2 Fist, in the CSV file, we consider $(x_i; y_i)$ (mod 6)
Let $A(u,v) = \{(x,y) \in CSV \mid x \equiv u \pmod 6$ and $y \equiv v$ (
We will run a Python script to find $|A(u,v)|$ for $0 \leq u$,
The script will be attached later. We obtain the following table for $|A(u,v)|$:

v \ u	0	1	2	3	4	5
0	8	9	11	2	16	8
1	7	22	7	3	6	9
2	25	5	8	5	7	4
3	9	18	5	10	4	7
4	10	7	9	2	20	6
5	7	7	4	24	9	4

We try to choose a small set S that contains all 90 v
Note that if we choose $(x;y) \in A(u,v)$, then we will choose $(x';y') \in A(u,v')$ for $v' \neq v$. Therefore, for eac
$u \in \{0;1;..;5\}$, we choose exacly one v, and let S cor
all elements in $A(u,v)$.
Now we just need to choose v for each u. We will tr
the easiest way. For each u, we choose v such that $|A(u,v)|$ is maximal. For example, when $u=0$, we
$v = 2$, when $u = 1$, we choose $v = 1$.
Let mod6list = {(0;4); (1;1); (2;0); (3;~~1~~); (4;4)

then for each u, mod6list contain (u,v) such that $|A(u,v)|$ is maximal.

$\mathrm{od}6)\}$ Our set S will be: $S = \{(x,y) \in CSV \mid (x,y) \in \text{mod6list} \pmod 6\}$

$\leq 5.$

Note that $\forall (x,y)$ and $(x',y') \in S$ we have:

$2 \mid x'-x \Rightarrow 2 \mid y'-y$ and $3 \mid x'-x \Rightarrow 3 \mid y'-y$

This increases the chance of S containing all the 90 correct values.

The script that print all elements in S will be attached later. Now, we have $|S| = 125$ and we need to choose 2 monic polynomials $A(x)$ and $B(x)$ such that $y_i = \dfrac{A(x_i)}{B(x_i)}$ for at least 90 indices of i in S in $\mathbb{Z}_{2022}$. I will use A, B instead of f, g.

es. Consider $A(x)$, $B(x)$ in $\mathbb{Z}_{337}[x]$. We will use something similar to Berlekamp-Welch algorithm to restore A and B in $\mathbb{Z}_{337}[x]$

ns To do so, let's find 2 polynomials A' and B' of degree 51 and monic such that $y_i = \dfrac{A'(x_i)}{B'(x_i)}$ $\forall$ $(x_i; y_i) \in S$ in $\mathbb{Z}_{337}$.

ose Let $P(x) = \prod_{\substack{(u,v)\in S \\ v \neq \frac{A(u)}{B(u)}}} (x-u) . A(x)$

$,3)\}$ $Q(x) = \prod_{\substack{(u,v)\in S \\ v \neq \frac{A(u)}{B(u)}}} (x-u) . B(x)$

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Note that $\frac{P(x)}{Q(x)} = \frac{A(x)}{B(x)}$ and $y_i Q(x_i) = P(x_i)\ \forall$ $(x_i; y_i) \in S$ in $\mathbb{Z}_{337}$.

We will claim that: If there exists such A, B, then $\frac{A'(x)}{B'(x)} = \frac{P(x)}{Q(x)}$ in $\mathbb{Z}_{337}[x]$

Proof of claim:

First, note that $\#\{(u,v) \in S \mid v \neq \frac{A(u)}{B(u)}$ in $\mathbb{Z}_{337}\}$ thus deg P and deg $Q \leq 51$.

Suppose $\frac{A'(x_i)}{B'(x_i)} = \frac{P(x_i)}{Q(x_i)} = y_i\ \forall (x_i; y_i) \in S$. Then $A'(x_i)Q(x_i) - B'(x_i)P(x_i) = 0$ in $\mathbb{Z}_{337}\ \forall (x_i; y_i) \in S$. However, the degree of $A'(x)Q(x) - B'(x)P(x)$ is at most 102, thus by Lagrange's theorem, $A'(x)Q(x) = B'$ in $\mathbb{Z}_{337}[x]$. In the other hand, we can choose $A'($ $= P(x)$ and $B'(x) = Q(x)$, then it is always possible find $A'(x)$ and $B'(x)$. Thus we proved our claim.

