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2.2 MERGESORT

- *mergesort*
- *bottom-up mergesort*
- *sorting complexity*
- *comparators*
- *stability*

Two classic sorting algorithms: mergesort and quicksort

Critical components in the world's computational infrastructure.

- Full scientific understanding of their properties has enabled us to develop them into practical system sorts.
- Quicksort honored as one of top 10 algorithms of 20^{th} century in science and engineering.

Mergesort. [this lecture]

…

Quicksort. [next lecture]

…

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2.2 Mergesort

- *mergesort*
- *bottom-up mergesort*
- *sorting complexity*
- *comparators*
- *stability*

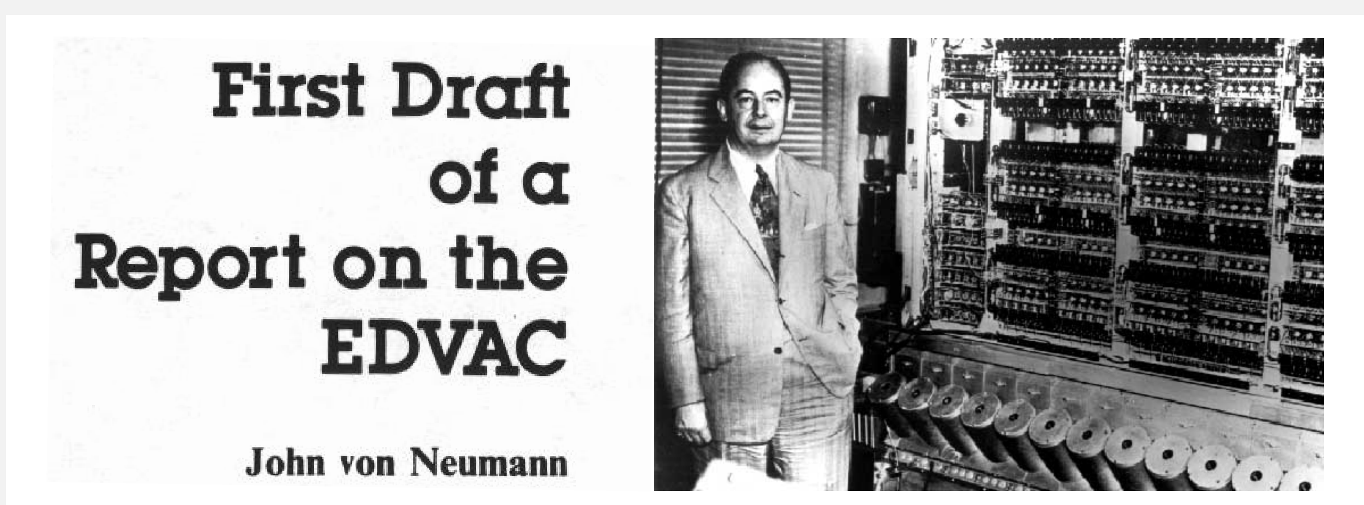
Mergesort

Basic plan.

- Divide array into two halves.
- Recursively sort each half.
- Merge two halves.

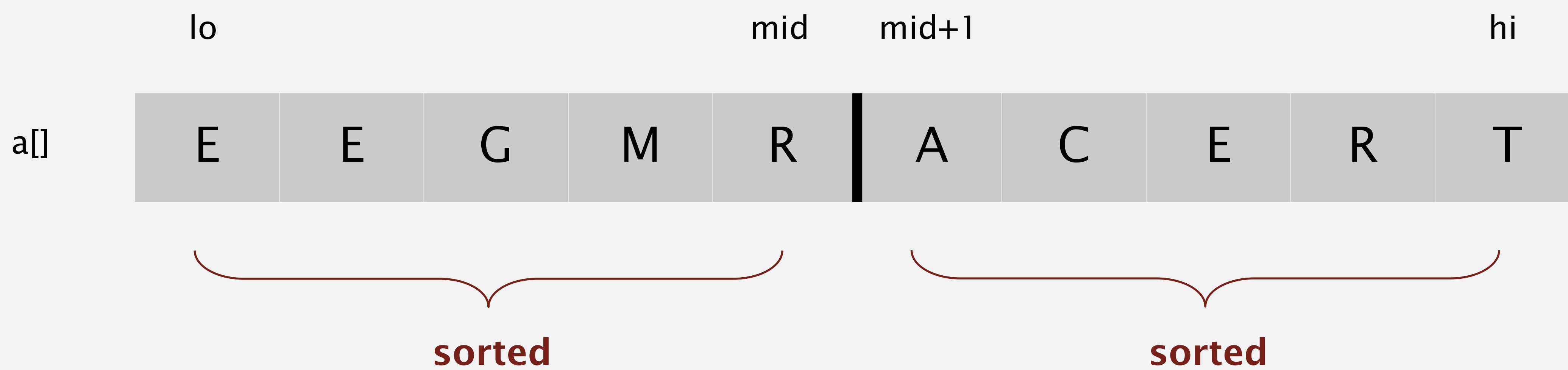
input	M	E	R	G	E	S	O	R	T	E	X	A	M	P	L	E
sort left half	E	E	G	M	O	R	R	S	T	E	X	A	M	P	L	E
sort right half	E	E	G	M	O	R	R	S	A	E	E	L	M	P	T	X
merge results	A	E	E	E	E	G	L	M	M	O	P	R	R	S	T	X

