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Trung Kien Pham

Gruppe 4

Blatt 1 · Mathematische Grundlagen

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1	2
2	2,5
3	1,5
4	3
5	2
6	2
7	2,5
8	2
9	2,5
10	4,5
Σ	26,5/25

HA 1:

(a) $n \mapsto A(n) = n^2$

$S \to T = \{0, 1, 4\}$

Die Abbildung $A: S \to T$ ist surjektiv ✓

(b) $n \mapsto B(n) = \sqrt{n}$

$T \to U = \{0, 1, 2\}$ ✓

(c) $C = B \circ A : n \mapsto B(A(n)) = \sqrt{n^2} = |n|$ ✓

(d) A ist surjektiv

B ist bijektiv

C ist surjektiv ✓

2/2

HA 2

(a) $+: \mathbb{Z}_5 \times \mathbb{Z}_5 \to \mathbb{Z}_5 \quad (p, p') \mapsto p + p' \equiv (p + p') \bmod 5$

Verknüpfungstabelle für Gruppe $(\mathbb{Z}_5, +)$

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

✓

Das neutrale Element von $(\mathbb{Z}_5, +)$ ist 0 ✓

Das Inverse von $n \in \mathbb{Z}$ ist $-n$. $5-n$

(b) $\circ: R_{72} \times R_{72} \to R_{72} \quad (r(\phi), r(\phi')) \mapsto r(\phi) \cdot r(\phi') \equiv r(\phi + \phi')$

Verknüpfungstabelle für Gruppe $(R_{72}, \circ)$

∘	0	72	144	216	288
0	0	72	144	216	288
72	72	144	216	288	0
144	144	216	288	0	72
216	216	288	0	72	144
288	288	0	72	144	216

✓

2,5/3

Das neutrale Element ist $r(0)$

Das Inverse von $r(\phi)$ ist $r(360° - \phi)$ ✓

(c) Abbildung $T: \mathbb{Z}_5 \to R_{72}$

$a \mapsto T(a) = a \cdot 72° \quad (a \in \mathbb{Z}_5)$

T ist homomorph: $T(a+b) = a\cdot 72^\circ + b\cdot 72^\circ = T(a) + T(b)$ — Trung Kien Pham, Gruppe 4

T ist bijektiv: $T(a) = T(b) \Rightarrow a \equiv b \bmod 5 \Rightarrow$ injektiv

$\exists a \in \mathbb{Z}_5 \Rightarrow T(a) = a \cdot 72^\circ \Rightarrow$ surjektiv

$\Rightarrow$ T ist bijektiv homomorph

$\Rightarrow (\mathbb{Z}_5, +) \cong (R_{72}, \circ)$

(d) Wie (c) bewiesen. $(\mathbb{Z}_n, +) \cong (R_{360^\circ/n}, \circ)$

HA3

1,5/2

(a) $123 \xrightarrow{[132]} 132$ ungerade $[132] = [132]$

(b) $123 \xrightarrow{[321]} 321 \xrightarrow{[132]} 231$ gerade $[231] = [132] \circ [321]$

(c) $1234 \xrightarrow{[3214]} 3214 \xrightarrow{[1432]} 3412$ gerade $[3412] = [1432] \circ [3214]$

(d) $1234 \xrightarrow{[3214]} 3214 \xrightarrow{[1432]} 3412 \xrightarrow{[2134]} 3421$ ungerade $[3421] = [2134] \circ [1432] \circ [3214]$

(e) $12345 \xrightarrow{[15324]} 15324 \xrightarrow{[13245]} 15234$ ~~gerade~~ $[15234] = [13245] \circ [15324]$

HA4

(a) $(z+i)^2 = z^2 + 2zi - 1$
$= x^2 - y^2 + 2ixy + 2xi - 2y - 1$
$= (x^2 - y^2 - 2y - 1) + i \cdot 2x(y+1)$

(b) $$\frac{z}{z+1} = \frac{x + i\cdot y}{(x+1) + i\cdot y} = \frac{(x + i\cdot y)(x + 1 - i\cdot y)}{(x+1)^2 + y^2}$$
$$= \frac{x(x+1) + y^2 + i[y(x+1) - xy]}{(x+1)^2 + y^2}$$
$$= \frac{x^2 + y^2 + x}{x^2 + y^2 + 2x + 1} + i \cdot \frac{y}{(x+1)^2 + y^2}$$

