# Real-Time Motion Estimation: Optical Flow - Feature Point Tracking

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# Major Issues

**1. Template matching**

1) 0 order/moment = texture (grayvalue or color)

2) $1^{st}$ order/moment/derivation = gradient component

> Optical flow for sub-pixel matching

**2. Optimization Using Sum of Squared Difference (SSD)**

min $E = \sum [ Ax - b]^2$ **using 1st order Taylor series expansion**

$Ax = b'$: estimation value. b: ground truth,

min $E =$ min $\sum [ b - b']^2$

3. **Hessian matrix**

1) Aperture problem

2) Texture or textureless judgment

3) Uncertainty



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Local optimization (0 moment): 1 and 2 =>
Global optimization: 3 => 1) LM 2) EM 3) SGD (Stochastic Gradient Decent)

# Solution/Optimization of Homogenous Matrix

1. **Closed Form Solution:**
   $Ax = 0$ => $A^tA$ = Covariance Matrix => PCA => Smallest >= 0 eigenvalue => eigenvector
2. **Pseudo Inverse:**
   **$Ax = b$, $x= (A^TA)^{-1} (A^Tb)$**
3. **Sum of Squared Difference: (max likelihood – exponential term)**
   **min $E = \sum [ Ax - b]^2$**

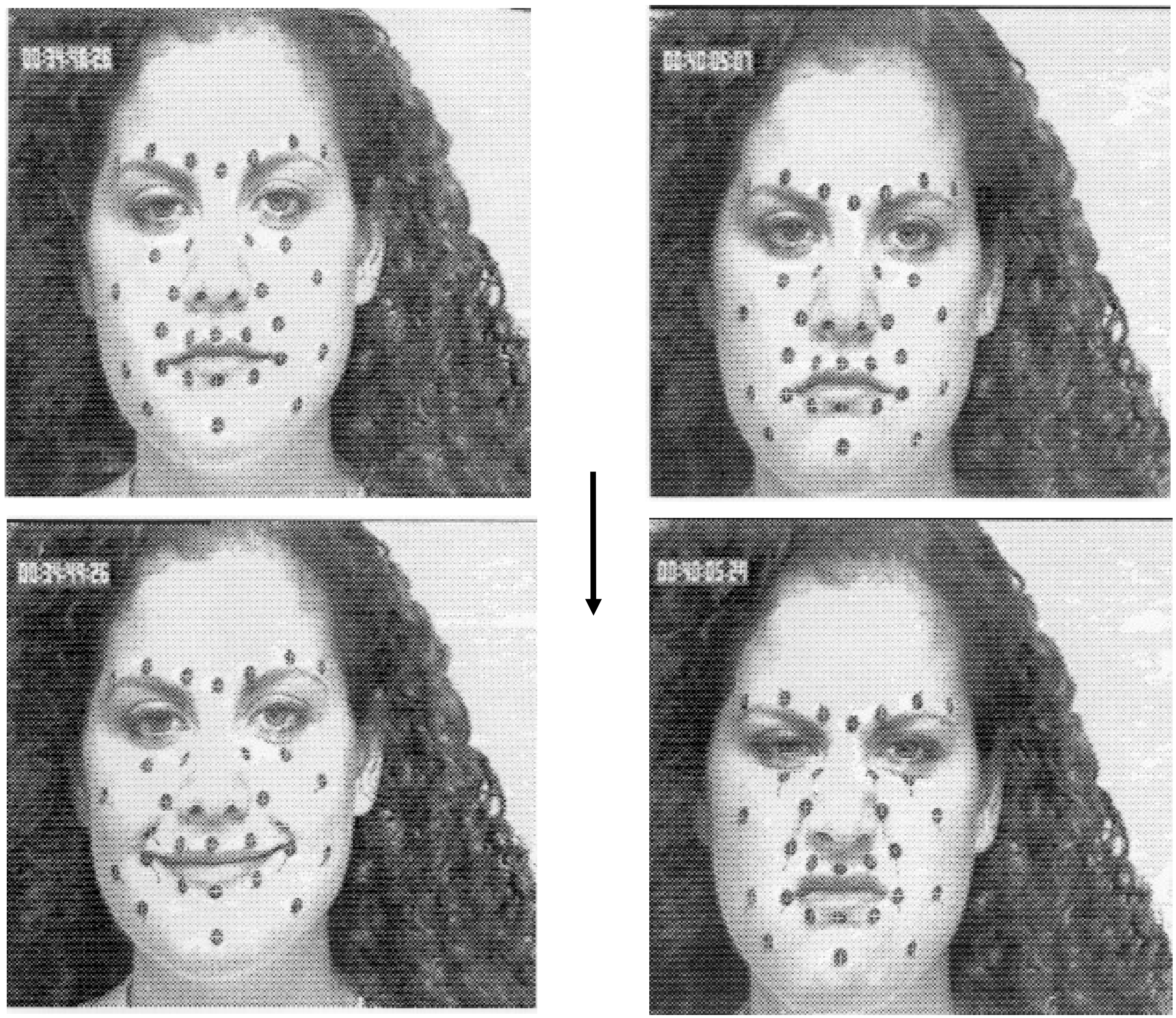
   **3.1 $Ax = b'$: estimation value. b: ground truth, $E = \sum [ b - b']^2$**
   - a. Initial value estimation => Pseudo Inverse (linear approach)
   - b. L-M (non-linear approach)
     - b.1 First order Taylor series expansion
     - **b.2** $2^{nd}$ order Taylor series expansion (sensitive to noise)

   **3.2 $Ax = b'$: estimation value. b'': estimation value, $E = \sum [ b'' - b']^2$**
   - **a. EM (Expected-Maximization), initial b = average value**

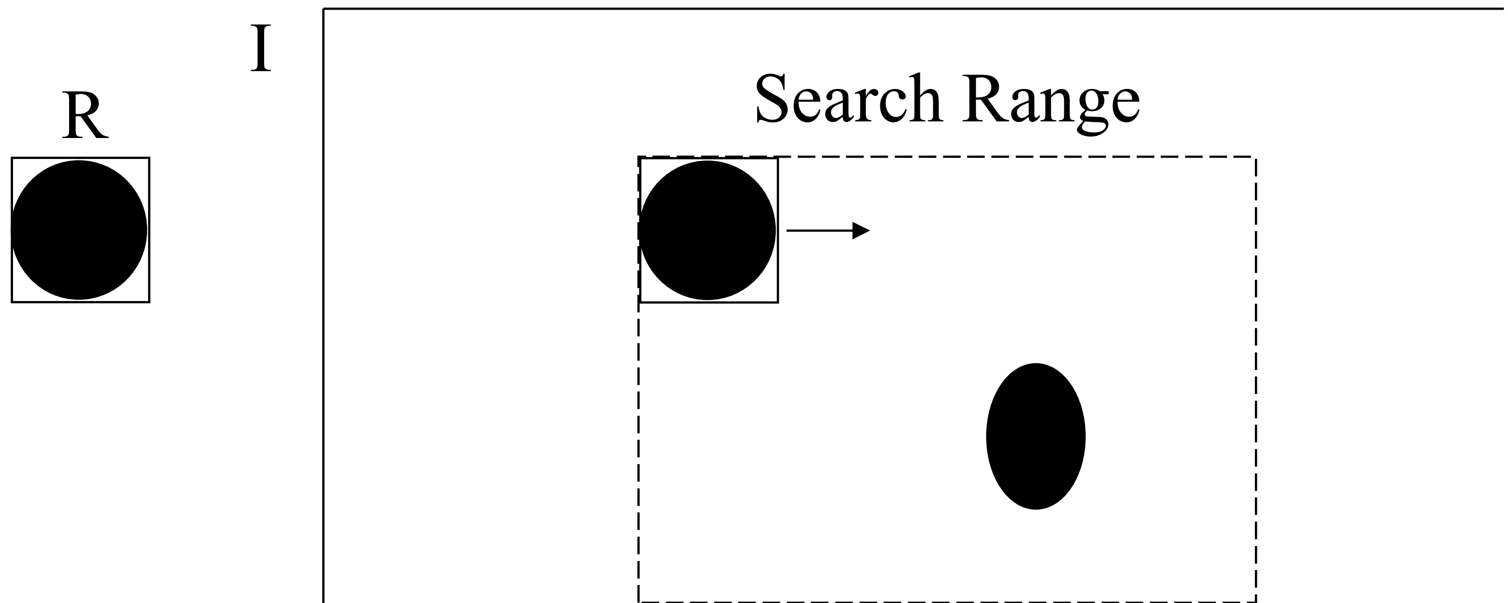
   Machine learning for prediction

4. **Lagrange Approach (outlier)**
   **min $E = \sum [ Ax - b]^2 + \underbrace{\lambda\ (x^2+y^2)^2}_{\text{constraint}}$**

# 1.0 Feature/Dot Tracking: Plastic Surgery



# Template Matching: Correlation Coefficient




$$\rho(x) = \left| \frac{\sum_{r \in R} [F(r) - \overline{F}][I(x+r) - \overline{I}(x)]}{\sqrt{\sum_{r \in R} [F(r) - \overline{F}]^2} \sqrt{\sum_{r \in R} [I(x+r) - \overline{I}(x)]^2}} \right| \qquad 0.0 \le \rho = \cos\theta \le 1.0$$

Why minus $\overline{I}(x)$ and minus $\overline{F}$ ?

(Ex: Gaussian distribution Normalization)

Why divided standard deviation ?

# Disadvantage/Inaccurate of Feature/Dot Tracking

- Feature/Dot is easily deformed.
- Feature/Dot is affected by the reflections/specular due to lighting.
- Computational time is slow when the number of feature/dots and search regions are increased.
- Mismatching when features/dots are closer to each other.

# Template Matching

- **Matching (similarity measure) => Likelihood Probability**

1. Texture ➔ Feature maps extracted from Deep Learning
   - 0 order
   - Pro: Stable
   - Con: Sensitive to lighting
2. Shape
   - $1^{st}$ order
   - Pro: Not sensitive to lighting
   - Con: Sensitive to noise
3. Color
   - RGB => HSV => Histogram domain
   - Pro: Fast to extract ROI (Region of interest)
   - Con: Confuse by similar background color
4. RGB-D

# Template Matching: Search Range, Window Size and Pyramid

- **Search range**
- **Window size**
  - Large window size
    - Contain more texture + structure information
    - Sensitive to distorted information
    - Slow
  - Small window size
    - Contain less texture + structure information
    - Not sensitive to distorted information
    - Fast
- **Pyramid**
  - Speed up from $N^2$ to N log N
  - Large tracking distance by extending search range

# Matching Methods:

**1.1 Sum-Square-Difference**

- ❑ **SIFT**
- ❑ **HOG**

**1.2.1 Correlation Matrix:**

**1) X1*X2**

**2) X1*X2 / (Lo_X1*Lo_X2)**

**1.2.2 Covariance Matrix:**

**1) (X1-mX1)*(X2-mX2)**

**2) (X1-mX1)*(X2-mX2)/(Lo_X1*Lo_X2)**

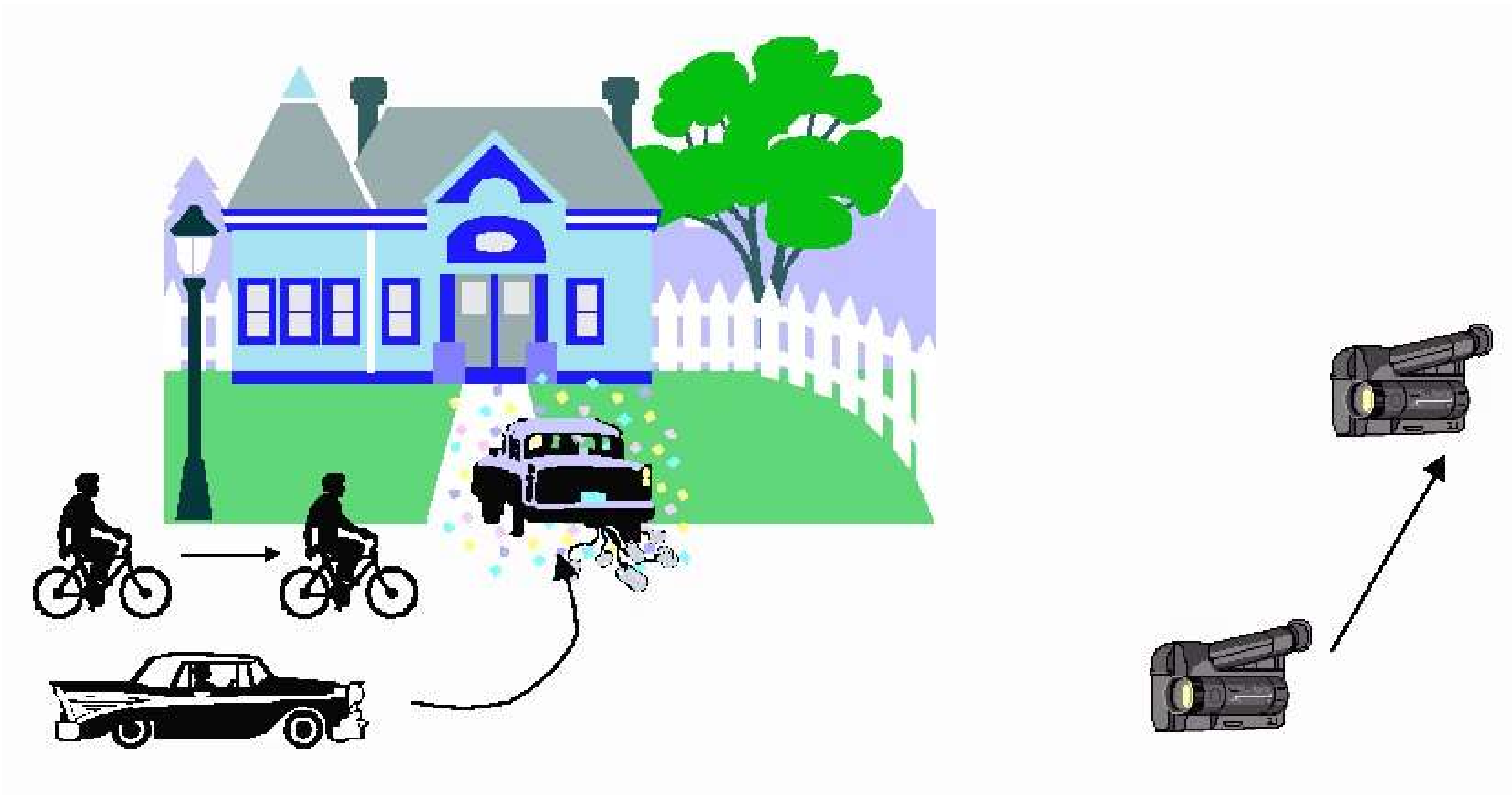
**2.1 Optical Flow**

❑ **Strong correlation/covariance (matrix):**

- ➢ Diagonal components of correlation/covariance matrix has brightest grayvalues – strong correation

# Template Matching:
# Sub-Pixel Accuracy and One Pixel Match

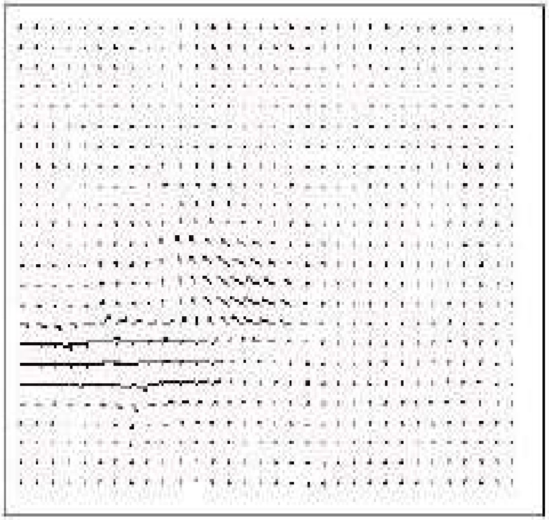
# 2.0 Movement between Camera and Object



# Definition of Optical Flow

- **Optical flow is the 2-D velocity field (u, v) induced in an image due to the projection of 3-D moving objects onto the image plane**





# Applications

- **Corresponding points Vs. Optical flow**
  - 2D -> 3D reconstruction
  - Mosaic
  - Stabilization
  - …

  - SIFT
  - SURF
  - HOG

# Methods of Computing Optical Flow: Matching

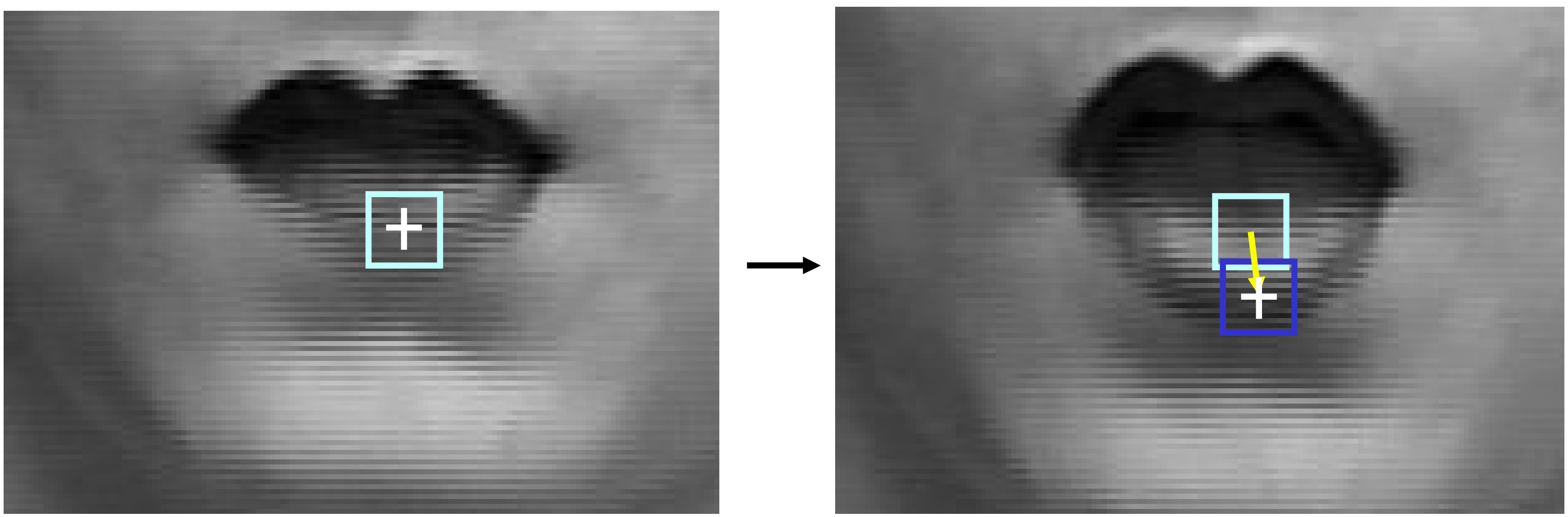
- **Three prevalent approaches to compute optical flow:**
  1) Token matching or correlation
     - » Extract features from each frame (gray level windows, edge detection)
     - » Match them from frame to frame
  2) Gradient techniques
     - » Relate optical flow to spatial and temporal image derivatives
  3) Velocity sensitive
     - » Frequency domain models of motion estimation