Mergesort overview



Abstract in-place merge demo

Goal. Given two sorted subarrays `a[lo]` to `a[mid]` and `a[mid+1]` to `a[hi]`, replace with sorted subarray `a[lo]` to `a[hi]`.



Abstract in-place merge demo

Goal. Given two sorted subarrays `a[lo]` to `a[mid]` and `a[mid+1]` to `a[hi]`, replace with sorted subarray `a[lo]` to `a[hi]`.

	lo									hi
a[]	A	C	E	E	E	G	M	R	R	T

sorted

Merging: Java implementation

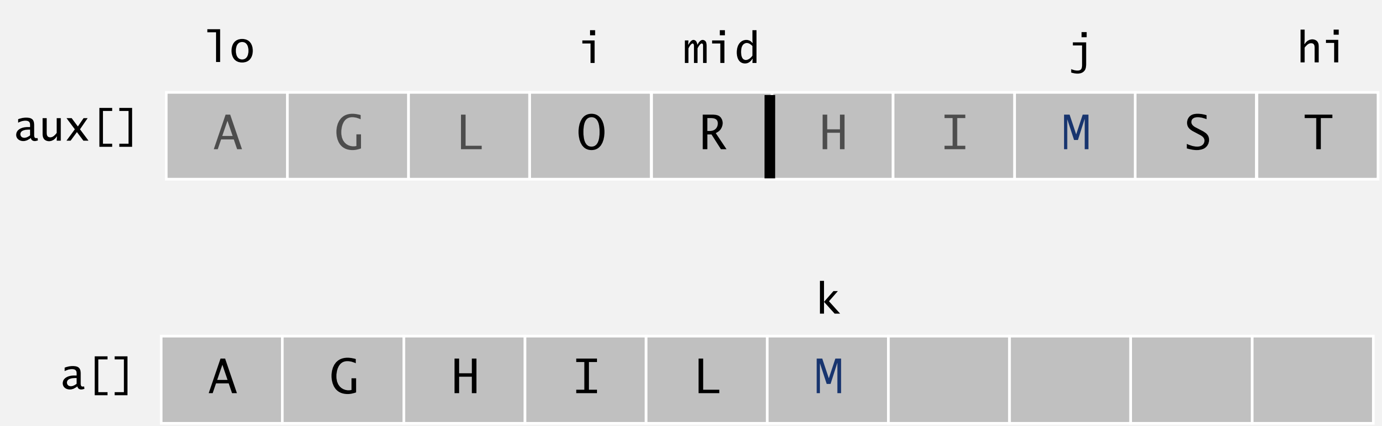
```
private static void merge(Comparable[] a, Comparable[] aux, int lo, int mid, int hi)
{

   for (int k = lo; k <= hi; k++)
      aux[k] = a[k];

   int i = lo, j = mid+1;
   for (int k = lo; k <= hi; k++)
   {
      if      (i > mid)              a[k] = aux[j++];
      else if (j > hi)               a[k] = aux[i++];
      else if (less(aux[j], aux[i])) a[k] = aux[j++];
      else                           a[k] = aux[i++];
   }
}
```

