http://algs4.cs.princeton.edu

2.2 Mergesort

- mergesort
- bottom-up mergesort
- sorting complexity
- comparators
- stability

Complexity of sorting

Computational complexity. Framework to study efficiency of algorithms for solving a particular problem X.

Model of computation. Allowable operations.
Cost model. Operation count(s).
Upper bound. Cost guarantee provided by some algorithm for X.
Lower bound. Proven limit on cost guarantee of all algorithms for X.
Optimal algorithm. Algorithm with best possible cost guarantee for X.

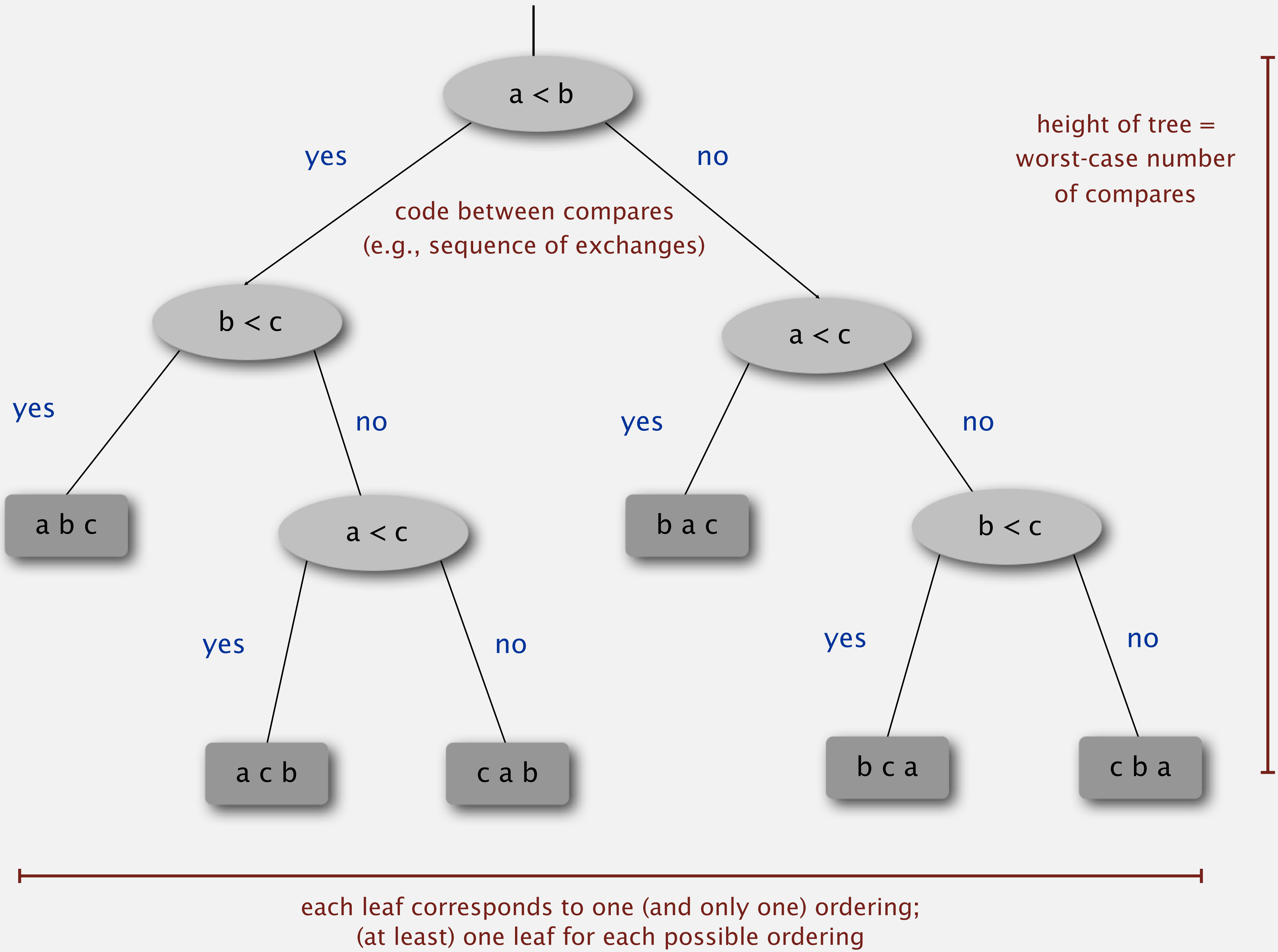
lower bound ~ upper bound

Example: sorting.

