Bài 1:

a) $$f(x) = \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)$$

Let $\int_{-\infty}^{\infty} f(x)\,dx = I$

We have:

$$I = \int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2\sigma^2}(x-\mu^x)^2\right)dx$$

Suppose that, we shift the distribution where $\mu = 0$, so the area stays the same:

$$I = \int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2\sigma^2}x^2\right)dx$$

Squaring both side:

$$I^2 = \int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2\sigma^2}x^2\right)dx \cdot \int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2\sigma^2}x^2\right)dx$$

$$I^2 = \int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2\sigma^2}x^2\right)dx \cdot \int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2\sigma^2}y^2\right)dy$$

$\Rightarrow$ 2 tọa độ và tích phân giống nhau $\Rightarrow$ areas same

$$I^2 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2\sigma^2}x^2\right)\cancel{dx}\,\exp\left(-\frac{1}{2\sigma^2}y^2\right)dx\,dy$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2\sigma^2}(x^2+y^2)\right)dx\,dy$$

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Then, conversion to polar coordinates leads to

$$\cancel{I^2 = \int_0^{2\pi}\int_0^{\infty} e^{r^2} r\,dr\,\theta}$$

$$x' = r\cos\theta$$
$$y = r\sin\theta$$

We have:

$$\cos^2\theta + \sin^2\theta = 1.$$
$$x^2 + y^2 = r^2$$

$$\Rightarrow I^2 = \int_0^{2\pi}\int_0^{\sigma} \exp\left(-\frac{1}{2\sigma^2} r^2\right) \cdot r \cdot dr\, d\theta$$

We antiderivative : $\exp\left(-\frac{1}{2\sigma^2} r^2\right)$

$$\Rightarrow \int_0^{2\pi} \left[-\sigma^2 \cdot \exp\left(\frac{-1}{2\sigma^2} r^2\right)\right]_0^{\infty}$$

$$\Rightarrow \int_0^{2\pi} \left(-\sigma^2 \cdot 0 - (-\sigma^2 \cdot 1)\right) d\theta$$

$$\Rightarrow \int_0^{2\pi} \sigma^2 d\theta = \sigma^2\, 2\pi$$

$$\Rightarrow I = \sigma\sqrt{2\pi}$$

To prove the case when mean is non zero, we suppose $t = x - \mu$ so that:

$$\int_{-\infty}^{\infty} P(x \mid \mu, \sigma^2)\,dx = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \exp\left(-\frac{t^2}{2\sigma^2}\right) dt.$$

$$= 1 \Big/ \sqrt{2\pi\sigma^2}$$

$$= \frac{\sigma\sqrt{2\pi}}{\sigma\sqrt{2\pi}}$$

$$= 1.$$

$\Rightarrow$ normalized.

b, We have probability density function of X variable:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

And, the expected value is:

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

Let $t = \dfrac{x-\mu}{\sqrt{2}\,\sigma}$ and we have $x = t\sigma\sqrt{2} + \mu$

$$\Rightarrow dt = \frac{1}{\sqrt{2}\,\sigma}\, dx$$

$$\Rightarrow dx = \sigma\sqrt{2}\, dt$$

Substitute t to $E(X)$:

$$E(X) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma\sqrt{2}\,t + \mu) \cdot \exp(t^2) \cdot \sigma\sqrt{2}\, dt$$

$$= \frac{1}{\sqrt{\pi}} \left[\sqrt{2}\sigma \int_{-\infty}^{\infty} t \cdot \exp(-t^2)\, dt + \mu \int_{-\infty}^{\infty} \exp(-t^2)\, dt \right]$$

$$= \frac{1}{\sqrt{\pi}} \left(\sqrt{2}\,\sigma \left[-\frac{1}{2} \exp(-t^2) \right]_{-\infty}^{+\infty} + \mu\sqrt{\pi} \right)$$

$$= \frac{\mu\sqrt{\pi}}{\sqrt{\pi}}$$

$$= \mu$$

c, Similarly to the expectation, from the probability density function, the Variance:

$$\mathrm{Var}(X) = \int_{-\infty}^{\infty} x^2 f(x)\, dx - (E[X])^2$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 \cdot \exp\left(-\frac{(x-\mu)^2}{2\sigma^2} \right) dx - \mu^2$$

