N3. Find all function f N*->N* such that:
$f(xy)+f(x)f(y)+x+y$ is a prime for all integers x and y

Solution:
$f(0)^2+f(0)$ is a prime
We have $f(0)=1$ or $f(0)=-2$
If $f(0)=-2$:
$-2-2f(x)+x$ is a prime, therefore $-2f(x)+x-2=2$ for all even x.
$f(x)=(x-4)/2$ for all even x.
But it is impossible for $(xy-4)/2+(x-4)(y-4)/4+x+y$ to be a prime for all even x,y.
Therefore $f(0)=1$
$f(y)+y+1$ is a prime for y
$f(1)=0$ or $f(1)=-1$, notice that $f(-1)^2+f(1)-2$ is a prime and $f(-1)$ is a prime
Therefore $f(1)=0$ and $f(-1)^2-2$ is a prime
Main claim: For all even integers x,y satisfy $x<0,y<0$, at least one of $f(xy)+1+xy$, $f(y)+y+1$, $f(x)+x+1$ is equal to 2.
Important collary: for all even $x<0$, at least one of $f(x)+x+1$ or $f(x^2)+x^2+1$ is equal to 2
Consider an even $x<-1$ such that $f(x)+1+x$ is odd
We have $f(x^2)=1-x^2$
We have $f(x^3)+(1-x^2)f(x)+x+x^2$ is a prime. If $f(x^3)=1-x^3$ then $f(x)=1-x$, contradiction. Therefore $f(x^3)=2-x-x^2+(x^2-1)f(x)$
Therefore $f(x^4)=1-x^4$. Now with this, we induct: $f(x^{2k+1})$ is even and $f(x^{2k})=1-x^{2k}$ for all k.
Indeed, $f(x^{2k+1})+(1-x^{2k})f(x)+x^{2k}+x$ is a prime.
If $f(x^{2k+1})=1-x^{2k+1}$, by induction we have $f(x^{2k})=1-x^{2k}$, then it is easy to see that $1-x^{2k+1}+(1-x^{2k})f(x)+x^{2k}+x<2$ since $f(x)>1-x$, contradiction.
Therefore $f(x^{2k+1})$ is even and $f(x^{2k+2})=1-x^{2k+2}$ since $f(x)$ and $f(x^{2k+1})$ is even.
Induction complete.
We have $f(x^{2k+1})+(1-x^{2k})f(x)+x^{2k}+x$ is a prime, but it is even, therefore $f(x^{2k+1})+(1-x^{2k})f(x)+x^{2k}+x=2$ for all k.
Therefore $A(k)=(x^{2k}-1)f(x)-x^{2k}-x+3+x^{2k+1}$ is a prime for all k. Since $f(x)>1-x$, $\lim_{k->\infty} A(k)=+\infty$ and consider a large k, if $A(k)=p$ is a prime, consider $A(k+p-1)$, it is not a prime, contradiction.
Therefore we have $f(x)=1-x$ for all even $x<-1$.
Now consider a positive integer x, we have $1-xy+f(x)(1-y)+x+y=$ $y(1-f(x)-x)+f(x)+x+1$ is a prime for all even $y<0$. If $f(x)$ is not equal to $1-x$, we can choose an even $y<0$ such that $y(1-f(x)-x)+f(x)+x+1$ is not a prime.
Therefore $f(x)=1-x$ for all positive integers x. And finally, consider an odd $y<0$, we have $-xy+(1-x)f(y)+x+y$ is a prime for all even $x>0$. Similarly we have $f(y)=1-y$ for all odd $y<0$.