Assignment: 2026329157

7. Show that if $X_n \to c$ in probability and if g is a continuous function, then $g(X_n) \to g(c)$ in probability.

Figure 1: Problem 7 J.Rice

Question

7. Show that if $X_n \to c$ in probability and if g is a continuous function, the, $g(X_n) \to g(x)$ in probability.

Solution

Assume:

$X_n \to c$ in probability for $\varepsilon > 0$

$$\lim_{n\to\infty} P(X_n - c| \geq \ \varepsilon) = 0$$

G is a continuous function at c

To Prove:

Show that $g(X_n) \to g(c)$ in probability for every $\varepsilon > 0$:

$$\lim_{n\to\infty} P(|g(X_n) - g(c)| \geq \ \varepsilon) = 0$$

Continuity of g:

Since g is continuous at c, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that:

$$|x - c| < \delta \implies |g(x) - g(c)| < \varepsilon$$

Equivalently:

$$|g(x) - g(c)| \geq \varepsilon \implies |x - c| \geq \delta$$

Relate $g(X_n)$ to X_n:

$$P(|g(X_n) - g(c)| \geq \varepsilon\ ) \leq P(|X_n - c| \geq \ \delta)$$

This is because if $|g(X_n - g(c)| \geq \varepsilon, then\ |X_n - c| \geq \delta$

Take the limit as $n \to \infty$:

Since $X_n \to c$ in probability:

$$\lim_{n\to\infty} P(|X_n - c| \geq \delta) = 0$$

Therefore:

$$\lim_{n\to\infty} P(|g(X_n) - g(c)| \geq \varepsilon) \leq \lim_{n\to\infty} P(|X_n - c| \geq \delta$$

Since probabilities are > 0

$$\lim_{n\to\infty} P(|g(X_n) - g(c)| \geq \varepsilon) = 0$$

This shows that $g(X_n) \to g(c)$ in probability

2.Why the problem is interesting.

This problem seemed the most interesting to me because it looked short and so assumed that it could have a short solution. However, while attempting it, it surely was not a brief question. Now that I was committed to doing it, I knew that I have to add it to my semester assignment as it adds on the principle of the WLLN.

3. What I learnt when typing the questions and Problems.

This problem shows how continuous functions carry on with convergence in probability. I initially did not think that this was the case when it comes to this WLLN. One more thing I learned for sure is that typing equations is a daunting task, I won't lie and say it was not. While the use of AI might be prohibited to some extent, I learnt a lot from it pertaining this problem. In practice this could be used to ensure that if the original sequence converges in probability, then the transformed sequence will also converge in probability.