(c) $$\frac{\bar{z}}{z - i} = \frac{x - i\cdot y}{x + i(y-1)} = \frac{(x - i\cdot y)[x - i(y-1)]}{x^2 + (y-1)^2}$$
$$= \frac{x^2 - y(y-1) - i[x(y-1) + xy]}{x^2 + (y-1)^2}$$
$$= \frac{x^2 - y^2 + y}{x^2 + (y-1)^2} - i \cdot \frac{2xy - x}{x^2 + (y-1)^2}$$

3/3

HA5

$z_1 = \frac{1}{\sqrt{8}}(1+i) = \frac{1}{\sqrt{8}} \cdot \sqrt{2} \cdot e^{i\cdot\frac{\pi}{4}} = \frac{1}{2} e^{i\cdot\frac{\pi}{4}}$

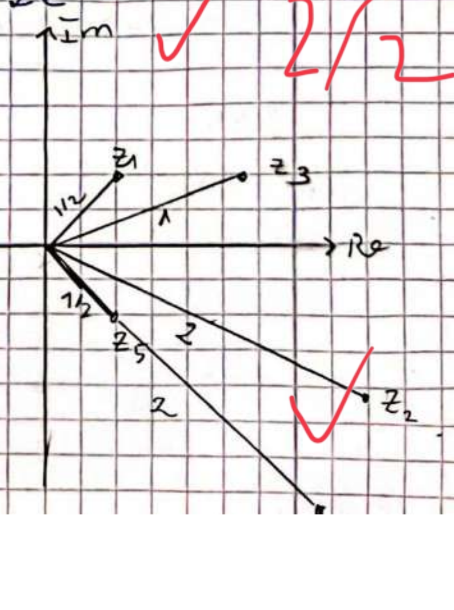
$z_2 = \sqrt{3} - i = 2\left(\cos\left(-\frac{\pi}{6}\right) + \sin\left(-\frac{\pi}{6}\right) i\right) = 2e^{-i\cdot\frac{\pi}{6}}$

2/2

$z_3 = z_1 \cdot z_2 = e^{i\cdot\frac{\pi}{12}}$

$z_5 = \bar{z}_1 = \frac{1}{2} e^{-i\cdot\frac{\pi}{4}}$

$z_4 = \frac{1}{z_1} = 2e^{-i\frac{\pi}{4}}$



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HA6

(a) $\cosh^2 x - \sinh^2 x$
$= \frac{1}{4}(e^x + e^{-x})^2 - \frac{1}{4}(e^x - e^{-x})^2$
$= \frac{1}{4}(e^{2x} + e^{-2x} + 2 - e^{2x} - e^{-2x} + 2)$
$= 1$ ✓

(b) $\frac{d}{dx}(\sinh x) = \frac{1}{2}\{\frac{d}{dx}(e^x) - \frac{d}{dx}(e^{-x})\}$
$= \frac{1}{2}(e^x + e^{-x}) \equiv \cosh x$ ✓

(d) $\frac{d}{dx}(\tanh x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)$
$= \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{(e^x + e^{-x})^2}$ ✓
$= 1 - \tanh^2 x = \operatorname{sech}^2 x$

(c) $\frac{d}{dx}(\cosh x) = \frac{1}{2}\frac{d}{dx}(e^x + e^{-x})$ ✓
$= \frac{1}{2}(e^x - e^{-x}) \equiv \sinh x$

2/2

(e) $\frac{d}{dx}(\coth x) = \frac{d}{dx}\left(\frac{e^x + e^{-x}}{e^x - e^{-x}}\right)$
$= \frac{(e^x - e^{-x})^2 - (e^x + e^{-x})^2}{(e^x - e^{-x})^2}$ ✓
$= 1 - \coth^2 x = -\operatorname{csch}^2 x$

HA7

(a) $f(x) = \sqrt[3]{x^2} = x^{2/3} \Rightarrow f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$ ✓