# 2.1 Optical Flow: Difference Minimization

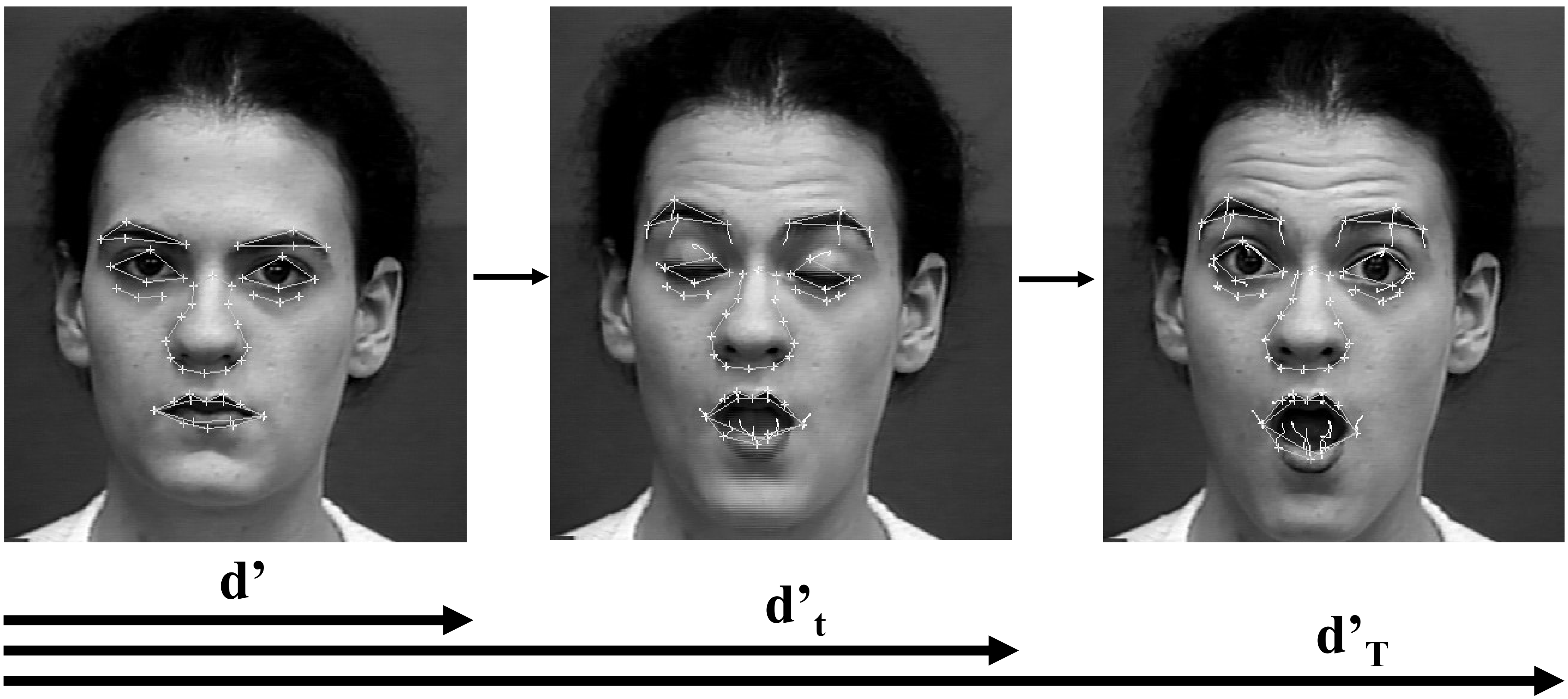
$$\min \; E = \sum_{x,y \in R} \left[ I_t(x-u(x,y),\, y-v(x,y)) - I_{t+1}(x,y) \right]^2$$

SSD problem

E: Energy or Cost Function



# Feature Point Tracking

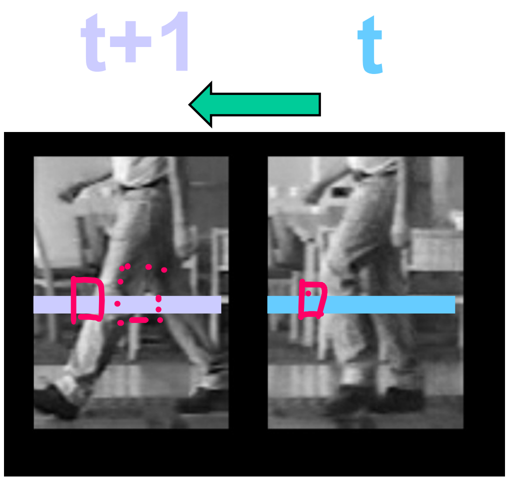


# A 1-D Gradient Technique

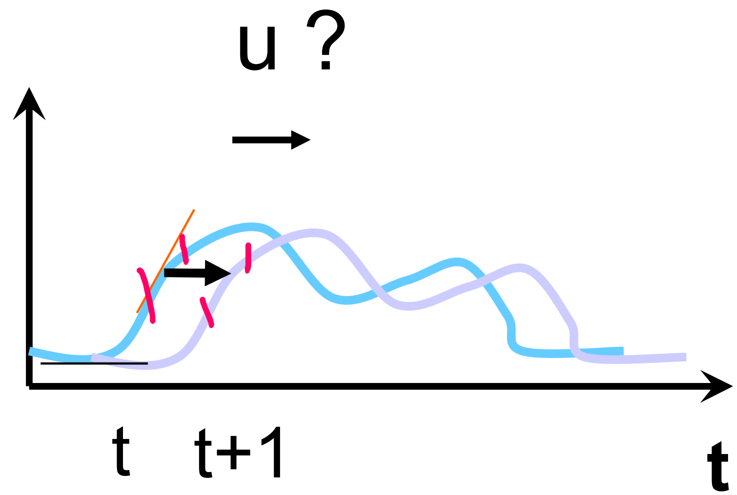
- **Suppose we have 1-D image that changes over time due to a translation of the image**
- **Suppose we also assume that the image function is, at least over small neighborhoods, well approximated by a linear function (continue, Taylor series expansion)**
  - Completely characterized by its value and slope
- **Can we estimate the motion of the image by comparing its spatial derivative (texture/edge) at a point to its temporal derivative ?**
  - Example: Spatial derivative (gradient as edge) is 10 units/pixel and temporal derivative is 20 units/frame

    Then motion is (20 units/frame)/(10 units/pixel)=2 pixels/frame

# 2.2.1 1-D Lucas-Kanade Flow




I: Intensity

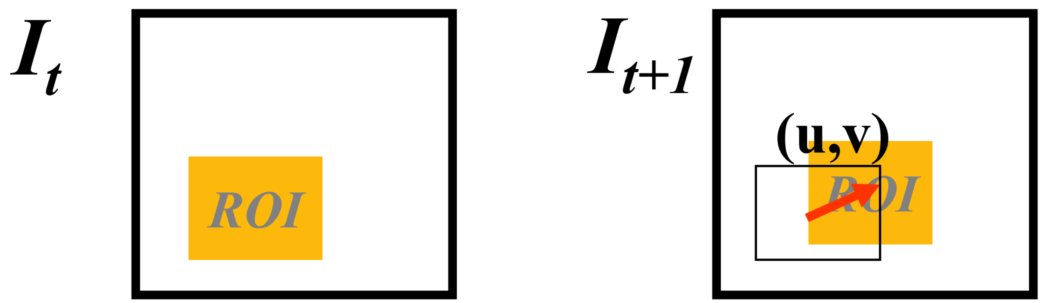



$$\min_{x \in R} \; E = \sum_{x \in R} [\, I_t(x-u) - I_{t+1}(x) \,]^2$$

?

$||x||$

# 2.2.2 2-D Lucas-Kanade Flow




$$\min_{\ } \; E = \sum_{x,y \in R} \left[ I_t(x-u(x,y),\, y-v(x,y)) - I_{t+1}(x,y) \right]^2$$

11x11

# Flow Computation: Physical Interpretation

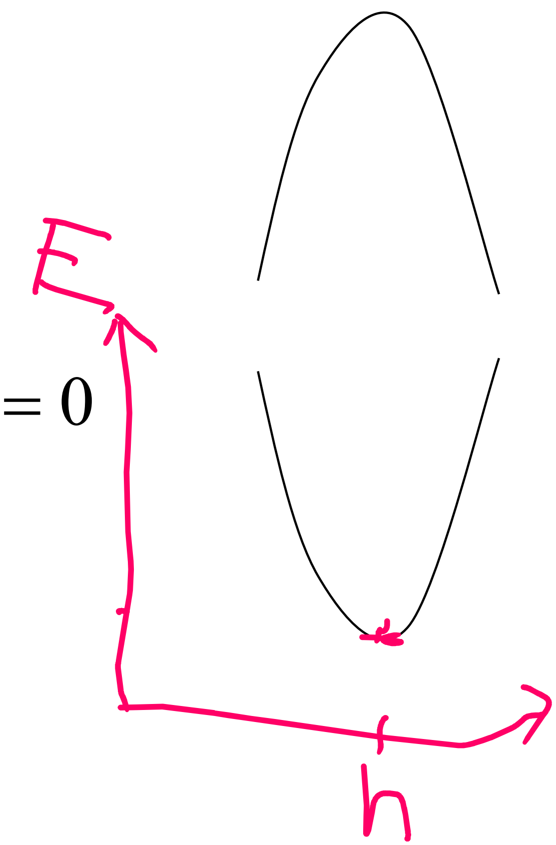
- Q: what is $(\frac{\partial I}{\partial x}, \frac{\partial I}{\partial y})$?
- A: image brightness gradient direction
  - known (spatial derivatives) (texture, edge)
- Q: what is $(u, v)$?
- A: local motion vector
  - unknown
- Q: what is $\frac{\partial I}{\partial t}$?
- A: change of brightness at a location w.r.t time
  - known (temporal derivative)

# 2.2.2-1 Lucas-Kanade Flow: SSD Minimization

$$\min \ E = \sum_{x \in R} \left[ I(x+h) - F(x) \right]^2$$

template

$$\frac{\partial E}{\partial h} = \sum_{x \in R} 2\left[ I(x+h) - F(x) \right] * \frac{\partial}{\partial h} I(x+h) = 0$$




By first order Taylor series expansion

$$I(x+h) \approx I(x) + h \frac{\partial I(x)}{\partial x}$$

then

$$\sum_{x \in R} 2\left[ I(x) + h \frac{\partial I(x)}{\partial x} - F(x) \right] * \frac{\partial}{\partial h} \left[ \underbrace{I(x)}_{0} + h \frac{\partial I(x)}{\partial x} \right] = 0$$

$$\sum_{x\in R}[I(x)-F(x)]\left(\frac{\partial h}{\partial h}\frac{\partial I(x)}{\partial x}\right)+\sum_{x\in R} h\frac{\partial I(x)}{\partial x}(\frac{\partial h}{\partial h}\frac{\partial I(x)}{\partial x})=0$$

$$h=\left[\sum_{x\in R}(\frac{\partial I(x)}{\partial x})\left(F(x)-I(x)\right)\right]\left[\sum_{x\in R}\left(\frac{\partial I(x)}{\partial x}\right)\left(\frac{\partial I(x)}{\partial x}\right)\right]^{-1}$$

Iteration

$$h_0=0$$

$$h_{n+1}=h_n+\left[\sum_{x\in R}\left(\frac{\partial I(x)}{\partial x}\right)\Big|_{x+h_n}[F(x)-I(x+h_n)]\right]\left[\sum_{x\in R}\left(\frac{\partial I(x)}{\partial x}\right)\left(\frac{\partial I(x)}{\partial x}\right)\Big|_{x+h_n}\right]^{-1}$$

$$h_{n+1}=h_n+e*G^{-1} \quad \text{until} \quad \left|h_{n+1}-h_n\right|<\varepsilon \quad 10^{-4}$$

# Lucas-Kanade Flow Equation

$$h_{n+1} = h_n + e * G^{-1} \quad \text{until} \quad \left| h_{n+1} - h_n \right| < \varepsilon$$

where

$$e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} \sum_{x \in R} \frac{\partial I}{\partial x_1}\Big|_{x+h_n} \left[ F(x_1) - I(x_1 + h_n) \right] \\ \sum_{x \in R} \frac{\partial I}{\partial x_2}\Big|_{x+h_n} \left[ F(x_2) - I(x_2 + h_n) \right] \end{bmatrix} \qquad G^{-1} = \frac{\begin{bmatrix} G_{22} & -G_{12} \\ -G_{12} & G_{11} \end{bmatrix}}{G_{11}G_{22} - G_{12}{}^2}$$

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{12} & G_{22} \end{bmatrix} = \begin{bmatrix} \sum_{x \in R} \left( \frac{\partial I(x)}{\partial x_1} \right)^2 \Big|_{x+h_n} & \sum_{x \in R} \left( \frac{\partial I(x)}{\partial x_1} \right) \left( \frac{\partial I(x)}{\partial x_2} \right) \Big|_{x+h_n} \\ \sum_{x \in R} \left( \frac{\partial I(x)}{\partial x_1} \right) \left( \frac{\partial I(x)}{\partial x_2} \right) \Big|_{x+h_n} & \sum_{x \in R} \left( \frac{\partial I(x)}{\partial x_2} \right)^2 \Big|_{x+h_n} \end{bmatrix}$$

# 2.2.2-2 Lucas-Kanade Flow: General Case

**Minimize a cost function *E* of the sum of squared differences (SSD)**

**min** $$E(d(x)) = \sum_{x \in R} [I_t(x - d(x)) - I_{t+1}(x)]^2 w(x)$$

$$d(x) = d^i(x) + \Delta d(x)$$

$$I_t(x - d) = I_t(x - (d^i + \Delta d))$$

The first order Taylor's expansion:

$$I_t(\underline{x - d^i} - \Delta d) \approx I_t(x - d^i) - I_t'(x - d^i)^T \Delta d = 0$$

**The incremental change $\Delta d$ in the SSD cost function:**

$$E(\Delta d) = E(d^i + \Delta d) - E(d^i)$$

$$\approx \sum_{x \in R} [I_t(x - d^i) - I_t^{'}(x - d^i)^T \Delta d - I_{t+1}(x)]^2 w(x) - \sum_{x \in R} [I_t(x - d^i) - I_{t+1}(x)]^2 w(x)$$

$$= \sum_{x \in R} [I_t^{'}(x - d^i)^T \Delta d]^2 w(x) - 2 \sum_{x \in R} [I_t(x - d^i) - I_{t+1}(x)] I_t^{'}(x - d^i)^T \Delta d w(x)$$

$$= \Delta d^T G \Delta d - 2 e^T \Delta d$$

where

$$G = \sum_x I_t^{'}(x - d^i) I_t^{'}(x - d^i)^T w(x)$$

$$e^T = \sum_x [I_t(x - d^i) - I_{t+1}(x)] I_t^{'}(x - d^i)^T w(x)$$

$$G = \sum_x I_t^{'}(x - d^i) I_t^{'}(x - d^i)^T w(x)$$

**The Hessian matrix of the gradients of $I_t$ with a window function $w(x)$**

$$e^T = \sum_x [I_t(x - d^i) - I_{t+1}(x)] I_t^{'}(x - d^i)^T w(x)$$

**The difference-gradient row vector which is the product of the difference (or error) between the regions in the two consecutive images and the gradient of the gray-value $I_t$ together with a window function $w(x)$**

**The maximum decrement E($\Delta d$) occurs when its gradient with respect to $\Delta d$ is zero**

$$\frac{\partial E(\Delta d)}{\partial(\Delta d)} = G\Delta d + (\Delta d^T G)^T - 2e = 2(G\Delta d - e) = 0$$

**Hence,**

$$\Delta d(x) = G^{-1} e$$

**Initializing $d^{(0)}(x) = [0,0]^T$ and following above equations, the optical flow $d(x)$ can be robustly estimated through iterations yielding the sub-pixel accuracy.**

**The motion estimate $d(x)$ is more accurate when the gradients of both $I_t$(x) and $I_{t+1}$(x) are large and nearly equal**

texture

not deform

# 2.2.3 LM Optimization: Optical Flow

- **1. Initialization by 0 order approach:**
  - Correlation Coefficient Match:
  - Lx = 0 or Ax =b

- **2. LM by 1st and 2nd order deviations (iteration): Taylor Extension**

$$h_{n+1} = h_n + e * G^{-1} \quad \text{until} \quad \left| h_{n+1} - h_n \right| < \varepsilon$$

$$e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} \sum_{x \in R} \frac{\partial I}{\partial x_1}\Big|_{x+h_n} \left[ F(x) - I(x+h_n) \right] \\ \sum_{x \in R} \frac{\partial I}{\partial x_2}\Big|_{x+h_n} \left[ F(x) - I(x+h_n) \right] \end{bmatrix} \qquad G^{-1} = \frac{\begin{bmatrix} G_{22} & -G_{12} \\ -G_{12} & G_{11} \end{bmatrix}}{G_{11}G_{22} - G_{12}{}^2}$$

