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Mario Ponce & Patricio Santibáñez

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On Equidistant Sets and Generalized Conics: The Old and the New

Mario Ponce and Patricio Santibáñez

Abstract. This article is devoted to the study of classical and new results concerning equidistant sets, both from the topological and metric point of view. We include a review of the most interesting known facts about these sets in Euclidean space and we prove two new results. First, we show that equidistant sets vary continuously with their focal sets. We also prove an error estimate result about approximative versions of equidistant sets that should be of interest for computer simulations. Moreover, we offer a viewpoint in which equidistant sets can be thought of as a natural generalization for conics. Along these lines, we show that the main geometric features of classical conics can be retrieved from more general equidistant sets.

1. INTRODUCTION. The set of points that are equidistant from two given sets in the plane appears naturally in many classical geometric situations. Namely, the classical conics, defined as the level sets of a real polynomial equation of degree 2, can always be realized as the equidistant set to two circles (see Section 2). The study of conics has played a significant role in the development of mathematics, with each new progression representing a breakthrough: the determination of bounded area by Archimides, their idea as plane curves by Apollonius, their occurrence as solutions for movement equations by Kepler, the development of projective and analytic geometry by Desargues and Descartes, and so on.

In another, let us say, less academic field, we find equidistant sets as conventionally defined frontiers in territorial domain controversies. For instance, the United Nations Convention on the Law of the Sea (Article 15) establishes that, in the absence of any previous agreement, the delimitation of the territorial sea between countries occurs exactly on the *median line every point of which is equidistant of the nearest points* to each country. The significance of this stresses the necessity of understanding the geometric structure of equidistant sets. The study of equidistant sets, other than conics, arose many decades ago, principally with the works of Loveland [**4**] and Wilker [**10**] (in Section 4 we review their main contributions concerning topological properties of equidistant sets).

In the literature, we can find many generalizations of conics. For instance, Groß and Strempel [**3**] start from the the usual definition of conics as the set of points in the plane that have a constant weighted sum of distances to two points (the *focal points*), and gave a generalization by allowing more than two focal points, weights other than ± 1, and point sets in higher dimensions. Recently, Vincze and Nagy [**7**] proposed that a generalized conic is a set of points with the same average distance from a point set $\Gamma \subset \mathbb{R}^n$. However, the study of generalized conics is not only motivated by theoretical reasons, but also has interesting real applications in approximation theory, optimization problems, and geometric tomography [**5, 8**].

Here, we present equidistant sets as a natural generalization of conics, making use of the fact that classical conics are equidistant sets to plane circles (*focal sets*). By admitting more complicated focal sets, we obtain more complicated equidistant sets

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(*generalized conics*). Our purpose is to show that these generalizations share many geometrical features with their classical ancestors. For instance, we show that the equidistant set to two disjoint connected compact sets looks like the branch of a hyperbola, since near infinity it is asymptotic to two rays. We also discuss possible generalizations of ellipses and parabolas. Additionally, we propose some further research directions.

Computational simulations are a useful tool for treating equidistant sets. This task faces two main theoretic issues. The first issue is that a computer manipulates only discrete approximations of plane sets (limited by the memory or screen resolution). In this direction, our result concerning the Hausdorff continuity of equidistant sets (cf. Theorem 11) is central, since it implies that the equidistant set computed by the machine approaches (by increasing the screen resolution or dedicated memory) to the genuine equidistant set. Secondly, remember that the plane is represented on the screen by a finite array of points (pixels). To determine precisely what points belong to the equidistant set, we need to compute the respective distances to the underlying sets (focal sets) and then decide whether this difference is equal to zero. Strictly speaking, a pixel belongs to the equidistant set if and only if this difference vanishes. Nevertheless, due to the discrete character of this computer screen plane, this absolute zero is virtually impossible. Thus, in order to obtain a good picture of the equidistant set, we need to introduce a more tolerant criterion. The general situation is as follows: We find a pixel for which the difference of the distances to the focal sets is very small, and we ask whether this means that we can assume the presence of a point of the equidistant set inside the region represented by this pixel. Theorem 12 provides us with a useful criterion, since it gives a sharp bound on the distance from a quasi-equidistant point to an authentic equidistant point (see the definitions following).

Definitions and notations. We consider $\mathbb{R}^n$ endowed with the classical Euclidean distance $\operatorname{dist}(\cdot, \cdot)$. The definition easily extends to admit the distance between a point $\tilde{x}$ and a set $X \subset \mathbb{R}^2$ as $\operatorname{dist}(\tilde{x}, X) = \inf_{x \in X} \operatorname{dist}(\tilde{x}, x)$. Given two nonempty sets $A, B \subset \mathbb{R}^n$, we define the *equidistant set* to A and B as

$$\{A = B\} := \{x \in \mathbb{R}^n : \operatorname{dist}(x, A) = \operatorname{dist}(x, B)\}.$$

This notation is due to Wilker [**10**]. We also use the word *midset*, as proposed by Loveland [**4**]. We say that A and B are the *focal sets* of the midset $\{A = B\}$.

For $x \in \mathbb{R}^n$, we write $\mathcal{P}_x(A) = \{p \in A : \operatorname{dist}(x, A) = \operatorname{dist}(x, p)\}$ the set of *foot points from x to A*.

