4. Find all function f N->N such that:
f(0)=0
There is a real number z such that $f(n^{2022})<n^{2022}+z$ for all n ∈ N
For all a,b,c ∈ N, we have $af^b(a)+bf^c(b)+cf^a(c)$ is a square

Solution:
We have af(a)+1 is a square for all a **(1)**
We have $3af^a(a)$ is a square for all a **(2)**
We have $af(a)+f^c(1)+cf^a(c)$ is a square for all c,a **(3)**
We have $cf^a(c)+a^2$ is a square for all a,c **(4)**

Case 1: There exists a u such that $f^u(1)=0$
Then for all large a and b we have $af^b(a)+bf(b)$ is a square.
For a large a, if f(a)>0 then af(a)+f(1) and af(a)+1 is a square then f(1)=1, but then 3 is a square, contradiction.
Then f(a)=0 for all large a. Now from **(3)** consider a large c we have af(a) is a square for all a, but from **(1)** we have f(a)=0 for all a.
We conclude that in case 1 the only solution is f(a)=0 for all a.

Case 2: $f^u(1)\neq 0$ for all u. Let $S=\{p^{2022} \mid p\in \mathbb{p}\}$
Fix a positive integer a.

Case 2.1: $f^a(c)=0$ for a large c∈S.
Then from **(3)** $bf(b)+f^c(1)$ is a square for all b>a and a large c∈S. For all b>>c, if f(b) is not equal to 0, then from **(1)** we have $f^c(1)=1$, since c is large enough we have f(1)=1 and we reach a contradiction. Therefore f(b)=0 for all large b, making f bounded, but we have $a^2+f^a(1)$ is a square for all a, then $f^a(1)$ is unbounded, contradiction.

Case 2.2: For all large c∈S, $f^a(c)\neq 0$
We have cf(c)+1 is a square for all c∈S. For a large c, if f(c)=0, then it is **case 2.1**. Otherwise, we have f(c)=c-2 or f(c)=c+2 for all large c∈S.
If f(a)=0 then $cf(c)+f^a(1)$ is a square for all large c∈S, thus $f^a(1)=1$, but we also have $cf(c)+f(1)+f^c(1)$ is a square for all large c ∈ S, thus $f(1)+f^c(1)=1$, contradiction.

Case 2.2.1: f(c)=c−2 for all large c∈S
We have $c(c-2)+f^c(1)+f(1)$ and $c^2+f^c(1)$ is a square for all large c∈S. We will prove that
$c(c-2)+f^c(1)+f(1) > (\sqrt{c^2+f^c(1)}-2)^2$
This is equivalent to $f(1)+4\sqrt{c^2+f^c(1)}>2c+4$ which is true for all large c∈S
Thus $c(c-2)+f^c(1)+f(1) = (\sqrt{c^2+f^c(1)}-1)^2$ for all large c∈S
This is to $2\sqrt{c^2+f^c(1)}=2c+1-f(1)$
Thus $f^c(1)=c(1-f(1))+\frac{1}{4}(1-f(1))^2$ for all large c∈S
Let $M=\frac{1}{2}(1-f(1))$ then M<0 since f(1)≠1
From **(3)** we have $af(a)+2Mc+M^2+cf^a(c)$ is a square for all c∈S
We have $cf^a(c)+a^2$ is a square for all c∈S, thus $cf^a(c)+a^2=(k_c c\pm a)^2$ for all c∈S

Case 2.2.1.1: $cf^a(c)+a^2=(k_cc+a)^2$ for infinitely many $c\in S$
Then $f^a(c)=k_c^2c+2ak_c$ for infinitely many $c\in S$ and $k_c>0$.
We have $g(c)=af(a)+M^2+k_c^2c^2+2ak_cc+2Mc$ is a square for infinitely many $c\in S$. It is easy to see that if $k_c>M$ then $(k_cc+a-1)^2<g(c)<(k_cc+a)^2$ therefore $k_c\leq M$ for infinitely many $c\in S$. Thus there are infinitely many $c\in S$ such that k_c attain the same value, let it be $T\leq M$.
We have $g(c)= af(a)+M^2+T^2c^2+2aTc+2Mc$ for infinitely many c, thus g(c) is a square of a polynomial in c, therefore we have $aT+M=\pm T\sqrt{af(a)+M^2}$
Thus for each positive integer a, there is a positive $T\leq M$ such that$(af(a)+M^2)T^2=a^2T^2+2aTM+M^2$, it is easy to see that if a is large enough, we have T=1 and $f(a)=a+2M$.
Plugging $a\in S$ we have $M=-1$. Now for a=1 we have T=1 and $M=-1$, thus f(1)=3 and $3+1=1+2(-1)+1$, contradiction.

Case 2.2.1.2: $cf^a(c)+a^2=(k_cc-a)^2$ for infinitely many $c\in S$
Then $f^a(c)=k_c^2c-2ak_c$ for infinitely many $c\in S$.
$af(a)+M^2+k_c^2c^2-2ak_cc+2Mc$ is a square for many $c\in S$. Similarly, for all a there is a positive $T\leq M$ such that $(af(a)+M^2)T^2=a^2T^2-2aTM+M^2$, if a is large enough we have T=1 and $f(a)=a-2M$. Plugging $a\in S$ then M=1, thus $f(1)<0$, contradiction.
Thus in case 2.2.1 we have no such function.

Case 2.2.2: f(c)=c+2 for all infinitely many $c\in S$
Similarly we have $c(c+2)+f^c(1)+f(1) < (\sqrt{c^2+f^c(1)}+2)^2$
Thus $c(c+2)+f^c(1)+f(1) = (\sqrt{c^2+f^c(1)}+1)^2$ for infinitely many $c\in S$
This is equivalent to $2\sqrt{c^2+f^c(1)}=2c+f(1)-1$
Thus $f^c(1)=c(f(1)-1)+\frac{1}{4}(f(1)-1)^2$ for infinitely many $c\in S$
Let $M=\frac{1}{2}(f(1)-1)$ then $M>0$ since $f(1)\neq 1$
From **(3)** we have $af(a)+2Mc+M^2+cf^a(c)$ is a square for infinitely many $c\in S$
We have $cf^a(c)+a^2$ is a square for all $c\in S$, thus $cf^a(c)+a^2=(k_cc\pm a)^2$ for infinitely many $c\in S$

Case 2.2.2.1: $cf^a(c)+a^2=(k_cc+a)^2$ for infinitely many $c\in S$
Do exacly what we did in **Case 2.2.1.1,** we see that for each positive integer a, there is a positive $T\leq M$ such that $(af(a)+M^2)T^2=a^2T^2+2aTM+M^2$, plugging a large a such that f(a)=a+2, we have T=1 and M=1, thus f(a)=a+2 for all a.

Case 2.2.2.2: $cf^a(c)+a^2=(k_cc-a)^2$ for infinitely many $c\in S$
Do exacly what we did in **Case 2.2.1.1,** we see that for each positive integer a, there is a positive $T\leq M$ such that $(af(a)+M^2)T^2=a^2T^2-2aTM+M^2$, plugging a large a such that f(a)=a+2, we have T=1 and $M=-1$, thus $f(1)=-3$, contradiction.
Thus in case 2.2.2 we have f(a)=a+2 for all a.
We conclude that in case 1 the only solution is f(a)=a+2 for all a.