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CHAPTER 8
LINEAR PROGRAMMING

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SCIENTIFIC COMPUTING

Contents

1) Simplex method
    1. The canonical and standard form of linear programming problems
    2. Basic feasible solution
    3. Formula for incremental change of the objective function. Optimality test
    4. The algebra of the simplex method
    5. The simplex method in tabular form
    6. The simplex method: termination
    7. Two-phase simple method
2) Duality theory
    1. Dual problem
    2. Duality theory
    3. Some applications of duality theory

THE SIMPLEX METHOD

The simplex method is the basic algorithm to solve the Linear programming problems

Content

1. The canonical and standard form of linear programming problems
2. Basic feasible solution
3. Formula for incremental change of the objective function. Optimality test
4. The algebra of the simplex method
5. The simplex method in tabular form
6. The simplex method: termination
7. Two-phase simple method

The canonical and standard form of the linear programming problems

The general form of the linear programming problem

- The standard form of the linear programming problem is the optimization problem in which we have to find the maximum (minimum) of a linear objective function with the condition that the variables must satisfy a number of equations and linear inequalities. The mathematical model of the problem can be stated as follows:

$$f(x_1, x_2, \ldots, x_n) = \sum_{j=1}^{n} c_j x_j \to \min(\max), \qquad (1)$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j = b_i, i = 1, 2, \ldots, p \quad (p \le m) \qquad (2)$$

$$\sum_{j=1}^{n} a_{ij} x_j \ge b_i, i = p+1, p+2, \ldots, m \qquad (3)$$

$$x_j \ge 0, j = 1, 2, \ldots, q \quad (q \le n) \qquad (4)$$

$$x_j <> 0, j = q+1, q+2, \ldots, n \qquad (5)$$

- Notation $x_j <> 0$ shows that variable x_j is unrestricted in sign.

The general form of the linear programming problem

- Constraints:

$$\sum_{j=1}^{n} a_{ij} x_j = b_i, i = 1,..., p$$

is called as basic functional constraints in equation form

- Constraints:

$$\sum_{j=1}^{n} a_{ij} x_j \geq b_i, i = p+1,..., m$$

is called as basic functional constraints in inequality form

- Constraints:

$$x_j \geq 0, j = 1,..., q$$

is called as nonnegativity constraints on variables

The canonical form of the linear programming problem

$$f(x_1, x_2, \dots, x_n) = \sum_{j=1}^{n} c_j x_j \to \min,$$

$$\sum_{j=1}^{n} a_{ij} x_j = b_i, i = 1, 2, \dots, m$$

$$x_j \geq 0, j = 1, 2, \dots, n$$

The standard form of the linear programming problem

$$f(x_1, x_2, \ldots, x_n) = \sum_{j=1}^{n} c_j x_j \to \min,$$

$$\sum_{j=1}^{n} a_{ij} x_j \geq b_i, i = 1, 2, \ldots, m$$

$$x_j \geq 0, j = 1, 2, \ldots, n$$

Transform general form to canonical form

- Obviously, the canonical form is the special case of the general form

$$f(x_1, x_2, ..., x_n) = \sum_{j=1}^{n} c_j x_j \rightarrow \min(\max), \qquad (1)$$

$$\sum_{j=1}^{n} a_{ij} x_j = b_i, i = 1, 2, ..., p \quad (p \le m) \qquad (2)$$

$$\sum_{j=1}^{n} a_{ij} x_j \ge b_i, i = p+1, p+2, ..., m \qquad (3)$$

$$x_j \ge 0, j = 1, 2, ..., q \quad (q \le n) \qquad (4)$$

$$x_j <> 0, j = q+1, q+2, ..., n \qquad (5)$$

$$f(x_1, x_2, ..., x_n) = \sum_{j=1}^{n} c_j x_j \rightarrow \min,$$

$$\sum_{j=1}^{n} a_{ij} x_j = b_i, i = 1, 2, ..., m$$

$$x_j \ge 0, j = 1, 2, ..., n$$

- On the other hand, any linear programming problem can always be transformed to the canonical form by using the following transformations:

Transform general form to canonical form

a) Change the inequality form "≤" *to the form* "≥". Linear inequality

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i$$

is equivalent to the following linear inequality:

$$-\sum_{j=1}^{n} a_{ij} x_j \ge -b_i$$

Transform general form to canonical form

b) Change the equality form "=" to the form "≥".

Linear equation

$$\sum_{j=1}^{n} a_{ij} x_j = b_i$$

is equivalent to following two linear inequalities:

$$\sum_{j=1}^{n} a_{ij} x_j \geq b_i$$

$$-\sum_{j=1}^{n} a_{ij} x_j \geq -b_i$$

Transform general form to canonical form

c) Change the equality form "=" to the form "≥". Linear inequality

$$\sum_{j=1}^{n} a_{ij} x_j \geq b_i \qquad (1)$$

is "equivalent" to an inequality constraint and a nonnegativity constraint on variable

$$\sum_{j=1}^{n} a_{ij} x_j - y_i = b_i \qquad (2)$$

$$y_i \geq 0$$

- "Equivalent" means that: If $(x_1, x_2, ..., x_n, y_i)$ is solution to (2) then $(x_1, x_2, ..., x_n)$ is solution to (1).
- Variable y_i is called as artificial variable.

Transform general form to canonical form

d) Replace each unrestricted sign variable x_j by two restricted sign variables:

$$x_j = x_j^+ - x_j^-,$$

$$x_j^+ \geq 0,\ x_j^- \geq 0.$$

e) Transform maximization problem to minimization problem.

$$\max \{ f(x) : x \in D \}$$

is equivalent to optimization problem

$$\min \{ -f(x) : x \in D \}$$

with the meaning that: Solution to a problem is also the solution to other problem, and vice versa. We have the equation:

$$\max \{ f(x) : x \in D \} = -\min \{ -f(x) : x \in D \}$$

Transform general form to canonical form

f) Change the form "≤" to the form "=". Linear inequality

$$\sum_{j=1}^{n} a_{ij}x_j \le b_i \qquad (1)$$

is "equivalent" to an equality constraint and a nonnegativity constraint on variable

$$\sum_{j=1}^{n} a_{ij}x_j + y_i = b_i \qquad (2)$$
$$y_i \ge 0$$

- "Equivalent" means that: If $(x_1, x_2, ..., x_n, y_i)$ is solution to (2) then $(x_1, x_2, ..., x_n)$ is solution to (1).
- Variable y_i is called as artificial variable.

Example

- Linear programming problem

$$x_1 + 2x_2 - 3x_3 + 4x_4 \to \max,$$
$$x_1 + 5x_2 + 4x_3 + 6x_4 \leq 15,$$
$$x_1 + 2x_2 - 3x_3 + 3x_4 = 9,$$
$$x_1, x_2, x_4 \geq 0, \; x_3 <> 0,$$

is equivalent to linear programming problem in canonical form:

$$
\begin{aligned}
&-x_1 - 2x_2 + 3(x_3^+ - x_3^-) - 4x_4 \to \min,\\
&x_1 + 5x_2 + 4(x_3^+ - x_3^-) + 6x_4 + x_5 = 15,\\
&x_1 + 2x_2 - 3(x_3^+ - x_3^-) + 3x_4 \qquad = 9,\\
&\quad x_1, x_2, x_3^+, x_3^-, x_4, x_5 \geq 0,
\end{aligned}
$$

It means if $(\overline{x_1}, \overline{x_2}, \overline{x_3}^+, \overline{x_3}^-, \overline{x_4}, \overline{x_5})$ is optimal solution to linear programming problem canonical form then $(\overline{x_1}, \overline{x_2}, \overline{x_3}, \overline{x_4})$ where $\overline{x_3}, = \overline{x_3}^+ - \overline{x_3}^-$ is optimal solution to original problem.

Solve linear programming problem graphically

Solve linear programming problem graphically

- Linear programming problem in standard form with 2 variables

$$f(x_1, x_2) = c_1x_1 + c_2x_2 \rightarrow \min,$$

subject to

$$a_{i1}\, x_1 + a_{i2}\, x_2 \geq\ b_i,\ i = 1, 2, ..., m$$

- Denote

$$D = \{(x_1,x_2):\ a_{i1}\, x_1 + a_{i2}\, x_2 \geq\ b_i,\ i = 1, 2, ..., m\}$$

the feasible region.

Solve linear programming problem graphically

- Each linear inequality

$$a_{i1}\, x_1 + a_{i2}\, x_2 \geq\ b_i,\ i = 1, 2, ..., m$$

corresponds to a line that forms the boundary of what is permitted by the constraint

$\Rightarrow$ Feasible region *D* determined as intersection of *m* lines will be a convex polygon on the plane.

Solve linear programming problem graphically

- Equation

$$c_1x_1 + c_2x_2 = \alpha$$

has normal vector (c_1,c_2)

when α changes, it will determine parallel lines that we call contour lines (with value α).

