Dimensionality Reduction

Question 1: Note: In this question, all columns will be written in their transposed form, as rows, to make the typography simpler. Matrix M has three rows and three columns, and the columns form an orthonormal basis. One of the columns is [2/7,3/7,6/7], and another is [6/7, 2/7, -3/7]. Let the third column be [x,y,z]. Since the length of the vector [x,y,z] must be 1, there is a constraint that $x^2+y^2+z^2 = 1$. However, there are other constraints, and these other constraints can be used to deduce facts about the ratios among x, y, and z. Compute these ratios.

Answer:

Let C1 be [2/7,3/7,6/7], C2 be [6/7, 2/7, -3/7] and C3 be [x, y, z]
The dot product of any two columns must be zero.
C1.C2 = (2/7 * 6/7) + (3/7 * 2/7) + (6/7 * -3/7) = 0
C2.C3 = (6/7 * x) + (2/7 * y) + (-3/7 * z) = 0 → 6x +2y -3z = 0 – Eq 1
C3.C1 = (x * 2/7) + (y * 3/7) + (z * 6/7) = 0 → 2x + 3y + 6z = 0 – Eq 2
2 * Eq 1 + Eq 2 → 12x + 4y -6z + 2x + 3y +6z = 0 → 14x + 7y = 0 → y = -2x
3 * Eq 2 – Eq 1 → 6x + 9y + 18z – 6x – 2y + 3z = 0 → 7y + 21z = 0 → y = -3z
x: y: z = -2: 1: -3

Question 2: Find the eigenvalues and eigenvectors of the following matrix:

You should assume the first component of an eigenvector is 1. Then, find out One eigenvalue and One eigenvector.

2	3
3	10

Answer:

Let the given matrix be A = $\begin{matrix} 2 & 3 \\ 3 & 10 \end{matrix}$ **and the eigen vector be of the form** $\begin{matrix} 1 \\ e \end{matrix}$

Ax = λx ➔ $\begin{matrix} 2 & 3 \\ 3 & 10 \end{matrix} * \begin{matrix} 1 \\ e \end{matrix}$ **= λ *** $\begin{matrix} 1 \\ e \end{matrix}$ **→ 2 + 3e = λ and 3 + 10e = λe → 3 + 10e = (2 + 3e)e**

$3e^2$ – 8e + 3 = 0 → e = 3, -1/3

The eigen vectors are $\begin{matrix} \mathbf{1} \\ \mathbf{3} \end{matrix}\ \boldsymbol{and}\ \begin{matrix} \mathbf{1} \\ \mathbf{-1/3} \end{matrix}$

The eigen values are 2 + 3e = λ → λ = 2 + 3*3 = 11 and λ = 2 + 3*(-1/3) = 1

Question 3: Suppose [1,3,4,5,7] is an eigenvector of some matrix. What is the unit eigenvector in the same direction? Find out the components of the unit eigenvector.

Answer:

Given the eigen vector of some matrix be M = [1,3,4,5,7]
To get the unit eigen vector of given matrix, we need to divide each component by square root of sum of squares in the same direction.
Sum of squares = $1^2 + 3^2 + 4^2 + 5^2 + 7^2$ = 100 and its square root is 10
Unit Eigen Vector = [1/10,3/10,4/10,5/10,7/10]

Question 4: Suppose we have three points in a two-dimensional space: (1,1), (2,2), and (3,4). We want to perform PCA on these points, so we construct a 2-by-2 matrix, call it N, whose eigenvectors are the directions that best represent these three points. Construct the matrix N and identify, its elements.

Answer:

The given three points in a 2- D space are (1,1), (2,2), and (3,4).

We should construct a matrix whose rows correspond to points and columns correspond to dimensions of the space.

Then the matrix will be M = $\begin{matrix} 1 & 1 \\ 2 & 2 \\ 3 & 4 \end{matrix}$ **$M^T M$ =** $\begin{matrix} 1 & 2 & 3 \\ 1 & 2 & 4 \end{matrix}$ ***** $\begin{matrix} 1 & 1 \\ 2 & 2 \\ 3 & 4 \end{matrix}$ **=** $\begin{matrix} 14 & 17 \\ 17 & 21 \end{matrix}$

Question 5: Consider the diagonal matrix M =

1	0	0
0	2	0
0	0	0

Compute its Moore-Penrose pseudoinverse.

Answer:

Moore-Penrose pseudoinverse means the matrix having diagonal elements replaced by 1 and divided by corresponding elements of given matrix and the other elements will be zero. Moore-Penrose pseudoinverse of given matrix is $\begin{matrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1/2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{matrix}$

Question 6: When we perform a CUR decomposition of a matrix, we select rows and columns by using a particular probability distribution for the rows and another for the columns. Here is a matrix that we wish to decompose:

1	2	3
4	5	6
7	8	9
10	11	12

Calculate the probability distribution for the rows.

Answer:

Probability with which we choose now = $\frac{\text{sum of squares of elements in the rows}}{\text{sum of squares of elements in the matrix}}$

Sum of squares of elements in the matrix = 12*13*25/6 = 3900/6 = 650

P(R1) = $\frac{1^2+2^2+3^2}{650}$ **= 14/650 = 0.02**

P(R2) = $\frac{4^2+5^2+6^2}{650}$ **= 77/650 = 0.12**

P(R3) = $\frac{7^2+8^2+9^2}{650}$ **= 194/650 = 0.298**

P(R4) = $\frac{10^2+11^2+12^2}{650}$ **= 365/650 =0.56**