On the finite field Kakeya problem in two dimensions

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Abstract

A two-dimensional Besicovitch set over a finite field is a subset of the finite plane containing a line in each direction. In this paper, we conjecture a sharp lower bound for the size of such a subset and prove some results toward this conjecture.
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1. Introduction

The classical Kakeya problem, posed in 1917 by Kakeya, asks for a compact region in $\mathbb{R}^2$ of minimum Lebesgue measure in which one can continuously turn a unit length segment through a full 360° rotation. By 1928 Besicovitch had proved that such a region exists with arbitrarily small Lebesgue measure. Prior to this result, he also constructed a compact subset of *zero Lebesgue measure* containing a unit length segment in any direction. (Of course, one cannot continuously turn the segment in this set.) See [1] for a pleasant exposition.

The finite field Kakeya problem, originally posited by Wolff in [8], asks for the smallest subset of $\mathbb{F}_q^n$ that contains a line in each direction, where $\mathbb{F}_q$ denotes the finite field with q elements. A subset containing a line in each direction is called a *Besicovitch set*. Wolff conjectured that there is a positive constant $C = C(n)$ such that $\#B \geqslant Cq^n$ for any Besicovitch set $B \subset \mathbb{F}_q^n$. For $n = 2$ he immediately proved that $\#B \geqslant q^2/2$; Wolff's method actually gives $\#B \geqslant q(q+1)/2$.

The finite field Kakeya problem has also been investigated in [2,5–7]. These authors have concentrated their efforts toward obtaining satisfactory asymptotic lower bounds for a Besicovitch set in $\mathbb{F}_q^n$ for $n \geqslant 3$.

In this paper, we focus our attention exclusively on Besicovitch sets in $\mathbb{F}_q^2$ and sharpen Wolff's lower bound by combinatorial methods. The next section will be devoted to explaining new results. All of the proofs will be given in Section 3.

It should also be noted that the recent work [3] builds upon the techniques in the present article in order to improve one of the results. See the remark following the statement of Theorem 1.

2. Results

In $\mathbb{F}_q^2$ a line is the set of solutions of an equation $ax + by = c$ with $a, b, c \in \mathbb{F}_q$. Write $\ell(m, b)$ and $\ell(\infty, a)$ for the lines $y = mx + b$ and $x = a$, respectively.

Definition 1. A *Besicovitch set in* $\mathbb{F}_q^2$ is a set of points $B \subset \mathbb{F}_q^2$ such that for each $i \in \mathbb{F}_q \cup \{\infty\}$ there exists $b_i \in \mathbb{F}_q$ so that $\ell(i, b_i) \subset B$.

The smallest Besicovitch sets will be those that are a union of lines with distinct slopes. Regarding the size of such a set, we have the following:

Incidence Formula. *Suppose B is a Besicovitch set with $B = \bigcup_{i \in \mathbb{F}_q \cup \{\infty\}} \ell(i, b_i)$. For $P \in \mathbb{F}_q^2$, let m_P be the number of these lines passing through P. Then*

$$\#B = \frac{q(q+1)}{2} + \sum_{P \in B} \frac{(m_P - 1)(m_P - 2)}{2}.$$

In particular, every Besicovitch set contains at least $\frac{q(q+1)}{2}$ points.

Our approach will be to study the intersections of lines in a Besicovitch set in order to show that the sum in the Incidence Formula cannot be too small.

Example. Consider the set

$$B_0 = \left(\bigcup_{i \in \mathbb{F}_q} \ell\left(i, -i^2\right) \right) \cup \ell(\infty, 0).$$

One can calculate that

$$\sum_{P \in B_0} \frac{(m_P - 1)(m_P - 2)}{2} = \begin{cases} 0 & \text{if } q \text{ is even}, \\ \frac{q-1}{2} & \text{if } q \text{ is odd}. \end{cases}$$

We will perform this calculation in Section 3. If q is even, the set B_0 achieves the minimum cardinality allowed by the Incidence Formula. When q is odd one might guess that $\#B_0$ gives a sharp lower bound as well.

Conjecture 1. *If q is odd, a Besicovitch set $B \subset \mathbb{F}_q$ must have*

$$\sum_{P \in B} \frac{(m_P - 1)(m_P - 2)}{2} \geqslant \frac{q-1}{2}.$$

That is,

$$\#B \geqslant \frac{q(q+1)}{2} + \frac{q-1}{2}.$$

Our first main result is an improvement on the trivial lower bound given by the Incidence Formula.