- Model of computation: decision tree.
- Cost model: # compares.
- Upper bound: $\sim N \lg N$ from mergesort.
- Lower bound:
- Optimal algorithm:

can access information only through compares (e.g., Java Comparable framework)

Decision tree (for 3 distinct keys a, b, and c)

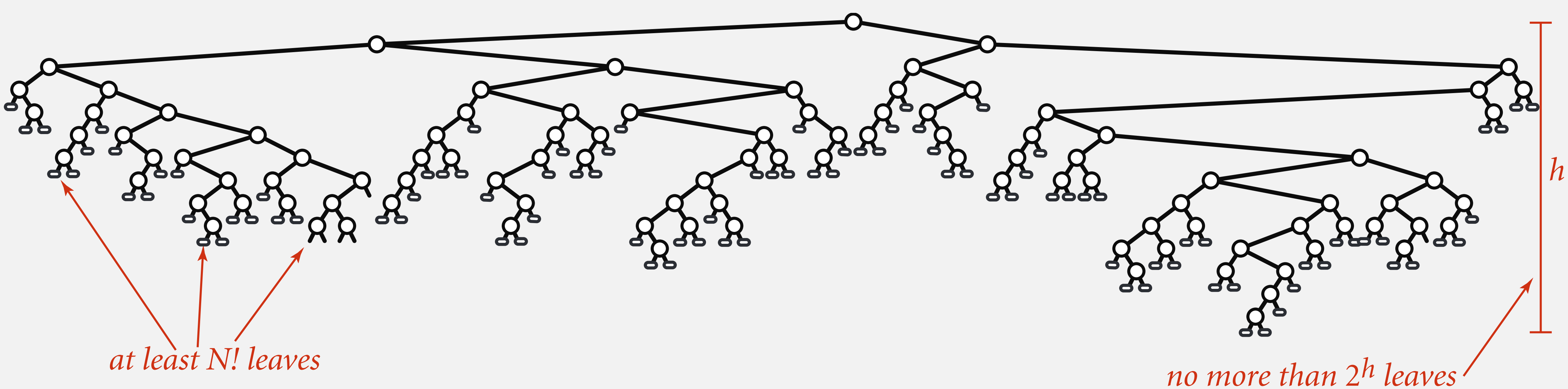


Compare-based lower bound for sorting

Proposition. Any compare-based sorting algorithm must use at least $\lg(N!) \sim N \lg N$ compares in the worst-case.

Pf.

- Assume array consists of N distinct values a_1 through a_N.
- Worst case dictated by height h of decision tree.
- Binary tree of height h has at most 2^h leaves.
- $N!$ different orderings $\Rightarrow$ at least $N!$ leaves.



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$$2^h \geq \#\,\text{leaves} \geq N!$$

$$\Rightarrow\ h \geq \lg(N!) \sim N \lg N$$

Stirling's formula

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Example: sorting.

- Model of computation: decision tree.
- Cost model: # compares.
- Upper bound: $\sim N \lg N$ from mergesort.
- Lower bound: $\sim N \lg N$.
- Optimal algorithm = mergesort.

First goal of algorithm design: optimal algorithms.

Complexity results in context

Compares? Mergesort is optimal with respect to number compares.

Space? Mergesort is not optimal with respect to space usage.

Lessons. Use theory as a guide.

Ex. Design sorting algorithm that guarantees $\frac{1}{2} N \lg N$ compares?

Ex. Design sorting algorithm that is both time- and space-optimal?

Complexity results in context (continued)

Lower bound may not hold if the algorithm can take advantage of:

- The initial order of the input.
  Ex: insert sort requires only a linear number of compares on partially-sorted arrays.
- The distribution of key values.
  Ex: 3-way quicksort requires only a linear number of compares on arrays with a constant number of distinct keys. [stay tuned]
- The representation of the keys.
  Ex: radix sort requires no key compares — it accesses the data via character/digit compares.