(b) $f(x) = \frac{x}{(x^2+1)^{1/2}} \Rightarrow f'(x) = \frac{(x^2+1)^{1/2} - x \cdot 2x \frac{1}{2}(x^2+1)^{-1/2}}{x^2+1}$
$f'(x) = \frac{1}{(1+x^2)^{3/2}}$ ✓

2,5/2

(c) $f(x) = -e^{1-x^2} \Rightarrow f'(x) = 2x\,e^{1-x^2}$ ✓

(d) $f(x) = 2^{x^2} \Rightarrow f'(x) = 2x \cdot \cancel{\log} \ln 2 \cdot 2^{x^2} = 2^{x^2+1} x \ln 2$ ✓

(e) $f(x) = 2\frac{\sqrt{\ln x}}{x} \Rightarrow f'(x) = 2 \cdot \frac{\frac{1}{x}\cdot\frac{1}{2\sqrt{\ln x}} \cdot x - \sqrt{\ln x}}{x^2}$
$= \frac{1 - 2\ln x}{x^2\sqrt{\ln x}}$ ✓

(f) $f(x) = \ln\sqrt{x^2+1} \Rightarrow f'(x) = \frac{1}{\sqrt{x^2+1}} \frac{2x}{2\sqrt{x^2+1}} = \frac{x}{x^2+1}$ ✓

HA8

(a) $y = \operatorname{arcsinh} x \Leftrightarrow x = \sinh y$
$\Leftrightarrow 1 = \frac{d}{dy}(\sinh y) \cdot \frac{dy}{dx}$
$\Leftrightarrow 1 = \cosh y \cdot \frac{dy}{dx}$
$\Leftrightarrow 1 = \sqrt{1 + \sinh^2 y} \cdot \frac{dy}{dx}$
$\Leftrightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}}$ ✓

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(b) $y = \operatorname{arccosh} x \Leftrightarrow x = \cosh y$

$\Leftrightarrow 1 = \sinh y \cdot \frac{dy}{dx}$

$\Leftrightarrow \frac{dy}{dx} = \frac{1}{\sinh y} = \frac{\pm 1}{\sqrt{x^2-1}}$ ✓

(c) $\frac{d}{dx}\, y = \operatorname{arctanh} x \Leftrightarrow x = \tanh y = \cancel{\frac{\sinh x}{\cos}}$

$\Leftrightarrow 1 = (1-\tanh^2 y)\cdot \frac{dy}{dx}$

$\Leftrightarrow \frac{dy}{dx} = \frac{1}{1-x^2}$

✓ GRAPH

✓ 2/2

MA 9

(a) $I(z) = \int_0^z x\sin(2x)\,dx$

$= \frac{1}{4}\int_0^{2z} u\sin(u)\,du = -\frac{1}{4}\int_0^{2z} u\,d(\cos u) \quad (u=2x)$

$= \frac{-1}{4}\left(\cancel{u\sin(u)}\; u\cos u\Big|_0^{2z} - \int_0^{2z}\cos u\,du\right)$

$= -\frac{1}{4}\left(2z\cos 2z \cancel{-2z} - \sin 2z\right)$ ✓

$= \frac{1}{4}(\sin 2z - 2z\cos 2z)$

(b) $I(z) = \int_0^z x^2\cos(2x)\,dx = \frac{1}{2}\int_0^z x^2\,d(\sin 2x)$

$= \frac{1}{8}\int_0^{2z} u^2\,d(\sin u) \quad (u=2x)$

$= \frac{1}{8}\left(u^2\sin u\Big|_0^{2z} - \int_0^{2z}\sin u \cdot 2u\,du\right)$

$= \frac{1}{8}\{4z^2\sin 2z - 2(\sin 2z - 2z\cos 2z)\}$

$= \frac{1}{8}(4z^2\sin 2z - 2\sin 2z + 4z\cos 2z)$ ✓

$= \frac{1}{4}[(2z^2-1)\sin 2z + 2z\cos 2z]$

(c) $I(z) = \int_0^z x\ln x\,dx = \cancel{\int_0^z x\,d\left(\frac{1}{x}\right)}\; \frac{1}{2}\int_0^z \ln x\,d(x^2)$