Given two points $x, y \in \mathbb{R}^n$, we write

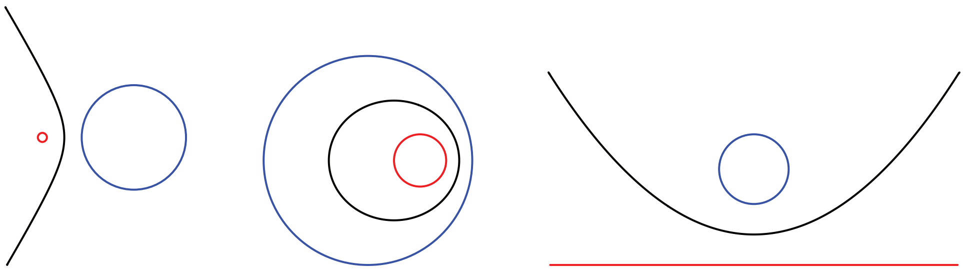
$$[x, y] := \{tx + (1 - t)y : 0 \leq t \leq 1\},$$

and we call it the *closed segment* between x and y (analogously for $[x, y)$, $(x, y]$, and (x, y)). For $r > 0$, we write $\overline{B}(x, r)$, $B(x, r)$, and $C(x, r)$, to represent the closed ball, the open ball, and the sphere centered at x with radius r, respectively. For $v \neq 0$ in $\mathbb{R}^n$, we write

$$[x, \infty)_v := \{x + tv : t \geq 0\}$$

for the *infinite ray starting at x in the direction of v*. We write $l_{a,v} := [a, \infty)_{-v} \cup [a, \infty)_v$ for the entire straight line passing through a in the direction v.

2. CONICS AS MIDSETS. In this section, we review the definition of the classical conics as the equidistant set to two circles (possibly degenerating into points or straight lines), as illustrated by Figure 1. In the sequel, we use complex notation for points in the plane.



(a) Hyperbola (b) Ellipse (c) Parabola

Figure 1. Classical conics

Hyperbola. Let $A = C(0, R)$ and $B = C(1, r)$ with $0 \leq r, R$ and $R < 1 - r$ (this implies $A \cap B = \emptyset$). The midset $\{A = B\}$ is composed of points $z \in \mathbb{C}$ such that

$$\operatorname{dist}(z, A) = \operatorname{dist}(z, B),$$

$$|z| - R = |z - 1| - r,$$

and

$$|z - 0| - |z - 1| = R - r.$$

Thus, the midset $\{A = B\}$ is exactly the locus of points z in the plane such that the difference of the distance from z to 0 and 1 is constant, that is, the branch of a hyperbola. In the case $R = r$, we obtain a straight line.

Ellipse. We now consider two circles $A = C(0, R)$ and $B = C(1, r)$ with $R > 1 + r$ (this implies that B lies inside A). The midset $\{A = B\}$ is composed of points $z \in \mathbb{C}$ such that

$$\operatorname{dist}(z, A) = \operatorname{dist}(z, B),$$

$$R - |z| = |z - 1| - r,$$

and

$$|z - 0| + |z - 1| = R + r.$$

Thus, the midset $\{A = B\}$ is an ellipse with focal sets $\{0\}$ and $\{1\}$.

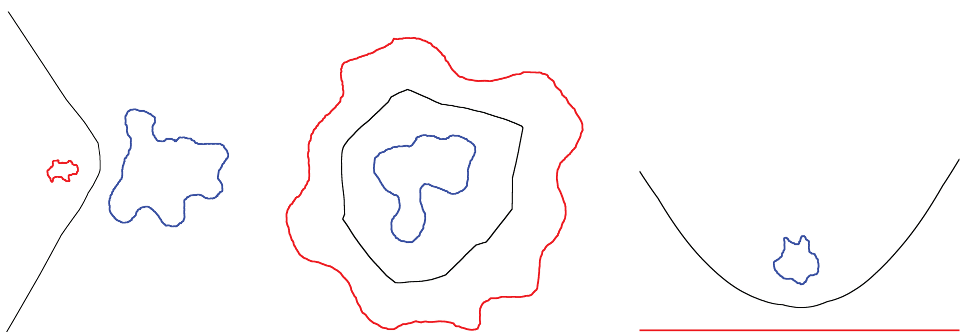
Parabola. The intermediate construction when one of the circles degenerates into a straight line and the other into a point is one of the most classical examples of an equidistant set. Namely, a parabola is the locus of points from where the distances to a fixed point (focus) and to a fixed line (directrix) are equal. Let us carry out the explicit computations for a simple example.

We construct a parabola as the equidistant set between the line $y = -1$ (the set A), and the circle with center in $(0, 2)$ and radius 1 (the set B). Hence, a point (x, y) belongs to the equidistant set if and only if

$$\begin{aligned} \operatorname{dist}((x,y),A) &= \operatorname{dist}((x,y),B), \\ y+1 &= \sqrt{x^2+(2-y)^2}-1, \\ (y+2)^2 &= x^2+(2-y)^2, \text{ and} \\ y &= \frac{x^2}{8}. \end{aligned}$$

Conversely, we leave it as an exercise to the reader to show that every ellipse or branch of a hyperbola can be constructed as the midset of two conveniently chosen circles.

3. MIDSETS AS GENERALIZED CONICS. In section 2, we saw how the classical conics can be realized as equidistant sets with circular focal sets. In this section, we want to interpret equidistant sets as natural generalizations of conics when admitting focal sets that are more complicated than circles. We concentrate on recovering geometric properties from conics for more general midsets. Figure 2 shows three (approximate versions of) midsets obtained by using an exhaustive algorithm for checking every pixel on the screen.



(a) Hyperbola (b) Ellipse (c) Parabola

Figure 2. Generalized conics

Generalized hyperbolas (see Figure 2(a)). In section 2, we saw that a branch of a hyperbola can be realized as the midset of two disjoint circles. In this section, we show that by replacing these two discs with two disjoint compact connected sets, we recover a midset that asymptotically resembles a branch of a hyperbola. Indeed, we show that far enough from these focal sets, the midset consists of two disjoint continuous curves that asymptotically approach two different directions in the plane in infinity. This is the content of Theorem 5 below.

We require some additional definitions and notation. Let $\vec{r} = [a, \infty)_v$ be a ray starting at $a \in \mathbb{R}^2$ in direction $v \in \mathbb{R}^2$, with $\|v\| = 1$. Pick $v^\perp$ such that $\{v, v^\perp\}$ is a positive orthonormal basis for $\mathbb{R}^2$. For $\varepsilon > 0$, we define the *tube* of width ε around $\vec{r}$ as

$$\operatorname{tub}_\varepsilon(\vec{r}) := \left\{a + tv + sv^\perp : t \geq 0, |s| \leq \varepsilon\right\}.$$

We say that a set M has an *asymptotic end in the direction of* $\vec{r}$ if there exists $\varepsilon > 0$ such that the set $M_{\varepsilon,\vec{r}} = M \cap \operatorname{tub}_\varepsilon(\vec{r})$ verifies the two following conditions.