Theorem 1. *Assume q is odd. For any Besicovitch set B, we have*

$$\sum_{P \in B} \frac{(m_P - 1)(m_P - 2)}{2} \geqslant \frac{q}{3}.$$

Observe that this estimate immediately implies Conjecture 1 for $q = 3, 5, 7$.

In [3], Cooper has refined the strategy for proving Theorem 1 and is able to obtain the stronger lower bound $(5q - 1)/14$.

On the other hand, Theorem 2 will give a sharp conditional form of Conjecture 1. Let us take a moment to motivate the hypothesis of the theorem before we state it.

Definition 2. *Let q be odd. A Besicovitch set $B \subset \mathbb{F}_q$ will be called* small *if #B does not exceed the lower bound in Conjecture 1.*

The proof of the Incidence Formula will yield the same result if B is a union of only q lines (as opposed to $q + 1$). Therefore any subset of $\mathbb{F}_q^2$ that is a union of q lines in distinct directions must contain at least $q(q + 1)/2$ points, with equality if and only if no three of the lines share a common point. If we consider the set $B_0' = \bigcup_{i \in \mathbb{F}_q} \ell(i, -i^2)$, neglecting the vertical line in the above example, one can see that $\#B_0' = q(q+1)/2$. Thus B_0' has minimum cardinality among all sets consisting of the union of q lines in distinct directions. It seems plausible that such a set has the best chance of yielding a small Besicovitch set when we adjoin one more line. Note that any set constructed in this way will have all of its points of multiplicity three lying on one line—the final line adjoined to the set. Indeed, we can prove that a Besicovitch set with this last property satisfies Conjecture 1:

Theorem 2. *Assume q is odd. Let $B = \bigcup_{i \in \mathbb{F}_q \cup \{\infty\}} \ell(i, b_i)$ be a Besicovitch set. Suppose there is $j \in \mathbb{F}_q \cup \{\infty\}$ such that every point $P \in B$ with $m_P \geqslant 3$ lies on the line $\ell(j, b_j)$. Then*

$$\sum_{P \in B} \frac{(m_P - 1)(m_P - 2)}{2} \geqslant \frac{q-1}{2}.$$

Equality holds if and only there are $(q - 1)/2$ points $P \in B$ with $m_P = 3$ and no points with $m_P > 3$.

It seems natural to state another conjecture in light of the above discussion.

Conjecture 2. *If q is odd and $B = \bigcup_{i \in \mathbb{F}_q \cup \{\infty\}} \ell(i, b_i)$ is a small Besicovitch set, then there is $j \in \mathbb{F}_q \cup \{\infty\}$ such that every point $P \in B$ with $m_P \geqslant 3$ lies on the line $\ell(j, b_j)$.*

Now that we have two conjectures it seems reasonable to think about testing them via computer calculation. By checking every Besicovitch set that is a union of $q + 1$ lines, we have learned that Conjectures 1 and 2 hold for $q \leqslant 13$ odd. Unfortunately, I have not been able to construct an algorithm for finding small Besicovitch sets that requires any fewer than about $O(q^q)$ steps. In order to try to disprove Conjecture 1, one might randomly select a collection of lines with distinct slopes and hope that it will yield a small Besicovitch set. The following theorem shows that one is unlikely to get so lucky.

Theorem 3.

(a) *The expected cardinality of a Besicovitch set formed by the union of $q + 1$ randomly chosen lines with distinct slopes is*

$$\left(1 - \left(1 - \frac{1}{q} \right)^{q+1} \right) q^2 = \left(1 - \frac{1}{e} \right) q^2 + O(q), \quad as \ q \to \infty.$$

(b) *For q sufficiently large, a Besicovitch set B formed by the union of $q + 1$ randomly chosen lines with distinct slopes will satisfy*

$$\left| \#B - \left(1 - \frac{1}{e} \right) q^2 \right| < 2q \log q,$$

with probability $1 - O((\log q)^{-2})$. In particular, the probability of randomly constructing a small Besicovitch set tends to zero as $q \to \infty$.

As $1 - 1/e \approx 0.632$, we see that the average randomly chosen Besicovitch set will contain around $0.632q^2$ points, whereas we expect a small Besicovitch set to consist of about $0.5q^2$ points.