Each point $u=(u_1,u_2)\in D$ is on the contour line with value

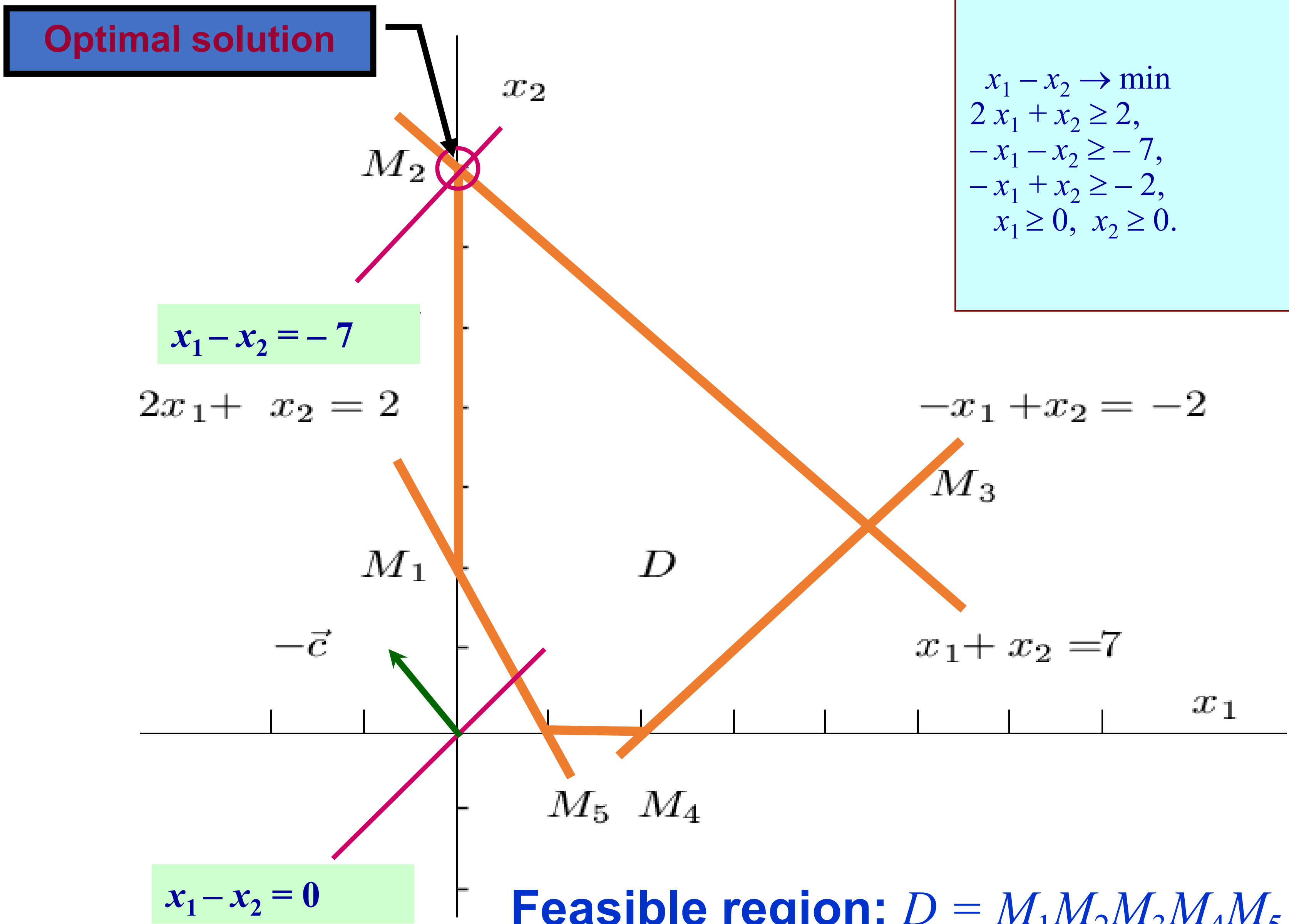
$$\alpha_u = c_1u_1 + c_2u_2 = f(u_1,u_2)$$

Example 1

- Solve the following linear programming problem:

$$
\begin{aligned}
& x_1 - x_2 \to \min \\
& 2x_1 + x_2 \ge 2, \\
& -x_1 - x_2 \ge -7, \\
& -x_1 + x_2 \ge -2, \\
& x_1 \ge 0,\ x_2 \ge 0.
\end{aligned}
$$

$$
\begin{aligned}
& x_1 - x_2 \to \min \\
& 2x_1 + x_2 \geq 2, \\
& -x_1 - x_2 \geq -7, \\
& -x_1 + x_2 \geq -2, \\
& x_1 \geq 0,\ x_2 \geq 0.
\end{aligned}
$$



Feasible region: $D = M_1M_2M_3M_4M_5$

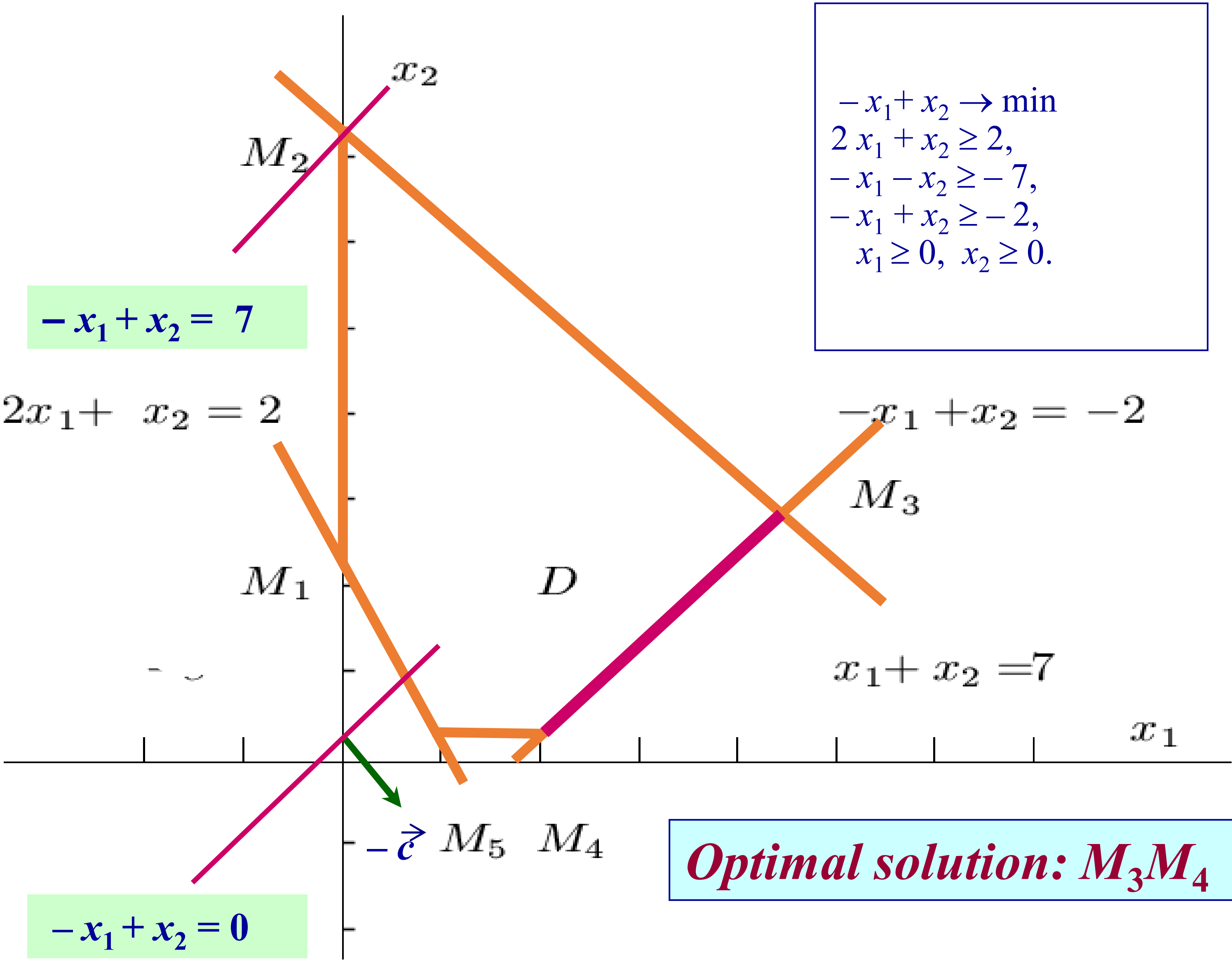
Example 1

- Using geometric concepts, we get optimal solution of the problem is the point $M_2(0,7)$: $x^* = (0,7)$, with the optimal value of $f^* = -7$.
- If we replace the objective function of the problem by

$$-x_1+x_2 \rightarrow \min,$$

then the optimal value will be -2, and all points on the segment M_3M_4 are optimal solutions of the problem. For example, the optimal solution of the problem can be taken as $x^*=(2,0)$ (corresponding to the point M4).

$$\begin{aligned} &-x_1 + x_2 \to \min \\ &2x_1 + x_2 \geq 2, \\ &-x_1 - x_2 \geq -7, \\ &-x_1 + x_2 \geq -2, \\ &x_1 \geq 0,\ x_2 \geq 0. \end{aligned}$$