(i) The orthogonal projection from $M_{\varepsilon,\vec{r}}$ to $\vec{r}$ is a bijection.

(ii) If we write $M_{\varepsilon,\vec{r}}$ using the parameters (t, s) of the tube $\mathrm{tub}_\varepsilon(\vec{r})$, then the point (i) above yields a function

$$\begin{aligned} s : [0, \infty) &\longrightarrow [-\varepsilon, \varepsilon], \\ t &\longmapsto s(t) \end{aligned}$$

in such a way that $M_{\varepsilon,\vec{r}}$ coincides with the graph of s. The second requirement is that $\lim_{t\to\infty} s(t) = 0$.

Remark.

(i) Notice that the function s defined above is continuous, since its graph is a closed set.

(ii) Notice also that every ray $\vec{p} \subset \vec{r}$ induces an asymptotic end just by considering the suitable restriction. Even though we can properly formalize using an equivalence relation, we are going to consider all these ends as equal.

Let $K \subset \mathbb{R}^2$ be a compact set. We say that the straight line $l = l_{b,w}$ is a *supporting line* for K if $l \cap K \neq \emptyset$, and that K is located entirely in one of the two half-planes defined by l. We say that $b \in l \cap K$ is a *right extreme point with respect to* l if $l \cap K$ is contained in $[b, \infty)_{-w}$ (analogously, we define a *left extreme point*). A supporting line always has both types of extreme points, and they coincide if and only if the intersection $l \cap K$ contains only one point.

Lemma 1. *Let $\varepsilon > 0$. Assume that $K \subset \{(x, y) | x \leq \varepsilon, y \leq 0\}$. For $h > 0$, we define $f_h(x) = \mathrm{dist}\,((x, h), K)$. The function f_h is strictly increasing for $x \geq \varepsilon$.*

Proof. Let $x_2 > x_1 \geq \varepsilon$ and let $p_2 \in \mathcal{P}_{(x_2,h)}(K)$ be a foot point. We have

$$f_h(x_1) \leq \mathrm{dist}((x_1, h), p_2) < \mathrm{dist}((x_2, h), p_2) = f_h(x_2).$$

Indeed, the first inequality comes from the definition of $f_h(x_1)$ and the strict inequality is due to $x_2 > x_1$. ■

In what follows, we are going to consider two disjoint compact sets A, B and a common supporting line l such that both sets are located in the same half-plane determined by l. For simplicity, we assume that l is the real line and $(-1, 0)$ is the right extreme point of A and that $(1, 0)$ is the left extreme point of B. Let $\varepsilon > 0$ be small enough. We assume that

$$A \subset \{(x, y) : x \leq -1 + \varepsilon, y \leq 0\}, \tag{1}$$

and

$$B \subset \{(x, y) : x \geq 1 - \varepsilon, y \leq 0\}. \tag{2}$$

Lemma 2. *Under the above hypotheses, for every $h > 0$ there exists a unique $x(h) \in [-1, 1]$ such that*

$$\mathrm{dist}((x(h), h), A) = \mathrm{dist}((x(h), h), B). \tag{3}$$

Moreover, $x(h)$ belongs to $(-\varepsilon, \varepsilon)$.

Proof. Since $(-1, 0) \in A$, we have

$$\operatorname{dist}((x, y), A) < \operatorname{dist}((x, y), B)$$

for every $(x, y) \in \{x \leq -\varepsilon, y \geq 0\}$.

Similarly, we have

$$\operatorname{dist}((x, y), A) > \operatorname{dist}((x, y), B)$$

for every $(x, y) \in \{x \geq \varepsilon, y \geq 0\}$.

The continuity of the function f_h defined in Lemma 1 then gives at least one point $x(h) \in (-\varepsilon, \varepsilon)$ satisfying the equality (3). Applying the conclusion of Lemma 1, we see that the function

$$x \longmapsto \operatorname{dist}((x, h), A) - \operatorname{dist}((x, h), B)$$

is strictly increasing for $x \in [-1 + \varepsilon, 1 - \varepsilon]$. We then deduce the unicity of $x(h)$ as required. ■

We apply the above lemma to characterize asymptotically the midset of two focal sets with a common supporting line. Notice that in the hypotheses of the next proposition, we drop conditions (1) and (2).

Proposition 3. *Consider two disjoint compact sets A, B and a common supporting line l such that both sets are located in the same half-plane determined by l. For simplicity, we assume that l is the real line, that $(-1, 0)$ is the right extreme point of A, and that $(1, 0)$ is the left extreme point of B. For every $\varepsilon > 0$, there exists $\tilde{h} = \tilde{h}(\varepsilon) > 0$ such that for every $h > \tilde{h}$ there exists $x(h) \in (-\varepsilon, \varepsilon)$ so that the following holds:*

$$\{A = B\} \cap \{(x, h) : x \in [-1, 1]\} = \{(x(h), h)\}.$$

Proof. In order to apply Lemma 2, we need to show that we can recover conditions (1), (2). Since $(-1, 0)$ is in A, for every $h > 0$ the foot points $\mathcal{P}_{(0,h)}(A)$ belong to the closed ball D_h centered at $(0, h)$ and passing through $(-1, 0)$ (the same happens for $\mathcal{P}_{(0,h)}(B)$, for the same ball D_h, since it also passes through $(1, 0)$). In other words, we have

$$\mathcal{P}_{(0,h)}(A) \cup \mathcal{P}_{(0,h)}(B) \subset D_h \cap \{(x, y) | y \leq 0\}.$$

We define

$$A_h := D_h \cap A, \quad B_h := D_h \cap B.$$

With these definitions it is clear that

$$\operatorname{dist}((0, h), A) = \operatorname{dist}((0, h), A_h), \text{ and}$$
$$\operatorname{dist}((0, h), B) = \operatorname{dist}((0, h), B_h).$$

