Dimensionality Reduction

Question 1: Note: In this question, all columns will be written in their transposed form, as rows, to make the typography simpler. Matrix M has three rows and three columns, and the columns form an orthonormal basis. One of the columns is [2/7,3/7,6/7], and another is [6/7, 2/7, -3/7]. Let the third column be [x,y,z]. Since the length of the vector [x,y,z] must be 1, there is a constraint that $x^2+y^2+z^2=1$. However, there are other constraints, and these other constraints can be used to deduce facts about the ratios among x, y, and z. Compute these ratios.

Answer:

Let C1 be [2/7,3/7,6/7], C2 be [6/7, 2/7, -3/7] and C3 be [x, y, z] The dot product of any two columns must be zero. C1.C2 = (2/7 * 6/7) + (3/7 * 2/7) + (6/7 * -3/7) = 0 C2.C3 = $(6/7 * x) + (2/7 * y) + (-3/7 * z) = 0 \rightarrow 6x + 2y - 3z = 0 - Eq 1$ C3.C1 = $(x * 2/7) + (y * 3/7) + (z * 6/7) = 0 \rightarrow 2x + 3y + 6z = 0 - Eq 2$ 2 * Eq 1 + Eq 2 \rightarrow 12x + 4y -6z + 2x + 3y +6z = 0 \rightarrow 14x + 7y = 0 \rightarrow y = -2x 3 * Eq 2 - Eq 1 \rightarrow 6x + 9y + 18z - 6x - 2y + 3z = 0 \rightarrow 7y + 21z = 0 \rightarrow y = -3z x: y: z = -2: 1: -3

Question 2: Find the eigenvalues and eigenvectors of the following matrix:

You should assume the first component of an eigenvector is 1. Then, find out One eigenvalue and One eigenvector.



Answer:

Let the given matrix be A = $\begin{pmatrix} 2 & 3 \\ 3 & 10 \end{pmatrix}$ and the eigen vector be of the form $\begin{pmatrix} 1 \\ e \end{pmatrix}$

Ax =
$$\lambda x \Rightarrow \frac{2}{3} \quad \frac{3}{10} * \frac{1}{e} = \lambda * \frac{1}{e} \Rightarrow 2 + 3e = \lambda \text{ and } 3 + 10e = \lambda e \Rightarrow 3 + 10e = (2 + 3e)e$$

 $3e^2 - 8e + 3 = 0 \Rightarrow e = 3, -1/3$

The eigen vectors are $\frac{1}{3}$ and $\frac{1}{-1/3}$

The eigen values are $2 + 3e = \lambda \rightarrow \lambda = 2 + 3*3 = 11$ and $\lambda = 2 + 3*(-1/3) = 1$

Question 3: Suppose [1,3,4,5,7] is an eigenvector of some matrix. What is the unit eigenvector in the same direction? Find out the components of the unit eigenvector.

Answer:

Given the eigen vector of some matrix be M = [1,3,4,5,7]

To get the unit eigen vector of given matrix, we need to divide each component by square root of sum of squares in the same direction.

Sum of squares = $1^2 + 3^2 + 4^2 + 5^2 + 7^2 = 100$ and its square root is 10 Unit Eigen Vector = [1/10,3/10,4/10,5/10,7/10]

Question 4: Suppose we have three points in a two-dimensional space: (1,1), (2,2), and (3,4). We want to perform PCA on these points, so we construct a 2-by-2 matrix, call it N, whose eigenvectors are the directions that best represent these three points. Construct the matrix N and identify, its elements.

Answer:

The given three points in a 2- D space are (1,1), (2,2), and (3,4).

We should construct a matrix whose rows correspond to points and columns correspond to dimensions of the space.

Then the matrix will be
$$\mathbf{M} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 4 \end{bmatrix}$$
 $\mathbf{M}^\mathsf{T} \mathbf{M} = \begin{bmatrix} 1 & 2 & 3 & * & 1 & 1 \\ 1 & 2 & 4 & * & 2 & 2 \\ 3 & 4 & & 17 & 21 \end{bmatrix}$

$$\mathbf{M}^{\mathsf{T}} \mathbf{M} = \begin{pmatrix} 1 & 2 & 3 & 1 & 1 \\ 1 & 2 & 4 & 2 & 2 \\ 1 & 2 & 4 & 2 \end{pmatrix} = \begin{pmatrix} 14 & 17 \\ 17 & 21 \end{pmatrix}$$

Question 5: Consider the diagonal matrix M =

Compute its Moore-Penrose pseudoinverse.



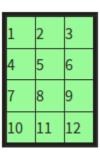
Answer:

Moore-Penrose pseudoinverse means the matrix having diagonal elements replaced by 1 and divided by corresponding elements of given matrix and the other elements will be

zero. Moore-Penrose pseudoinverse of given matrix is
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Question 6: When we perform a CUR decomposition of a matrix, we select rows and columns by using a particular probability distribution for the rows and another for the columns. Here is a matrix that we wish to decompose:

Calculate the probability distribution for the rows the rows.



Answer:

Probability with which we choose now = $\frac{\text{sum of squares of elements in the rows}}{\text{sum of squares of elements in the matrix}}$

Sum of squares of elements in the matrix = 12*13*25/6 = 3900/6 = 650

$$P(R1) = \frac{1^2 + 2^2 + 3^2}{650} = 14/650 = 0.02$$

$$P(R2) = \frac{4^2 + 5^2 + 6^2}{650} = 77/650 = 0.12$$

$$P(R3) = \frac{7^2 + 8^2 + 9^2}{650} = 194/650 = 0.298$$

$$P(R4) = \frac{10^2 + 11^2 + 12^2}{650} = 365/650 = 0.56$$