Optimal solution: M_3M_4

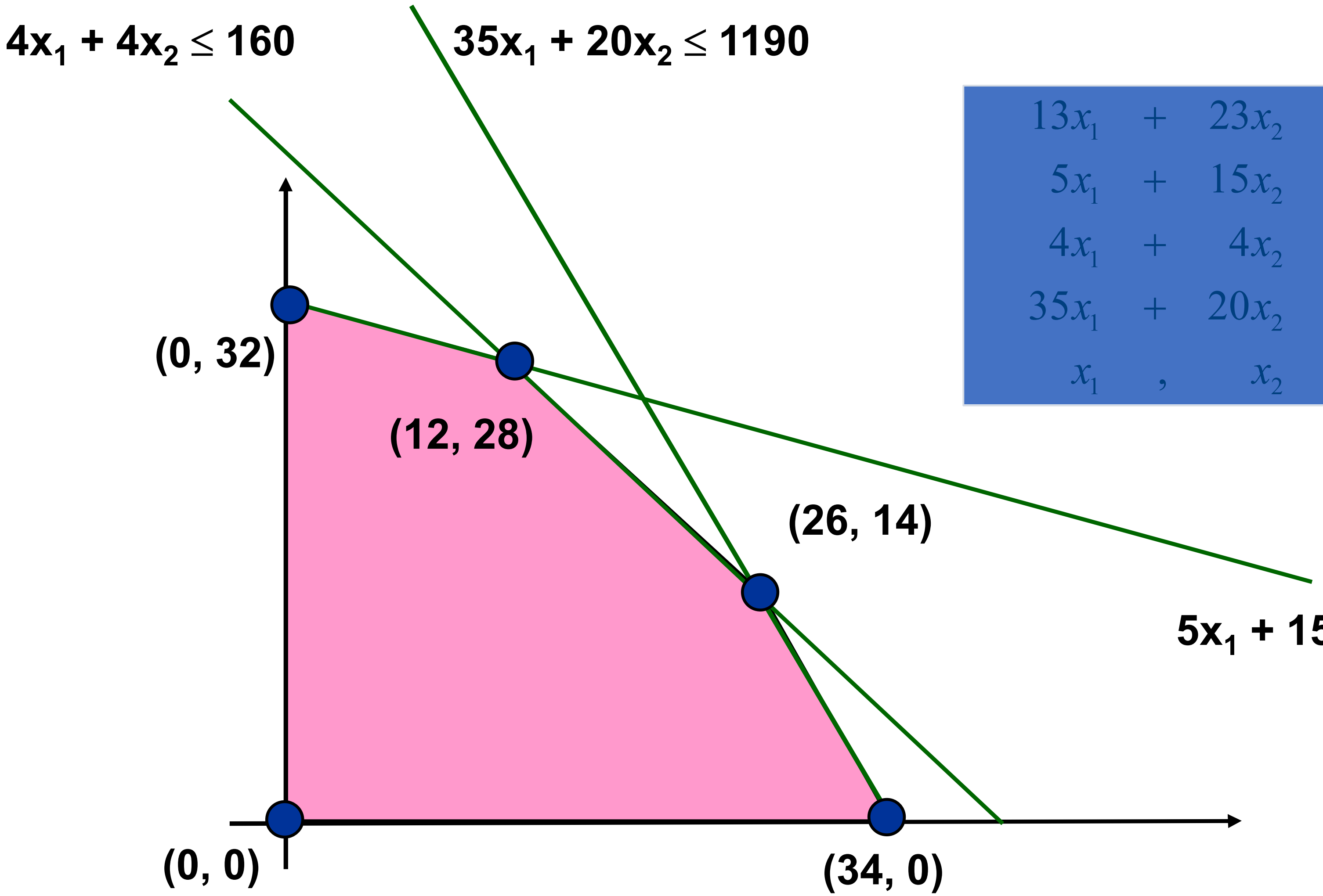
Comments

- In both cases, we always find the optimal solution as a corner point of the feasible region (a point of intersections between constraint boundaries).
- "The linear programming problem in the plane always has the optimal solution as a corner point of the feasible region."
- This important geometrical remark has led to the proposal of the simplex method to solve the linear programming problem (LP).

Example 2

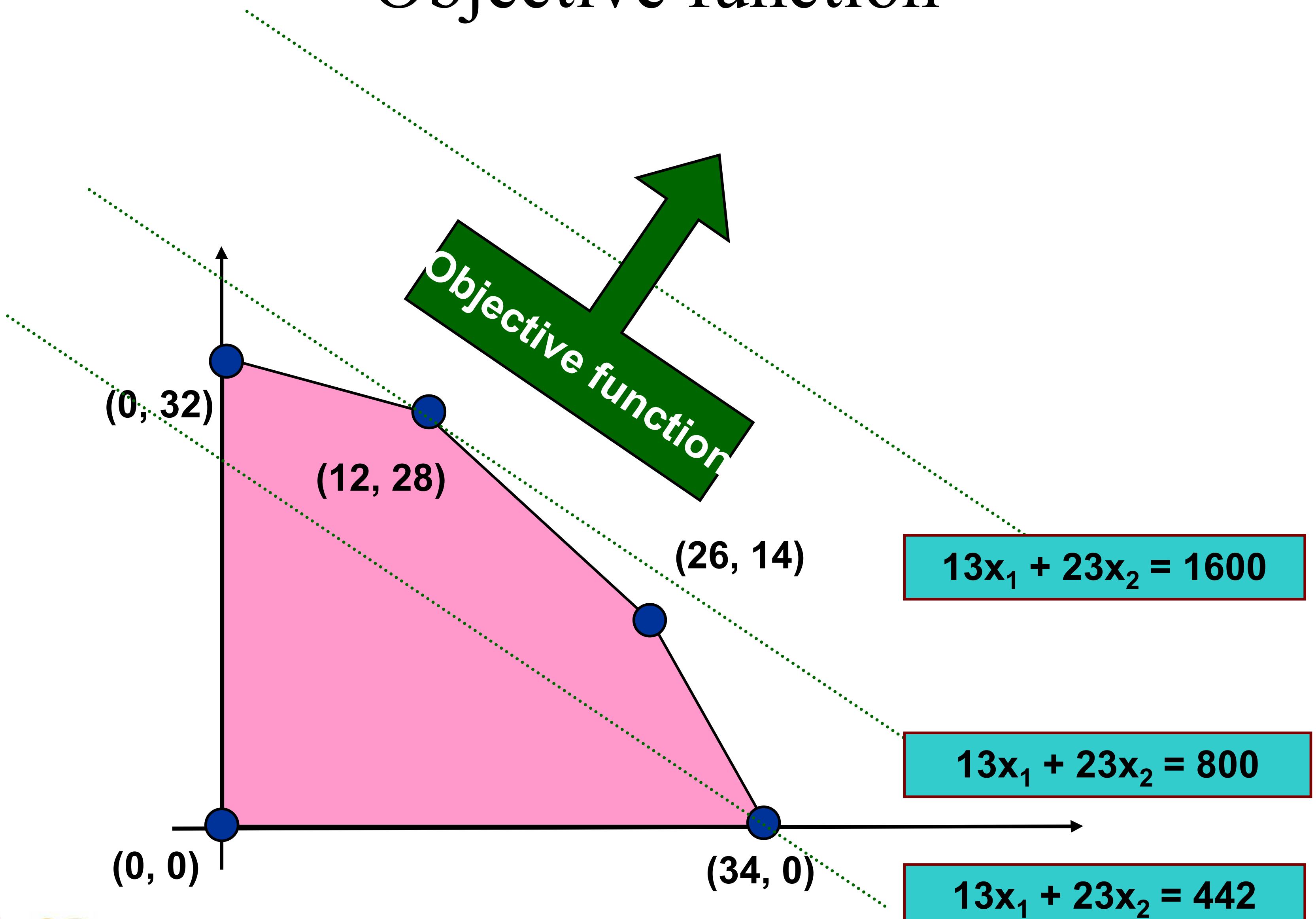
$$
\begin{array}{rcrcr}
13x_1 & + & 23x_2 & \rightarrow & \max \\
5x_1 & + & 15x_2 & \leq & 480 \\
4x_1 & + & 4x_2 & \leq & 160 \\
35x_1 & + & 20x_2 & \leq & 1190 \\
x_1 & , & x_2 & \geq & 0
\end{array}
$$

Feasible region



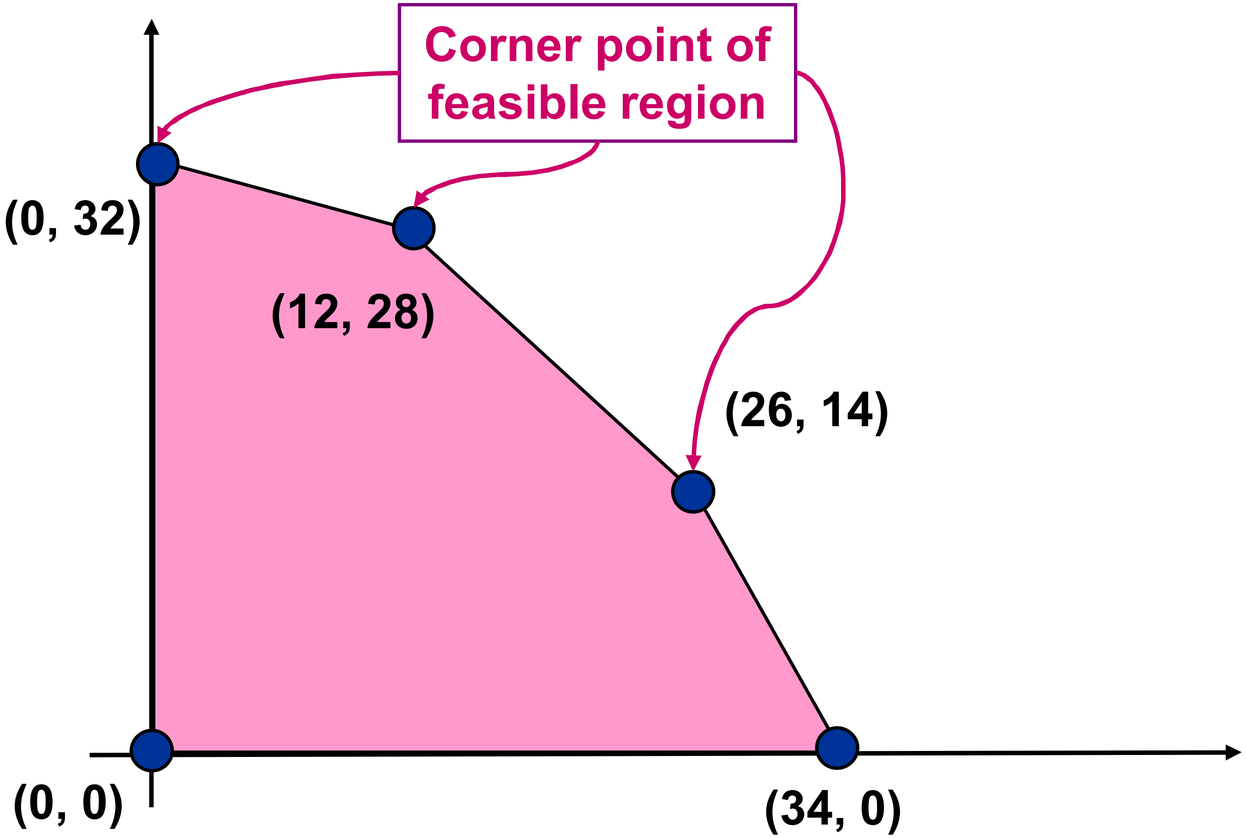
$$
\begin{array}{rcrcr}
13x_1 & + & 23x_2 & \rightarrow & \max \\
5x_1 & + & 15x_2 & \leq & 480 \\
4x_1 & + & 4x_2 & \leq & 160 \\
35x_1 & + & 20x_2 & \leq & 1190 \\
x_1 & , & x_2 & \geq & 0
\end{array}
$$