We claim that for every $\varepsilon > 0$ there exists $\tilde{h} > 0$, such that for every $h > \tilde{h}$ we have

$$A_h \subset \{(x, y) \mid x \leq -1 + \varepsilon, y \leq 0\}, \text{ and}$$
$$B_h \subset \{(x, y) \mid x \geq 1 - \varepsilon, y \leq 0\}.$$

Assume, on the contrary, that there exists $\tilde{\varepsilon} > 0$ and a sequence $(x_n, y_n) \in A_n$ with $x_n > -1 + \tilde{\varepsilon}$. Notice that since $(x_n, y_n) \in D_n$, we then have

$$-y_n + n \leq \sqrt{n^2 + 1}.$$

This, and the classical undergraduate limit $\lim_{n\to\infty} \sqrt{n^2+1} - n = 0$, implies that $y_n \to 0$ (see Figure 3 for a geometric meaning). Since A is a compact set, there exists a subsequence (x_n, y_n) converging to a point $(\tilde{x}, 0) \in A$, with $\tilde{x} \geq -1 + \tilde{\varepsilon} > -1$. This contradicts the fact that $(-1, 0)$ is the right extreme point of A. We then apply Lemma 2 to find $x(h) \in (-\varepsilon, \varepsilon)$ in the midset $\{A = B\}$. It is easy to see for fixed $\varepsilon > 0$ and h large enough, that $(-\varepsilon, h)$ is closer to A and (ε, h) is closer to B, thus concluding the proof. ■

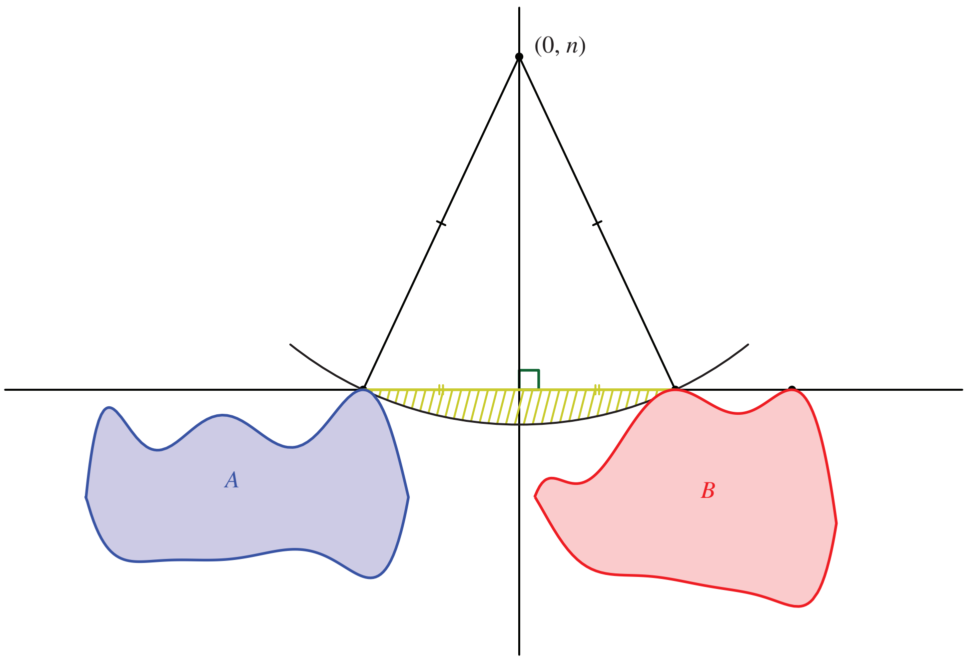


Figure 3. Foot points to $(0, n)$ lie inside the small shaded region

Given two disjoint nonempty compact connected sets A, B, we want to discuss the existence of a common supporting line leaving both sets in the same half-plane. For this, we need to remember the concept of the convex hull $\mathrm{ch}(K)$ of a compact set $K \subset \mathbb{R}^2$, defined as the smallest convex set containing K. The convex hull $\mathrm{ch}(K)$ is a convex compact set. Given two disjoint compact convex sets $\mathcal{A}, \mathcal{B} \subset \mathbb{R}^2$, it is an interesting exercise to show that there exist four common supporting lines. Two of them are called *interior common tangents*, and each one leaves the sets $\mathcal{A}, \mathcal{B}$ in a different half-plane. The remaining two supporting lines are called *exterior common tangents*, and each one leaves both sets in the same half-plane.

Two disjoint nonempty compact connected sets A, B are called ch-*disjoint* if $\mathrm{ch}(A) \cap \mathrm{ch}(B) = \emptyset$. It is easy to see that supporting lines and common supporting lines of $\mathrm{ch}(A)$, $\mathrm{ch}(B)$ are also supporting lines and common supporting lines of A, B, respectively. The above discussion directly yields the following, which we present without proof.

Lemma 4. *Two nonempty compact connected sets A, B that are* ch*-disjoint have two distinct common supporting lines, each of which leaves both sets A, B in the same half-plane.*

Now we can state the main theorem of this section.

Theorem 5 (Generalized hyperbola). *Let A, B be two nonempty compact connected sets that are* ch*-disjoint. There exists $R > 0$ and two disjoint rays $\vec{r}_1, \vec{r}_2$ such that*

$$\{A = B\} \cap B(0, R)^c$$

consists of exactly two asymptotic ends in the directions $\vec{r}_1$ and $\vec{r}_2$, respectively.

