IMC Problems

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Problem 1.

We have $f''(7x+1) = f''(x)$ for all $x \in \mathbb{R}$, and we also have $f''(x) = f''((x-1)/7)$ for all $x \in \mathbb{R}$.

For any $x \in \mathbb{R}$ consider

$$x_n = \frac{x}{7^n} - \frac{1}{7^{n-1}} - \frac{1}{7^{n-2}} - \cdots - \frac{1}{7}$$

Then we have $f''(x) = f''(x_n)$ for all n. Letting $n \to \infty$, $f''(x)$ is constant for all $x \in \mathbb{R}$. Hence $f(x) = ax^2 + bx + c$ for some a, b, c. Finally, we get $b = \dfrac{1}{3}a, c = \dfrac{1}{36}a$.

Problem 2.

We have $AB^3 = B^3C + 2A$ and $B^3C = AB^3 + 2C$ hence $C = -A$

Thus $B^3 - I = I - ABA$ and $B^3 + I = 3I - ABA$. Multiplying these, we get $B^6 - I = 3I - 4ABA + ABA^2BA = 3I - 4ABA + AB^4A = 3I - 4(2I - B^3) + A^6 = -5I + 4B^3 + B^6$.

Therefore we have $B^3 = I$, thus $B^6 = A^6 = I$, as desired.

Problem 3.

We see that $P(x, y) = x^2 + y^2$ works. First, we see that the equality holds for all $x, y, z, t \in \mathbb{R}$, so we will just extend to $\mathbb{C}$. Thus we have $P(x, y)P(z, t) = P(xz - yt, xt + yz) \;\forall\; x, y, z, t \in \mathbb{C}$.

WLOG, P is not constant, then $x = y = 0$ gives us $P(0, 0) = 0$. P has zero when $x^2 + y^2 = 0$, so let's take $y = ix$ and $t = -iz$, we have $P(x, ix)P(z, -iz) = P(0, 0) = 0$. Hence, we must have $P(x, ix) = 0$ or $P(z, iz) = 0$.

WLOG $P(x, ix) = 0$, then $P(x,y) = (x^2+y^2)Q(x,y) + yR(x) + S(x)$. Letting $y = ix$ then $ixR(x) + S(x) = 0$, but R and S have real coefficients, we have $R(x) = S(x) = 0$, or $P(x,y) =)(x^2+y^2)Q(x,y)$. Repeating with $Q(x,y)$ we get $P(x,y) = (x^2+y^2)^n$ for some non negative integer n.

Problem 4.

We can either take the sum, or take the product of a_is. It seems that summation does not lead us to anywhere, so let's take the product.

By taking the product $a_i \cdot a_2 \dots a_{p-1}$, we have

$$\prod_{i=1}^{p-1}(i^{k-1}+1) \equiv 1 \pmod p$$

Let g to be the generator $\pmod p$ then it holds that

$$\prod_{i=1}^{p-2}(g^{(k-1)i}+1) \equiv 1 \pmod p \tag{1}$$

.

Let $k-1 = dx$ and $p-1 = dy$, then we have:

$$\prod_{i=1}^{p-2}(g^{(k-1)i}+1) \equiv \prod_{i=1}^{y-1}(g^{dxi}+1)^d \equiv \prod_{i=1}^{y-1}(g^{di}+1)^d \pmod p$$

In the other hand, in $\mathbb{F}_p[X]$ we have

$$X^y - 1 = \prod_{i=0}^{y-1}(X - g^{di})$$

Hence in (1), LHS is equal to $(1-(-1)^y)^d \pmod p$, thus we must have $y \equiv 1 \pmod 2$ and thus $2^d \equiv 1 \pmod p$. Thus, we conclude that $a_2 \equiv 4 \pmod p$.

Problem 6.

The answer is no.

We consider the map: $2^a \cdot 3^b \mapsto a + b \cdot i$. Then the problem can be reformulated as follow:

Initially we have the matrix $A = \begin{pmatrix} 1 & i \\ 1 & 2 \end{pmatrix}$. Each step, we choose a row or a column of A and add of subtract the chosen row entry-wise by the other row. Can we obtain $A = \begin{pmatrix} 1 & 2 \\ 1 & i \end{pmatrix}$?
.
Suppose it is possible. It is trivial to see that the operation does not change the determinant of A. The first and the last state of A have different determinant, contradiction.

Problem 7.

