Bài toán: Đặt $A=\sum_{\substack{1\le i,j\le (p-1)/2\\ i+j>(p-1)/2}}\frac{1}{ij}$. Cmr A=0 (mod p)

Cm: Đặt $H_i=\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{i}$

Để ý $\binom{p-1}{i}\equiv(-1)^{i-1}pH_i+(-1)^i \pmod{p^2}$ nên $H_i\equiv\frac{\binom{p-1}{i}(-1)^{i-1}+1}{p} \pmod{p}$ (1)

Cũng để ý $\frac{\binom{p}{i}}{p}\equiv\frac{(-1)^{i-1}}{i} \pmod{p}$ (2)

Ta có: $A\equiv H^2_{(p-1)/2}-(\frac{1}{1}H_{(p-3)/2}+\frac{1}{2}H_{(p-5)/2}+\ldots+\frac{1}{\frac{p-3}{2}}H_1) \pmod{p}$

Sử dụng (1) và (2) ta được:

$$A\equiv H^2_{(p-1)/2}+\sum_{i=1}^{(p-3)/2}\frac{(\binom{p-1}{i}(-1)^{(p+1)/2}+(-1)^{(p-1)/2-i})\binom{p}{(p-1)/2-i}}{p^2} \pmod{p}$$

Để ý $\sum_{i=0}^{(p-1)/2}\binom{p-1}{i}\binom{p}{(p-1)/2-i}=\binom{2p-1}{(p-1)/2}$

Tiếp tục để ý $\sum_{i=0}^{b}\binom{a}{i}(-1)^i=\binom{a-1}{b}(-1)^b$ với mọi a,b ko âm.

Vậy ta có:

$$A\equiv H^2_{(p-1)/2}+\frac{(-1)^{(p+1)/2}(\binom{2p-1}{(p-1)/2}-\binom{p-1}{(p-1)/2}-\binom{p}{(p-1)/2})+((-1)^{(p-3)/2}\binom{p-1}{(p-3)/2}-1)}{p^2} \pmod{p}$$

$$\Rightarrow A\equiv H^2_{(p-1)/2}+\frac{(-1)^{(p+1)/2}(\binom{2p-1}{(p-1)/2}-2\binom{p-1}{(p-1)/2})-1}{p^2} \pmod{p}\ (4)$$

Đặt $B=\sum_{1\le i<j\le (p-1)/2}\frac{1}{ij}$

Để ý:

$H^2_{(p-1)/2}\equiv 2B \pmod{p}$ (5)

$\binom{2p-1}{(p-1)/2}(-1)^{(p+1)/2}\equiv -4p^2B+2pH_{(p-1)/2}-1 \pmod{p^3}$ (6)

$\binom{p-1}{(p-1)/2}(-1)^{(p+1)/2} \equiv -p^2B+pH_{(p-1)/2}-1 \pmod{p^3}$ (7)

Từ (4), (5), (6), (7) ta có $p^2A \equiv 0 \pmod{p^3}$ hay $A \equiv 0 \pmod{p}$ nên ta có ĐPCM.