Proof. The existence of the two different asymptotic ends is due to Proposition 3 and Lemma 4. The remaining part of the proof consists of showing that there is no other piece of the midset going to infinity. This can be directly deduced from Theorem 9, part (i), which ensures that the midset $\{A = B\}$ is homeomorphic to the real line. We also present a self-contained proof. Assume that $\vec{r}_1, \vec{r}_2$ are not parallel and suppose that there exists a sequence $x_n \in \{A = B\}$ with $|x_n| > n$. Let $x_n = |x_n|e^{i\theta_n}$ be the complex notation for x_n. Taking a subsequence if needed, we can assume that there exists $\tilde{\theta} \in [0, 2\pi]$, such that $\theta_n \to \tilde{\theta}$. Let $l_{\tilde{\theta}\perp}$ be a supporting line for $A \cup B$ that is orthogonal to the direction $\tilde{\theta}$, such that $A \cup B$ and infinitely many elements from $\{x_n\}_{n\in\mathbb{N}}$ are located in different half-planes. We claim that $l_{\tilde{\theta}\perp}$ is a common supporting line for A and B. Assume, for instance, that $l_{\tilde{\theta}\perp} \cap A = \emptyset$. In this case, it is easy to see that for n large enough we should have $\operatorname{dist}(x_n, A) > \operatorname{dist}(x_n, B)$, which is impossible since x_n belongs to $\{A = B\}$. Hence, $\tilde{\theta}$ coincides with the direction of $\vec{r}_1$ (or $\vec{r}_2$) and we deduce that $\{x_n\}$ is necessarily a subset of the union of the two asymptotic ends. To prove the existence of exactly two asymptotic ends in case the rays $\vec{r}_1, \vec{r}_2$ are parallel, we can consider a slight perturbation A_ε of A in the Hausdorff topology in such a way that the above arguments can be applied to A_ε and B. We can conclude using Theorem 11 from Section 4.1. ■

Remark. For simplicity, we stated this theorem for *ch*-disjoint sets, even though it holds for every pair of compact connected disjoint sets having two supporting lines, each of which leaves both sets in the same half-plane.

Generalized parabolas (see Figure 2(c)). A remarkable geometric property of parabolas is that they are *strictly convex* in the sense that for any supporting line, the parabola becomes asymptotically more and more separated from the supporting line. In other words, the parabola can be seen as the graph of a continuous function over the supporting line (a tangent) such that the values of this function tend to infinity with the parameter of the line (check, for instance, the parabola $y = x^2$, and see how the derivatives grow to infinity). We do not give a definition for generalized parabolas, but merely say that midsets sharing various properties like *strict convexity* should be considered as some kind of generalization for parabolas.

Along the lines of the generalized hyperbolas treated in the previous paragraphs, we want to consider a midset defined by a compact connected focal set A (instead of the classical focus point) and some disjoint unbounded closed set B playing the role of the directrix. We also need to require some additional properties such as: $\operatorname{ch}(B)$ does not intersect A (in order to obtain an unbounded midset); there is no common supporting line for A and B (in order to avoid the existence of an asymptotic ray), and so on. For simplicity, we are going to keep B as a straight line, even though the reader will be able to treat more general situations.

Proposition 6. *Let $A \subset \mathbb{R}^2$ be a nonempty connected compact set and B be a disjoint straight line. There exists $\overline{R} > 0$ so that for every $R \geq \overline{R}$ and every supporting line l for $\{A = B\} \cap B(0, R)^c$, we have*

$$\lim_{s\to\infty} \operatorname{dist}(l, \{A = B\} \cap B(0, s)^c) = \infty.$$

Sketch of the proof. For this special case where B is a straight line, the proof can be obtained easily from the fact that the midset $\{A = B\}$ is actually the graph of a continuous function over B. We leave as an exercise to the reader to show that this function grows faster than any linear map.

We now outline a proof that fits more general situations. Assume for simplicity that B is the real line. The idea is to truncate B and consider the midset $\{A = B_R\}$, where $B_R = [-R, R] \subset \mathbb{R}$. As seen before, this is a generalized hyperbola that is asymptotic (let's say, to the right) to a ray $\vec{r}_R$ that is perpendicular to a segment $[a_R, (R, 0)]$ for some point $a_R \in A$, and passes through its midpoint. Since A is compact, the slope of $\vec{r}_R$ grows to infinity with R and the reader can easily complete the details. ■

Generalized ellipses (see Figure 2(b)). In this case we don't have much to say. However, part (ii) of Theorem 9 (see Section 4) serves to recognize some topological reminiscences of ellipses when the focal sets of the midset are a convex compact set *inside* a compact set. Indeed, given a connected compact set $B \subset \mathbb{R}^2$ and a convex compact set $A \subset \mathbb{R}^2$ that lies *inside* B, the midset is homeomorphic to $\mathcal{S}^1$.

4. TOPOLOGICAL PROPERTIES OF MIDSETS. The first part of this section corresponds to a survey of some topological properties of midsets in $\mathbb{R}^n$. We mainly concentrate on the articles [**10**] and [**4**]. In Sections 4.1 and 4.2, we present new results.

Since the closure $\overline{A}$ of a nonempty set $A \subset \mathbb{R}^n$ satisfies $\operatorname{dist}(x, A) = \operatorname{dist}(x, \overline{A})$ for every $x \in \mathbb{R}^n$, we can easily conclude that $\{A = B\} = \{\overline{A} = \overline{B}\}$, for every $A, B \subset \mathbb{R}^n$. Hence, we can consider closed sets as focal sets of midsets. The function

$$\begin{aligned} d_A : \mathbb{R}^n &\longrightarrow \mathbb{R}, \\ x &\longmapsto d_A(x) := \operatorname{dist}(x, A) \end{aligned}$$

is continuous and

$$\{A = B\} = d_{A,B}^{-1}(0),$$

where $d_{A,B}(x) := d_A(x) - d_B(x)$. We conclude that midsets are always closed sets. Furthermore, a midset is never empty. Indeed, we can compute the function $d_{A,B}$ over a continuous path joining A and B in order to obtain a zero for $d_{A,B}$ (in fact, it can be shown that every midset is nonempty if and only if the ambient space is connected). The main theorem in [**10**] is the following.

Theorem 7 (see Theorem 4 in [10]). *If A and B are nonempty connected sets, then $\{A = B\}$ is connected.*

The following is a simple property that, at least from the point of view of applications to sea frontiers, provides the politically correct fact that there is no region inside the UN's definition of the sea boundary between two disjoint countries.