3. Proofs of the results

Proof of the Incidence Formula. Let us arbitrarily assign an ordering to the lines that comprise B: $\ell_0, \ldots, \ell_q$. We use the fact that each pair of lines with distinct slopes must intersect in exactly one point, and we argue essentially by inclusion–exclusion.

Fix $0 \leqslant j \leqslant q$. For P a point on ℓ_j, define $m_P(j)$ to be the number of lines ℓ_i that contain P with $i \leqslant j$. We wish to consider the intersections of ℓ_j with ℓ_i for $i < j$. If all of these intersections are distinct, then there are $q - j$ points on the line ℓ_j that do not lie on any ℓ_i with $i < j$. For $P \in \ell_j$, we see $m_P(j) - 1$ of these lines meet at P; if $m_P(j) - 1 \geqslant 2$, then we have undercounted the points on ℓ_j that do not lie on any ℓ_i with $i < j$ by $m_P(j) - 2$ points. That is,

$$\#\left(\ell_j \setminus \bigcup_{i=0}^{j-1} \ell_i \right) = q - j + \sum_{P \in \ell_j} \max\{0, m_P(j) - 2\}.$$

Summing over all j we get

$$\#B = \sum_{j=0}^{q} \#\left(\ell_j \setminus \bigcup_{i=0}^{j-1} \ell_i\right) = \sum_{j=0}^{q}(q-j) + \sum_{j=0}^{q}\sum_{P\in\ell_j} \max\{0, m_P(j)-2\}$$

$$= \frac{q(q+1)}{2} + \sum_{\substack{P\in B}} \sum_{\substack{j=0 \\ P\in\ell_j}}^{q} \max\{0, m_P(j)-2\}$$

$$= \frac{q(q+1)}{2} + \sum_{P\in B} \max\{0, 1+2+\cdots+(m_P-2)\}$$

$$= \frac{q(q+1)}{2} + \sum_{P\in B} \frac{(m_P-1)(m_P-2)}{2}. \qquad \square$$

Example. Recall that we defined

$$B_0 = \left(\bigcup_{i\in\mathbb{F}_q} \ell(i, -i^2)\right) \cup \ell(\infty, 0).$$

For i, j distinct elements of $\mathbb{F}_q$, one can easily see that $\ell(i, -i^2) \cap \ell(j, -j^2) = \{(i+j, ij)\}$. Thus the lines $\ell(i, -i^2), \ell(j, -j^2), \ell(k, -k^2)$ cannot share a common point if i, j, k are distinct. It follows that no point P has multiplicity $m_P > 3$, and if $m_P = 3$, then P must lie on the line $\ell(\infty, 0)$. In fact, $\ell(i, -i^2) \cap \ell(\infty, 0) = \{(0, -i^2)\}$.

If $i \neq 0$ and q is odd, then precisely two of our lines with nonzero slope pass through $(0, -i^2)$, namely $\ell(i, -i^2)$ and $\ell(-i, -i^2)$. There are $\frac{q-1}{2}$ nonzero squares in $\mathbb{F}_q$, so

$$\sum_{P\in B_0} \frac{(m_P-1)(m_P-2)}{2} = \frac{q-1}{2}.$$

If q is even, then $\ell(i, -i^2) = \ell(-i, -i^2)$. There are no points of multiplicity $m_P = 3$ in this case, and

$$\sum_{P\in B_0} \frac{(m_P-1)(m_P-2)}{2} = 0.$$

One can also prove the Incidence Formula in a fancier way using intersection theory on algebraic surfaces. Roughly speaking, we consider the divisor on $\mathbb{P}^2$ arising from B consisting of $q+1$ lines and compute its arithmetic genus in two ways: (1) using the adjunction formula for divisors, and (2) by blowing up $\mathbb{P}^2$ at all of the multiple points of B to get a surface on which the lines in B become pairwise disjoint. We leave the details to the interested reader. (See [4, Exercise V.1.3 and Corollary V.3.7].)

