Sinusoidal Steady-State Analysis

$\boxed{0}$ Introduction

This section will concentrate on the steady-state response of circuits driven by sinusoidal sources;and absolutely, the steady-state response will also be sinusoidal. A source that can be described by a periodic function can be replaced by an equivalent combination (Fourier series) of sinusoids; therefore, for a linear circuit, the assumption of a sinusoidal source represents no real restriction.

$\boxed{1}$ The Sinusoidal Time-Varying Source

- A sinusoidally time-varying voltage(or current) source (independent or dependent) produces a voltage(or current) that varies sinusoidally with time.
- A sinusoidal signal can be expressed with either the sine function or the cosine function. Throughout this section, the cosine function will be used.
- A sinusoidal voltage source $v(t)$ is given by $v(t) = \widehat{V}cos(\omega t + \phi)$, where $\widehat{V}$ is the amplitude, ω is the angular velocity, or angular frequency [rad/s], and ϕ is the phase angle [rad]. So, the period of the function $T = 1/f = 2\pi/\omega$ [s].
- Suppose there are two sinusoidal signal with the same angular frequency ω:

  $f_1(t) = \widehat{F_1}cos(\omega t + \phi_1)$, and $f_2(t) = \widehat{F_2}cos(\omega t + \phi_2)$

  _The value $\Delta\phi_{12} = (\omega t + \phi_1) - (\omega t + \phi_2) = \phi_1 - \phi_2$ is called the phase shift of the signal f_1 compared to the signal f_2.

  + If $\Delta\phi_{12} = 0$, then we say f_1 and f_2 are in phase.
  + If $\Delta\phi_{12} \neq 0$, then we say f_1 and f_2 are out of phase.
  + If $\Delta\phi_{12} > 0$, then we say f_1 leads f_2 by an advance of $\Delta\phi_{12}$, or by a phase angle of $\Delta\phi_{12}$.
  + If $\Delta\phi_{12} < 0$, then we say f_1 lags f_2 by a phase lag of $|\Delta\phi_{12}|$, or by a phase angle of $|\Delta\phi_{12}|$.
- Recall a concerned characteristic of a periodic function, which will be certainly applied to sinusoidal signals as well:

  _A periodic function $f(t)$, with a period T, has an average or mean value F_{avg} given by $F_{avg} = \dfrac{1}{T}\displaystyle\int_t^{t+T} f(t)dt = \dfrac{1}{T}\int_0^T f(t)dt = \text{const}$

  $\rightarrow$ The mean value of a sinusoidal function is 0.

  _The rectified average value F_{avg} of a periodic function $f(t)$ with a period T is the mean value of the absolute value of the function,

  $$F_{avg} = \frac{1}{T}\int_t^{t+T} |f(t)|dt = \frac{1}{T}\int_0^T |f(t)|dt = \text{const}$$

  _The effective or rms value, which arises from the need to measure the effectiveness of a voltage or current source in delivering power to a resistive load, of a periodic function is defined as the square root of the mean value of the squared function. Therefore, for any periodic function $i(t)$ in general, the rms value of the function is given by $\quad I_{rms} = \sqrt{\dfrac{1}{T}\displaystyle\int_t^{t+T} i(t)^2dt} = \sqrt{\dfrac{1}{T}\displaystyle\int_0^T i(t)^2dt} = \text{const}$

  $\rightarrow$ Consider the average power absorbed in a period T by the resistor through whom a sinusoidal current $i(t)$ passes in an ac circuit,

  $$P_{avg} = \frac{1}{T}\int_0^T p(t)dt = \frac{1}{T}\int_0^T Ri(t)^2dt = \text{const}$$

  Based on the definition of the rms value, $I_{rms}^2 = \dfrac{1}{T}\displaystyle\int_0^T i(t)^2dt$

$=> RI_{rms}^2 = \dfrac{1}{T}\displaystyle\int_0^T Ri(t)^2 dt = P_{avg}$, is equivalent to the power absorbed by the resistor with a constant current I_{rms} in a dc circuit.

$\rightarrow$ The effective value of the voltage V_{rms} is found in the same way.

$\rightarrow$ For every sinusoidal function $v(t) = \widehat{V}cos(\omega t + \phi)$; $V_{rms} = \widehat{V}/\sqrt{2}$.

$\boxed{2}$ The Response Of A Sinusoidal Steady-State Circuit

• In general, voltage responses and current responses of an electric circuit are solutions of a corresponding system of differential equations which are direct applications of Kirchhoff's laws. It's not easy in most cases to obtain the solutions by straightly solving this kind of systems.

• However, for a circuit driven by a sinusoidally time-varying source, in a steady state, all voltages or currents vary sinusoidally with time at the same angular frequency as the source, mathematically typified by

$$f(t)_i = \widehat{F_i}cos(\omega t + \phi_i) = Re\{\widehat{F_i}\angle(\omega t + \phi_i)\} = Re\{\widehat{F_i}e^{j(\omega t + \phi_i)}\}$$
$$= Re\{(\widehat{F_i}\angle(\phi_i))(\widehat{F_i}\angle(\omega t))\} = Re\{(\widehat{F_i}e^{j\phi_i})(\widehat{F_i}e^{j\omega t})\}$$

Therefore, at any given time, all signals are only different from each other in their amplitudes and in their phase angles; so, a phasor with the same amplitude and phase angle as a particular signal in the circuit represent the signal, for example, known $v(t) = \widehat{V}cos(\omega t + \phi_v)$, $\rightarrow$ responsible phasor $\underline{\mathrm{V}} = \widehat{V}\angle\phi_v = \widehat{V}e^{j\phi_v}$.

• Based on this charateristic, solving directly the system of differential equations can be avoided, and, the solutions are obtained by calculation in terms of phasors, which are derived from correspondent sinusoidal signals. In particular, a sinusoidally time-varying signal will be now represented by a phasor with the same amplitude and phase angle, for example, given $v(t) = \widehat{V}cos(\omega t + \phi_v)$, $=>$ phasor $\underline{\mathrm{V}} = \widehat{V}\angle\phi_v = \widehat{V}e^{j\phi_v}$.