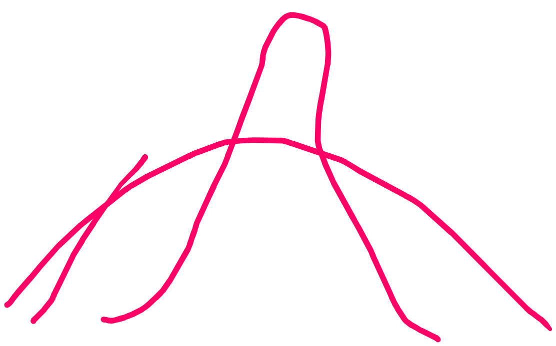
where

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{12} & G_{22} \end{bmatrix} = \begin{bmatrix} \sum_{x \in R} \left( \frac{\partial I(x)}{\partial x_1} \right)^2 \Big|_{x+h_n} & \sum_{x \in R} \left( \frac{\partial I(x)}{\partial x_1} \right) \left( \frac{\partial I(x)}{\partial x_2} \right) \Big|_{x+h_n} \\ \sum_{x \in R} \left( \frac{\partial I(x)}{\partial x_1} \right) \left( \frac{\partial I(x)}{\partial x_2} \right) \Big|_{x+h_n} & \sum_{x \in R} \left( \frac{\partial I(x)}{\partial x_2} \right)^2 \Big|_{x+h_n} \end{bmatrix}$$

# Hessian Matrix

$$G=\begin{bmatrix} G_{11} & G_{12} \\ G_{12} & G_{22} \end{bmatrix}=\begin{bmatrix} \sum_{x\in R}\left(\frac{\partial I(x)}{\partial x_1}\right)^2\Big|_{x+h_n} - \lambda_1 & \sum_{x\in R}\left(\frac{\partial I(x)}{\partial x_1}\right)\left(\frac{\partial I(x)}{\partial x_2}\right)\Big|_{x+h_n} \\ \sum_{x\in R}\left(\frac{\partial I(x)}{\partial x_1}\right)\left(\frac{\partial I(x)}{\partial x_2}\right)\Big|_{x+h_n} & \sum_{x\in R}\left(\frac{\partial I(x)}{\partial x_2}\right)^2\Big|_{x+h_n} - \lambda_2 \end{bmatrix} = 0$$

11x11



$\lambda = \sigma$

# Correlation Matrix Vs. Covariance Matrix Vs. Hessian Matrix

1. **Covariance Matrix A Vs. Correlation Matrix A'**

2. **$A=(A'-m)=UWV^T$**
   - **Gaussian distribution N(m, Lamda$^2$),**
   - **as affine transform**

3. **Affine transform –**

   m: as the translation terms

   U: Eigenvector matrix as the rotation matrix

   W: Eigenvalues (Lamda$^2$) as the variance/scaling terms at denomination for normalization.

   V: Coordinate rotation matrix

# 2.2.4 Demo: Feature Point Tracking

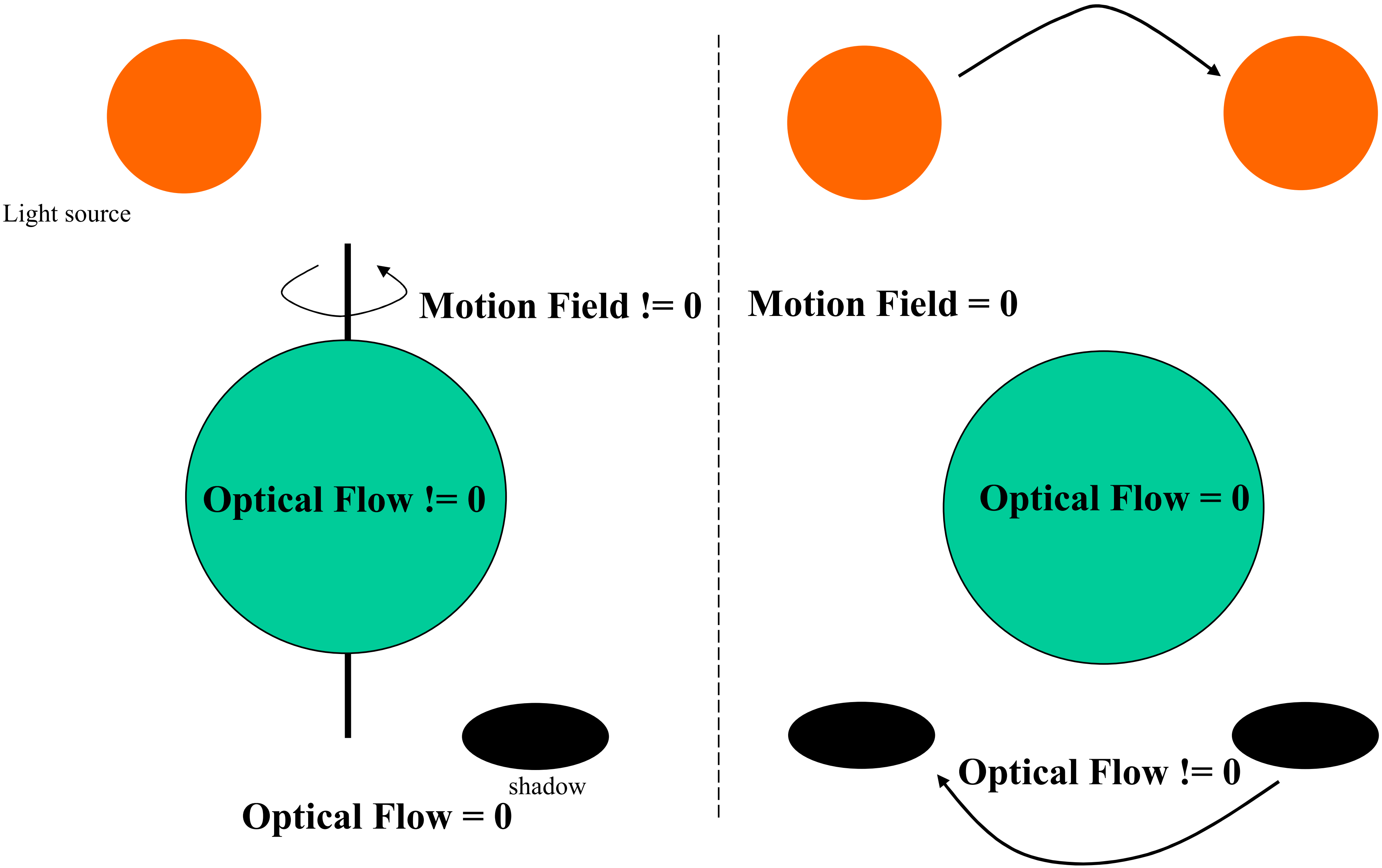
# Good Feature to Track

# 1-Pixel Tracking

# 2.3 Affine Flow

$$\min \ E = \sum_{x \in R} \left[ I(x + (ax + h)) - F(x) \right]^2$$

# 3.0 Motion Field and Optical Flow



# 3.1.1 Optical Flow Constraint Equation

$$I(x+\delta x, y+\delta y, t+\delta t) = I(x,y,t)$$

1. Optical flow (velocity) vector (u(x,y),v(x,y))

$$\delta x = u\delta t,\ \ \delta y = v\delta t$$

$$I(x+u\delta t, y+v\delta t, t+\delta t) = I(x,y,t)$$

For small time interval $\delta t \to 0$
and the motion field is continuous almost everywhere

2. And by the first order Taylor's expansion:

$$I(x,y,t) + \delta x\frac{\partial I}{\partial x} + \delta y\frac{\partial I}{\partial v} + \delta t\frac{\partial I}{\partial t} = I(x,y,t)$$

$$\frac{\partial I}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial I}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial I}{\partial t}\frac{\partial t}{\partial t} = 0$$

$$\frac{\partial I}{\partial x}u + \frac{\partial I}{\partial y}v + \frac{\partial I}{\partial t}1 = 0$$

**Optical flow constraint equation:**

(express a constraint on the components u and v of the optical flow)

(Or called constraint line L)

$$I_x u + I_y v + I_t = 0$$

$$(I_x, I_y) \cdot (u, v) = -I_t \neq 0$$

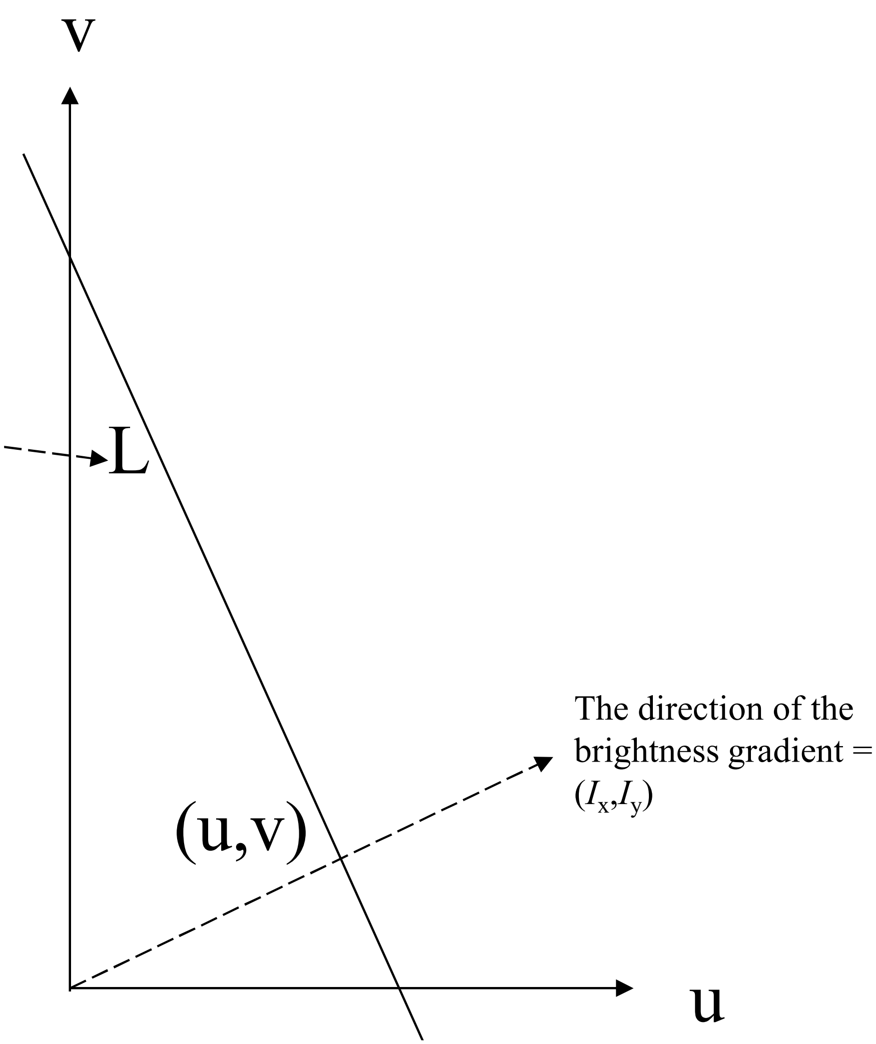
Spatial change =texture => Hessian Matrix

Temporal change =motion flow

J: Result => Feature/patch moving?

The component of optical flow in the direction of the brightness gradient $(I_x, I_y)^T$ is thus

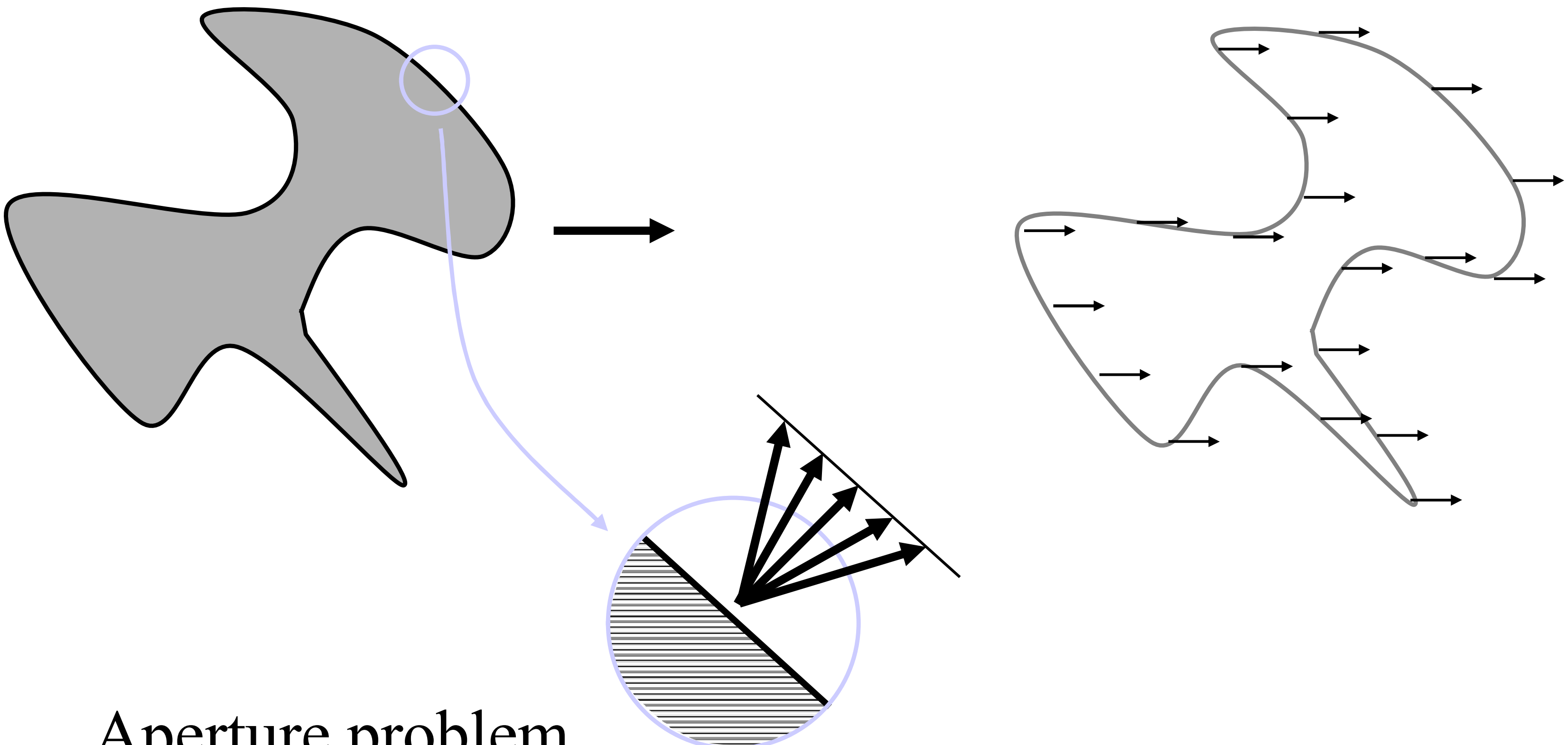
$$d(0, L) = \frac{I_t}{\sqrt{I_x^2 + I_y^2}}$$




We can only determine the component in the direction of the brightness gradient.

We cannot, however, determine the component of the optical flow at right angles ($90^0$) to this direction, that is, along the isobrightness contour. This ambiguity is also known as the aperture problem.