In order to prove Theorems 1 and 2, we require the following lemma:

Triple Point Lemma. *Let q be odd. Suppose B is a Besicovitch set with $B = \bigcup_{i \in \mathbb{F}_q \cup \{\infty\}} \ell(i, b_i)$. Then with at most one exception, for any choice of $i \in \mathbb{F}_q \cup \{\infty\}$ there exists a point $P \in \ell(i, b_i)$ with $m_P \geqslant 3$.*

Proof. Suppose ℓ and ℓ' are two lines in B such that no point P with $m_P \geqslant 3$ lies on either one. Without loss of generality, we may apply a translation followed by a linear automorphism of $\mathbb{F}_q^2$ so that it suffices to assume $\ell(0, 0)$ and $\ell(\infty, 0)$ are the two lines. Note that translations and linear automorphisms carry lines to lines and respect the multiplicities m_P.

As i varies through $\mathbb{F}_q^\times$, it must be true that the y-intercepts of $\ell(i, b_i)$ are distinct. For if not, there would exist a triple point on the line $\ell(\infty, 0)$. Similarly, the x-intercepts of these lines must be distinct. Note that b_i cannot be zero for any $i \neq 0$ since that would imply the existence of a triple point at the origin. The x- and y-intercepts of $\ell(i, b_i)$ are $-i/b_i$ and b_i, respectively. We conclude that

$$\{i \colon i \in \mathbb{F}_q^\times\} = \{-i/b_i \colon i \in \mathbb{F}_q^\times\} = \{b_i \colon i \in \mathbb{F}_q^\times\},$$

since each set is a collection of $q - 1$ distinct nonzero elements of $\mathbb{F}_q$. Using the fact that the product of all nonzero elements of $\mathbb{F}_q$ is -1 when q is odd, we see that

$$-1 = \prod_{i \in \mathbb{F}_q^\times} i = \prod_{i \in \mathbb{F}_q^\times} \left(-\frac{i}{b_i} \right) = (-1)^{q-1} \frac{\prod_{i \in \mathbb{F}_q^\times} i}{\prod_{i \in \mathbb{F}_q^\times} b_i} = 1.$$

Evidently this is a contradiction, so we are forced to accept the statement of the lemma. $\quad\square$

Proof of Theorem 1. We may suppose that B consists of $q + 1$ lines, arbitrarily labelled $\ell_0, \ldots, \ell_q$. For a point $P \in B$, there are m_P lines passing through it; we make the trivial observation

$$\frac{(m_P - 1)(m_P - 2)}{2} = \frac{1}{m_P} \sum_{\substack{j=0 \\ P \in \ell_j}}^{q} \frac{(m_P - 1)(m_P - 2)}{2}.$$

It follows that

$$\sum_{P \in B} \frac{(m_P - 1)(m_P - 2)}{2} = \sum_{P \in B} \sum_{\substack{j=0 \\ P \in \ell_j}}^{q} \frac{m_P^2 - 3m_P + 2}{2m_P}$$

$$= \sum_{j=0}^{q} \sum_{P \in \ell_j} \frac{m_P^2 - 3m_P + 2}{2m_P}$$

$$\geqslant \sum_{j=0}^{q} \sum_{\substack{P \in \ell_j \\ m_P \geqslant 3}} \frac{1}{3}. \tag{1}$$

For the inequality, note that the function $x \mapsto \frac{x^2-3x+2}{2x}$ is increasing for $x \geqslant 3$ and evaluates to $1/3$ for $x = 3$. By the Triple Point Lemma, we know that every line, except perhaps one, contains a point of multiplicity three or greater. Hence there are at least q terms in the final double sum in (1), and we obtain

$$\sum_{P \in B} \frac{(m_P - 1)(m_P - 2)}{2} \geqslant \frac{q}{3}. \qquad \square$$

Proof of Theorem 2. Without loss of generality, we may apply a linear automorphism of $\mathbb{F}_q^2$ and assume that all points $P \in B$ with $m_P \geqslant 3$ lie on the line $\ell(\infty, 0)$. Suppose the number of such points is T. Let us agree to write $\sum'$ for the sum over points P with $m_P \geqslant 3$. As every line $\ell(i, b_i)$ with $i \neq \infty$ must intersect $\ell(\infty, 0)$ exactly once, and there exists at most one line that does not contain a point P with $m_P \geqslant 3$, we find that the sum of the multiplicities m_P over all points with $m_P \geqslant 3$ must satisfy $\sum' m_P = q + T - \delta$, where δ is 0 or 1. We now have

$$\sum_{P \in B} \frac{(m_P - 1)(m_P - 2)}{2} = \frac{1}{2} \sum{}' m_P^2 - \frac{3}{2} \sum{}' m_p + \sum{}' 1$$

$$= \frac{1}{2} \sum{}' m_P^2 - \frac{3}{2}(q - \delta) - \frac{1}{2}T. \qquad (2)$$