Objective function



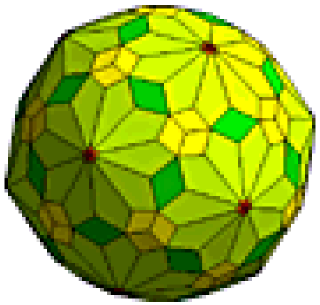
Geometric meaning

- There exists an optimal solution that is the corner point of the feasible region

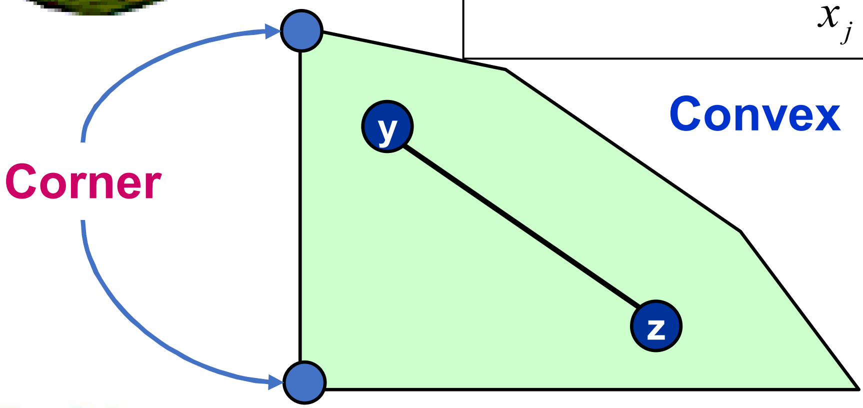


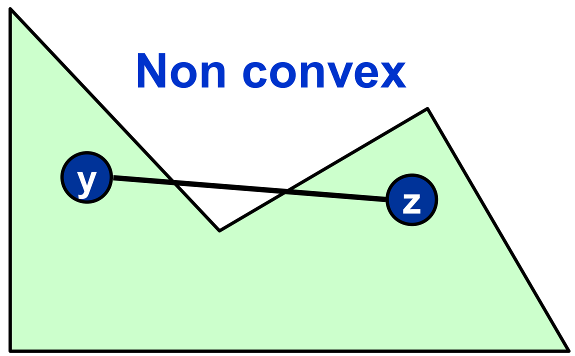
Geometric meaning of LP

- Feasible region is a a polyhedral convex set.
  - Convex: if y and z are feasible solutions, then $\alpha y + (1-\alpha)z$ is also feasible solution for all $0 \leq \alpha \leq 1$.
  - Corner: feasible solution x which can not be expressed as $\alpha y + (1-\alpha)z$, $0<\alpha<1$, for all distinct feasible solutions y and z.



$$
\begin{array}{llrcll}
\text{(P)} & \max & \sum_{j=1}^{n} c_j x_j & & & \\
 & & \sum_{j=1}^{n} a_{ij} x_j & \leq & b_i & 1 \leq i \leq m \\
 & & x_j & \geq & 0 & 1 \leq j \leq n
\end{array}
$$

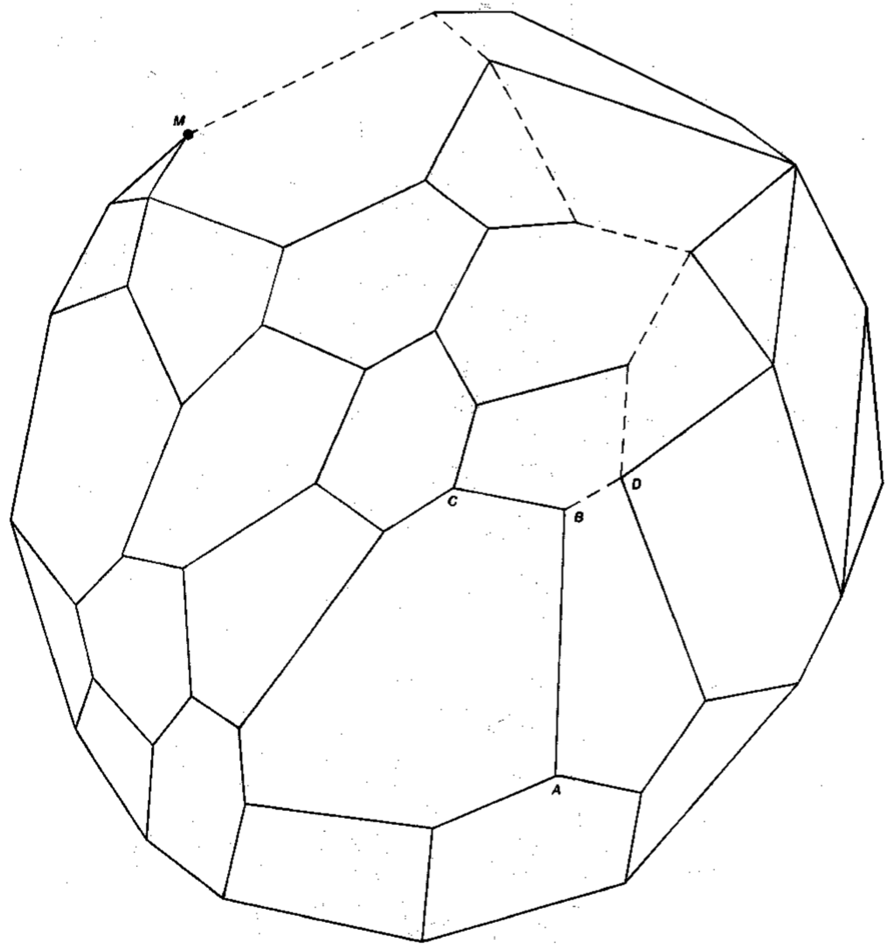




Geometric meaning of LP

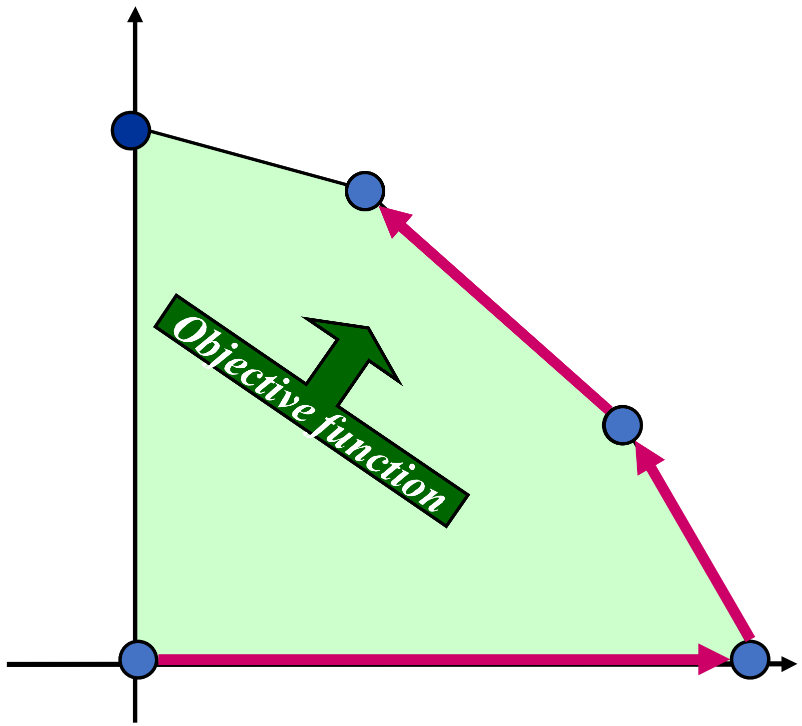
- Conclusion: If the problem has an optimal solution, then it always has an optimal solution that is the corner of the feasible region even for many dimensions.

➔ Just find the optimal solution among a finite number of feasible solutions.



Simplex algorithm

- Simplex Algorithm. (Dantzig 1947)
  - The algorithm starts from any corner of the feasible region and repeatedly moves to an adjacent corner of better objective value, if one exists. When it gets to a corner with no better neighbor, it stops: this is the optimal solution.
  - The algorithm is finite but has exponential complexity.



Some notations and definitions

- In the following sections, we will only work with the linear programming in canonical form (LP-C):

Find min:

$$f(x_1,x_2,\ldots,x_n)= \Sigma^n_{j=1} c_j x_j \rightarrow \min,$$

subject to

$$\Sigma^n_{i=1} a_{ij} x_j = b_i,\ i = 1,2,\ldots,m$$

$$x_j \geq 0,\ j = 1,2,\ldots,n.$$

Some notations and definitions

- Notations:
  - $x=(x_1, x_2, ..., x_n)^T$ – variable vector
  - $c=(c_1, c_2, ..., c_n)^T$ – objective function coefficient vector
  - $A = (a_{ij})_{m\times n}$ – constraint matrix
  - $b=(b_1,...,b_m)^T$- constraint vector (right side)

Some notations and definitions

- We can rewrite the problem in matrix form:

$$f(x) = c^{\mathrm{T}}x \to \min,$$
$$Ax = b,\ x \geq 0$$

or

$$\min\{\, f(x) = c^{\mathrm{T}}x : Ax = b,\ x \geq 0\}$$

- Vector inequality:

$$y = (y_1, y_2, \ldots, y_k) \geq 0$$

means that each component:

$$y_i \geq 0\ ,\ i = 1, 2, \ldots, k.$$

Some notations and definitions

- Denote index set:
  - $J = \{1,2,...,n\}$ indices for variables
  - $I = \{1,2,...,m\}$ indices for constraints

Then we use the following symbols

$x = x(J) = \{x_j: j\in J\}$ – variable vector;

$c = c(J) = \{c_j: j\in J\}$ – objective function coefficient vector;

$A = A(I, J) = \{a_{ij}: i\in I, j\in J\}$ – constraint matrix

$A_j = (a_{ij}: i\in I)$ – j^{th} column vector of matrix A.