copy

merge

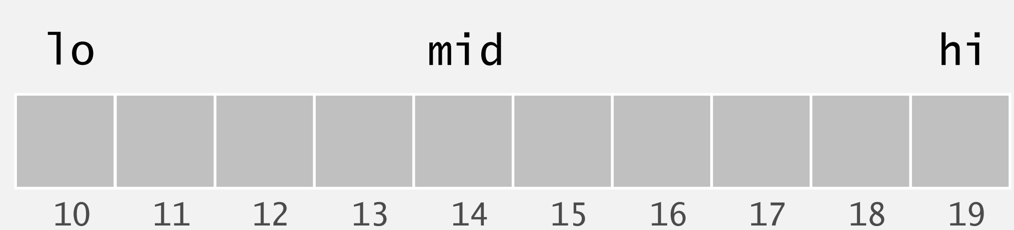


Mergesort: Java implementation

```java
public class Merge
{
   private static void merge(...)
   {  /* as before */  }

   private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi)
   {
      if (hi <= lo) return;
      int mid = lo + (hi - lo) / 2;
      sort(a, aux, lo, mid);
      sort(a, aux, mid+1, hi);
      merge(a, aux, lo, mid, hi);
   }

   public static void sort(Comparable[] a)
   {
      Comparable[] aux = new Comparable[a.length];
      sort(a, aux, 0, a.length - 1);
   }
}
```