By the Cauchy–Schwartz inequality, we find that

$$\left(\sum{}' m_p\right)^2 \leqslant \left(\sum{}' 1\right)\left(\sum{}' m_P^2\right) = T \sum{}' m_P^2. \qquad (3)$$

Combining (2) and (3), and again using the fact that $\sum' m_P = q + T - \delta$, we find that

$$\sum_{P \in B} \frac{(m_P - 1)(m_P - 2)}{2} \geqslant \frac{1}{2T}(q + T - \delta)^2 - \frac{3}{2}(q - \delta) - \frac{1}{2}T$$

$$= \frac{1}{2T}(q - \delta)^2 - \frac{1}{2}(q - \delta).$$

This last expression is a decreasing function of T. At least two non-vertical lines (slope $i \neq \infty$) pass through each point P with $m_P \geqslant 3$, and at most q non-vertical lines pass through these points in total. So $2T \leqslant q$, but since $q/2$ is not an integer, we obtain $T \leqslant (q-1)/2$. We now see that

$$\sum_{P \in B} \frac{(m_P - 1)(m_P - 2)}{2} \geqslant \frac{(q - \delta)^2}{q - 1} - \frac{1}{2}(q - \delta)$$

$$= \frac{q - 1}{2} + \frac{3}{2} - \frac{3}{2}\delta + \frac{(1 - \delta)^2}{q - 1}. \qquad (4)$$

The final three terms contribute a non-negative quantity for $\delta = 0$ or $\delta = 1$, which shows the desired inequality.

As for the final claim of the theorem, equality clearly holds if $m_P = 3$ for $(q - 1)/2$ points $P \in B$ and $m_P < 3$ otherwise. Conversely, if equality holds in the theorem, then equality must

hold in (4). Evidently this is equivalent to saying $T = (q-1)/2$. Now there are $(q-1)/2$ nonzero terms in the sum in (4), and their sum must be $(q-1)/2$. We conclude that $\frac{(m_P-1)(m_P-2)}{2} = 1$ for all P with $m_P \geqslant 3$. That is, $m_P = 3$ for exactly $(q-1)/2$ points, and $m_P < 3$ for all other points in B. $\quad\square$

To prove Theorem 3, we first formalize the underlying probability space. Let $\Omega = \bigoplus_{i \in \mathbb{F}_q \cup \{\infty\}} \mathbb{F}_q$. We can identify an element $\sum b_i \in \Omega$ with a Besicovitch set by setting $B = \bigcup_{i \in \mathbb{F}_q \cup \{\infty\}} \ell(i, b_i)$. We will use this identification without further comment. We make Ω into a probability space by assigning probability $q^{-(q+1)}$ to each Besicovitch set.

Proof of Theorem 3. We will proceed in three steps. The first is to calculate the mathematical expectation for the cardinality function $\#: \Omega \to \mathbb{R}$.

For $P \in \mathbb{F}_q^2$, let $f_P : \Omega \to \mathbb{R}$ be the characteristic function of P; i.e.,

$$f_P(B) = \begin{cases} 1, & \text{if } P \in B, \\ 0, & \text{otherwise.} \end{cases}$$

It follows that $\#B = \sum_{P \in \mathbb{F}_q^2} f_P(B)$.

For a given point $P \in \mathbb{F}_q^2$ we now calculate $\mathrm{P}\{f_P = 1\}$, the probability that P appears in a randomly chosen Besicovitch set. For fixed $i \in \mathbb{F}_q \cup \{\infty\}$, the probability that P does not lie on $\ell(i, b_i)$ is $1 - 1/q$, since there are q choices for the y-intercept b_i. The probability that P does not lie in a given Besicovitch set $B \in \Omega$ is the probability that it lies on none of the lines comprising B. As the y-intercepts of lines with distinct slopes are independent, we see that

$$\mathrm{P}\{f_P = 0\} = \prod_{i \in \mathbb{F}_q \cup \{\infty\}} \left(1 - \frac{1}{q}\right) = \left(1 - \frac{1}{q}\right)^{q+1}. \tag{5}$$