Proposition 8 (see Theorem 2 in [10]). *Let $A, B \subset \mathbb{R}^n$ be two disjoint nonempty closed sets. Then the midset $\{A = B\}$ has empty interior.*

Proof. Let $x \in \{A = B\}$ and let $p_A \neq p_B$ be foot points in $\mathcal{P}_x(A), \mathcal{P}_x(B)$, respectively. We claim that for any $\tilde{x} \in [p_A, x)$, we have

$$\operatorname{dist}(\tilde{x}, A) < \operatorname{dist}(\tilde{x}, B). \tag{4}$$

Indeed, the closed ball $\overline{B}(x, \operatorname{dist}(x, A))$ strictly contains the closed ball $\overline{B}(\tilde{x}, \operatorname{dist}(\tilde{x}, A))$, except for the foot point p_A (see Figure 4). But $\operatorname{dist}(\tilde{x}, p_A) = \operatorname{dist}(\tilde{x}, A)$, which implies that there is no point of B in $\overline{B}(\tilde{x}, \operatorname{dist}(\tilde{x}, A))$; this gives us (4). The inequality (4) tells us in particular that $\tilde{x} \notin \{A = B\}$, and the proposition follows by picking $\tilde{x}$ as close to x as necessary. ■

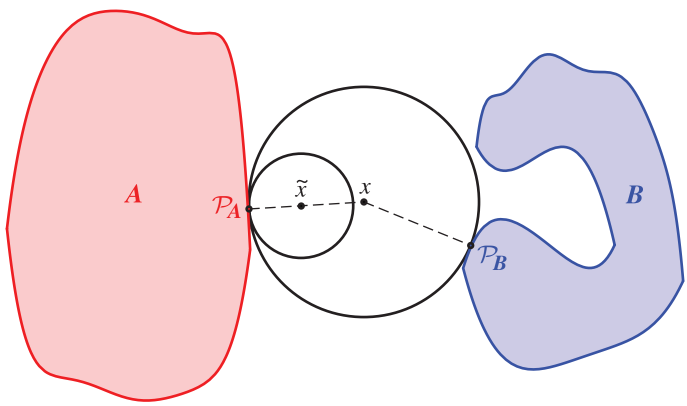


Figure 4.

Remark. Notice above that $\operatorname{dist}(\tilde{x}, A) = \operatorname{dist}(x, A) - \operatorname{dist}(x, \tilde{x})$, and (4) can be improved to

$$\operatorname{dist}(\tilde{x}, B) > \operatorname{dist}(x, A) - \operatorname{dist}(x, \tilde{x}). \tag{5}$$

Continuing with the topological properties of midsets, we concentrate on the case when the focal sets A, B are disjoint compact connected nonempty sets. In that case we have the following.

Theorem 9 (see Theorem 3.2 in [4]). *Let $A, B \subset \mathbb{R}^n$ be two disjoint compact connected nonempty sets.*

(i) *If $n = 2$, then the midset $\{A = B\}$ is a topological 1-manifold.*

(ii) *For $n > 2$, the above result is no longer true in general. However, for every n, if A is convex then $\{A = B\}$ is topologically equivalent to an open set of the sphere $\mathcal{S}^{n-1}$. Furthermore, the midset $\{A = B\}$ is homeomorphic to the sphere $\mathcal{S}^{n-1}$ if and only if A is convex and lies in the interior of the convex hull of B.*

A geometrical object that is closely related to midsets is the so called ε-boundary of a set $A \in \mathbb{R}^n$, which is defined as

$$\partial_\varepsilon(A) := \{x \in \mathbb{R}^n : \operatorname{dist}(x, A) = \varepsilon\}.$$

In fact, we have the relation

$$\{A = B\} = \bigcup_{\varepsilon \geq 0} (\partial_\varepsilon(A) \cap \partial_\varepsilon(B)).$$

These sets have been widely studied in [**1**], [**2**] and recently in [**9**]. A deeper relation between midsets and ε-boundaries is indebted to Loveland.

Theorem 10 (see Theorem 3.1 in [4]). *If A, B are disjoint closed sets of $\mathbb{R}^n$, A is convex, and $\varepsilon > 0$, then $\{A = B\}$ is homeomorphic to an open subset of $\partial_\varepsilon(A)$.*

4.1. Continuity of midsets. Let $(X, \operatorname{dist}_X)$ be a compact metric space. For $A \subset X$ and $\varepsilon > 0$, we denote by

$$B(A, \varepsilon) := \{x \in X : \operatorname{dist}_X(x, A) < \varepsilon\}$$

the ε-neighborhood of A. The *Hausdorff distance* between two compact sets $K_1, K_2 \subset X$ is

$$\operatorname{dist}_{\mathcal{H}}(K_1, K_2) := \inf\{\varepsilon > 0 : K_1 \subset B(K_2, \varepsilon) \text{ and } K_2 \subset B(K_1, \varepsilon)\}.$$

This distance defines a topology on the space $\mathcal{K}(X)$, of compact subsets of X. With this topology, $\mathcal{K}(X)$ is itself a compact space (see, for instance, [**6**]). Given a convergent sequence $A_n \in \mathcal{K}(X)$, the Hausdorff limit is characterized as the set of points that are limits of sequences $x_n \in A_n$.

In general, equidistant sets are closed but not necessarily bounded sets. In order to treat compact sets and use the Hausdorff topology, we are going to consider restrictions of equidistant sets to a large enough ball containing both focal sets. Let R be a large positive number and A, B be compact sets such that $A \cup B \subset B(0, R)$. We write

$$\{A = B\}_R := \{A = B\} \cap \overline{B}(0, R).$$

We are interested in the continuity of the mapping

$$\begin{aligned} \mathcal{M}id_R : \mathcal{K}(\overline{B}(0, R)) \times \mathcal{K}(\overline{B}(0, R)) &\longrightarrow \mathcal{K}(\overline{B}(0, R)), \\ (A, B) &\longmapsto \{A = B\}_R. \end{aligned}$$

Theorem 11. *If $A \cap B = \emptyset$, then $\mathcal{M}id_R$ is continuous at (A, B).*

Proof. Let $\{A_n\}_{n \in \mathbb{N}}, \{B_n\}_{n \in \mathbb{N}}$ be two sequences in $\mathcal{K}(\overline{B}(0, R))$ so that

$$A_n \to A \quad \text{and} \quad B_n \to B.$$

Define $E_n := \{A_n = B_n\}_R \in \mathcal{K}(\overline{B}(0, R))$. A compactness argument (and a suitable subsequence) allows us to assume that there exists $E \in \mathcal{K}(\overline{B}(0, R))$ such that $E_n \to E$. We affirm that $\{A = B\}_R = E$.