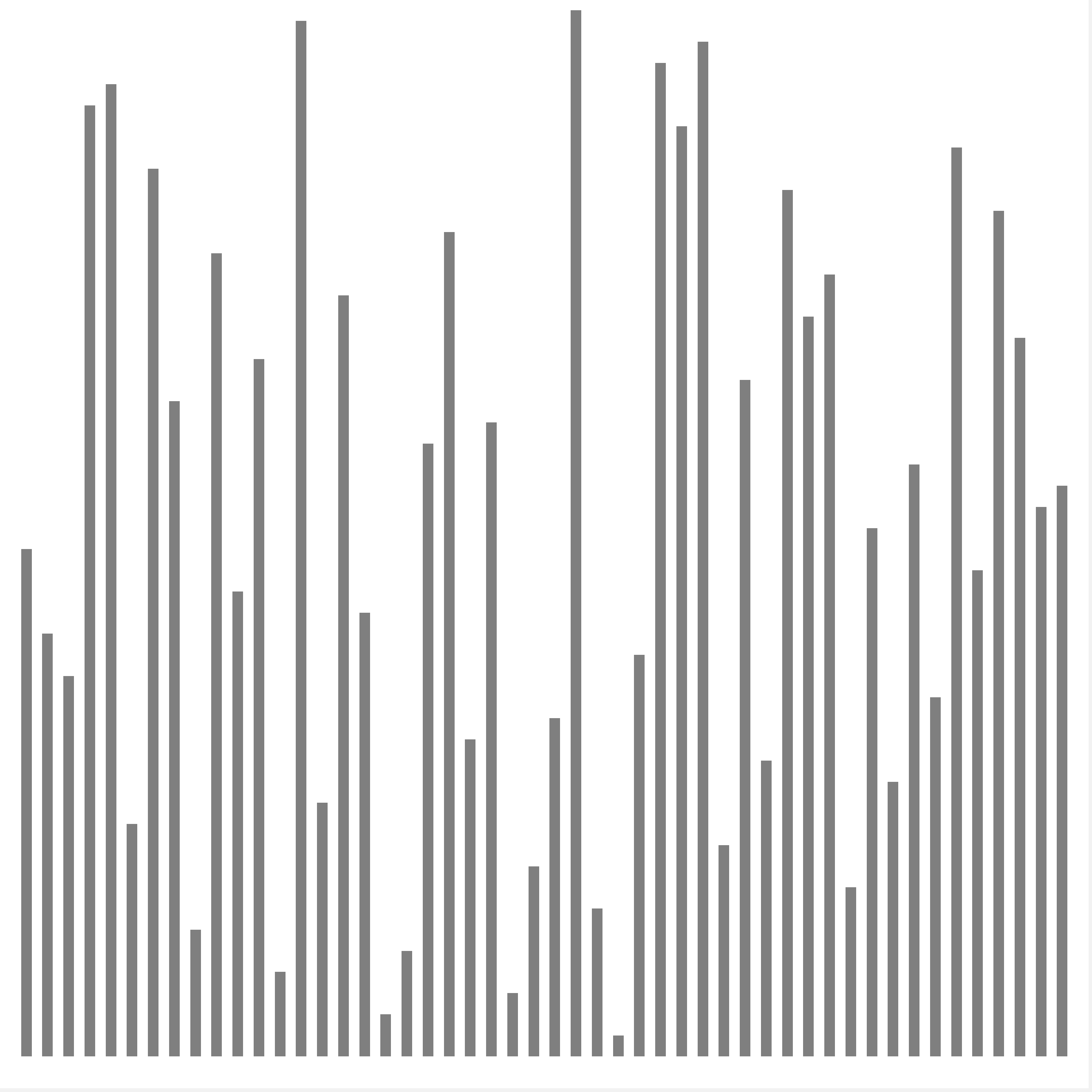
Mergesort: trace

lo … hi	a[] 0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
	M	E	R	G	E	S	O	R	T	E	X	A	M	P	L	E
`merge(a, aux, 0, 0, 1)`	E	M	R	G	E	S	O	R	T	E	X	A	M	P	L	E
`merge(a, aux, 2, 2, 3)`	E	M	G	R	E	S	O	R	T	E	X	A	M	P	L	E
`merge(a, aux, 0, 1, 3)`	E	G	M	R	E	S	O	R	T	E	X	A	M	P	L	E
`merge(a, aux, 4, 4, 5)`	E	G	M	R	E	S	O	R	T	E	X	A	M	P	L	E
`merge(a, aux, 6, 6, 7)`	E	G	M	R	E	S	O	R	T	E	X	A	M	P	L	E
`merge(a, aux, 4, 5, 7)`	E	G	M	R	E	O	R	S	T	E	X	A	M	P	L	E
`merge(a, aux, 0, 3, 7)`	E	E	G	M	O	R	R	S	T	E	X	A	M	P	L	E
`merge(a, aux, 8, 8, 9)`	E	E	G	M	O	R	R	S	E	T	X	A	M	P	L	E
`merge(a, aux, 10, 10, 11)`	E	E	G	M	O	R	R	S	E	T	A	X	M	P	L	E
`merge(a, aux, 8, 9, 11)`	E	E	G	M	O	R	R	S	A	E	T	X	M	P	L	E
`merge(a, aux, 12, 12, 13)`	E	E	G	M	O	R	R	S	A	E	T	X	M	P	L	E
`merge(a, aux, 14, 14, 15)`	E	E	G	M	O	R	R	S	A	E	T	X	M	P	E	L
`merge(a, aux, 12, 13, 15)`	E	E	G	M	O	R	R	S	A	E	T	X	E	L	M	P
`merge(a, aux, 8, 11, 15)`	E	E	G	M	O	R	R	S	A	E	E	L	M	P	T	X
`merge(a, aux, 0, 7, 15)`	A	E	E	E	E	G	L	M	M	O	P	R	R	S	T	X