Basic constraint equations of the LP-C can also be written in the form:

$$A_1x_1 + A_2x_2 + \ldots + A_nx_n = b$$

Some notations and definitions

- Set

$$D = \{x: Ax = b, x \geq 0\}$$

is called constraint region (feasible region)

x is called feasible solution.

- Feasible solution x^* giving the smallest value of objective function, i.e.,

$$c^T x^* \leq c^T x \quad \text{for all} \quad x \in D$$

is called optimal solution of the problem, and the value

$$f^* = c^T x^*$$

is called optimal value of the problem

BASIC FEASIBLE SOLUTION

The concept of an basic feasible solution (BF solution) is a main concept in simplex algorithm

Basic feasible solution

- Firstly, we assume that

$$\text{rank}\,(A) = m \qquad\qquad (*)$$

that is, the system of basic constraint equations consists of m linearly independent equations.

- Note: In fact, the assumption (*) is equivalent to the assumption that the system of linear equations $Ax = b$ has solution.
- We will remove these assumptions later.

Basic feasible solution

- **Definition 1**. ***Basis*** of matrix A is a set of m linearly independent column vectors $B = \{A_{j_1}, A_{j_2}, \ldots, A_{j_m}\}$.

Assume $B = A(I, J_B)$, such that $J_B = \{j_1, \ldots, j_m\}$ is a basis of matrix A.

Then vector $x = (x_1, .., x_n)$ such that:

$x_j = 0, j \in J_N = J \setminus J_B$;

x_{j_k} is the k^{th} element of vector $B^{-1}b$ (k=1,..,m).

is called as basic feasible solution corresponding to basis B.

Variables $x_j, j \in J_B$ are called basic variables, and $x_j, j \in J_N$ are called nonbasic variables.

Basic feasible solution

- Thus, if we denote

  $x_B = x(J_B)$, $x_N = x(J_N)$

  then the basic solution x corresponding to basic B could be determined by the following procedure:

  1. Set x_N=0.

  2. Determine x_B from equations $Bx_B = b$.

- Using the assumption (*), we see that the problem always has the basic solution.

Basic feasible solution

- Assume $x = (x_B, x_N)$ is basic solution corresponding to basis B. Then LP-C could be rewritten as following:

$$f(x_B, x_N) = c_B x_B + c_N x_N \rightarrow \min$$

$$B x_B + N x_N = b,$$

$$x_B,\ x_N \geq 0,$$

where $N = (A_j : j \in J_N)$ is called non-basic matrix of A.

Basic feasible solution

- Consider the LP:

$$6x_1 + 2x_2 - 5x_3 + x_4 + 4x_5 - 3x_6 + 12x_7 \rightarrow \min$$

$$\begin{aligned}
x_1 + x_2 + x_3 + x_4 \qquad\qquad\qquad &= 4 \\
x_1 \qquad\qquad + x_5 \qquad\qquad &= 2 \\
x_3 \qquad\qquad + x_6 \qquad &= 3 \\
3x_2 + x_3 \qquad\qquad\qquad + x_7 &= 6
\end{aligned}$$

$$x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6,\ x_7 \geq 0$$

Basic feasible solution

- In the matrix form:

  $c = (6, 2, -5, 1, 4, -3, 12)^T$;

  $b = (4, 2, 3, 6)^T$;

$$A = \left\| \begin{array}{ccccccc} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 3 & 1 & 0 & 0 & 0 & 1 \\ A_1 & A_2 & A_3 & A_4 & A_5 & A_6 & A_7 \end{array} \right\|$$

Basic feasible solution

- Consider basis

$$B = \{A_4, A_5, A_6, A_7\} = E_4$$

- Basic solution $x = (x_1, x_2, ..., x_7)$ corresponding to B could be obtained by setting:

$$x_1 = 0, x_2 = 0, x_3 = 0$$

  and values of $x_B = (x_4, x_5, x_6, x_7)$ could be obtained by solving equations

$$Bx_B = b \text{ or } E_4 x_B = b$$

  Then we get: $x_B = (4, 2, 3, 6)$.

- Thus basic solution corresponding to basis B is

$$x = (0, 0, 0, 4, 2, 3, 6)$$

Basic feasible solution

- Consider basis

$$B_1 = \{A_2, A_5, A_6, A_7\}$$

- Basic solution $y = (y_1, y_2, ..., y_7)$ corresponding to B_1 could be obtained by setting:

$$y_1 = 0;\ y_3 = 0,\ y_4 = 0$$

and values of $y_B = (y_2, y_5, y_6, y_7)$ could be obtained by solving equations

$$B_1 y_B = b$$

or

$$\begin{aligned} y_2 \qquad\qquad\qquad\qquad &= 4 \\ y_5 \qquad\qquad\qquad &= 2 \\ y_6 \qquad\quad &= 3 \\ 3y_2 \qquad\qquad\quad + y_7 &= 6 \end{aligned}$$

Then we get: $y_B = (4, 2, 3, -6)$.

- Thus, basic solution corresponding to basis B_1 is

$$y = (0, 4, 0, 0, 2, 3, -6)$$

Basic feasible solution

- We can see that:
  - Basic solution corresponding to basis B

$$x = (0, 0, 0, 4, 2, 3, 6)$$

  is feasible solution

  - Basic solution corresponding to basis B_1

$$y = (0, 4, 0, 0, 2, 3, -6)$$

  is non feasible solution.

Definition. *Basic solution is called as **basic feasible solution** if it is feasible solution.*

Basic feasible solution

- How many basic feasible solutions are there in the LP?

- The number of basic feasible solutions ≤ the number of basis ≤ $C(n,m)$

- Thus, there is a limited number of basic feasible solutions in a LP

Basic feasible solution

- The LP always has basic feasible solution?
- Not correct!
- Example: If the LP does not have feasible solution, then it also does not have basic feasible solution!
- However, we can prove the following theorem:

- *Theorem 1. If the LP has feasible solution, then it also has basic feasible solution*

Formula for incremental change of the objective function

Formula for incremental change of the objective function

- Assume x is a basic feasible solution corresponding to basis $B=(A_j: j\in J_B)$. Denote:

  $J_B=\{j_1,j_2, ..., j_m\}$ – indices of basic variables;

  $J_N=J \setminus J_B$ – indices of non-basic variables;

  $B=(A_j: j\in J_B)$ – basis matrix;

  $N=(A_j: j\in J_N)$ – non-basic matrix;

  $x_B = x(J_B) = \{x_j: j\in J_B\}$, $x_N = x(J_N) = \{x_j: j\in J_N\}$ – basic variable vector and non-basic variable vector;

  $c_B = c(J_B) = \{c_j: j\in J_B\}$, $c_N = c(J_N) = \{c_j: j\in J_N\}$ – objective function coefficient vectors of basic variables and non-basic variables;

Formula for incremental change of the objective function

- Consider basic feasible solution $z=x+\Delta x$, where $\Delta x = (\Delta x_1, \Delta x_2,..., \Delta x_n)$ – incremental change vector in variables. We find formula to calculate the incremental change of objective function:

$$\Delta f = c'z - c'x = c'\Delta x.$$

- Since x, z are both feasible solution, so $Ax=b$ and $Az=b$. Therefore, the Δx incremental change must satisfy condition $A\Delta x = 0$, it means:

$$B\Delta x_B + N\Delta x_N = 0,$$

where $\Delta x_B = (\Delta x_j: j\in J_B)$, $\Delta x_N = (\Delta x_j: j\in J_N)$.

Formula for incremental change of the objective function

- Thus

$$\Delta x_B = -B^{-1}N\Delta x_N\,. \qquad (1.10)$$

- So we have

$$c'\Delta x = c'_B\Delta x_B + c'_N\Delta x_N = -(c'_B B^{-1}N - c'_N)\,\Delta x_N\,.$$

- Denote:

$u = c'_B B^{-1}$ – transpose vector

$\Delta_N = (\Delta_j: j\in J_N) = uN - c'_N$ – estimate vector.

we get the formula:

$$\Delta f = c'z - c'x = -\Delta_N\Delta x_N = -\sum_{j\in J_N} \Delta_j\,\Delta x_j$$

- The obtained formula is called **formula for incremental change of the objective function**

Optimality criterion (Optimality test)

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- **Theorem 2.** (Optimality criterion) Inequality

$$\Delta_N \leq 0 \ (\Delta_j \leq 0, j \in J_N) \qquad (1.13)$$

is a sufficient condition and in the non-degenerate case is also a necessary condition for basic feasible solution x to be optimal.

Sufficient condition so that the objective function is unbounded

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Sufficient condition for the objective function to be unbounded

- **Theorem 3**. If among Δ_j values of basic feasible solution x, there is positive value Δ_{j_0}>0 and the corresponding elements of vector $B^{-1}A_{j_0} \leq 0$, then the objective function of the problem is unbounded.

SIMPLEX METHOD IN THE MATRIX FORM

Simplex iterations

- We continue to analyze the basic feasible solution x corresponding to the basis B. Consider the case when the optimal criterion and the sufficient condition for the objective function to be unbounded are not fulfilled. Then we must find the j_0 such that $\Delta_{jo}>0$.

- From the proof of the necessary conditions of the optimal criterion, we can see that it is possible to build a feasible solution $\bar{x}$ with smaller value of objective function. We have the formula to build $\bar{x}$:

$$\bar{x} = x + \Delta x,$$

where vector Δx is determined as follows

$$\Delta x_{jo} = \theta,\ \Delta x_j = 0, j \neq j_0, j \in J_N,$$

$$\Delta x_B = -\ \theta B^{-1} A_j$$

Simplex iterations

- Then we get

$$\bar{x}_N = \Delta x_N, \quad \bar{x}_B = x_B - \theta B^{-1} A_{jo} ,$$

  and the value of objective function at $\bar{x}$ is

$$c\bar{x} = cx - \theta \Delta_{j_0}$$

- Clearly: the larger θ, the more objective function value can be reduced. We are interested in the largest possible value of θ. Since we need to choose θ such that $\bar{x}$ is feasible solution, it must satisfy the following linear inequalities:

$$\bar{x}_B = x_B - \theta B^{-1} A_{jo} \geq 0, \ \theta \geq 0.$$

- Denote $B^{-1}A_{jo} = \{x_{i,jo}: i \in J_B\}$, we rewrite the linear inequalities as follows:

$$x_i - \theta x_{i,jo} \geq 0, \ i \in J_B,$$

$$\theta \geq 0.$$

Simplex iterations

- Denote

$$\theta_i = \begin{cases} x_i / x_{ij_0}, \text{when } x_{ij_0} > 0 \\ +\infty, \quad \text{when } x_{ij_0} \le 0 \end{cases}, \; i \in J_B$$

$$\theta_0 = \theta_{i_0} = x_{i_0} / x_{i_0 j_0} = \min \; \{ \theta_i : i \in J_B \}$$

- We have the solution to inequalities is

$$0 \le \theta \le \theta_0 \, .$$

- From that, it is deduced that the largest possible value of θ is θ_0. Then if replace x by $\bar{x}(\theta_0) = x + \Delta x,$ where the vector $\theta = \theta_0$, the value of objective function is reduced by an amount of $\theta_{io} \Delta_{jo} > 0$.