- Let $e \in E$. There exist sequences $e_n \in E_n$, $a_n \in A_n$, $b_n \in B_n$, and two points $a \in A$, $b \in B$ such that

$$\operatorname{dist}(e_n, a_n) = \operatorname{dist}(e_n, A_n) = \operatorname{dist}(e_n, B_n) = \operatorname{dist}(e_n, b_n) \tag{6}$$

with $e_n \to e$, $a_n \to a$, and $b_n \to b$. We claim that $\operatorname{dist}(e, A) = \operatorname{dist}(e, a)$. Assume otherwise that there is a point $\tilde{a} \in A$ such that $\operatorname{dist}(e, \tilde{a}) < \operatorname{dist}(e, a)$. There exists a sequence $\tilde{a}_n \in A_n$ with $\tilde{a}_n \to \tilde{a}$. But (6) implies that

$$\operatorname{dist}(e_n, a_n) \leq \operatorname{dist}(e_n, \tilde{a}_n),$$

which leads to $\operatorname{dist}(e, a) \leq \operatorname{dist}(e, \tilde{a})$. In a similar way, we show that $\operatorname{dist}(e, B) = \operatorname{dist}(e, b)$. Taking the limit in (6), we get $\operatorname{dist}(e, A) = \operatorname{dist}(e, B)$ and $E \subset \{A = B\}_R$.
- Let $m \in \{A = B\}_R$ and $a_n \in A_n$, $b_n \in B_n$, satisfying

$$\operatorname{dist}(m, A_n) = \operatorname{dist}(m, a_n), \tag{7}$$

and

$$\operatorname{dist}(m, B_n) = \operatorname{dist}(m, b_n). \tag{8}$$

Passing to a subsequence if necessary, there exist $a \in A$, $b \in B$ such that $a_n \to a$ and $b_n \to b$. Then

$$\operatorname{dist}(m, a_n) = \operatorname{dist}(m, A_n) \to \operatorname{dist}(m, a), \tag{9}$$

and

$$\operatorname{dist}(m, b_n) = \operatorname{dist}(m, B_n) \to \operatorname{dist}(m, b). \tag{10}$$

From (7), (8), (9), and (10) we have

$$\lim_{n\to\infty} \operatorname{dist}(m, A_n) - \operatorname{dist}(m, B_n) = 0.$$

Passing to a subsequence (or interchanging the roles of A_n and B_n), we can assume that $\operatorname{dist}(m, A_n) - \operatorname{dist}(m, B_n)$ increases to zero. Let $t \geq 0$. We define $m_t \in [m, b]$ so that $\operatorname{dist}(m, m_t) = t$. Define $f_n(t) = \operatorname{dist}(m_t, A_n) - \operatorname{dist}(m_t, B_n)$. Let $\varepsilon > 0$. We claim that there exists $\tilde{n} \in \mathbb{N}$ such that $f_n(\varepsilon) > 0$ for every $n \geq \tilde{n}$. Indeed, we know that

$$\operatorname{dist}(m_\varepsilon, B) = \operatorname{dist}(m, B) - \varepsilon, \text{ and}$$
$$\operatorname{dist}(m_\varepsilon, A) > \operatorname{dist}(m, A) - \varepsilon.$$

Notice that the second inequality above follows from (5) (here we use the hypothesis $A \cap B = \emptyset$). From these facts we obtain $\operatorname{dist}(m_\varepsilon, A) - \operatorname{dist}(m_\varepsilon, B) > 0$. Since $f_n(\varepsilon) \to \operatorname{dist}(m_\varepsilon, A) - \operatorname{dist}(m_\varepsilon, B) > 0$, our claim holds. Using that $f_n(0) \leq 0$ for every $n \geq \tilde{n}$, we can pick $m_n \in [m, m_\varepsilon]$ such that $f_n(m_n) = 0$, that is, $m_n \in \{A_n = B_n\}$. This construction holds for every $\varepsilon > 0$. A diagonal sequence argument then allows us to construct a sequence $m_n \in E_n$ with $m_n \to m$. That is, $m \in \lim E_n = E$, and finally $\{A = B\}_R \subset E$. ■

4.2. Error estimates for quasi-equidistant points. Given two nonempty closed disjoint sets A, B and $\varepsilon > 0$, we define the set of ε-equidistant points to A and B as

$$\{|A - B| < \varepsilon\} := \{x \in \mathbb{R}^n : |\mathrm{dist}(x, A) - \mathrm{dist}(x, B)| < \varepsilon\}.$$

This notion is crucial when we deal with computer simulations. Recall that finding an equidistant point is equivalent to finding a zero of a continuous function. In the case of computer simulations, this function is no longer continuous since it is evaluated in pixels (a discrete set). In fact, this function in general may have no zeros at all. Then, in order to draw a good picture of the equidistant set, we need to check for points (pixels) such that the difference between the distances to the focal sets is small enough to guarantee that inside a small neighborhood there is a zero for the continuous function that defines the midset. In conclusion, we look for a set $\{|A - B| < \varepsilon\}$ for some positive ε that depends on the screen resolution, computer capabilities, and so on. As we will see, the theorem we present here requires a very specific configuration of the focal sets. Nevertheless, the reader should notice that the result can be applied to more general situations.

Let $x \notin A \cup B$. We say that *A and B are separated by an angle of measure α at x* if there exist two supporting lines l_A, l_B passing through x such that

1. l_A is a supporting line for A, and B lies in a different half-plane than A,
2. l_B is a supporting line for B, and A lies in a different half-plane than B, and
3. the angle formed at x by l_A and l_B measures α.