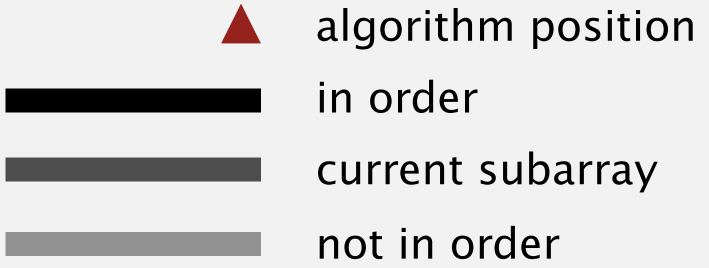
result after recursive call

Mergesort: animation

50 random items

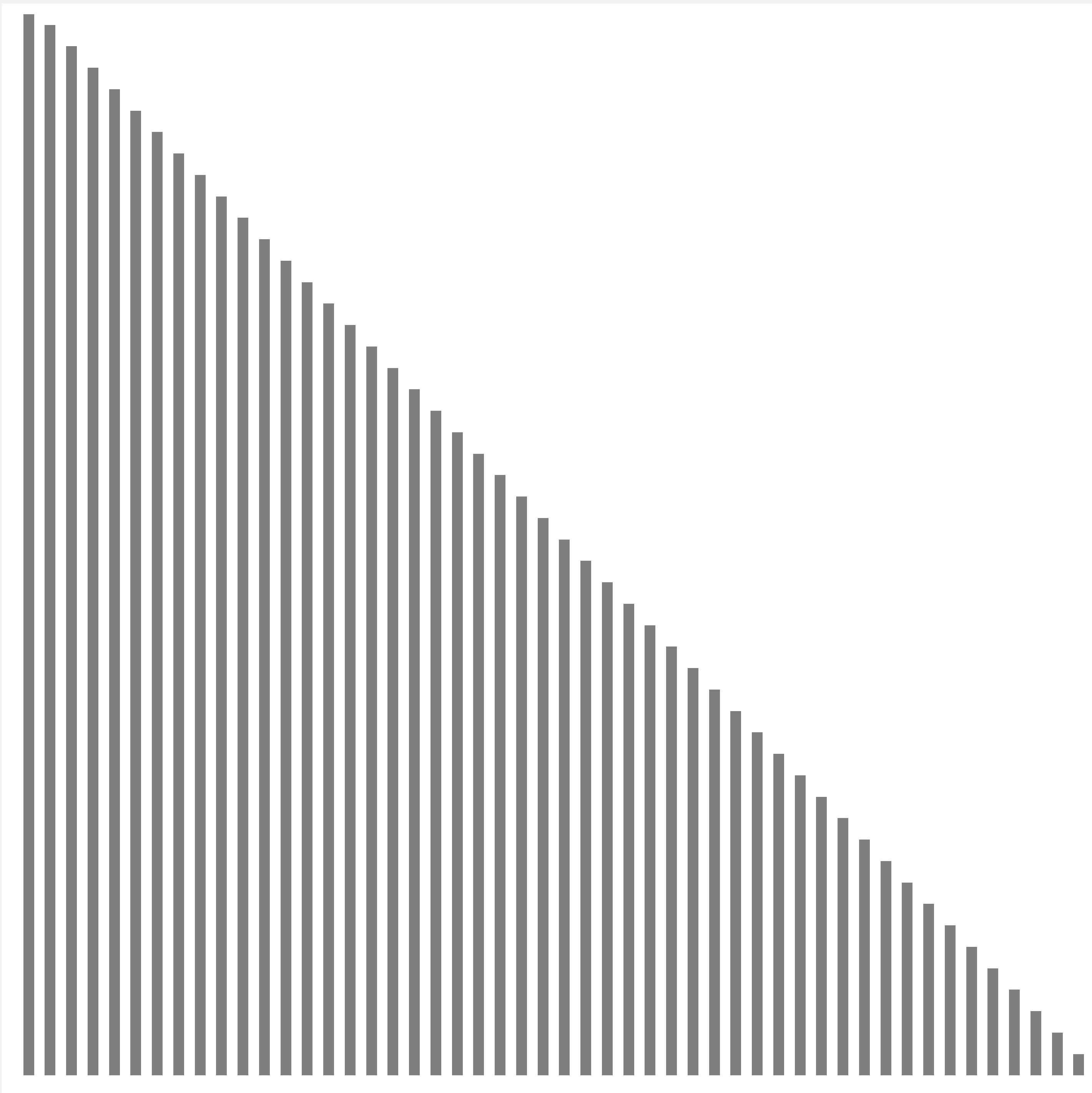


http://www.sorting-algorithms.com/merge-sort



Mergesort: animation

50 reverse-sorted items



http://www.sorting-algorithms.com/merge-sort

algorithm position

in order

current subarray

not in order

Mergesort: empirical analysis

Running time estimates:

- Laptop executes 10^8 compares/second.
- Supercomputer executes 10^{12} compares/second.

	insertion sort (N^2)			mergesort (N log N)		
computer	thousand	million	billion	thousand	million	billion
home	instant	2.8 hours	317 years	instant	1 second	18 min
super	instant	1 second	1 week	instant	instant	instant

Bottom line. Good algorithms are better than supercomputers.

Mergesort: number of compares

Proposition. Mergesort uses $\le N \lg N$ compares to sort an array of length N.

Pf sketch. The number of compares $C(N)$ to mergesort an array of length N satisfies the recurrence:

$$C(N) \le \underbrace{C(\lceil N/2 \rceil)}_{\text{left half}} + \underbrace{C(\lfloor N/2 \rfloor)}_{\text{right half}} + \underbrace{N}_{\text{merge}} \quad \text{for } N > 1, \text{ with } C(1) = 0.$$