Now, to find $A'(x)$ and $B'(x)$, note that for each $(x_i; y_i) \in S$ give us an equation $y_i = \frac{A'(x_i)}{B'(x_i)}$. Since there are 125 equation and 102 variables (the coefficients of A' and B'), we can easily find A and $B'(x)$ by solving a system of equation.

The scipt for finding $A'(x)$ and $B'(x)$ will be atta

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After finding $A'(x)$ and $B'(x)$, we will factor them and find $A(x)$ and $B(x)$ in $\mathbb{Z}_{337}[x]$

The result is :

$A(x) = x^{16} + 30x^{15} + 262x^{14} + 35x^{13} + 186x^{12} + 218x^{11} + 129x$
$229x^{8} + 184x^{7} + 336x^{6} + 70x^{5} + 145x^{4} + 182x^{3} + 79x^{2} + 146x +$

$B(x) = x^{16} + 272x^{15} + 36x^{14} + 76x^{13} + 57x^{12} + 296x^{11} + 316x$
$165x^{8} + 42x^{7} + 178x^{6} + 125x^{5} + 140x^{4} + 239x^{3} + 87x^{2} + 104x +$

This is the resutt of $A(x)$ and $B(x)$ in $\mathbb{Z}_{337}[x]$
to find $A(x)$ and $B(x)$ in $\mathbb{Z}_{2022}[x]$, we need to find $A(x)$ and $B(x)$ in $\mathbb{Z}_2[x]$ and $\mathbb{Z}_3[x]$

In $\mathbb{Z}_2[x]$, we can choose $B(x) = x^{15}(x+1)+1$, and in $\mathbb{Z}_3[x]$, we can choose $B(x) = x^{14}(x^2-1)+1$

In $\mathbb{Z}_2[x]$, we can choose $A(x) = x^{15}(x+1)+x$ and in $\mathbb{Z}_3[x]$, we can choose $A(x) = x^{14}(x^2-1)+x^2-x+1$

The reason we choose A and B like that in $\mathbb{Z}_2[x]$ ar $\mathbb{Z}_3[x]$ is because of considering (mod 6) in the list mod6list and all $(x,y) \in S$ are in mod6list (mod 6).

Ve have chosen $A(x)$ and $B(x)$ in $\mathbb{Z}_2[x]$, $\mathbb{Z}_3[x]$ and ow we just use CRT to find $A(x)$ and $B(x)$ in $\mathbb{Z}_{20}$

he script for finding $A(x)$ and $B(x)$ will be attac

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The final result of $A(x)$ and $B(x)$ in $\mathbb{Z}_{2022}[x]$ is:

$$A(x) = x^{16} + 1041x^{15} + 1610x^{14} + 372x^{13} + 186x^{12} + 1566x^{11} + 1140x^{10} + 1734x^{9} + 1914x^{8} + 858x^{7} + 336x^{6} + 744x^{5} + 1830x^{4} + 1530x^{3} + 1090x^{2} + 1157x + 1912$$

$$B(x) = x^{16} + 609x^{15} + 710x^{14} + 750x^{13} + 1068x^{12} + 1644x^{11} + 990x^{10} + 456x^{9} + 1176x^{8} + 42x^{7} + 852x^{6} + 462x^{5} + 1488x^{4} + 576x^{3} + 1098x^{2} + 1452x + 679.$$

The polynomial $A(x)$ and $B(x)$ satisfy:

$y = \dfrac{A(x)}{B(x)}$ for at least 90 values of (x, y) in CSV file.

We conclude that we can choose:

$f(x) = A(x)$ and $g(x) = B(x)$ in $\mathbb{Z}_{2022}[x]$

Note: the script for finding $|A(u,v)|$ and all elements of S will be in process_data_mod_6.py

the script for finding $A(x)$ and $B(x)$ in $\mathbb{Z}_{337}[x]$ will be in find_A(x)_and_B(x)_in_Z337[x].sage.

similarly for finding $A(x)$ and $B(x)$ in $\mathbb{Z}_{2022}[x]$