Simplex iterations

- We will show that $\bar{x}$ is also basic feasible solution.

Clearly, $\bar{x}_j = 0$ with $j \in \bar{J}_N = J_N \backslash j_0 \cup i_0$. Let $\bar{J}_B = J_B \backslash i_0 \cup j_0, \bar{B} = \{A_j : j \in \bar{J}_B\}$. ($\bar{B}$ is obtained from B by replacing column A_{j_0} by column A_{i_0}). We have

$$B^{-1}\bar{B} = (B^{-1}A_{j_1}, \ldots, B^{-1}A_{j_0}, \ldots, B^{-1}A_{j_m}),$$

so

$$B^{-1}\bar{B} = V = \begin{pmatrix} 1 & \ldots & x_{j_1 j_0} & \ldots & 0 \\ & & \ldots\ldots & & \\ 0 & \ldots & x_{i_0 j_0} & \ldots & 0 \\ & & \ldots\ldots & & \\ 0 & \ldots & x_{j_m j_0} & \ldots & 1 \end{pmatrix}. \qquad (1.21)$$

Therefore $\det(B^{-1}\ \bar{B}) = \det V = x_{io,jo} \neq 0$, so $\det\ \bar{B} \neq 0$ or $\bar{B}$ is the basis of the problem. Consequently, $\bar{x}$ is basic feasible solution.

We call the conversion from the basic feasible solution *x* to the basic feasible solution $\bar{x}$ according to the procedure described above as a simplex iteration on the basic feasible solution *x*.

Simplex method in the matrix form

- In the calculations of a simplex iteration, the B^{-1} matrix plays an important role. The algorithm below will be described using the the inverse matrix B^{-1}
- Initialization step.

  Find a basic feasible solution x with corresponding basis B.

  Calculate B^{-1}.

Simplex method in the matrix form

- **Step** k=1,2,… At the beginning of each step, we have matrix $B_k^{-1}(B_1^{-1}= B^{-1})$, basic variable index set $J_B^k, J_N^k = J \setminus J_B^k$, basic feasible solution $x^k = (x_B^k, x_N^k) = B_k^{-1}b, 0)$.

1) Calculate $u = c'_{B_k} B_k^{-1}$ (corresponding to solve equations $u' B_k = c_{B_k}$)
2) Calculate $\Delta_j = u'^{A_j} - c_j, j \in J_N^k$
3) If $\Delta_j \leq 0, \forall j \in J_N^k$ then the algorithm finishes and x^k is the optimal solution.
4) If among values $\Delta_j, j \in J_N^k$, there is still positive value, then select $\Delta_{j_0} > 0$ $(j_0 = j_0(k))$
5) Calculate $x_{j j_0}, j \in J_B^k$ are elements of vector $y = B_k^{-1} A_{j_0}$(corresponding to solve equations $B_k y = A_{j_0}$)
6) If $x_{j j_0} \leq 0, \forall j \in J_B^k$ then the objective function of the problem is not unbounded. The algorithm finishes.

Simplex method in the matrix form

7) Calculate

$$\theta_i = \begin{cases} x_i / x_{ij_0}, & \text{if } x_{ij_0} > 0,\ i \in J_B^k, \\ +\infty, & \text{if } x_{ij_0} \le 0,\ i \in J_B^k. \end{cases}$$

$$\theta_0 = \theta_{i_0} = x_{i_0} / x_{i_0 j_0} = min\{\theta_i :\ i \in J_B^k\}.$$

$(i_0 = i_0(k))$.

8) Set $J_B^{k+1} = J_B^k \backslash i_0 \cup j_0;\ J_N^{k+1} = J_N^k \backslash j_0 \cup i_0,$

$$B_{k+1} = \{A_j :\ j \in J_B^{k+1}\}.$$

Calculate inverse matrix B_{k+1}^{-1} of B_{k+1}, go to step k+1.

Simplex method in the matrix form

Note. In 4), Δ_{j_0} is any positive value. However, in case $|J_N^k|$ is not too large, we can choose its value as following rule:

$$\Delta_{j_0} = \max\{\Delta_j : j \in J_N^k\}.$$

To calculate inverse matrix B_{k+1}^{-1} of $\bar{B} = B_{k+1}$, from inverse matrix B_k^{-1} of $B = B_k$, looking at its relationship in the formula (1.21) $B^{-1}\bar{B} = V$, we have:

$$\bar{B}^{-1} = V^{-1}B^{-1}. \tag{1.22}$$

Simplex method in the matrix form

- As

$$V^{-1} = \begin{pmatrix} 1 & \dots & -x_{j_1 j_0}/x_{i_0 j_0} & \dots & 0 \\ & & \dots. & & \\ 0 & \dots & 1/x_{i_0 j_0} & \dots & 0 \\ & & \dots. & & \\ 0 & \dots & -x_{j_m j_0}/x_{i_0 j_0} & \dots & 1 \end{pmatrix}, \qquad (1.23)$$

We denote $B_{k+1}^{-1} = \{\, \bar{u}_{ij} : i \in J_B^{k+1},\ j \in I\}, B_k^{-1} = \{\, u_{ij} : i \in J_B^k,\ j \in I\}$

Then, from (1.22), (1.23) we get the following formula to calculate elements of $A = B_{k+1}^{-1}$

$$\bar{u}_{i_0 j} = u_{i_0 j}/x_{i_0 j_0},\ j \in I,$$

$$\bar{u}_{ij} = u_{ij} - x_{i j_0} u_{i_0 j}/x_{i_0 j_0},\ i \neq i_0,\ i \in J_B^{k+1},\ j \in I.$$

THE SIMPLEX METHOD IN TABULAR FORM

The simplex method in Tabular form

- When we need to solve the problem by hand, we recommend the tabular form described in this section.
- Suppose we have feasible basic solution x corresponding to basis B. The notations are still the same as before
- We call ***simplex tableau corresponding to the basic feasible (BF) solution x*** is the following table

Simplex tableau

c_j basic	Basis	Solution	c_1 / A_1	… / …	c_j / A_j	… / …	c_n / A_n	θ
c_{j_1}	A_{j_1}	x_{j_1}			$x_{j_1 j}$			θ_{j_1}
…	…	…			…			…
c_i	A_i	x_i			x_{ij}			θ_i
…	…	…			…			…
c_{j_m}	A_{j_m}	x_{j_m}			$x_{j_m j}$			θ_{j_m}
Δ			Δ_1	…	Δ_j	…	Δ_n	

Simplex tableau

- The first column is the objective function coefficients of basic variables
- The second column is the name of basis column.
- The third column is the values of basic variables (elements of vector $x_B = \{x_j : j \in J_B\} = B^{-1}b$).
- Elements x_{ij}, $i \in J_B$ written on the next columns are calculated based on the formular:

$$\{x_{ij},\ i \in J_B\} = B^{-1}A_j,\ j=1,2,\dots,n.$$

- The last column is values of ratios θ_i.
- The first row: the objective function coefficients of variables (c_j).
- Next rows contains name of columns $A_1, \dots, A_n$.
- The last row is called estimate line:

$$\Delta_j = \sum_{i \in J_B} c_i x_{ij} - c_j,\ j=1,2,\dots,n.$$

- We can see that $\Delta_j = 0$, $j \in J_B$.

The simplex method in Tabular form

With the simplex tableau, we can run a simplex iteration with the current feasible solution *x* as following:

1. *Optimality test*: If elements on estimate line are not positive (Δ_j <0, j=1,..,n), then the current feasible solution x is optimal, the algorithm finishes.

2. *Test the sufficient condition for the objective function to be unbounded*: if there is $\Delta_{j_0} > 0$ while its corresponding elements in the simplex tableau $x_{j\,j_0} \leq 0, j \in J_B$, then the objective function is unbounded, the algorithm finishes.

3. *Find the pivot column*: Find $\Delta_{j_0} = \max\{\Delta_j \ : j = 1,..,n\} > 0$.