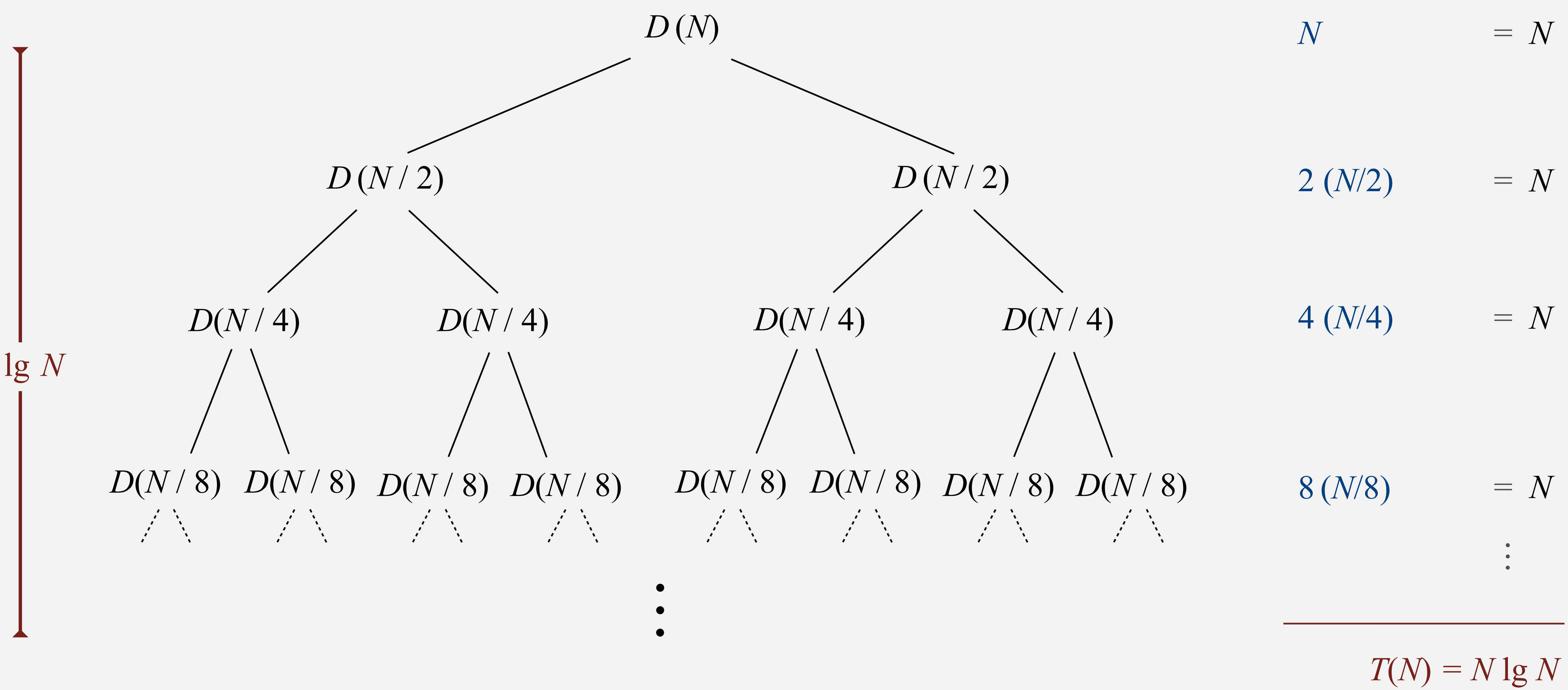
We solve the recurrence when N is a power of 2: ← result holds for all N (analysis cleaner in this case)

$$D(N) = 2\,D(N/2) + N, \text{ for } N > 1, \text{ with } D(1) = 0.$$

Divide-and-conquer recurrence: proof by picture

Proposition. If $D(N)$ satisfies $D(N) = 2\,D(N/2) + N$ for $N > 1$, with $D(1) = 0$, then $D(N) = N \lg N$.

Pf 1. [assuming N is a power of 2]



Divide-and-conquer recurrence: proof by induction

Proposition. If $D(N)$ satisfies $D(N) = 2\,D(N/2) + N$ for $N > 1$, with $D(1) = 0$, then $D(N) = N \lg N$.

Pf 2. [assuming N is a power of 2]

- Base case: $N = 1$.
- Inductive hypothesis: $D(N) = N \lg N$.
- Goal: show that $D(2N) = (2N) \lg (2N)$.

$$
\begin{aligned}
D(2N) &= 2\,D(N) + 2N && \text{given} \\
&= 2\,N \lg N + 2N && \text{inductive hypothesis} \\
&= 2\,N(\lg(2N) - 1) + 2N && \text{algebra} \\
&= 2\,N \lg(2N) && \text{QED}
\end{aligned}
$$

Mergesort: number of array accesses

Proposition. Mergesort uses $\leq 6 N \lg N$ array accesses to sort an array of length N.

Pf sketch. The number of array accesses $A(N)$ satisfies the recurrence:

$$A(N) \leq A(\lceil N/2 \rceil) + A(\lfloor N/2 \rfloor) + 6N \text{ for } N > 1, \text{ with } A(1) = 0.$$

Key point. Any algorithm with the following structure takes $N \log N$ time:

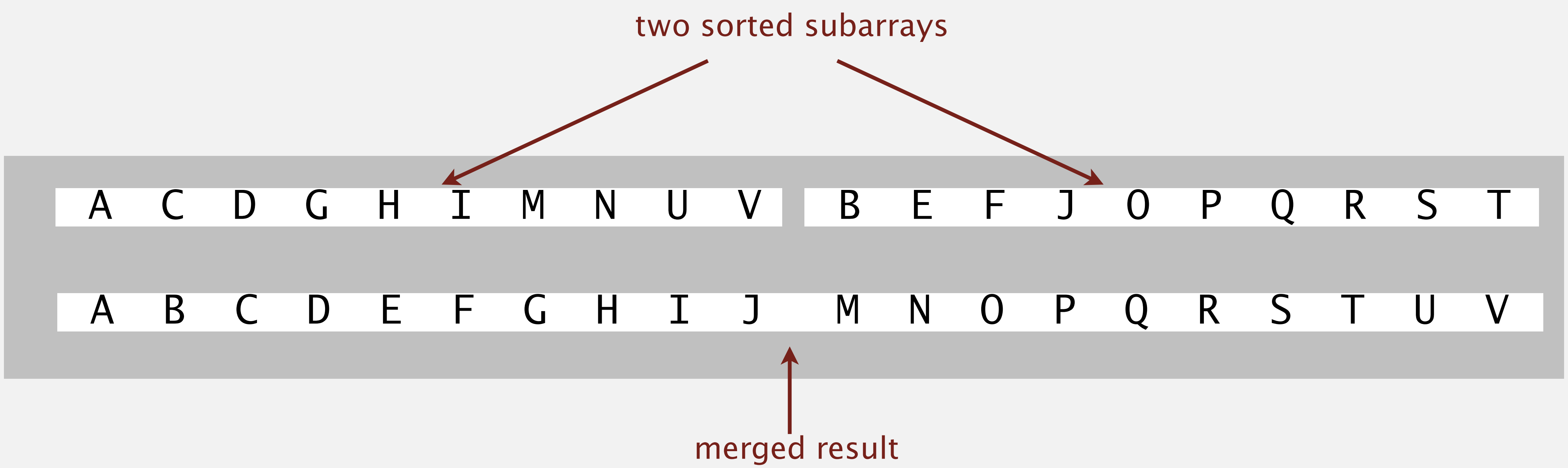
```
public static void linearithmic(int N)
{
    if (N == 0) return;
    linearithmic(N/2);    ← solve two problems
    linearithmic(N/2);    ← of half the size
    linear(N);            ← do a linear amount of work
}
```

Notable examples. FFT, hidden-line removal, Kendall-tau distance, …

Mergesort analysis: memory

Proposition. Mergesort uses extra space proportional to N.

Pf. The array `aux[]` needs to be of length N for the last merge.



Def. A sorting algorithm is in-place if it uses $\leq c \log N$ extra memory.

Ex. Insertion sort, selection sort, shellsort.

Challenge 1 (not hard). Use `aux[]` array of length $\sim \frac{1}{2} N$ instead of N.

Challenge 2 (very hard). In-place merge. [Kronrod 1969]