$= \cancel{x - \int_0 \frac{1}{x}\,dx}\;\; \frac{1}{2}\left(x^2\ln x\Big|_0^z - \int_0^z x^2\frac{1}{x}\cdot dx\right)$

$= \frac{1}{2}\left(z^2\ln z - \frac{1}{2}z^2\right)$ ✓

$= z^2\frac{1}{4}(2\ln z - 1)$

(d) $I(z)^{n\neq -1} = \int_0^z x^n\ln x\,dx = \frac{1}{n+1}\int_0^z \ln x\,d(x^{n+1})$

$= \frac{1}{n+1}\left(x^{n+1}\ln x\Big|_0^z - \int_0^z x^{n+1}\cdot\frac{1}{x}\,dx\right)$

$= \frac{1}{n+1}\left(z^{n+1}\ln z - \frac{1}{n+1}z^{n+1}\right)$

$= \frac{z^{n+1}[(n+1)\ln z - 1]}{(n+1)^2}$ ✓

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(e) $I(z) = \int_0^z \cos^2 x\,dx$

$= \frac{1}{2}\int_0^z (\cos 2x + 1)\,dx$

$= \frac{1}{4}\int_0^{2z} (\cos u + 1)\,du \quad (u = 2x)$

$= \frac{1}{4}(\sin 2z + 2z)$

$= \frac{1}{2}(\sin z \cos z + z)$

By Parts $\Rightarrow -0{,}5$P

(f) $I(z) = \int_0^z \cos^4 x\,dx$

$= \frac{1}{4}\int_0^z (\cos 2x + 1)^2\,dx$

$= \frac{1}{4}\int_0^z (\cos^2 2x + 1 + 2\cos 2x)\,dx$

$= \frac{1}{4}\left[\int_0^z \frac{1}{2}(\cos 4x + 1)\,dx + \sin 2z + z\right]$

$= \frac{1}{4}\left[\frac{1}{8}(\sin 4z + 4z) + \sin 2z + z\right]$

$= \frac{1}{32}(\sin 4z + 8\sin 2z + 12z)$

4,5/4

MA 10

(a) $I(z) = \int_0^z x^2\sqrt{x^3+1}\,dx = \int_0^{z^3} \frac{1}{3}\sqrt{u+1}\,du \quad (u = x^3)$

$= \frac{1}{3}\cdot\frac{2}{3}(u+1)^{3/2}\Big|_0^{z^3}$

$= \frac{2}{9}\left[(1+z^3)^{3/2} - 1\right]$

(b) $I(z) = \int_0^z \sin x\, e^{\cos x}\,dx = -\int_1^{\cos z} e^u\,du \quad (u = \cos x)$

$= e^1 - e^{\cos z}$

(c) $I(z) = \int_0^z \cos^3 x\,dx = +\int_0^z (1 - \sin^2 x)\,d(\sin x)$

$= \int_0^{\sin z} (1 - u^2)\,du \quad (u = \sin x)$

$= \sin z - \frac{1}{3}\sin^3 z$

4,5/3

(d) $I(z) = \int_0^z \sinh^3 x\,dx = \int_0^z (\cosh^2 x - 1)\,d(\cosh x)$

$= \int_1^{\cosh z} (u^2 - 1)\,du \quad (u = \cosh x)$

$= \frac{1}{3}(\cosh^3 z - 1) - (\cosh z - 1)$

$= \frac{1}{3}(\cosh^3 z - 3\cosh z + 2)$

(e) $I(z) = \int_0^z \frac{\sin\sqrt{\pi x}}{\sqrt{x}}\,dx = \int_0^{\sqrt{\pi z}} \frac{2}{\sqrt{\pi}}\sin u\,du \quad (u = \sqrt{\pi x})$

$= \frac{2}{\sqrt{\pi}}(1 - \cos(\sqrt{\pi x}))$

(f) $I(z) = \int_0^z e^{\sqrt{x^3}}\sqrt{x}\,dx = \frac{2}{3}\int_0^{z^{3/2}} e^u\,du \quad (u = x^{3/2})$

$= \frac{2}{3}(e^{\sqrt{z^3}} - 1)$