# 3.1.2 Aperture Problem



Aperture problem

$$h_{n+1} = h_n + e * G^{-1} \quad \text{until} \quad \left| h_{n+1} - h_n \right| < \varepsilon$$

where

$$e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} \sum_{x \in R} \frac{\partial I}{\partial x_1}\Big|_{x+h_n} \left[ F(x) - I(x + h_n) \right] \\ \sum_{x \in R} \frac{\partial I}{\partial x_2}\Big|_{x+h_n} \left[ F(x) - I(x + h_n) \right] \end{bmatrix} \qquad G^{-1} = \frac{\begin{bmatrix} G_{22} & -G_{12} \\ -G_{12} & G_{22} \end{bmatrix}}{G_{11} G_{22} - G_{12}{}^2}$$

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{12} & G_{22} \end{bmatrix} = \begin{bmatrix} \sum_{x \in R} \left( \frac{\partial I(x)}{\partial x_1} \right)^2 \Big|_{x+h_n} & \sum_{x \in R} \left( \frac{\partial I(x)}{\partial x_1} \right) \left( \frac{\partial I(x)}{\partial x_2} \right) \Big|_{x+h_n} \\ \sum_{x \in R} \left( \frac{\partial I(x)}{\partial x_1} \right) \left( \frac{\partial I(x)}{\partial x_2} \right) \Big|_{x+h_n} & \sum_{x \in R} \left( \frac{\partial I(x)}{\partial x_2} \right)^2 \Big|_{x+h_n} \end{bmatrix}$$

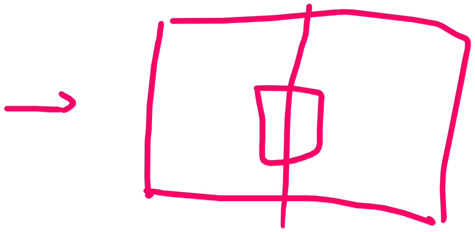
$$I_x u + I_y v + I_t = 0$$

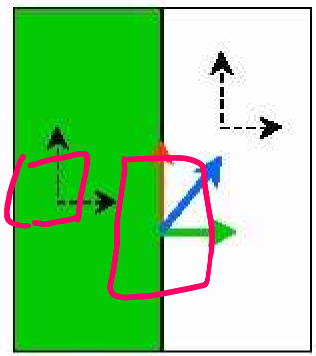
$$(I_x, I_y) \cdot (u, v) = -I_t \neq 0$$

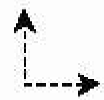
Spatial change =texture => Hessian Matrix

Temporal change =motion flow

Feature/patch moving



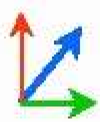




no spatial change in brightness
induce no temporal change in brightness
no discernible motion

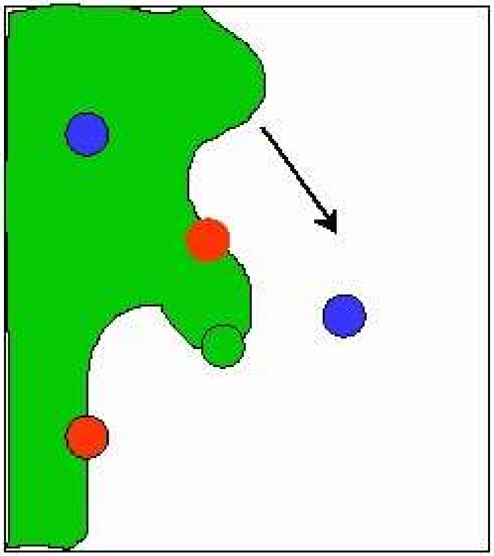
motion perpendicular to local gradient
induce no temporal change in brightness
no discernible motion

motion in the direction of local gradient
induce temporal change in brightness
discernible motion



only the motion component
in the direction of local gradient
induce temporal change in brightness
discernible motion

# Aperture Problem



- intensity gradient is zero
  no constraints on $(u,v)$ $(0,0)\cdot(u,v)=0$
  interpolated from other places

- intensity gradient is nonzero
  but is *constant*
  one constraints on $(u,v)$ $(\frac{\partial I}{\partial x},\frac{\partial I}{\partial y})\cdot(u,v)=-\frac{\partial I}{\partial t}$
  only the component along the gradient are recoverable

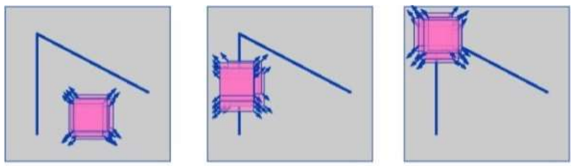
- intensity gradient is nonzero
  and *changing*
  multiple constraints on $(u,v)$
  motion recoverable

$$(\frac{\partial I}{\partial x_1},\frac{\partial I}{\partial y_1})\cdot(u,v)=-\frac{\partial I}{\partial t}_{(x_1,y_1)}$$

$$(\frac{\partial I}{\partial x_2},\frac{\partial I}{\partial y_2})\cdot(u,v)=-\frac{\partial I}{\partial t}_{(x_2,y_2)}$$

# 3.2 Addition: Harris Corner Detector (1/3)

- Basic Idea



"flat" region: no change in all directions | "edge": no change along the edge direction | "corner": significant change in all directions

$$E(u,v)=\sum_{x,y} w(x,y)\left[I(x+u,y+v)-I(x,y)\right]^2$$

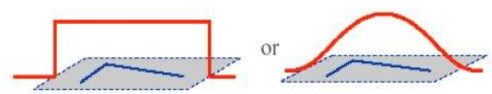
Window function | Shifted intensity | Intensity

textureless

For nearly constant patches, this will be near 0.
For very distinctive patches, this will be larger.
Hence... we want patches where E(u,v) is LARGE.

Smoothing, weight

Window function $w(x,y)$ =




1 in window, 0 outside | Gaussian

in SSD:

$$\sum\left[I(x+u,y+v)-I(x,y)\right]^2 \approx \sum\left[I(x,y)+uI_x+vI_y-I(x,y)\right]^2$$ First order approx

$$=\sum u^2I_x^2+2uvI_xI_y+v^2I_y^2$$

$$=\begin{bmatrix} u & v \end{bmatrix}\left(\sum\begin{bmatrix} I_x^2 & I_xI_y \\ I_xI_y & I_y^2 \end{bmatrix}\right)\begin{bmatrix} u \\ v \end{bmatrix}$$ Rewrite as matrix equation

$$=\sum\begin{bmatrix} u & v \end{bmatrix}\begin{bmatrix} I_x^2 & I_xI_y \\ I_xI_y & I_y^2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$$

Taylor Series for 2D function

$$f(x+u,y+v)=f(x,y)+uf_x(x,y)+vf_y(x,y)+$$

**First partial derivatives**

$$\frac{1}{2!}\left[u^2f_{xx}(x,y)+uvf_{xy}x,y+v^2f_{yy}(x,y)\right]+$$

**Second partial derivatives**

$$\frac{1}{3!}\left[u^3f_{xxx}(x,y)+u^2vf_{xxy}(x,y)+uv^2f_{xyy}(x,y)+v^3f_{yyy}(x,y)\right]$$

**Third partial derivatives**

$+\ \ldots$ (Higher order terms)

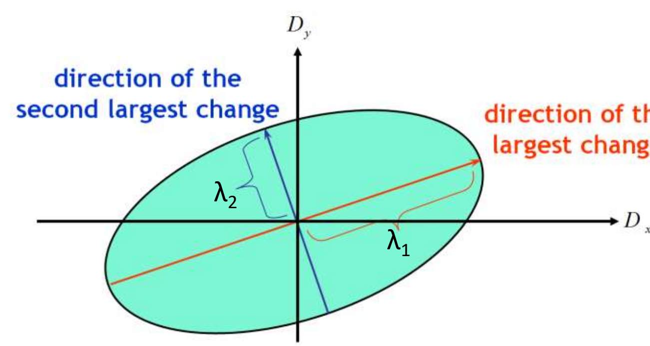
- For small shifts [u,v] we have a bilinear approximation:

$$E(u,v)\cong[u,v]\ \ M\begin{bmatrix} u \\ v \end{bmatrix}$$

Hessian Matrix: M

$$M=\sum_{x,y} w(x,y)\begin{bmatrix} I_x^2 & I_x \\ I_xI_y & I \end{bmatrix}$$

- Compute $\lambda_{1,2}$ (eigenvalues) of M

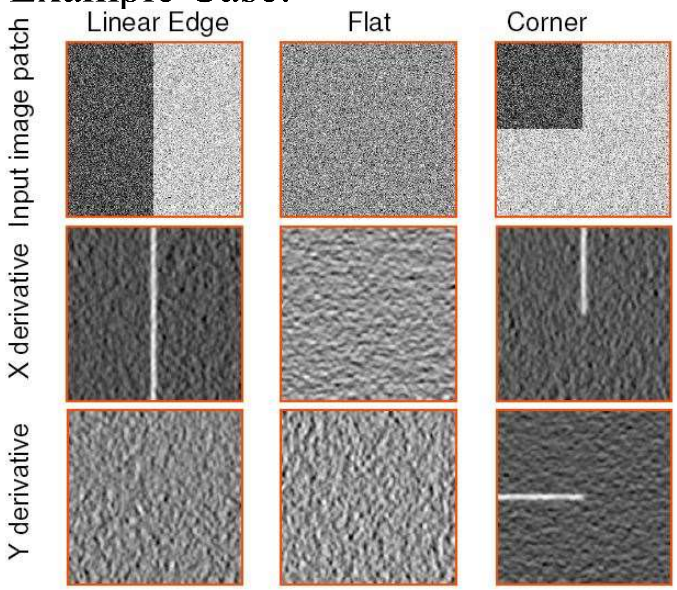




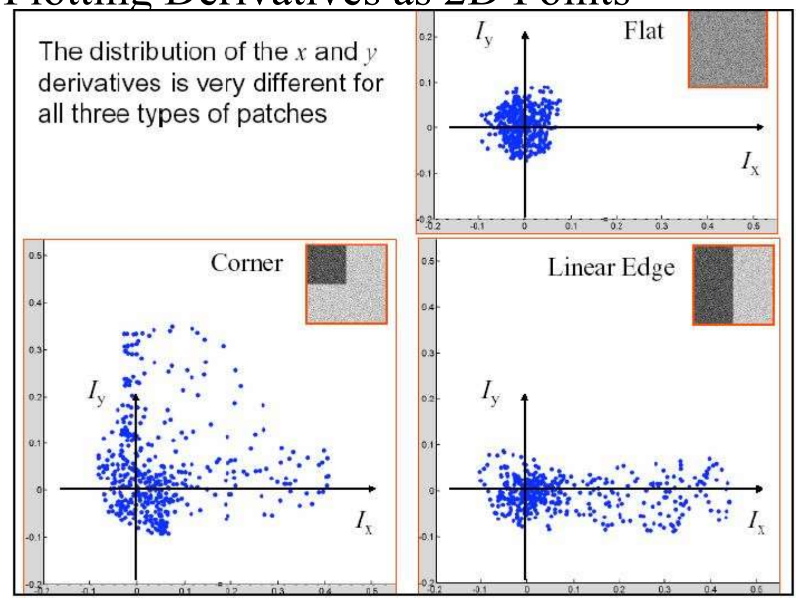

Reference
http://www.cse.psu.edu/~rcollins/CSE486/lecture06.pdf

# Addition: Harris Corner Detector (2/3)

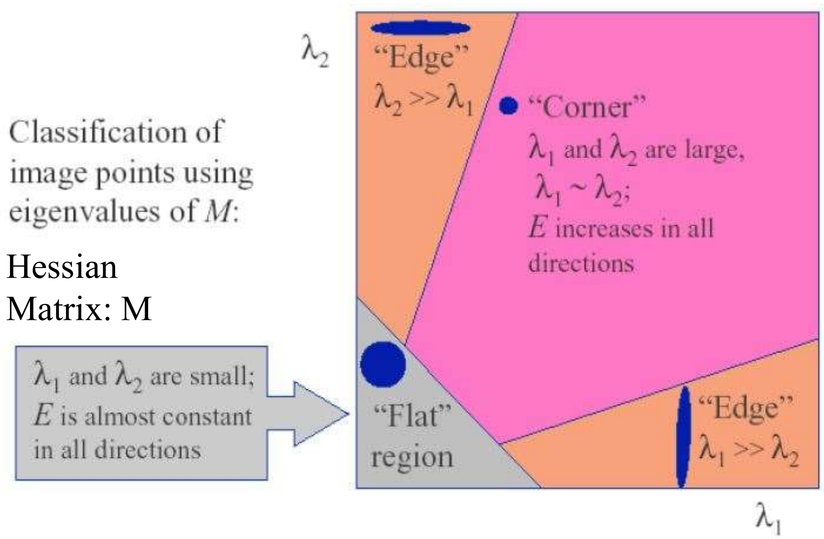
- Example Case:




- Plotting Derivatives as 2D Points

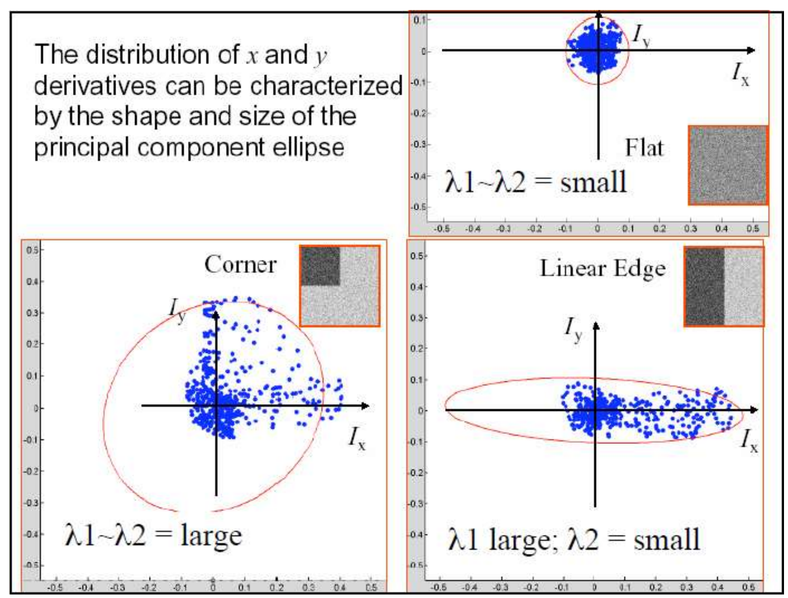



- Classification via Eigenvalues




Hessian Matrix: M

- Fitting Ellipse to each Set of Points

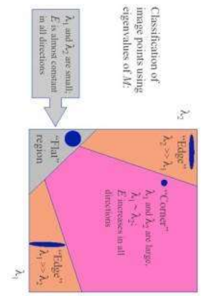


# Addition: Harris Corner Detector (3/3)

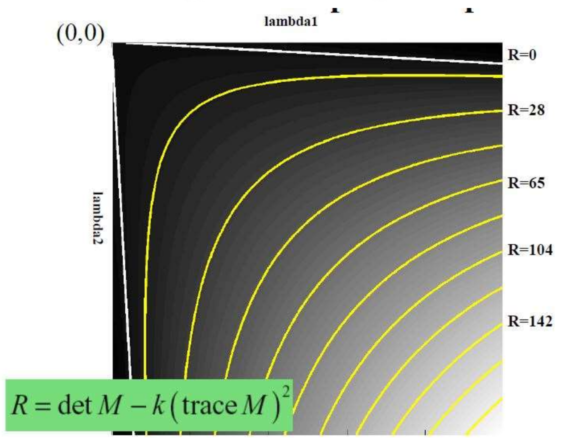
- Measure of corner response:

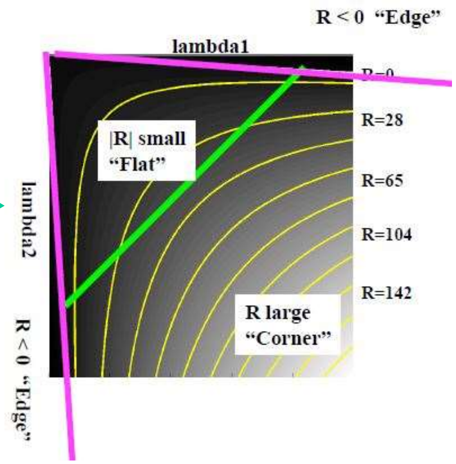
$$R = \det(M) - k\,trace(M)^2 = \lambda_0\lambda_1 - k(\lambda_0+\lambda_1)^2$$

where k is a determined constant; k = 0.04 - 0.06



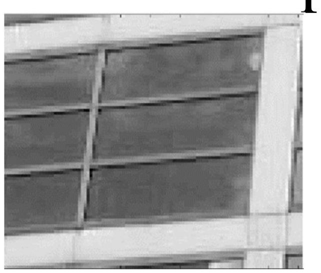
- Corner Response Map

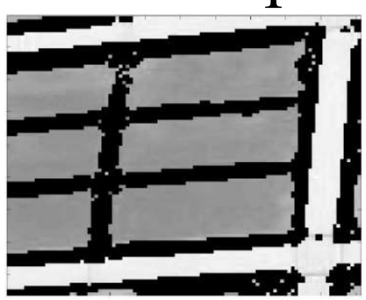


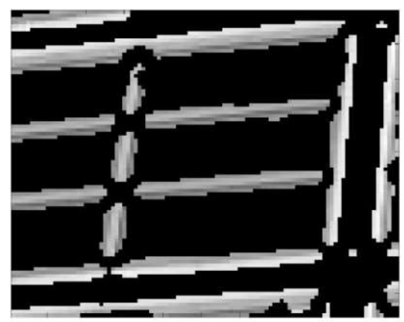

- $R$ depends only on eigenvalues of M
- $R$ is large for a corner
- $R$ is negative with large magnitude for an edge
- $|R|$ is small for a flat region

- Corner Response Example





Threshold: -10000 < R < 10000
(neither edges nor corners)



Threshold: R < -10000
(edges)

Threshold: > 10000
(corners)

# 3.3 If a Moving Object Is Planar

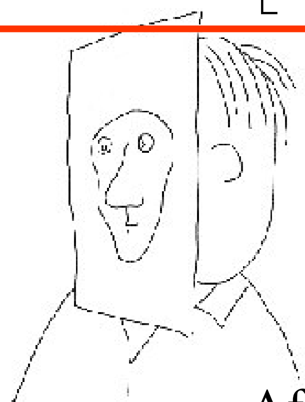
1. Velocity Base: $\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \dot{x} = \dot{X} / \dot{Z} \\ \dot{y} = \dot{Y} / \dot{Z} \end{pmatrix} =$

$$\begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} = \begin{pmatrix} a_0 + a_1 x + a_2 y + p_0 x^2 + p_1 xy \\ a_3 + a_4 x + a_5 y + p_1 xy + p_0 y^2 \end{pmatrix}$$

X+Y+Z=k

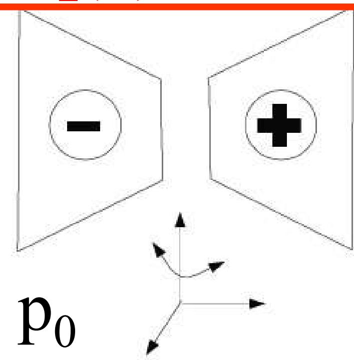
2. Displacement Base: $\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \overrightarrow{x' - x} \\ y' - y \end{bmatrix} =$

Flow=velocity
=位移 with delta_t(=1)

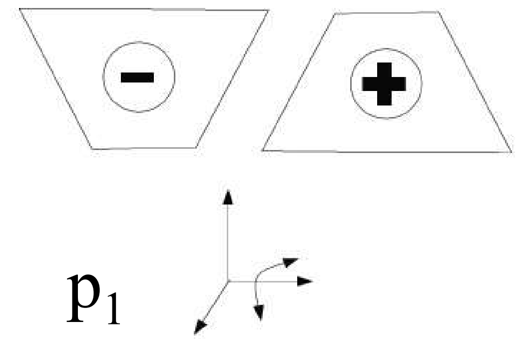


Perspective
Camera

(Pinhole
camera)