Theorem 12 (Error estimates). *Let A, B be two disjoint nonempty closed sets. Let $\varepsilon > 0$ and $x_0 \in \{|A - B| < \varepsilon\}$ such that A and B are separated by an angle of measure α at x_0. Then there exists $x_1 \in \{A = B\}$ verifying*

$$\mathrm{dist}(x_1, x_0) < \frac{\varepsilon}{2}\left(\frac{\varepsilon + 2d}{\varepsilon + d - d\cos\alpha}\right),$$

where $d = \min\{\mathrm{dist}(x_0, A), \mathrm{dist}(x_0, B)\}$.

Proof. Consider the function $f(x) := d_{B,A}(x) = \mathrm{dist}(x, B) - \mathrm{dist}(x, A)$ and assume that $d = \mathrm{dist}(x_0, A)$, that is,

$$0 \le f(x_0) < \varepsilon.$$

We look for x_1 such that $f(x_1) = 0$. Let $b \in \mathcal{P}_{x_0}(B)$. We write $x(t)$ as the point on $[x_0, b]$ such that $\mathrm{dist}(x_0, x(t)) = t$. Finally, we write

$$g(t) := f(x(t)) = \mathrm{dist}(x(t), B) - \mathrm{dist}(x(t), A).$$

Since $f(x_0) = g(0)$, we have $0 \le g(0) < \varepsilon$ and $g(\overline{d}) < 0$ where $\overline{d} = \mathrm{dist}(x_0, B)$. Although we know that there exists $\overline{t} \in (0, \overline{d})$ that satisfies $g(\overline{t}) = 0$, the function g is not differentiable in general and we cannot directly estimate the size of $\overline{t}$. We are going to construct an upper bound for g in order to get a good estimate. Let a be the intersection of the circle centered at x_0 and radius d with l_A as shown in Figure 5. For every t, we have

$$\mathrm{dist}(x(t), a) \le \mathrm{dist}(x(t), A). \tag{11}$$

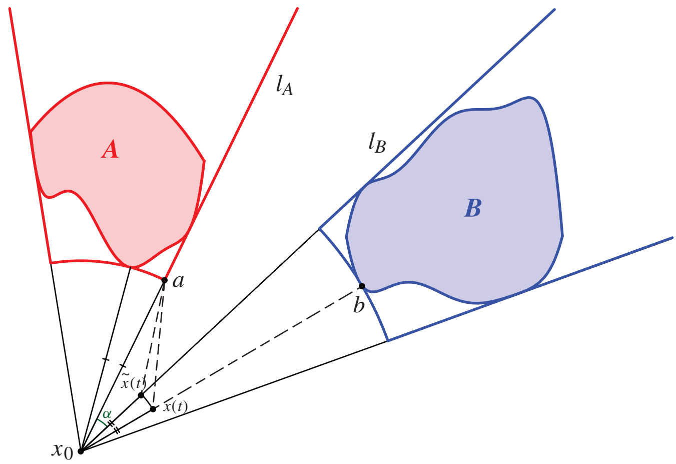


Figure 5. Construction for the proof of Theorem 12

Define $\tilde{x}(t) \in l_B$ so that $\operatorname{dist}(x_0, \tilde{x}(t)) = t$ (Figure 5). Thus, we have

$$\operatorname{dist}(\tilde{x}(t), a) \leq \operatorname{dist}(x(t), A). \tag{12}$$

The left term above can be explicitly computed, using elementary Euclidean geometry, as

$$\operatorname{dist}(\tilde{x}(t), a)^2 = d^2 + t^2 - 2dt\cos(\alpha). \tag{13}$$

Moreover, we know that

$$\begin{aligned} \operatorname{dist}(x(t), B) &= \operatorname{dist}(x_0, B) - t \\ &< d + \varepsilon - t. \end{aligned} \tag{14}$$

Using (11, 12, 13, and 14), we get (and define $\hat{g}$ by)

$$g(t) < d + \varepsilon - t - \sqrt{d^2 + t^2 - 2dt\cos(\alpha)} := \hat{g}(t). \tag{15}$$

Notice that $\hat{g}(0) = \varepsilon$, and that

$$\hat{t} = \frac{\varepsilon}{2}\left(\frac{\varepsilon + 2d}{\varepsilon + d - d\cos\alpha}\right)$$

verifies $\hat{g}(\hat{t}) = 0$. Finally, the inequality (15) helps us to find a point $\bar{t} \in (0, \hat{t})$ such that $f(\bar{t}) = 0$. ■

Concluding remarks. Section 3 concentrates on showing that equidistant sets look simple, at least asymptotically. As a future line of research, we suggest exploring how complicated an equidistant set can actually be. As mentioned in the introduction, this question should become crucial, since equidistant sets are meant to be used as regional boundaries for many real-life situations. For instance, a sea delimitation needs to have some *physical* properties in order to make its role in real life feasible. As equidistant sets become more intricate, it becomes more difficult to consider these sets as viable frontiers. For example, we suggest the following.

Question: Does there exist an equidistant set in the plane with connected disjoint focal sets, having Hausdorff dimension greater than 1? What about other notions of dimension? How does the dimension of an equidistant set depend on the dimension of its focal sets?

Problem: Characterize all closed sets of $\mathbb{R}^2$ that can be realized as the equidistant set of two connected disjoint closed sets.

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MARIO PONCE received his B.A. from the Universidad de Chile in 2001, his Mestrado from IMPA (Rio de Janeiro, Brazil) in 2003, and his Doctorat from the Université Paris Sud (Orsay, France) in 2007, under the supervision of Jean-Christophe Yoccoz. Currently he is an assistant professor at the Universidad Católica de Chile. His main mathematical interests are research in dynamical systems and coordinating the academic committee for the Chilean mathematical olympiad.
Facultad de Matemáticas, Universidad Católica de Chile, Casilla 306, Santiago 22, Chile
mponcea@mat.puc.cl

PATRICIO SANTIBÁÑEZ is an undergraduate student in the faculty of mathematics of the Universidad Católica de Chile. He is preparing himself to become a very good teacher for secondary students.
Facultad de Matemáticas, Universidad Católica de Chile, Casilla 306, Santiago 22, Chile
pjsantib@uc.cl