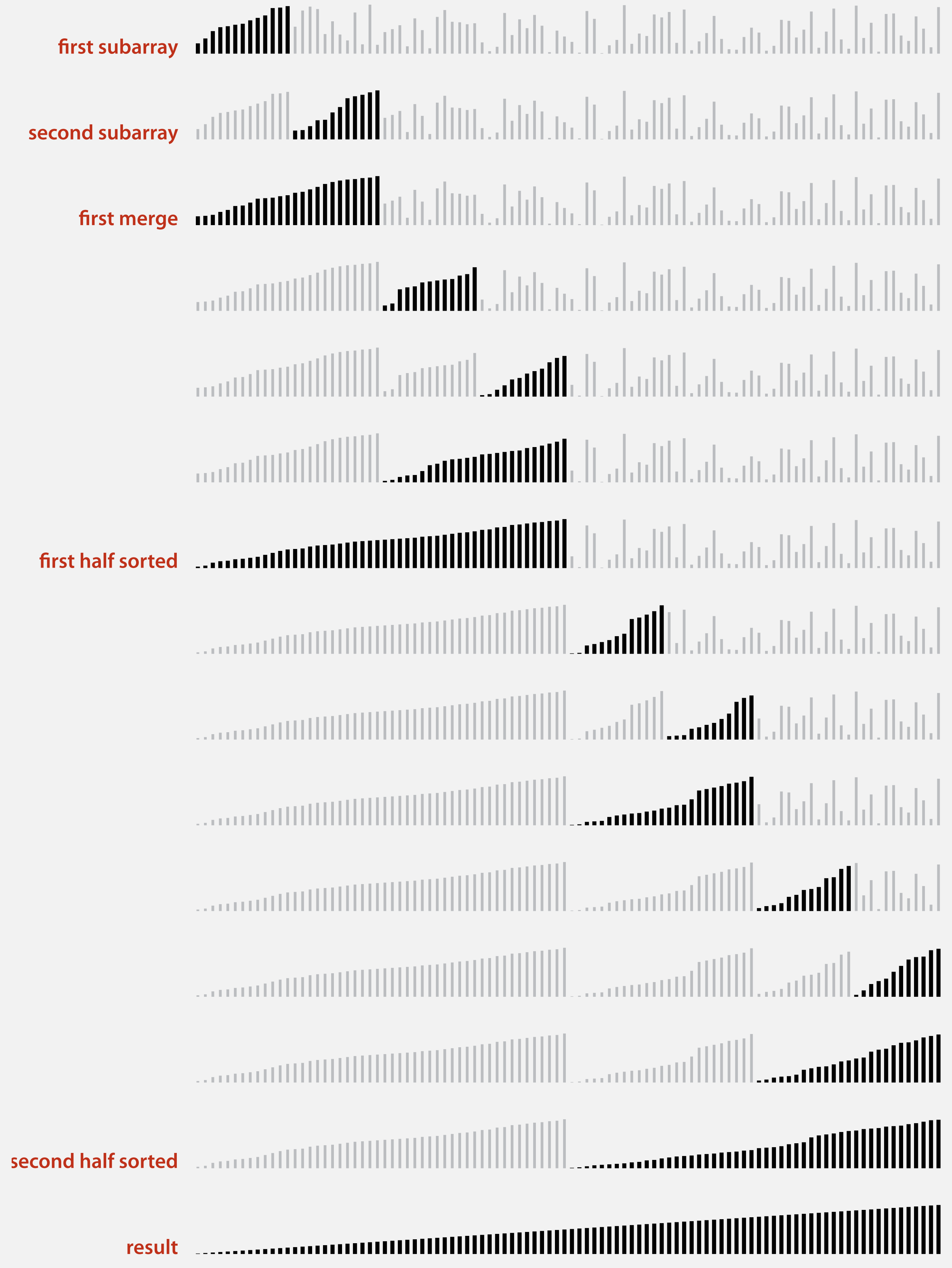
Mergesort: practical improvements

Use insertion sort for small subarrays.

- Mergesort has too much overhead for tiny subarrays.
- Cutoff to insertion sort for ≈ 10 items.

```
private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi)
{
   if (hi <= lo + CUTOFF - 1)
   {
      Insertion.sort(a, lo, hi);
      return;
   }
   int mid = lo + (hi - lo) / 2;
   sort (a, aux, lo, mid);
   sort (a, aux, mid+1, hi);
   merge(a, aux, lo, mid, hi);
}
```

Mergesort with cutoff to insertion sort: visualization



Mergesort: practical improvements

Stop if already sorted.

- Is largest item in first half ≤ smallest item in second half?
- Helps for partially-ordered arrays.

A	B	C	D	E	F	G	H	I	J	M	N	O	P	Q	R	S	T	U	V
A	B	C	D	E	F	G	H	I	J	M	N	O	P	Q	R	S	T	U	V

```
private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi)
{
   if (hi <= lo) return;
   int mid = lo + (hi - lo) / 2;
   sort (a, aux, lo, mid);
   sort (a, aux, mid+1, hi);
   if (!less(a[mid+1], a[mid])) return;
   merge(a, aux, lo, mid, hi);
}
```

Mergesort: practical improvements

Eliminate the copy to the auxiliary array. Save time (but not space) by switching the role of the input and auxiliary array in each recursive call.

```
private static void merge(Comparable[] a, Comparable[] aux, int lo, int mid, int hi)
{
   int i = lo, j = mid+1;
   for (int k = lo; k <= hi; k++)
   {
      if      (i > mid)          aux[k] = a[j++];
      else if (j > hi)           aux[k] = a[i++];
      else if (less(a[j], a[i])) aux[k] = a[j++];
      else                       aux[k] = a[i++];
   }
}

private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi)
{
   if (hi <= lo) return;
   int mid = lo + (hi - lo) / 2;
   sort (aux, a, lo, mid);
   sort (aux, a, mid+1, hi);
   merge(a, aux, lo, mid, hi);
}
```

merge from a[] to aux[]

assumes aux[] is initialize to a[] once, before recursive calls

switch roles of aux[] and a[]

Java 6 system sort

Basic algorithm for sorting objects = mergesort.

- Cutoff to insertion sort = 7.
- Stop-if-already-sorted test.
- Eliminate-the-copy-to-the-auxiliary-array trick.

Arrays.sort(a)

http://www.java2s.com/Open-Source/Java/6.0-JDK-Modules/j2me/java/util/Arrays.java.html