$p_0$

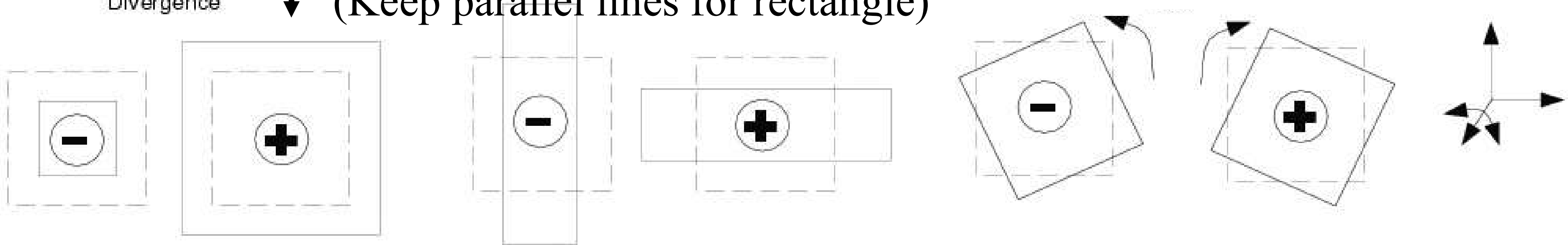


$p_1$

Planar parameters

Affine Camera
(Keep parallel lines for rectangle)

Divergence

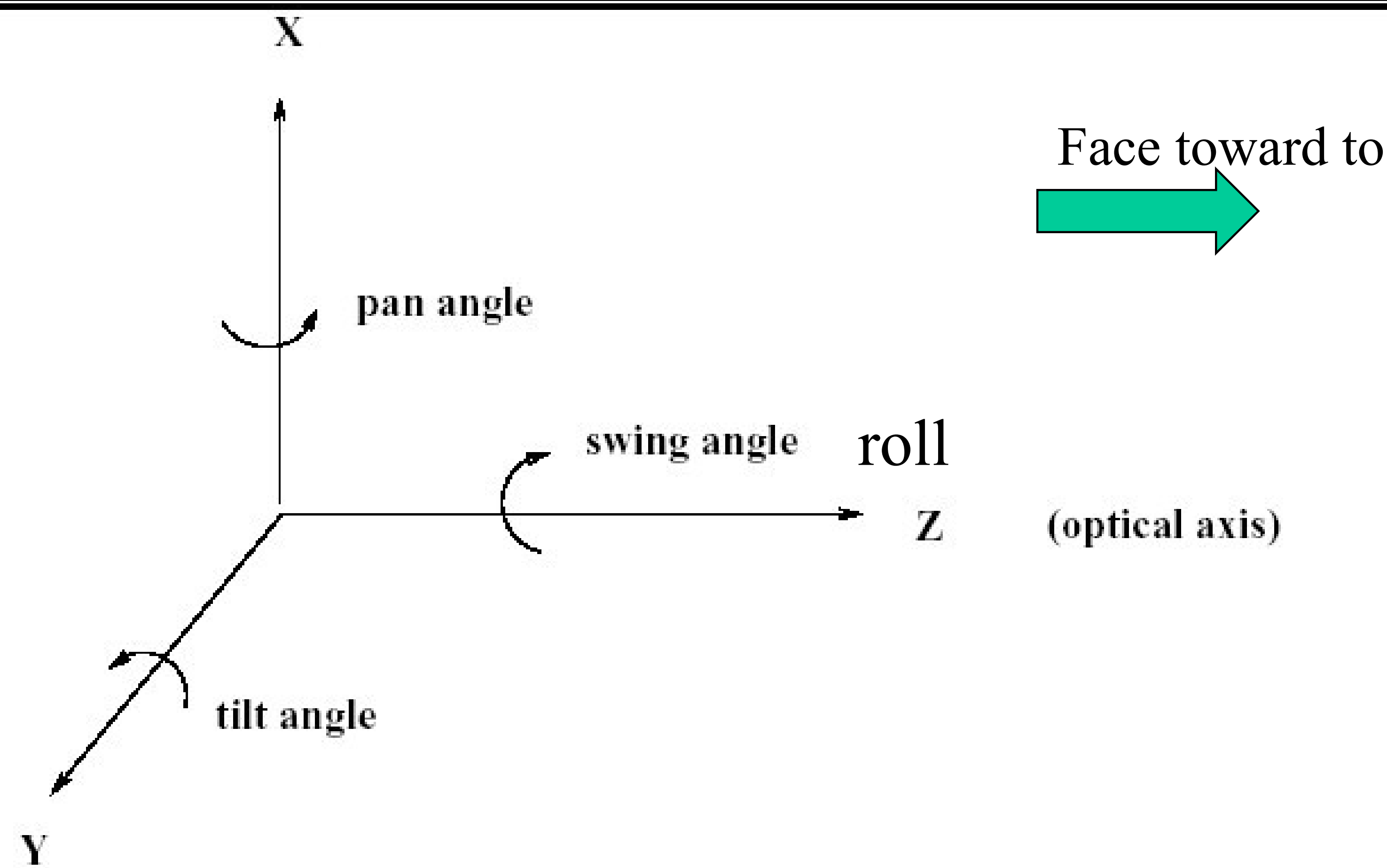


$$\text{Divergence}: a_1 + a_5 = u_x + v_y \qquad \text{Curl}: -a_2 - a_4 = -(u_y + v_x)$$

$$\text{Deformation}: a_1 - a_5 = u_x - v_y$$

# 3.3.1 In the Velocity-Based Scheme

Rotation Matrix Representation: Euler angles



# Rotation Matrix Representation: Euler angles

Assume rotation matrix $R$ results from successive Euler rotations of the camera frame around its $X$ axis by $\omega$, its once rotated $Y$ axis by $\phi$, and its twice rotated $Z$ axis by $\kappa$, then

omega, phi, kappa

$$R(\omega, \phi, \kappa) = R_X(\omega) R_Y(\phi) R_Z(\kappa)$$

where $\omega$, $\phi$, and $\kappa$ are often referred to as tilt, pan, and swing angles respectively.

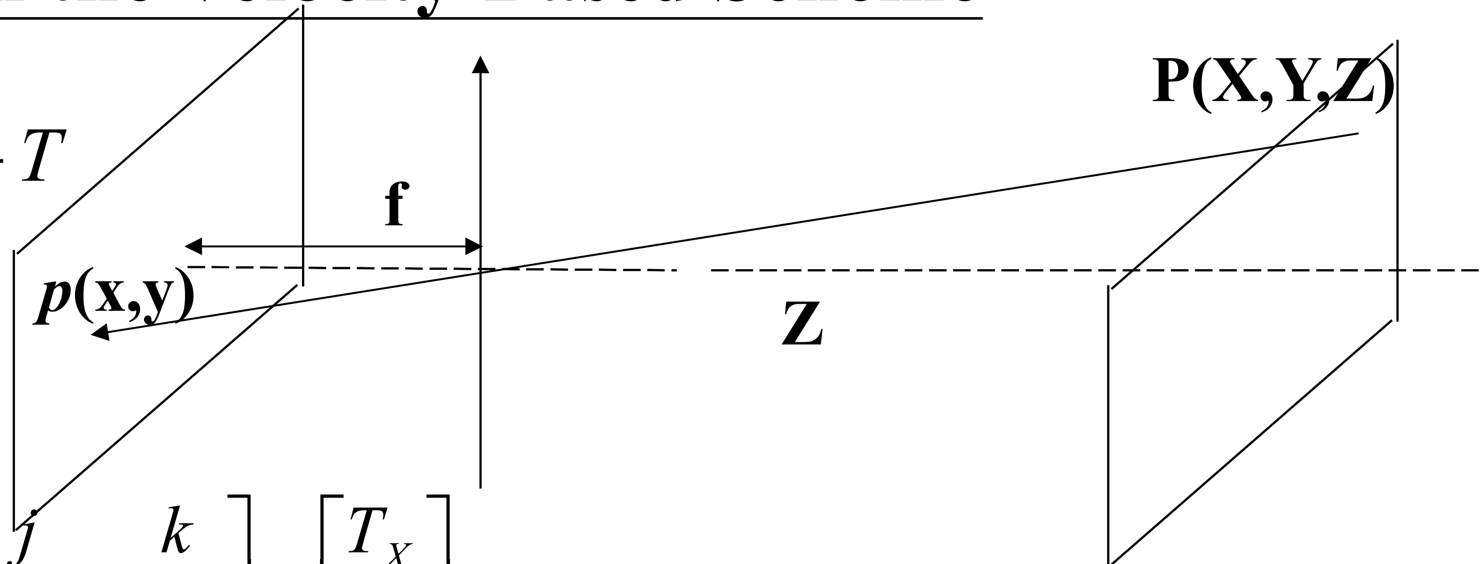
$$R_x(\omega) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\omega & \sin\omega \\ 0 & -\sin\omega & \cos\omega \end{pmatrix}$$

$$R_y(\phi) = \begin{pmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{pmatrix}$$

$$R_z(\kappa) = \begin{pmatrix} \cos\kappa & \sin\kappa & 0 \\ -\sin\kappa & \cos\kappa & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

# In the Velocity-Based Scheme

$$\dot{P} = \Omega \times P + T$$




omega

$$\begin{bmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{bmatrix} = \begin{bmatrix} i & j & k \\ \Omega_X & \Omega_Y & \Omega_Z \\ X & Y & Z \end{bmatrix} + \begin{bmatrix} T_X \\ T_Y \\ T_Z \end{bmatrix}$$

$$\begin{bmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{bmatrix} = \begin{bmatrix} \Omega_Y Z - \Omega_Z Y + T_X \\ \Omega_Z X - \Omega_X Z + T_Y \\ \Omega_X Y - \Omega_Y X + T_Z \end{bmatrix}$$

$$\left( \begin{matrix} \dfrac{x}{f=1} = \dfrac{X}{Z} \\ \dfrac{y}{f=1} = \dfrac{Y}{Z} \end{matrix} \right)$$

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \dot{x} = \dot{X}/\dot{Z} \\ \dot{y} = \dot{Y}/\dot{Z} \end{pmatrix} = \begin{pmatrix} \dfrac{\dot{X}}{Z} - \dfrac{X\dot{Z}}{Z^2} \\ \dfrac{\dot{Y}}{Z} - \dfrac{Y\dot{Z}}{Z^2} \end{pmatrix} = \frac{1}{Z^2}\begin{pmatrix} \dot{X}Z - X\dot{Z} \\ \dot{Y}Z - Y\dot{Z} \end{pmatrix}$$

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -\Omega_X xy + \Omega_Y(1+x^2) - \Omega_Z y + (T_X - T_Z x)/Z \\ -\Omega_X(1+y^2) + \Omega_Y xy + \Omega_Z x + (T_Y - T_Z y)/Z \end{pmatrix} = \begin{pmatrix} u_R + u_T \\ v_R + v_T \end{pmatrix}$$

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \left(\Omega_Y + \dfrac{T_X}{Z}\right) + \left(-\dfrac{T_Z}{Z}\right)x + (-\Omega_Z)y + \Omega_Y x^2 + (-\Omega_X)xy \\ \left(-\Omega_X + \dfrac{T_Y}{Z}\right) + \Omega_Z x + \left(-\dfrac{T_Z}{Z}\right)y + \Omega_Y xy + (-\Omega_X)y^2 \end{pmatrix}$$

X+Y+Z=k

Flow=velocity=位移

$$\longrightarrow \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \dot{x} = \dot{X}/\dot{Z} \\ \dot{y} = \dot{Y}/\dot{Z} \end{pmatrix} = \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} = \begin{pmatrix} a_0 + a_1 x + a_2 y + p_0 x^2 + p_1 xy \\ a_3 + a_4 x + a_5 y + p_1 xy + p_0 y^2 \end{pmatrix}$$

# 3.3.2 In the Displacement-Based Scheme

- **Convert a rotation vector a=[$a_x$,$a_y$,$a_z$] into a rotation matrix**
- **Skew symmetric matrix**
  - The cross product of two vectors can be written in terms of a skew symmetric matrix

$$a, b \in R^3$$

$$a \times b = \hat{a} b = J(a) b$$

$$\hat{a} = J(a) = \begin{pmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{pmatrix} \in so(3)$$

# Rotation Matrix Representation: Quaternion

$R = (dI + S)(dI - S)^{-1}$ where

$$S = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}$$

$$R = \begin{pmatrix} d^2 + a^2 - b^2 - c^2 & 2(ab - cd) & 2(ac + bd) \\ 2(ab + cd) & d^2 - a^2 + b^2 - c^2 & 2(bc - ad) \\ 2(ac - bd) & 2(bc + ad) & d^2 - a^2 - b^2 + c^2 \end{pmatrix}$$

where $a^2 + b^2 + c^2 + d^2 = 1$ and $a$, $b$, c, and $d$ are referred to as the quaternion parameters.

# Rotation Matrix Representation: $RR^t = 1$

Prove $RR'$=1

From the above definition of $S$, we have

$$
\begin{aligned}
(dI + S)^t &= (dI - S) \\
(dI - S)^t &= (dI + S)
\end{aligned}
$$

As a result, we have

$$
\begin{aligned}
RR^t &= (dI + S)(dI - S)^{-1}[(dI + S)(dI - S)^{-1}]^t \\
&= (dI + S)(dI - S)^{-1}(dI + S)^{-1}(dI - S) \\
&= (dI + S)[(dI + S)(dI - S)]^{-1}(dI - S)
\end{aligned}
$$

$$
\begin{aligned}
&= (dI + S)[(dI + S)(dI - S)]^{-1} \\
&\quad (dI - S)(dI + S)(dI + S)^{-1} \\
&= (dI + S)(dI + S)^{-1} \\
&= I
\end{aligned}
$$

Note $(dI + S)(dI - S) = (dI - S)(dI + S)$

# In the Displacement-Based Scheme

**In the Velocity-Based Scheme**

$$\dot{P} = \Omega \times P + T$$

$$\begin{bmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{bmatrix} = \begin{bmatrix} \Omega_Y Z - \Omega_Z Y + T_X \\ \Omega_Z X - \Omega_X Z + T_Y \\ \Omega_X Y - \Omega_Y X + T_Z \end{bmatrix}$$

$$\begin{pmatrix} \dfrac{x}{f=1} = \dfrac{X}{Z} \\ \dfrac{y}{f=1} = \dfrac{Y}{Z} \end{pmatrix}$$

**In the Dispacement-Based Scheme**

$$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = R \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + T = \begin{bmatrix} 1 & -\Omega_Z & \Omega_Y \\ \Omega_Z & 1 & -\Omega_X \\ -\Omega_Y & \Omega_X & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + \begin{bmatrix} T_X \\ T_Y \\ T_Z \end{bmatrix}$$

Position at time t

Position at time t-1

$$= \begin{bmatrix} \boxed{X} - \Omega_Z Y + \Omega_Y Z + T_X \\ \Omega_Z X + \boxed{Y} - \Omega_X Z + T_Y \\ -\Omega_Y X + \Omega_X Y + \boxed{Z} + T_Z \end{bmatrix} = Z \begin{bmatrix} x - \Omega_Z y + \Omega_Y + \dfrac{T_X}{Z} \\ \Omega_Z x + y - \Omega_x + \dfrac{T_Y}{Z} \\ -\Omega_Y x + \Omega_X y + 1 + \dfrac{T_Z}{Z} \end{bmatrix}$$

C •See Fang and Huang

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \left(\Omega_Y + \frac{T_X}{Z}\right) + \left(-\frac{T_Z}{Z}\right)x + (-\Omega_Z)y + \Omega_Y x^2 + (-\Omega_X)xy \\ \left(-\Omega_X + \frac{T_Y}{Z}\right) + \Omega_Z x + \left(-\frac{T_Z}{Z}\right)y + \Omega_Y xy + (-\Omega_X)y^2 \end{pmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \frac{X'}{Z'} \\ \frac{Y'}{Z'} \end{bmatrix} = \begin{bmatrix} \frac{x - \Omega_Z y + \Omega_Y + T_X / Z}{-\Omega_Y x + \Omega_X y + 1 + T_Z / Z} \\ \frac{\Omega_Z x + y - \Omega_X + T_Y / Z}{-\Omega_Y x + \Omega_X y + 1 + T_Z / Z} \end{bmatrix}$$

Flow=velocity=位移

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} x' - x \\ y' - y \end{bmatrix} = \begin{bmatrix} \frac{-\Omega_X xy + \Omega_Y (1 + x^2) - \Omega_Z y + (T_X - T_Z x)/Z}{1 + (\Omega_X y - \Omega_Y x) + T_Z / Z} \\ \frac{-\Omega_X (1 + y^2) + \Omega_Y xy + \Omega_Z x + (T_Y - T_Z y)/Z}{1 + (\Omega_X y - \Omega_Y x) + T_Z / Z} \end{bmatrix}$$

Flow from
t-1 to t over
delta_t=1

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \left(\Omega_Y + \frac{T_X}{Z}\right) + \left(-\frac{T_Z}{Z}\right)x + (-\Omega_Z)y + \Omega_Y x^2 + (-\Omega_X)xy \\ \left(-\Omega_X + \frac{T_Y}{Z}\right) + \Omega_Z x + \left(-\frac{T_Z}{Z}\right)y + \Omega_Y xy + (-\Omega_X)y^2 \end{pmatrix}$$

# 3.3.3 Motion: Displacement Field or Velocity Field ??

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -\Omega_X xy + \Omega_Y(1+x^2) - \Omega_Z y + (T_X - T_Z x)/Z \\ -\Omega_X(1+y^2) + \Omega_Y xy + \Omega_Z x + (T_Y - T_Z y)/Z \end{pmatrix} = \begin{pmatrix} u_R + u_T \\ v_R + v_T \end{pmatrix}$$

---

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} x' - x \\ y' - y \end{bmatrix} = \begin{bmatrix} \dfrac{-\Omega_X xy + \Omega_Y(1+x^2) - \Omega_Z y + (T_X - T_Z x)/Z}{\boxed{1 + (\Omega_X y - \Omega_Y x) + T_Z / Z}} \\ \dfrac{-\Omega_X(1+y^2) + \Omega_Y xy + \Omega_Z x + (T_Y - T_Z y)/Z}{\boxed{1 + (\Omega_X y - \Omega_Y x) + T_Z / Z}} \end{bmatrix}$$

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \left(\Omega_Y + \frac{T_X}{Z}\right) + \left(-\frac{T_Z}{Z}\right)x + (-\Omega_Z)y + \Omega_Y x^2 + (-\Omega_X)xy \\ \left(-\Omega_X + \frac{T_Y}{Z}\right) + \Omega_Z x + \left(-\frac{T_Z}{Z}\right)y + \Omega_Y xy + (-\Omega_X)y^2 \end{pmatrix}$$

X+Y+Z=k

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} x' - x \\ y' - y \end{bmatrix} =$$

$$\begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} = \begin{pmatrix} a_0 + a_1 x + a_2 y + p_0 x^2 + p_1 xy \\ a_3 + a_4 x + a_5 y + p_1 xy + p_0 y^2 \end{pmatrix}$$

Flow=velocity
=位移 with delta_t(=1)

- **Optical flow: Displacement field or Velocity field ?⇔**
  - If the time interval between two image frames is short enough (high sampling rate) or
  - the motion of the object is slow compared with frame rate
    - » $Z >> T_z$
    - » If small rotation parameters $\Omega_X, \Omega_Y, \Omega_Z$

# 4.1 2D (Planar) Motions (Transformation) (1/11)

## 6) 8-Parameter planar transformation Vs. perspective projection transformation

Motion flow

$$u(x, y) = a_0 + a_1 x + a_2 y + p_0 x^2 + p_1 xy$$
$$v(x, y) = a_3 + a_4 x + a_5 y + p_0 xy + p_1 y^2$$

Motion flow

$$u(x, y) = a_0 + a_1 x + a_2 y \qquad (8)$$
$$v(x, y) = a_3 + a_4 x + a_5 y + cx^2 \qquad (9)$$

$$\text{divergence} = a_1 + a_5 = (u_x + v_y), \qquad (3)$$
$$\text{curl} = -a_2 + a_4 = -(u_y - v_x), \qquad (4)$$
$$\text{deformation} = a_1 - a_5 = (u_x - v_y) \qquad (5)$$

Yaw = $p_0$,
Pitch = $p_1$ .