Only able to solve the first direction. The answer is $\alpha = \dfrac{1}{e-1}$.
Let us find a function f such that $f(0) = 0, f(1) = 1$ and

$$f(x)' = f(x) + \alpha \ \forall \ x \in \mathbb{R}$$

It is easy to check such a function has the form $f(x) = ae^x + b$. Hence we get $a = 1/(e-1)$ and $b = -1/(e-1)$. How choosing $f(x)$ we get $\alpha = 1/(e-1)$.

Problem 8.

Let $N = \dbinom{n}{2}$ and let $d_1, d_2, \ldots, d_N$ denote the $d(u,v)$s. Then the problem is equivalent to

$$\sum_{1 \leq i < j \leq N} \frac{(d_i - d_j)^2}{d_i d_j} \geq \frac{(n-1)^2(n-2)}{4}$$

Now we wish to bound LHS. The idea is to inspect the length between the vertices. First we consider $d((u,v)) = 1$, this happen if and only if (u,v) forms an edge and thus there are exactly $n-1$ such pairs (u,v).

WLOG, $d_1, \ldots, d_{n-1} = 1$, thus $d_i \geq 2$ for all $i \geq n-1$. Hence we have that LHS $\geq \sum_{i=1}^{n-1} \sum_{j=n}^{N} \dfrac{(2 \cdot (1)_j - 1 \cdot (1)_i)^2}{2 \cdot (1)_j} = \dfrac{(n-1)^2(n-2)}{4}$, as desired.

In addition, it is possible choose choose a tree for $d_i = 2$ for all $i \geq n-1$, which make equality happens, thus the inequality is tight.

Problem 10.

The answer is yes.
Choose a fixed and large inter k and consider a sufficient large N. Let $a_n = \sum_{i=1}^{n} \dfrac{n!}{i!}$

Let $\mathcal{A} = \{p \in \mathbb{P} \mid \exists N \le m < n \le N+k,\ p \text{ divides } 1+m+m\cdot(m+1)+\cdots+m\cdot(m+1)\dots(n-2)\}$

The main claim is that: $\forall\ p\ \notin\ \mathcal{A}$, then at most one of $a_N, \dots, a_{N+k}$ is divisible by p.

Indeed, suppose there are $N \le m < n < N+k$ s.t $p \mid a_m$ and $p \mid a_n$. then we have:

$p \mid a_m$ and $p \mid 1+m+m\cdot(m+1)+m\cdot(m+1)\dots(n-1)a_m$. Hence $p \mid 1+m+m\cdot(m+1)+m\cdot(m+1)+\dots(n-2)$ for some $N < m < n \le N+k$, contradiction.

Let us count $|\mathcal{A}|$. Since there are at most k^2 pairs (m,n) with $N < m < n < N+k$ and each pair give at most $O(log(n)/log(log(n)))$ primes divisors. Hence $|\mathcal{A}| = O(log(n)/log(log(n)))$.

Now we explain why each pair (m,n) give at most $O(log(n)/log(log(n)))$ prime divisors. It is well known that $\omega(M) = O(log(M)/log(log(M)))$ where $\omega(M)$ is the number of prime divisors of M. Since, each pair (m,n) gives us a value T in the form $T = 1+m+m\cdot(m+1)+m\cdot(m+1)+\dots(n-1)$, which is bounded by $O(N^k)$, hence the number of its prime divisor is also bounded by $O(log(T)/log(log(T))) = O(log(N)/log(log(N)))$

Finally, consider $\frac{a_N}{N!}, \frac{a_{N+1}}{(N+1)!}, \dots, \frac{a_{N+k}}{(N+k)!}$. For each $p \notin \mathcal{A}$ and $p \le N$, at most one value a_i is divisible by p, and for each j s.t a_j is not divisible by p, then $g(j)$ must be divisible by $p^{v_p(n!)}$, hence at least k values $g(j)$ must be divisible by $p^{v_p(N!)}$. Thus we have: $g(N)g(N+1)\dots g(N+k) \ge \frac{N!^k}{\prod_{p\in\mathcal{A}} p^{v_p(N!)}} \ge \frac{N!^k}{2^{N\cdot O(log(N)/log(log(N))}}$. Hence, one of $g(N), g(N+1), \dots, g(N+k)$ must be greater than $\frac{N!^{k/(k-1)}}{N^{o(N)}}$, as desired.