3. Let a_1 be a positive integer and for all n>0, a_{n+1} is the largest odd divisor of $a_n + 2^n - 1$. Let r be a positive integer greater than 1, determine if there exists infinitely many pair (u,v) with u>v and and a_u =ra_v

First, note that the number of 1s in binary of a_n is non increasing, therefore it will become a constant for sufficient large n. Let m be the number of 1s in binary of a_n when n is large enough and WLOG m>1 since it is trivial when m=1.
Suppose $a_u = ra_v$, then since both a_u *and* a_v has the same number of 1s in binary, consider both sides in binary, and it will be easy to consider the sum of digit in base 2 of both sides, we observe that if the distance between 2 consecutive 1s of a_n is too large, then we will reach a contradiction, since S_2(LHS)=m and S_2(RHS)=mS_2(r).
Now for a large n, suppose we have a_n=$1 + 2^{t_1} + 2^{t_2} + \ldots + 2^{t_m}$. Now we wish to calculate a_{n+1}, but in order to do that, we need to know if $t_1 < n$ or not. Thus, we try several values of a_1 to compare $t_1, t_2, \ldots, t_m$ *and* n. And we can see that a_n<2^n. The proof is easy, since $a_{n+1} \leq \frac{a_n+2^n-1}{2}$
Now for large n, we have c=$1 + 2^{t_1} + 2^{t_2} + \ldots + 2^{t_m}$, then:
a_{n+1}=$1 + 2^{t_2-t_1} + 2^{t_3-t_1} + \ldots + 2^{t_m-t_1} + 2^{n-t_1}$
a_{n+2}=$1 + 2^{t_3-t_2} + 2^{t_4-t_2} + \ldots + 2^{t_m-t_2} + 2^{n-t_2} + 2^{n+1-t_2+t_1}$
a_{n+3}=$1 + 2^{t_4-t_3} + 2^{t_5-t_3} + \ldots + 2^{t_m-t_3} + 2^{n-t_3} + 2^{n+1-t_3+t_1} + 2^{n+2-t_3+t_2}$
…….
a_{n+m}=$1 + 2^{n-t_m} + 2^{n+1-t_m+t_1} + 2^{n+2-t_m+t_2} + \ldots + 2^{n+m-1-t_m+t_{m-1}}$
a_{n+m+1}=$1 + 2^{t_1+1} + 2^{t_2+2} + \ldots + 2^{n+m+t_m}$
Look at a_n and a_{n+m+1}, if we let d_n= the minimal distance between 2 consecutive 1s in binary of a_n, in other word, d_n=min($t_2 - t_1$,$t_3 - t_2$,…,$t_m - t_{m-1}$), then d_{n+m+1}>d_n for all large n, thus $\lim_{n->+\infty} d_n = +\infty$, thus S_2(ra_v)=mS_2(r) for all large v, and since u>v we have $S_2(a_u)$=m, thus $a_u \neq$ra_v for all large u,v.
If v is not large, then it is easy to see that $\lim_{n->+\infty} a_n = +\infty$, thus $a_u >$ra_v for all large u and small v. And our conclusion follows.