$a_i$: Constants.
$u(x) = [u(x,y), v(x,y)]^T$ : Horizontal and vertical components of the flow at the image point $x = (x,y)$.

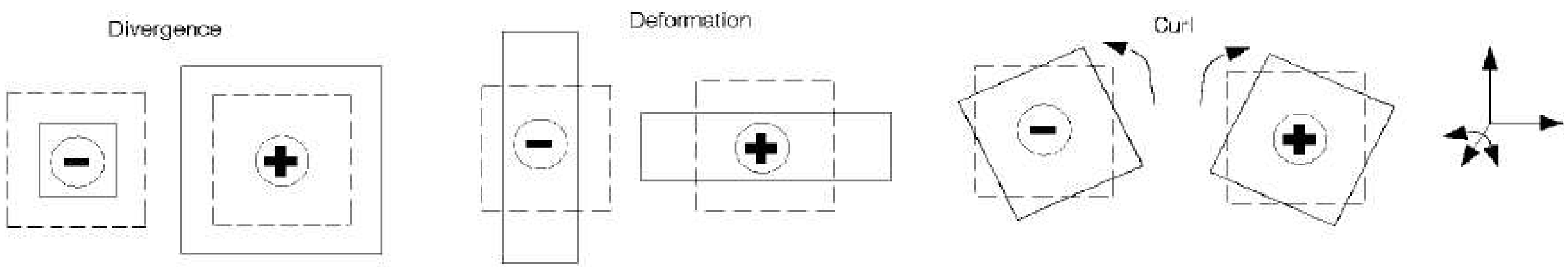



*Figure 3.* The figure illustrates the motion captured by the various parameters used to represent the motion of the regions. The solid lines indicate the deformed image region and the "−" and "+" indicate the sign of the quantity.

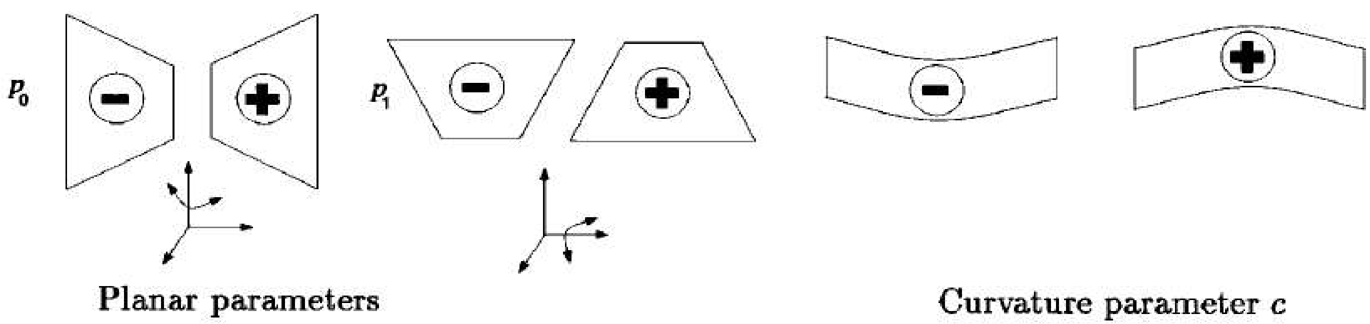



*Figure 4.* Additional parameters for planar motion and curvature.

M. Black and Y. Yaccob, "Recognizing Facial Expressions in Image Sequences Using Local Parameterized Models of Image Motion", IJCV1997.

# 2D (Planar) Motions (Transformation) (2/11)

$$\mathbf{X}(\mathbf{x}) = \begin{bmatrix} 1 & x & y & 0 & 0 & 0 & x^2 & xy & 0 \\ 0 & 0 & 0 & 1 & x & y & xy & y^2 & x^2 \end{bmatrix} \tag{10}$$

$$\mathbf{A} = [\, a_0 \;\; a_1 \;\; a_2 \;\; a_3 \;\; a_4 \;\; a_5 \;\; 0 \;\; 0 \;\; 0 \,]^T \tag{11}$$

$$\mathbf{P} = [\, a_0 \;\; a_1 \;\; a_2 \;\; a_3 \;\; a_4 \;\; a_5 \;\; p_0 \;\; p_1 \;\; 0 \,]^T \tag{12}$$

$$\mathbf{C} = [\, a_0 \;\; a_1 \;\; a_2 \;\; a_3 \;\; a_4 \;\; a_5 \;\; 0 \;\; 0 \;\; c \,]^T \tag{13}$$

such that $\mathbf{u}(\mathbf{x}; \mathbf{A}) = \mathbf{X}(\mathbf{x})\mathbf{A}$, $\mathbf{u}(\mathbf{x}; \mathbf{P}) = \mathbf{X}(\mathbf{x})\mathbf{P}$, and $\mathbf{u}(\mathbf{x}; \mathbf{C}) = \mathbf{X}(\mathbf{x})\mathbf{C}$ represent, respectively, the affine, planar, and affine + curvature flow models described above.

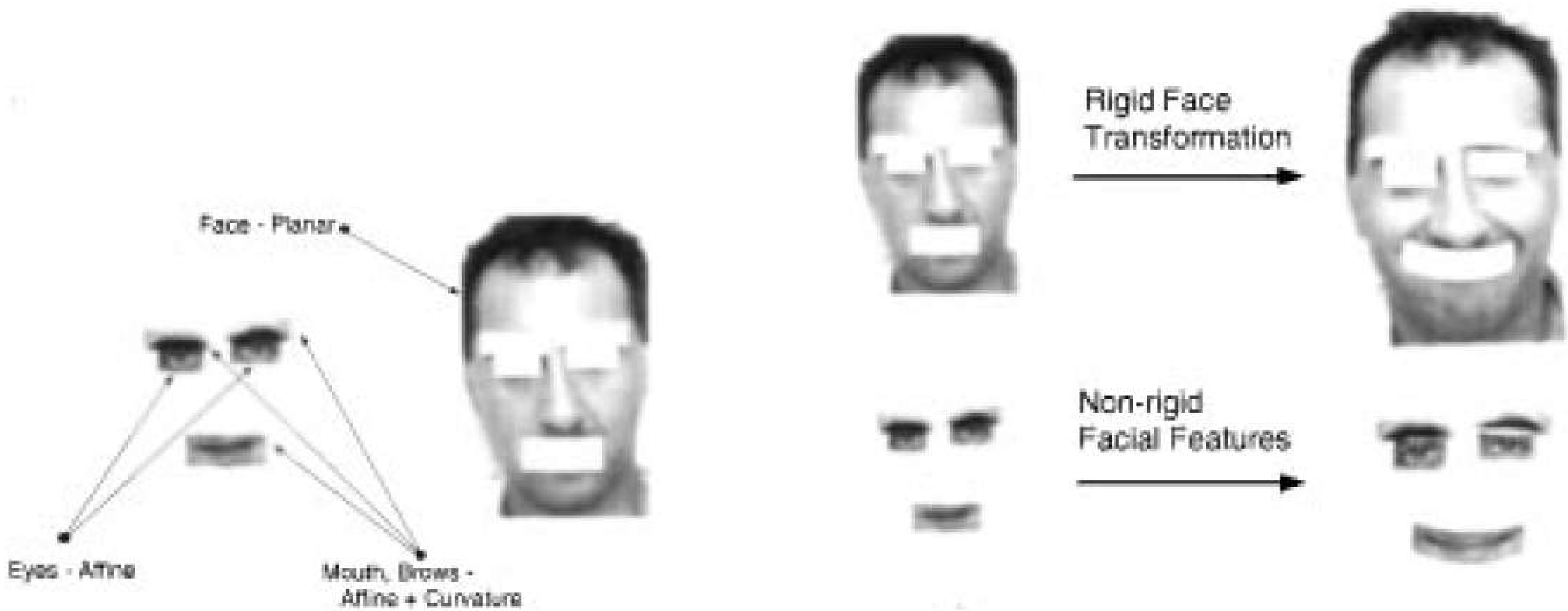



*Figure 1.* Illustration showing the parametric motion models employed and an example of a face undergoing a looming motion while smiling.

*Table 1.* The mid-level predicates derived from deformation and motion parameter estimates.

| Parameter | Threshold | Derived predicates (mouth) |
|---|---|---|
| $a_0$ | $>0.25$ | Rightward |
| | $<-0.25$ | Leftward |
| $a_3$ | $<-0.1$ | Upward |
| | $>0.1$ | Downward |
| *Div* | $>0.02$ | Expansion |
| | $<-0.02$ | Contraction |
| *Def* | $>0.005$ | Horizontal deformation |
| | $<-0.005$ | Vertical deformation |
| *Curl* | $>0.005$ | Clockwise rotation |
| | $<-0.005$ | Counter clockwise rotation |
| $c$ | $<-0.0001$ | Curving upward ('U' like) |
| | $>0.0001$ | Curving downward |

*Table 2.* The mid-level predicates derived from deformation and motion parameter estimates as applied to head motion.

| Parameter | Threshold | Derived predicates (head) |
|---|---|---|
| $a_0$ | $>0.5$ | Rightward |
| | $<-0.5$ | Leftward |
| $a_3$ | $<-0.5$ | Upward |
| | $>0.5$ | Downward |
| *Div* | $>0.01$ | Expansion |
| | $<-0.01$ | Contraction |
| *Def* | $>0.01$ | Horizontal deformation |
| | $<-0.01$ | Vertical deformation |
| *Curl* | $>0.005$ | Clockwise rotation |
| | $<-0.005$ | Counter clockwise rotation |
| $p_0$ | $<-0.00005$ | Rotate right about neck |
| | $>0.00005$ | Rotate left about neck |
| $p_1$ | $<-0.00005$ | Rotate forward |
| | $>0.00005$ | Rotate backward |

*Table 3.* The rules for classifying facial expressions (B = beginning, E = ending).

| Expr. | B/E | Satisfactory actions |
|---|---|---|
| Anger | B | Inward lowering of brows and mouth contraction |
| Anger | E | Outward raising of brows and mouth expansion |
| Disgust | B | Mouth horizontal expansion and lowering of brows |
| Disgust | E | Mouth contraction and raising of brows |
| Happiness | B | Upward curving of mouth and expansion or horizontal deformation |
| Happiness | E | Downward curving of mouth and contraction or horizontal deformation |
| Surprise | B | Raising brows and vertical expansion of mouth |
| Surprise | E | Lowering brows and vertical contraction of mouth |
| Sadness | B | Downward curving of mouth and upward-inward motion in inner parts of brows |
| Sadness | E | Upward curving of mouth and downward-outward motion in inner parts of brows |
| Fear | B | Expansion of mouth and raising-inwards inner parts of brows |
| Fear | E | Contraction of mouth and lowering inner parts of brows |

# 2D (Planar) Motions (Transformation) (4/11)

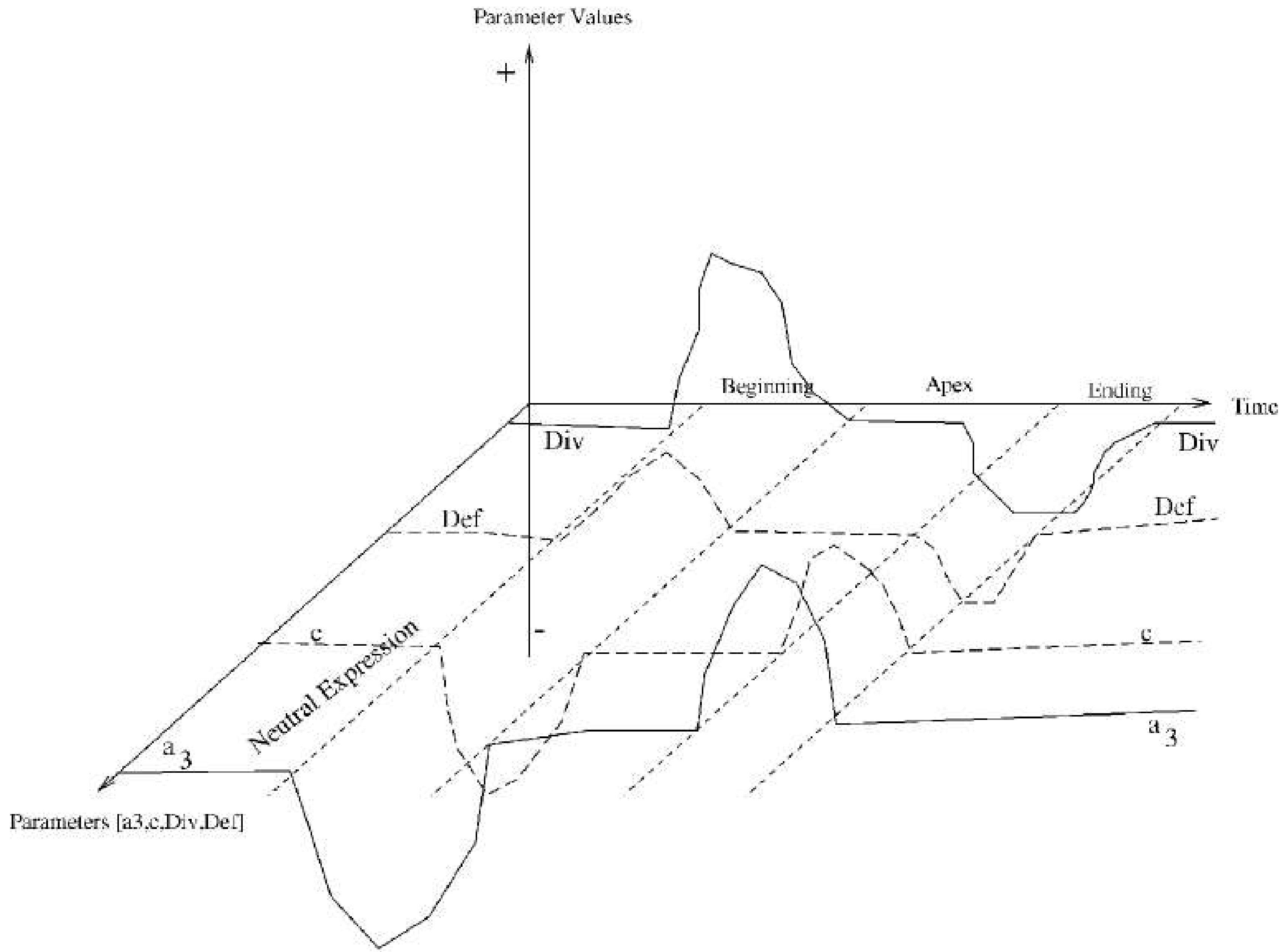



*Figure 5.* The temporal model of the "smile" expression.