Let $t = \frac{x-\mu}{\sqrt{2}\,\sigma} \Rightarrow dt = \frac{1}{\sqrt{2}\,\sigma}\, dx \Rightarrow dx = \sigma\sqrt{2}\, dt$

$$\Rightarrow \mathrm{Var}(X) = \cancel{\left(\frac{(x-\mu)^2}{2\sigma^2} \right)}.$$

$$\Rightarrow \cancel{\mathrm{Var}(X) = \frac{1}{\sigma\sqrt{2}\pi} \int_{-\infty}^{\infty} \frac{(x-\mu)^2}{2\sigma^2}} \cdot \exp(-t^2)$$

$$\Rightarrow \mathrm{Var}(X) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma\sqrt{2}t + \mu)^2 \cdot \exp(-t^2) \cdot \sigma\sqrt{2}\, dt - \mu^2$$

$$= \frac{1}{\sqrt{\pi}} \left[2\sigma^2 \int_{-\infty}^{\infty} t^2 \exp(-t^2)\, dt + 2\sqrt{2}\cdot\sigma\cdot\mu \cdot \int_{-\infty}^{\infty} t \exp(-t^2)\, dt + \mu^2 \int_{-\infty}^{\infty} \exp(-t^2)\, dt \right] - \mu^2$$

$$= \frac{1}{\sqrt{\pi}} \left[2\sigma^2 \int_{-\infty}^{+\infty} t^2 \cdot \exp(-t^2) dt + 2\sqrt{2}\sigma\mu \left[-\frac{1}{2} \exp(-t^2) \right]_{-\infty}^{\infty} + \mu^2 \sqrt{\pi} \right] - \mu^2$$

$$= \frac{1}{\sqrt{\pi}} \left[2\sigma^2 \int_{-\infty}^{\infty} t^2 \exp(-t^2) dt + 2\sqrt{2}\sigma\mu \cdot 0 \right] + \mu^2 - \mu^2$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} t^2 \exp(-t^2) dt.$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \left(\left[-\frac{t}{2} \exp(-t^2) \right]_{-\infty}^{+\infty} + \frac{1}{2} \int_{-\infty}^{\infty} \exp(-t^2) dt \right)$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \cdot \frac{1}{2} \int_{-\infty}^{\infty} \exp(-t^2) dt$$

$$= \frac{2\sigma^2 \sqrt{\pi}}{2\sqrt{\pi}}$$

$$= \sigma^2$$

Problem 2:

a, Let x is a Dimensional vector:

$$x = \begin{pmatrix} x_a \\ x_b \end{pmatrix}$$

And the mean vector μ:

$$\mu = \begin{pmatrix} \mu_a \\ \mu_b \end{pmatrix}$$

And the covariance matrix Σ given by:

$$\Sigma = \begin{pmatrix} \Sigma_{aa} & \Sigma_{ab} \\ \Sigma_{ba} & \Sigma_{bb} \end{pmatrix}$$

where $\Sigma^T = \Sigma$, Σ_{aa} and Σ_{bb} are symatriv, $\Sigma_{ba} = \Sigma_{ab}^T$

$$\Rightarrow A = \Sigma^{-1} = \begin{pmatrix} A_{aa} & A_{ab} \\ A_{ba} & A_{bb} \end{pmatrix}$$

We are looking for conditional distribution $p(x_a | x_b)$

$$-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)$$

$$= -\frac{1}{2}(x-\mu)^T A (x-\mu)$$

$$= -\frac{1}{2}(x_a-\mu_a)^T A_{aa}(x_a-\mu_a) - \frac{1}{2}(x_a-\mu_a)^T A_{aa}(x_b-$$

$$-\frac{1}{2}(x_b-\mu_b)^T A_{aa}(x_a-\mu_a) - \frac{1}{2}(x_b-\mu_b)^T A_{bb}(x_b-$$

$$= -\frac{1}{2}x_a^T A_{aa}^{-1} x_a + x_a^T(A_{aa}\mu_a - A_{ab}(x_b-\mu_b)) + \text{const}$$

It is quadratic form of x_a hence conditional distribution $p(x_a|x_b)$ will be Gaussian, because this distribution is characteriz by its mean and its variance, So, compare with Gaussian distribute

$$\Delta^2 = -\frac{1}{2}x^T \Sigma^{-1} x + x^T \Sigma^{-1} \mu + \text{const}$$

$$\Sigma_{a|b} = \Lambda_{aa}^{-1}$$

$$\mu_{a|b} = \Sigma_{a|b} \left(\Lambda_{aa}\mu_a - \Lambda_{ab}(x_b - \mu_b) \right)$$

$$= \mu_a - \Lambda_{aa}^{-1}\Lambda_{ab}(x_b - \mu_b)$$

By using Schur complement:

$$\Rightarrow \Lambda_{aa} = \left(\Sigma_{aa} - \Sigma_{ab}\Sigma_{bb}^{-1}\Sigma_{ba}\right)^{-1}$$

$$\Lambda_{ab} = -\left(\Sigma_{aa} - \Sigma_{ab}\Sigma_{bb}^{-1}\Sigma_{ba}\right)^{-1}\Sigma_{ab}\Sigma_{bb}^{-1}$$

As a result:

$$\mu_{a|b} = \mu_a + \Sigma_{ab}\Sigma_{bb}^{-1}(x_b - \mu_b)$$

$$\Sigma_{a|b} = \Sigma_{aa} - \Sigma_{ab}\Sigma_{bb}^{-1}\Sigma_{ba}$$

$$\Rightarrow p(x_a | x_b) = N(x_{a|b} \mid \mu_{a|b}, \Sigma_{a|b})$$