Column A_{j_0} is called pivot column (column entering basis), and x_{j_0} is called entering basic variable

Find the pivot column

c_j basic	Basis	Solution	c_1	…	c_{j_0}	…	c_n	θ
			A_1	…	A_{j_0}	…	A_n	
c_{j_1}	A_{j_1}	x_{j_1}			$x_{j_1 j_0}$			θ_{j_1}
…	…	…			…			…
c_i	A_i	x_i			$x_{i j_0}$			θ_i
…	…	…			…			…
c_{j_m}	A_{j_m}	x_{j_m}			$x_{j_m j_0}$			θ_{j_m}
Δ			Δ_1	…	Δ_{j_0}	…	Δ_n	

The simplex method in Tabular form

4. *Find the pivot row*: Calculate

$$\theta_i = \begin{cases} x_i / x_{ij_0}, & \text{if } x_{ij_0} > 0, \\ +\infty, & \text{if } x_{ij_0} \leq 0, \ i \in J_B, \end{cases}$$

$$\theta_0 = \theta_{i_0} = x_{i_0} / x_{i_0 j_0} = min\{\theta_i : \ i \in J_B\}.$$

Row A_{i_0} is called pivot row (row leaving basis), and x_{i_0} is called leaving basic variable. Put a box around this pivot row and a box around the pivot column. We call the number that is in *both* boxes the **pivot number.**

5. *Perform a transformation from a basic feasible solution x to a basic feasible solution* $\bar{x}$: the simplex tableau corresponding to $\bar{x}$ (referred to as new tableau) could be obtained from the simplex tableau corresponding to x (referred to as old tableau) based on the following transformation rules (get directly from formulations (1.22), (1.23):

Find the pivot row

c_j basic	Basis	Solution	c_1 / A_1	... / ...	c_{j_0} / A_{j_0}	... / ...	c_n / A_n	θ
c_{j_1}	A_{j_1}	x_{j_1}			$x_{j_1 j_0}$			θ_{j_1}
...	...	...			...			...
c_{i_0}	A_{i_0}	x_{i_0}			$x_{i_0 j_0}$			θ_{i_0}
...	...	...			...			...
c_{j_m}	A_{j_m}	x_{j_m}			$x_{j_m j_0}$			θ_{j_m}
Δ			Δ_1	...	Δ_{j_0}	...	Δ_n	

Simplex transformation rules

i. Elements on the pivot row of new tableau ($\bar{x}_{i_0 j}$) equal to corresponding elements of old tableau divided by the pivot number : $\bar{x}_{i_0 j} = x_{i_0 j} / x_{i_0 j_0}, j \in J$

ii. Elements on the pivot column of new tableau, excluding element on position of pivot number equals to 1, all remaining equal to 0.

iii. Remaining elements in new tableau: ($\bar{x}_{ij}, \bar{\Delta}_j$) can be calculated from the corresponding elements in the old table according to the following formula:

$$\bar{x}_{ij} = x_{ij} - x_{i_0 j} x_{i j_0} / x_{i_0 j_0},\ i \in J_B\ (i \neq i_0),\ j \in J\ (j \neq j_0),$$

$$\bar{\Delta}_j = \Delta_j - x_{i_0 j} \Delta_{j_0} / x_{i_0 j_0},\ j \in J\ (j \neq j_0).$$

The simplex method in Tabular form

- The above formulas can be easily memorized by the following rectangles.

$$\begin{tikzcd}
x_{ij} \arrow[r, dash] \arrow[d, dash] & x_{ij_0} \arrow[d, dash] \\
x_{i_0 j} \arrow[r, dash] & x_{i_0 j_0}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
x_{i_0 j} \arrow[r, dash] \arrow[d, dash] & x_{i_0 j_0} \arrow[d, dash] \\
\Delta_j \arrow[r, dash] & \Delta_{j_0}
\end{tikzcd}$$

Thus, above formulas could be also called as rectangle rule.

Note. Transformations i), ii), iii) could be totally determined if we known the pivot number $x_{i_0 j_0}$. We call them as simplex transformations with the pivot number $x_{i_0 j_0}$.

The rectangular rule

c_j basic	Basis	Solution	c_1	…	c_{j_0}	…	c_n	θ
			A_1	…	A_{j_0}	…	A_n	
c_{j_1}	A_{j_1}	x_{j_1}	$x_{j_1 1}$		$x_{j_1 j_0}$		$x_{j_1 n}$	θ_{j_1}
…	…	…			…			…
c_{i_0}	A_{i_0}	x_{i_0}	$x_{i_0 1}$		$x_{i_0 j_0}$		$x_{i_0 n}$	θ_{i_0}
…	…	…			…			…
c_{j_m}	A_{j_m}	x_{j_m}	$x_{j_m 1}$		$x_{j_m j_0}$		$x_{j_m n}$	θ_{j_m}
Δ			Δ_1	…	Δ_{j_0}	…	Δ_n	

Example

- Example: Solve the LP by using the simplex method

$$
\begin{array}{lllllllll}
x_1 & -6x_2 & +32x_3 & +x_4 & +x_5 & +10x_6 & +100x_7 & \longrightarrow min, \\
x_1 & & & +x_4 & & +6x_6 & & = 9, \\
3x_1 & +x_2 & -4x_3 & & & +2x_6 & +x_7 & = 2, \\
x_1 & +2x_2 & & & +x_5 & +2x_6 & & = 6,
\end{array}
$$

$$x_j \geq 0,\ j = 1, 2, ..., 7.$$

Starting from the basic feasible solution $x' = (0,0,0,9,6,0,2)$ corresponding to basis $B = (A_4, A_7, A_5)$. As the basis B is the unit matrix, we can easily build the simplex tableau with basic feasible solution x.

Simplex tableau

c_j basic	Basis	Solution	1	-6	32	1	1	10	100	θ
			A_1	A_2	A_3	A_4	A_5	A_6	A_7	
1	A_4	9	1	0	0	1	0	6	0	9
100	A_7	2	3	1	-4	0	0	2	1	2/3
1	A_5	6	1	2	0	0	1	2	0	6
Δ			301	108	-432	0	0	198	0	

Find the column pivot: Column pivot is the one with maximum value of θ

Simplex tableau

c_j basic	Basis	Solution	1	-6	32	1	1	10	100	θ
			A_1	A_2	A_3	A_4	A_5	A_6	A_7	
1	A_4	9	1	0	0	1	0	6	0	9
100	A_7	2	3	1	-4	0	0	2	1	2/3
1	A_5	6	1	2	0	0	1	2	0	6
Δ			301	108	-432	0	0	198	0	

Find row pivot: Calculate values of θ_i . Row pivot is the one with minimum value of θ

Change the simplex tableau

c_j basic	Basic	Solution	1	-6	32	1	1	10	100	θ
			A_1	A_2	A_3	A_4	A_5	A_6	A_7	
1	A_4									
1	A_1	2/3	1	1/3	-4/3	0	0	2/3	1/3	2
1	A_5									
Δ										

Change the tableau: Elements on the pivot row in new Tableau = corresponding elements in old tableau divided by the pivot

Change the simplex tableau

c_j basic	Basis	Solution	1	-6	32	1	1	10	100	θ
			A_1	A_2	A_3	A_4	A_5	A_6	A_7	
1	A_4									
1	A_1	2/3	1	1/3	-4/3	0	0	2/3	1/3	2
1	A_5									
Δ										

Change the tableau: The elements on the basis columns are unit vectors.

Change the simplex tableau

c_j basic	Basis	Solution	1	-6	32	1	1	10	100	θ
			A_1	A_2	A_3	A_4	A_5	A_6	A_7	
1	A_4		0			1	0			
1	A_1	2/3	1	1/3	-4/3	0	0	2/3	1/3	
1	A_5		0			0	1			
Δ			0			0	0			

Change the tableau: The elements on the basis columns are unit vectors.

Simplex tableau at Step 2

c_j basic	Basis	Solution	1	-6	32	1	1	10	100	θ
			A_1	A_2	A_3	A_4	A_5	A_6	A_7	
1	A_4	25/3	0	-1/3	4/3	1	0	16/3	-1/3	-
1	A_1	2/3	1	**1/3**	-4/3	0	0	2/3	1/3	2
1	A_5	16/3	0	5/3	4/3	0	1	4/3	-1/3	16/5
Δ			0	23/3	-92/3	0	0	-8/3	**-301/3**	

Change the tableau: The remaining elements are calculated according to the rectangular rule.

Simplex tableau at Step 3

c_j basic	Basis	Solution	1	-6	32	1	1	10	100	θ
			A_1	A_2	A_3	A_4	A_5	A_6	A_7	
1	A_4	9	1	0	0	1	0	6	0	
-6	A_2	2	3	1	-4	0	0	2	1	
1	A_5	2	-5	0	8	0	1	-2	-2	
Δ			-23	0	0	0	0	-18	-108	

The optimal criterion is satisfied. Algorithm is terminated.

Optimal solution: $x^* = (0, 2, 0, 9, 2, 0, 0)$. Optimal value: $f^* = -1$

THE TERMINATION OF SIMPLEX METHOD

The termination of simplex method

- **Theorem 1.4.** Assume that the simplex method is applied to an LP, and that each generated basic feasible solution is nondegenerate ($x_B > 0$ at each iteration). Then, in a finite number of iterations, the method finds an optimal solution or determines that the problem is unbounded.

THE TWO-PHASE SIMPLEX METHOD

The two-phase simplex method

- The simplex method described in previous section is used to solve the LP:

$$\min\{c'x: Ax = b, x \geq 0\} \quad (1.25)$$

and it is based on the following assumptions:

i) $rank\ A = m$;

ii) $D = \{x: Ax = b, x \geq 0\} \neq \emptyset$;

iii) Already know a basic feasible solution.

In this section, we build algorithm to solve the LP without using the above assumptions.

Auxiliary LP

- Original LP

$$\min\{\sum_{j=1}^{n} c_j x_j : \sum_{j=1}^{n} a_{ij} x_j = b_i,\ i = 1, 2, \ldots, m, x_j \geq 0,\ j = 1, 2, \ldots, n\} \qquad (1.25)$$

- Without loss of generality, we assume

$$b_i \geq 0,\ i = 1,\ 2,\ \ldots,\ m,$$

because: if $b_i < 0$ then just multiply the two sides of the corresponding equation by -1.

- According to the parameters of the given problem, we construct the following auxiliary LP:

$$\sum_{i=1}^{m} x_{n+i} \quad \to \quad \min,$$

$$\sum_{j=1}^{n} a_{ij} x_j + x_{n+i} = b_i,\ \ i = 1, 2, \ldots, m, \qquad (1.27)$$

$$x_j \geq 0,\ j = 1, 2, \ldots, n, n+1, \ldots, n+m.$$

- Vector $x_u = (x_{n+1}, x_{n+2}, \ldots, x_{n+m})$ is called auxiliary variables.

Auxiliary LP

The following lemma shows the relationship between problem (1.25) and problem (1.27).

Lemma 1. Problem (1.25) has feasible solution if and only if element x_u^* of optimal solution (x^*, x_u^*) of problem (1.27) is equal to 0.

Proof.