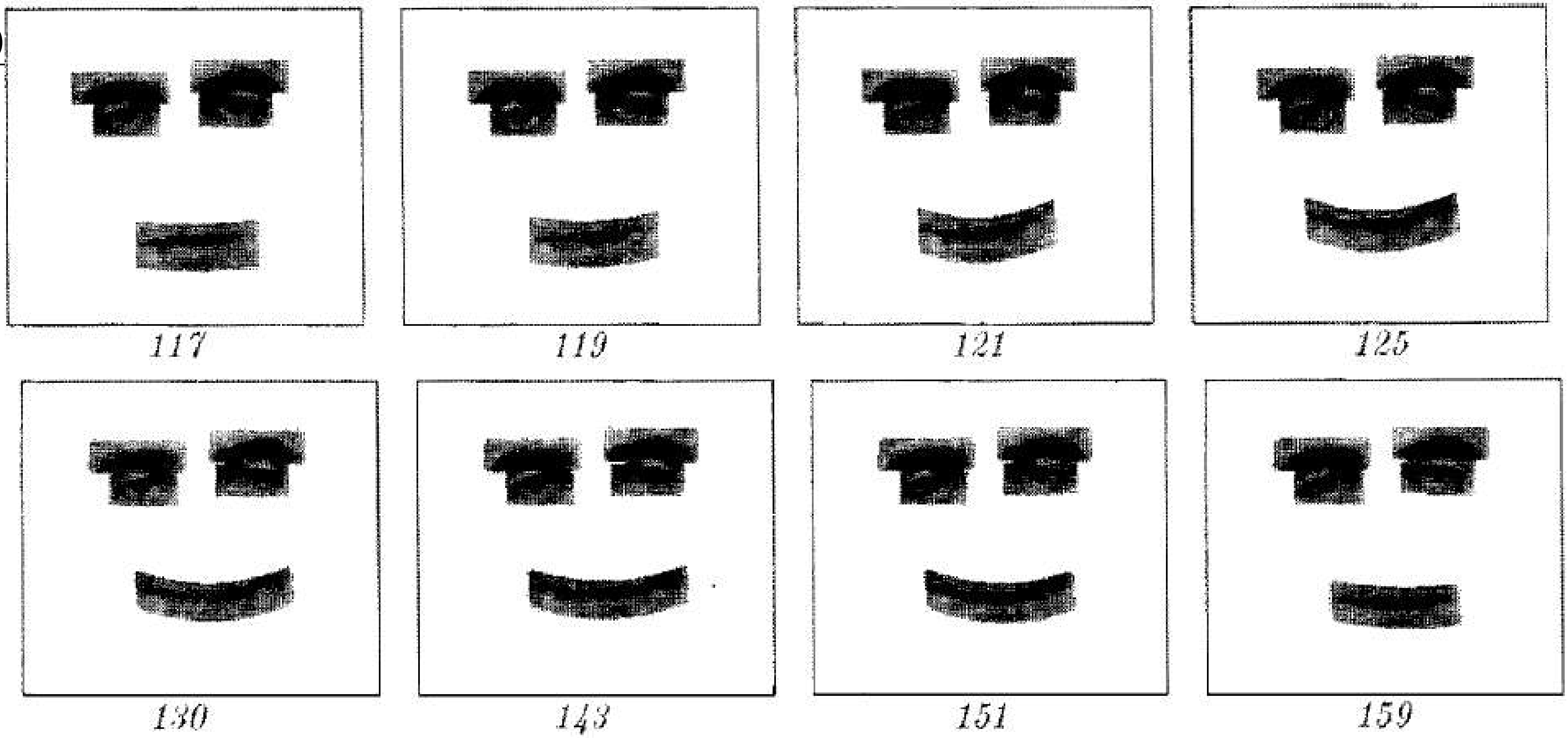



*Figure 6.* Smile experiment: facial expression tracking.

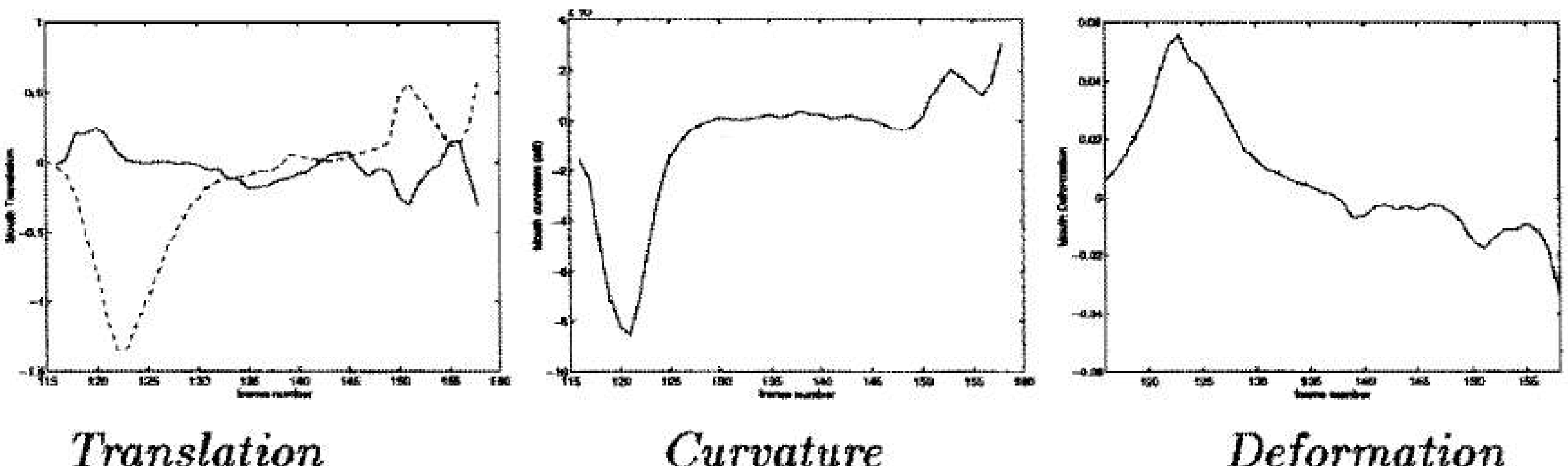



*Figure 7.* The smile mouth parameters. For translation, solid and dashed lines indicate horizontal and vertical motion respectively.

# 2D (Planar) Motions (Transformation) (6/11)

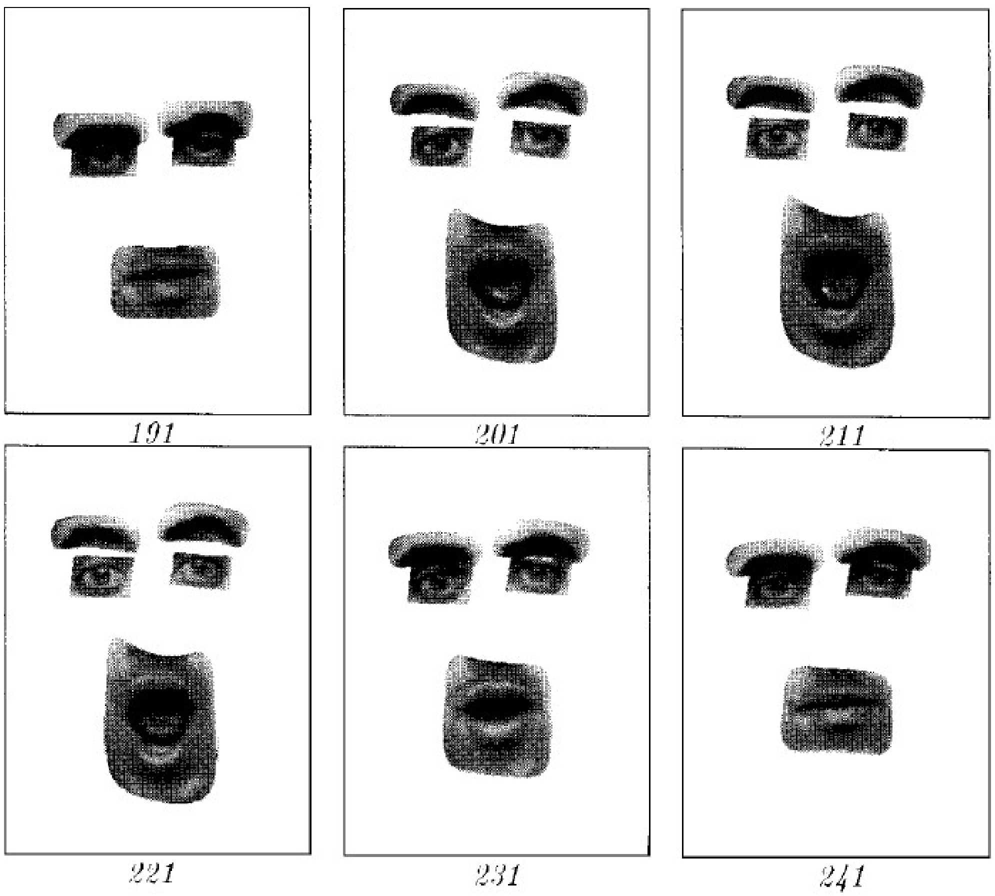



*Figure 10.* Surprise experiment: facial expression tracking. Features every 10 frames.

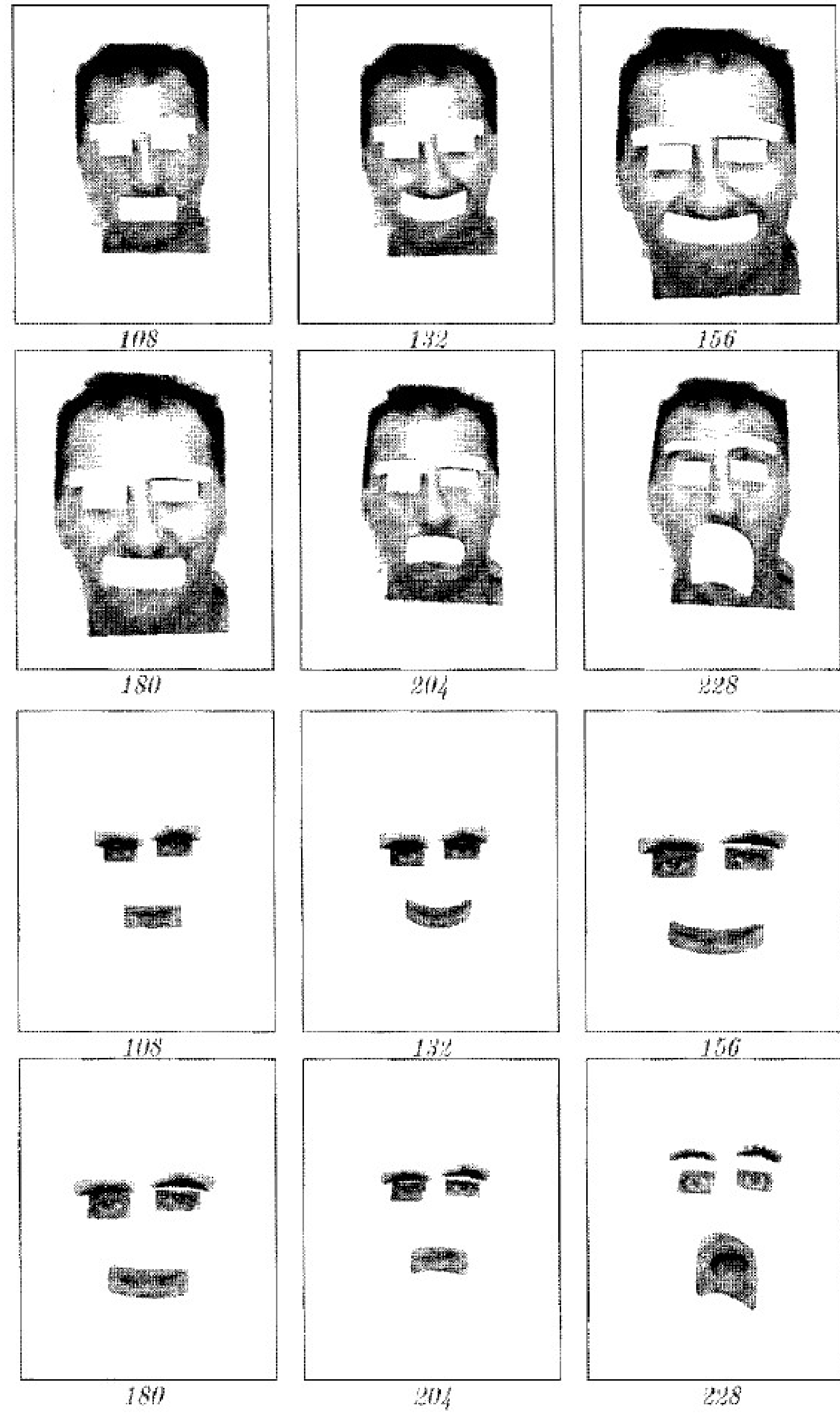



*Figure 14.* Looming experiment. Facial expression tracking with rigid head motion (every 24 frames).

# 2D (Planar) Motions (Transformation) (7/11)

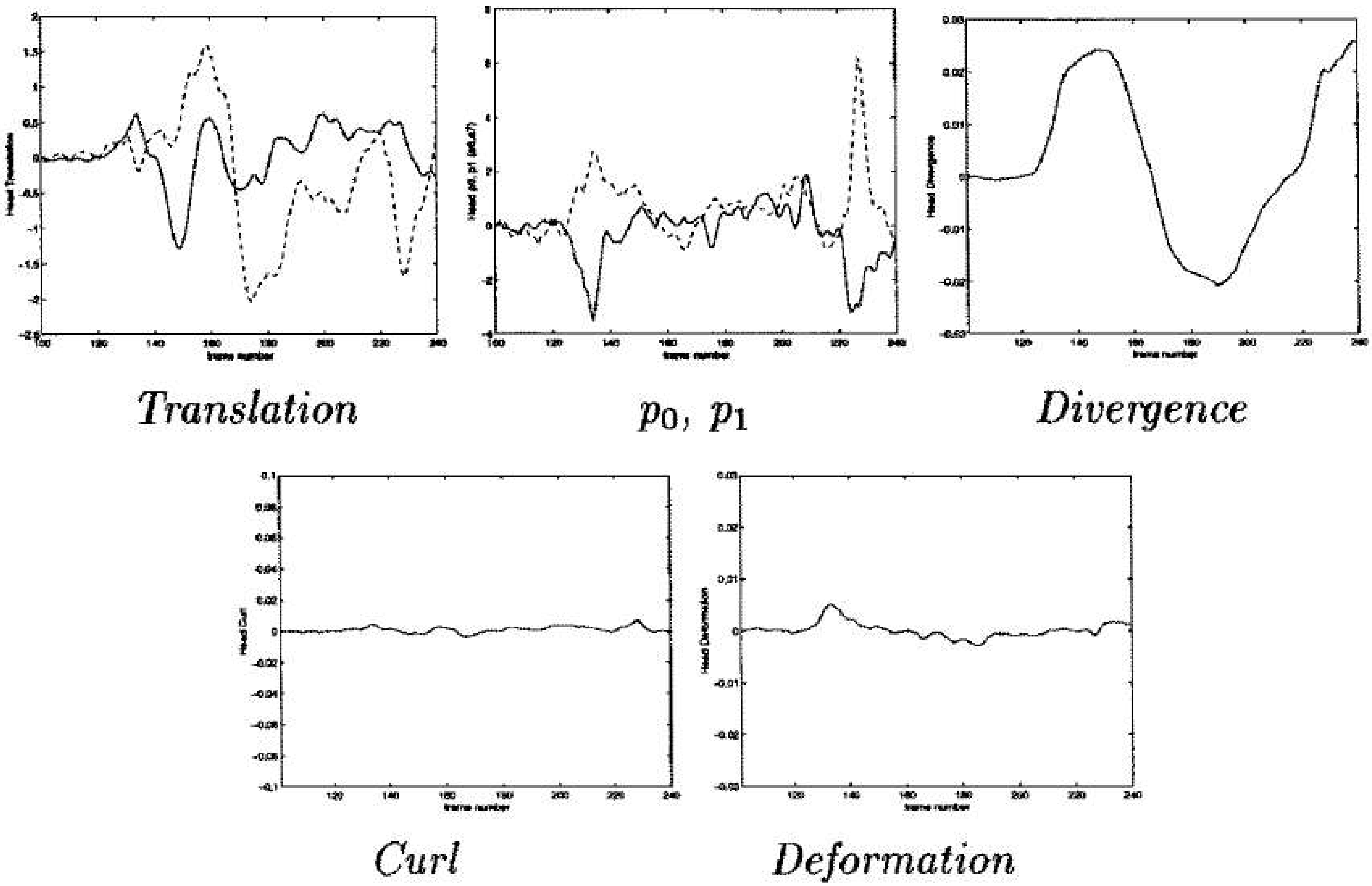



*Figure 15.* The looming face motion parameters. Translation: solid = horizontal, dashed = vertical. Quadratic terms: solid = $p_0$, dashed = $p_1$.

# 2D (Planar) Motions (Transformation) (8/11)

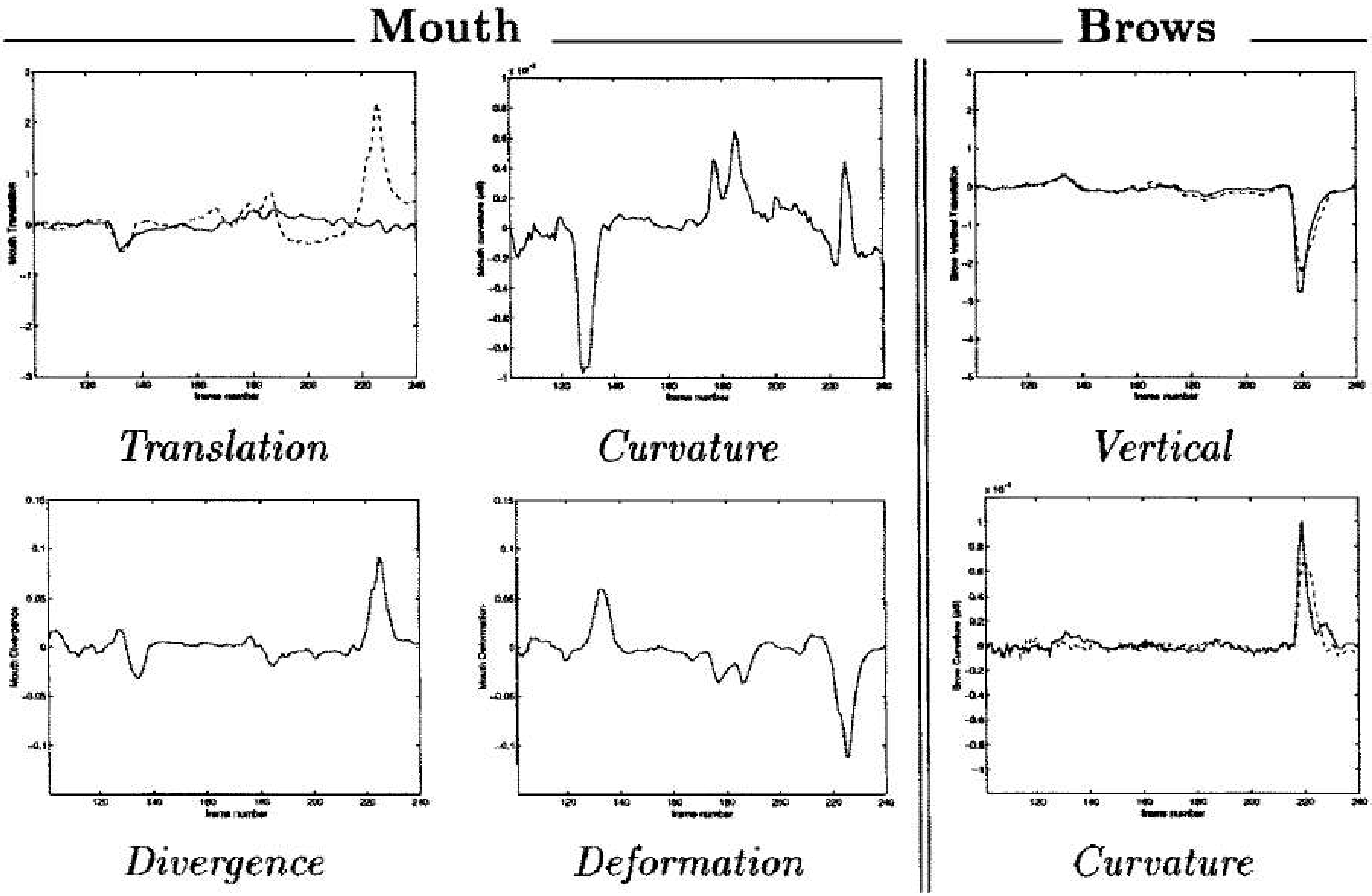



*Figure 16.* The looming sequence. Mouth translation: solid and dashed lines indicate horizontal and vertical motion respectively. For the brows, the solid and dashed lines indicate left and right brows respectively.