Hence

$$\mathrm{P}\{f_P = 1\} = 1 - \left(1 - \frac{1}{q}\right)^{q+1}. \tag{6}$$

We can now determine the expectation of the cardinality function:

$$\mathbb{E}(\#) = \sum_{B \in \Omega} \#B \cdot \mathrm{P}\{B\} = \sum_{B \in \Omega} \sum_{P \in \mathbb{F}_q^2} f_P(B) \cdot \mathrm{P}\{B\}$$

$$= \sum_{P \in \mathbb{F}_q^2} \sum_{B \in \Omega} f_P(B) \cdot \mathrm{P}\{B\} = \sum_{P \in \mathbb{F}_q^2} \mathrm{P}\{f_P = 1\} = \left(1 - \left(1 - \frac{1}{q}\right)^{q+1}\right) q^2$$

$$= \left(1 - \frac{1}{e}\right) q^2 + O(q), \quad \text{as } q \to \infty. \tag{7}$$

This completes part (a) of the theorem.

The second step in the proof is to compute the variance of the cardinality function. To this end, we will need to determine $P\{f_P = f_Q = 1\}$ for two distinct points $P, Q \in \mathbb{F}_q^2$. We can rewrite this probability as

$$P\{f_P = f_Q = 1\} = 1 - P\{f_P = f_Q = 0\}$$
$$- P\{f_P = 1, f_Q = 0\} - P\{f_P = 0, f_Q = 1\}$$
$$= 1 + P\{f_P = f_Q = 0\} - P\{f_P = 0\} - P\{f_Q = 0\}. \qquad (8)$$

The second term is the only one we do not know yet. There is precisely one line containing both P and Q, say $\ell(j, a)$. The probability that a line with slope j does not contain P or Q must be $1 - 1/q$. For any other slope $i \neq j$, there is precisely one line with slope i passing through P, and one through Q. The probability that a line with slope $i \neq j$ does not contain P or Q is $1 - 2/q$. Again by independence of y-intercepts it follows that $P\{f_P = f_Q = 0\} = (1 - 1/q)(1 - 2/q)^q$. We conclude from (8) and (5) that

$$P\{f_P = f_Q = 1\} = 1 + \left(1 - \frac{1}{q}\right)\left(1 - \frac{2}{q}\right)^q - 2\left(1 - \frac{1}{q}\right)^{q+1}. \qquad (9)$$

The variance of the cardinality function is given by

$$\mathrm{Var}(\#) = \mathbb{E}(\#^2) - \mathbb{E}(\#)^2 = \sum_{B \in \Omega} \sum_{P,Q \in \mathbb{F}_q^2} f_P(B) f_Q(B) \cdot P\{B\} - \mathbb{E}(\#)^2$$

$$= \sum_{P,Q \in \mathbb{F}_q^2} P\{f_P = f_Q = 1\} - \mathbb{E}(\#)^2$$

$$= \sum_{P \neq Q \in \mathbb{F}_q^2} P\{f_P = f_Q = 1\} + \sum_{P \in \mathbb{F}_q^2} P\{f_P = 1\} - \mathbb{E}(\#)^2$$

$$= q(q+1)(q-1)^2\left(1 - \frac{2}{q}\right)^q + q^2\left(1 - \frac{1}{q}\right)^{q+1}\left\{1 - q^2\left(1 - \frac{1}{q}\right)^{q+1}\right\}$$

$$= \left(\frac{1}{e} - \frac{5}{2e^2}\right)q^2 + O(q), \quad \text{as } q \to \infty.$$

The second to last expression follows from (9), (6), (7), and a bit of simplification.

The third and final step of the proof is an application of the Chebyshev inequality. Recall that the Chebyshev inequality asserts that for any function $g : \Omega \to \mathbb{R}$ and any $\varepsilon > 0$, we have

$$P\{B \in \Omega : |g(B) - \mathbb{E}(g)| \geqslant \varepsilon\} \leqslant \frac{1}{\varepsilon^2}\mathrm{Var}(g).$$

Applying this to our situation with $g = \#$ and $\varepsilon = q \log q$ shows

$$P\{B \in \Omega : |\#B - \mathbb{E}(\#)| \geqslant q \log q\} = O\big((\log q)^{-2}\big).$$

As $\mathbb{E}(\#)$ differs from $(1 - 1/e)q^2$ by $O(q)$, part (b) of the theorem follows. $\square$

Acknowledgments

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