)=2 for k, f(2k) | 2k+3 for all k and f(2)=1 is a solution
We will solve the case f(1)=1 and f(x) f(x) ≠ 1 for all x. We have $f(1)=1$, $f(m)\mid m+2$ therefore $f(3)=5$, $f(5)=7$. It is not possible for $f(2)=2$ therefore $f(2)=4$ thus $f^4(m) \mid m+3$, since $f(7)\mid 9$, if $f(7)=3$ then using $f^4(m) \mid m+3$, plugging $m=3$ we reach a contradiction, thus $f(7)=9$ and $f(9)=11$. However this is impossible because $f^4(m) \mid m+3$, plugging $m=3$ we reach a contradiction.

Case 2: $f(x) \neq 1$ for all x, f is injective because $f^{f(p-1-k)}(k)=p$ for all $k=1,2,\dots,p-2$ and large primes p . If f is not injective, there is a number c such that $f^{f(c)}(k)$ is bounded for all k (we have $f(c)=f(d)$ for some c and d, plugging $n=c$ and $n=d$ in the original equation), thus the sequence $1,f(1),f(f(1)),\dots$ is, bounded, but for large primes p, $f^{f(p-2)}(1)=p$, contradiction.
We prove that f is strictly increasing. Notice that for all large primes p, $f^{f(p-2)}(1)=p$, and $f^{f(p-1-k)}(k)=p$ since f is injective and $f(x) \neq 1$ for all x, we have

$f(p-1-k)<f(p-2)$ for all $k>1$ và $f^{f(p-2)-f(p-k-1)}(1)=k$ for all $1<k<p-1$. Since $f^{f(3)-f(1)}(1)=3$ and $f^{f(3)-f(2)}(1)=2$ and $f^{f(1)}(1)=3$ we have $f(3)=2f(1)$. Notice that $f^{f(2)}(1) \mid 4$ and $f^{f(1)}(2) \mid 4$. If $f^{f(2)}(1)=2$, since $f^{f(3)-f(2)}(1)=2$ we have that $f(3)=2f(2)$ but f is injective and $f(3)=2f(1)$, thus this case is impossible, thus $f^{f(2)}(1)=4$. If $f^{f(1)}(2)=2$ we have $f(3)=f(2)+f(1)$, however $f(3)=2f(1)$ thus this case is impossible. Thus $f^{f(1)}(2)=4$ and $f^{f(2)-f(1)}(1)=2$ or $f(2)=1.5f(1)$.
We have $f(2)=1.5f(1)$ and $f(3)=2f(1)$, therefore $f(1)<f(2)<f(3)$. Let p_k be the k-th prime number, we will prove that f is strictly increasing in $[1,p_k-2]$ for all k by induction.
Suppose f is strictly increasing in $[1,p_n-2]$ for some n. Notice that for all primes p,q and $p<q$ we have that $f(p-2)-f(p-k-1)=f(q-2)-f(q-k-1)$ for all $k=2,3,\ldots,p-2$. Also notice that for all $k>2$, we have that $p_{k+1}-p_k+1\leq p_k-2$ (If $p_k<29$, it is obviously true, if $p_{k>}30$ then by Nagell's theorem between 5/6 p and p there is at least one prime number) thus plugging $p=p_n,q=p_{n+1}$ and $k=2,3,\ldots,p_n-2$, by induction it is easy to see that f is strictly increasing in $[p_{n+1}-p_n+1,p_{n+1}-2]$, induction complete. We conclude that $f(n)>=n+1$ for all positive integers n, thus $f^{f(n)}(m)\geq m+n+1$, but $f^{f(n)}(m) \mid m+n+1$ equality happen if and only if $f(n)=n+1$.
Thus f(n)=n+1 is a solution.