Model

# 2D (Planar) Motions (Transformation) (9/11)

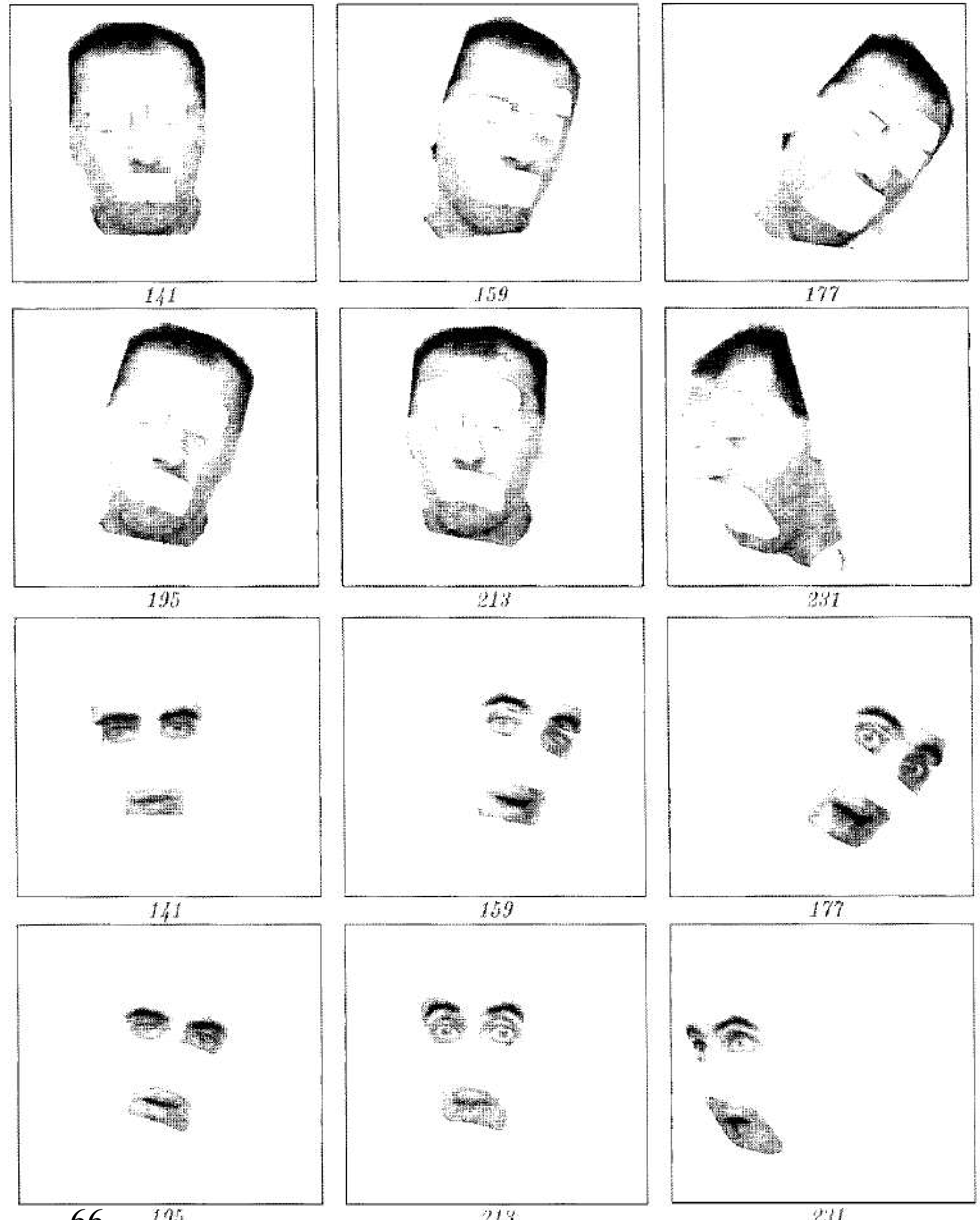



*Figure 17.* Rotation experiment. Rigid head tracking, every 18th frame.

# 2D (Planar) Motions (Transformation) (10/11)

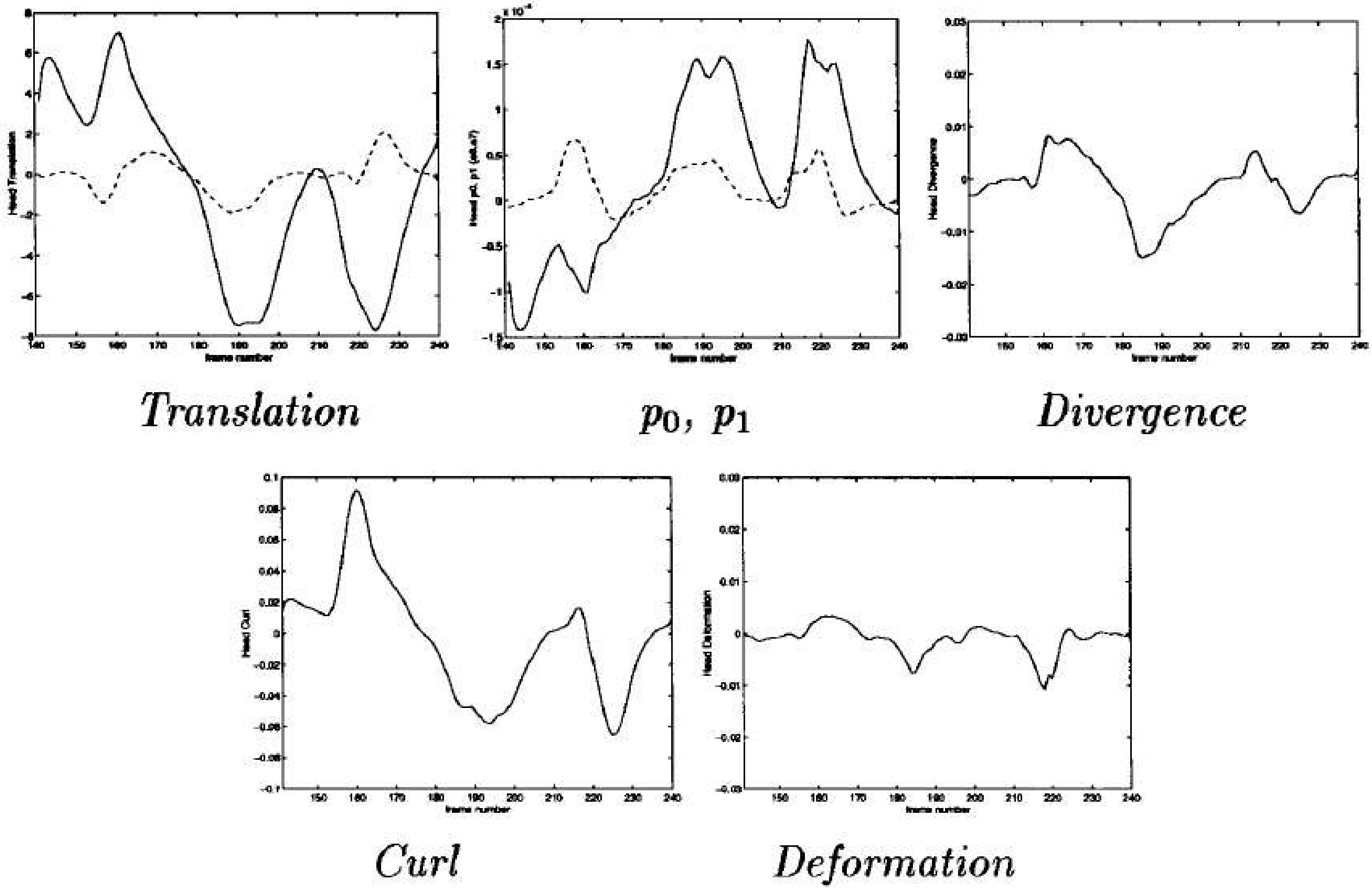



*Figure 18.* The rotate sequence face motion parameters. Translation: solid = horizontal, dashed = vertical. Quadratic terms: solid = $p_0$, dashed = $p_1$.

# 2D (Planar) Motions (Transformation) (11/11)

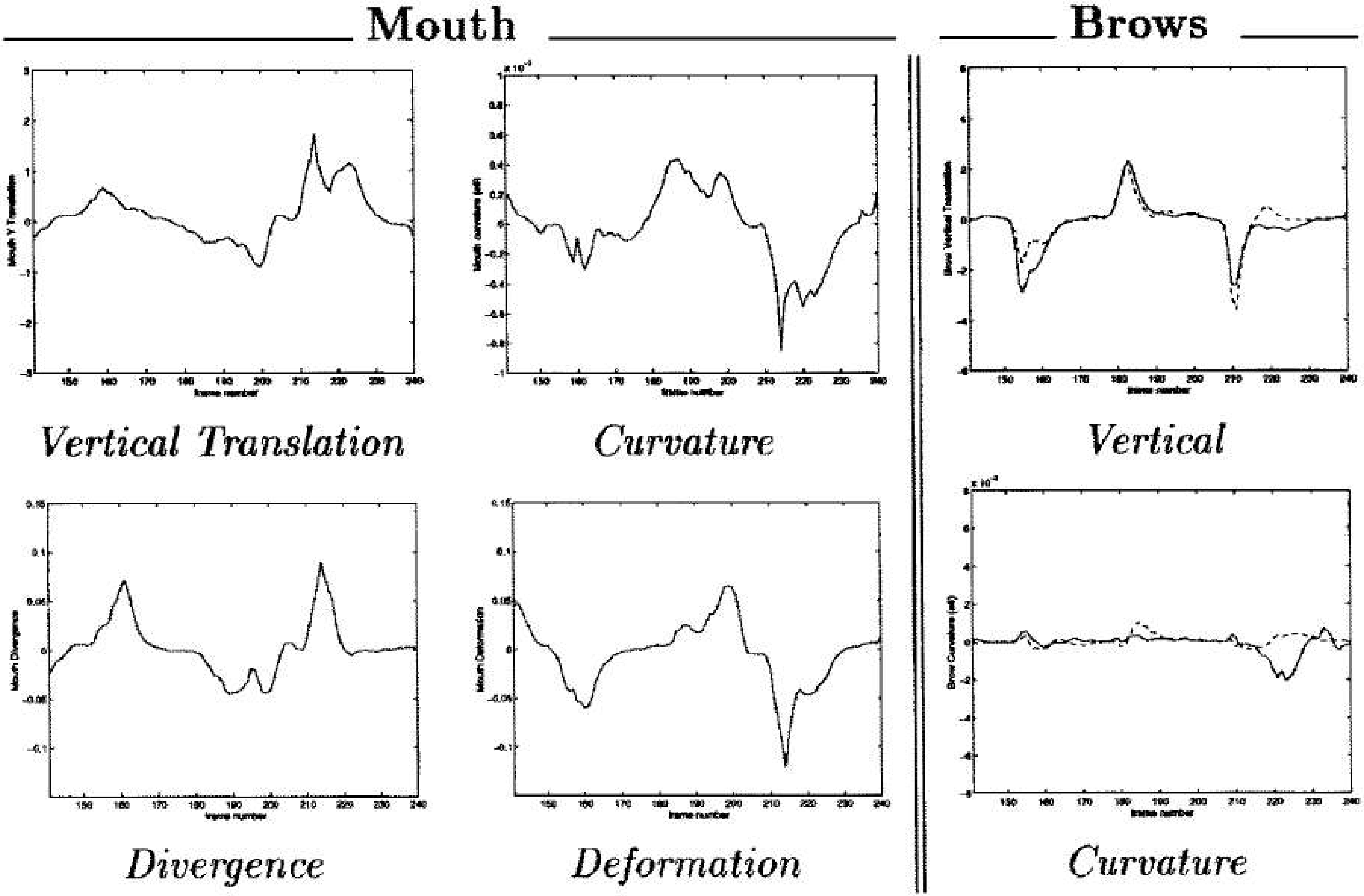



*Figure 19.* The rotate sequence. For the brows, the solid and dashed lines indicate left and right brows respectively.

# 2D (Planar) Motions (Transformation) jj

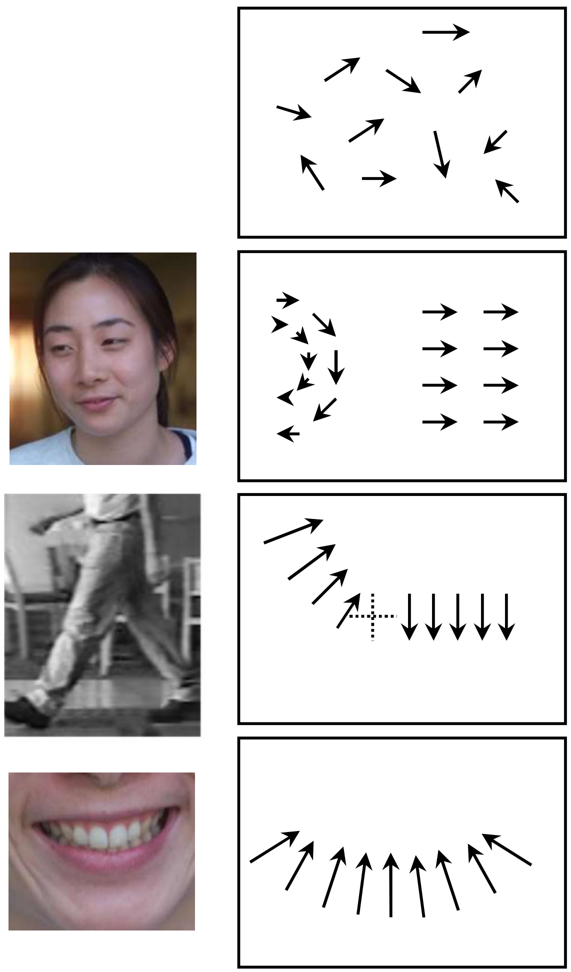
| Name | Matrix | # D.O.F. | Preserves: | Icon |
|---|---|---|---|---|
| translation | $\begin{bmatrix} I \mid t \end{bmatrix}_{2\times3}$ | 2 | orientation $+\cdots$ | |
| rigid (Euclidean) | $\begin{bmatrix} R \mid t \end{bmatrix}_{2\times3}$ | 3 | lengths $+\cdots$ | |
| similarity **$S_x$=$S_y$** | $\begin{bmatrix} sR \mid t \end{bmatrix}_{2\times3}$ | 4 | angles $+\cdots$ | |
| affine **$S_x$!= $S_y$** | $\begin{bmatrix} A \end{bmatrix}_{2\times3}$ | 6 | parallelism $+\cdots$ | |
| projective | $\begin{bmatrix} \tilde{H} \end{bmatrix}_{3\times3}$ | 8 | straight lines | |

**Table 1:** *Hierarchy of 2D coordinate transformations. The* $2\times3$ *matrices are extended with a third* $[0^T\ 1]$ *row to form a full* $3\times3$ *matrix for homogeneous coordinate transformations.*

**#D.O.F : Degrees Of Freedom**

1) **Translation :** $t_x, t_y$
2) **Euclidean :** $t_x, t_y, \theta$
3) **Similarity :** $t_x, t_y, \theta, s$
4) **Affine :** $a_{00}, a_{01}, a_{02}, a_{10}, a_{11}, a_{12}$
5) **Projective :** $h'_{00}, h'_{01}, h'_{02}, h'_{10}, h'_{11}, h'_{12}, h'_{20}, h'_{21}$
6) **8-Parameter planar transformation: a0, a1, a2, a3, a4, a5, p0 and p1**

# 4.2 Kinematic Models



- Optical Flow/Feature tracking:  no constraints
- Layered Motion:  rigid constraints
- Articulated:  kinematic chain constraints
- Nonrigid:  implicit / learned constraints

# References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- J. Shi and C. Tomasi, “Good Feature to Track,” IEEE Conference on Computer Vision and Pattern Recognition, pages 593-600, 1994.


- **Source Code:**
  - http://vision.stanford.edu/~birch/klt/