Necessary condition: Assume x* is feasible solution of (1.25). Then (x^*, x_u^*) is clearly the solution of (1.27). On the other hand, since $e'x_u^* = 0 \leq e'x_u$, for all (x, x_u) as the feasible solution of (1.27), so (x^*, x_u^*) is its optimal solution.

Sufficient condition: Clearly if (x^*, x_u^* =0) is optimal solution of (1.27) then x^* is feasible solution of (1.25)

First phase: Solve auxiliary LP

- The auxiliary LP has immediately a basic feasible solution

$$(x=0, x_u=b)$$

corresponding with basis

$$B = \{A_{n+1}, A_{n+2}, \ldots, A_{n+m}\} = E_m$$

where A_{n+i} is column vector corresponding with auxiliary variable x_{n+i}, i=1, 2, ..., m.

- So we can apply the simplex algorithm to solve the auxiliary LP. Solving the auxiliary LP using the simplex method is called **the first phase** of the two-phase simplex method that solves the canonical linear programming problem (1.25), and the auxiliary LP is also called the problem of the first phase problem.

First phase: Solve auxiliary LP

Once the first phase terminates, we get the optimal basic feasible solution (x^*, x_u^*) corresponding basis $B^* = \{A_j : j \in J_B^*\}$ of the problem (1.27). One of the following three possibilities can happen:

- i) $x_u^* \neq 0$;
- ii) $x_u^* = 0$ and B^* matrix does not contain column corresponding to auxiliary variables, that means this matrix only contains columns of constraint matrix of the problem (1.25):

$$B^* = \{A_j : \ j \in J_B^*\} \ , J_B^* \cap J_u = \emptyset.$$

- iii) $x_u^* = 0$ and B^* matrix contains column corresponding to auxiliary variables, that means

$$B^* = \{A_j : \ j \in J_B^*\} \ , J_B^* \cap J_u \neq \emptyset.$$

First phase: Solve auxiliary LP

We will consider each case one by one:

i) If $x_u^* \neq 0$, then according to the Lemma 1, the problem (1.25) has no feasible solution, the algorithm finishes.

ii) In this case, x^* is a basic feasible solution of the problem (1.25) corresponding to basis B^*. Starting from it, we can use simplex method to solve (1.25). It is called the 2nd phase of two-phase simplex method, and whole process just described is call two-phase simplex method to solve the problem (1.25).

iii) Denote $x_{i_*}, i_* \in J_B^* \cap J_U$ is an element of auxiliary variable in optimal basic feasible solution (x^*, x_u^*) of (1.27)

Denote $x_{i_* j}, j \in J \cup J_U$ are elements of row i_* in simplex tableau corresponding to optimal solution (x^*, x_u^*) of (1.27)

Simplex tableau

c_j basic	Basis	Solution	c^*_1	…	c^*_j	…	c^*_{n+m}	θ
			A_1	…	A_j	…	A_{n+m}	
$c^*_{j(1)}$	$A_{j(1)}$	$x_{j(1)}$			$x_{j(1)j}$			
…	…	…			…			
c^*_{i*}	A_{i*}	x_{i*}			x_{i*j}			
…	…	…			…			
$c^*_{j(m)}$	$A_{j(m)}$	$x_{j(m)}$			$x_{j(m)j}$			
Δ			Δ_1	…	Δ_j	…	Δ_{n+m}	

First phase: Solve auxiliary LP

- If we could find index $j_* \in J \setminus J_B$ such that $x_{i_* j_*} \neq 0$ then perform a transformation on simplex tableau with selected pivot number $x_{i_* j_*}$, we get the auxiliary variable x_{i_*} leaving the basis and in its place is variable x_{j_*}.
- If $x_{i_* j} = 0, \forall j \in J \setminus J_B^*$, then that means equation corresponding to it in the linearly equations *Ax=b* is a consequence of the remaining equations. Then, from the simplex tableau corresponding to optimal solution (x^*, x_u^*) of (1.27), we can delete the above row and also delete the column corresponding to auxiliary variable x_{i_*}.

Two-phase simplex method

- Pointing out all auxiliary variable in the basis according to the procedure we just did for x_{i*}, we will get to a new simplex tableau in which there is no auxiliary variable in the basis, that is we get the case ii), at the same time in this process we can also remove all linearly dependent constraints in $Ax=b$.
- From the obtained tableau, we can start the implementation of the second phase of the two-phase simplex method.

Two-phase simplex method

- Thus, the two-phase simplex method applied to any LP can only terminate in one of the following three situations:
  1) The problem has no feasible solution.
  2) The problem has objective function unbounded.
  3) Can find the optimal basic feasible solution to the problem.
- At the same time, in the process of implementing the method, we also detect and remove all linearly dependent constraints in the basic constraint system $Ax=b$.

Some theoretical results

- **Theorem 1.** Once the LP has feasible solution, it then also has basic feasible solution.
- **Prove.**

Applying the two-phase simplex method to the given LP, at the end of the first phase, we obtain a basic feasible solution to the problem.

Some theoretical results

- **Theorem 1.6.** Once the LP has optimal solution, it also has optimal basic feasible solution.
- **Prove.**

Assume the problem has optimal solution. Then, the two-phase simplex method applied to solve the given LP can only terminate in situation 3), that is, obtain the optimal basic feasible solution.

Some theoretical results

- **Theorem 1.7.** *The necessary and sufficient condition for the LP to have an optimal solution is that its objective function is lower bounded below on the non-empty feasible region.*
- **Prove.**
- **Necessary condition.** Assume x^* is optimal solution to the problem. Then, $f(x) \geq f(x^*)$, $\forall x \in D$, that means the objective function is lower bounded.
- **Sufficient condition.** If the problem has the objective function is lower bounded on the non-empty feasible solution, then the two-phase simplex method applied to solve the given LP can only terminate in situation 3), that is, obtain the optimal basic feasible solution.

Example

- Solve the following LP by using the two-phase simplex method

$$
\begin{array}{llllll}
2x_1 & +x_2 & +x_3 & & & \longrightarrow min, \\
x_1 & +x_2 & +x_3 & +x_4 & +x_5 & = 5, \\
x_1 & +x_2 & +2x_3 & +2x_4 & +2x_5 & = 8, \\
x_1 & +x_2 & & & & = 2, \\
 & & x_3 & +x_4 & +x_5 & = 3,
\end{array}
$$

$$x_j \geq 0,\ j = 1, 2, ..., 5.$$

Example

- The auxiliary problem is

$$
\begin{array}{llllllllll}
 & & & & & x_6 & +x_7 & +x_8 & +x_9 & \to min \\
x_1 & +x_2 & +x_3 & +x_4 & +x_5 & +x_6 & & & & = 5 \\
x_1 & +x_2 & +2x_3 & +2x_4 & +2x_5 & & +x_7 & & & = 8, \\
x_1 & +x_2 & & & & & & +x_8 & & = 2, \\
 & & x_3 & +x_4 & +x_5 & & & & +x_9 & = 3,
\end{array}
$$

$$x_j \geq 0, \;\; j = 1, 2, ..., 9.$$

The basic feasible solution to the auxiliary problem is

$$(x, x_u) = (0, 0, 0, 0, 0, 5, 8, 2, 3),$$

Example: First phase

c_j basic	Basis	Solution	0	0	0	0	0	1	1	1	1	θ
			A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	
1	A_6	5	1	1	1	1	1	1	0	0	0	5
1	A_7	8	1	1	2	2	2	0	1	0	0	4
1	A_8	2	1	1	0	0	0	0	0	1	0	–
1	A_9	3	0	0	$1*$	1	1	0	0	0	1	3
Δ			3	3	4	4	4	0	0	0	0	

1	A_6	2	$1*$	1	0	0	0	1	0	0	-1	2
1	A_7	2	1	1	0	0	0	0	1	0	-2	2
1	A_8	2	1	1	0	0	0	0	0	1	0	2
0	A_3	3	0	0	1	1	1	0	0	0	1	$-$
Δ			3	3	0	0	0	0	0	0	-4	
0	A_1	2	1	1	0	0	0	1	0	0	-1	
1	A_7	0	0	0	0	0	0	-1	1	0	-1	
1	A_8	0	0	0	0	0	0	-1	0	1	1	
0	A_3	3	0	0	1	1	1	0	0	0	1	
Δ			0	0	0	0	0	-3	0	0	-1	

1	A_6	2	1*	1	0	0	0	1	0	0	−1	2
1	A_7	2	1	1	0	0	0	0	1	0	−2	2
1	A_8	2	1	1	0	0	0	0	0	1	0	2
0	A_3	3	0	0	1	1	1	0	0	0	1	–
Δ			3	3	0	0	0	0	0	0	−4	
0	A_1	2	1	1	0	0	0	1	0	0	−1	
1	A_8	0	0	0	0	0	0	−1	0	1	1	
0	A_3	3	0	0	1	1	1	0	0	0	1	
Δ			0	0	0	0	0	−3	0	0	−1	

1	A_6	2	1*	1	0	0	0	1	0	0	-1	2
1	A_7	2	1	1	0	0	0	0	1	0	-2	2
1	A_8	2	1	1	0	0	0	0	0	1	0	2
0	A_3	3	0	0	1	1	1	0	0	0	1	–
Δ			3	3	0	0	0	0	0	0	-4	
0	A_1	2	1	1	0	0	0	1	0	0	-1	
0	A_3	3	0	0	1	1	1	0	0	0	1	
Δ			0	0	0	0	0	-3	0	0	-1	

Example: 2^{nd} phase

c_j Basic	Basis	Solution	2	1	1	0	0	θ
			A_1	A_2	A_3	A_4	A_5	
2	A_1	2	1	1*	0	0	0	2
1	A_3	3	0	0	1	1	1	—
$\triangle$			0	1	0	1	1	
1	A_2	2	1	1	0	0	0	—
1	A_3	3	0	0	1	1*	1	3
$\triangle$			-1	0	0	1	1	
1	A_2	2	1	1	0	0	0	
0	A_4	3	0	0	1	1	1	
$\triangle$			-1	0	-1	0	0	

Solution

- Optimal solution:

$$x^* = (0, 2, 0, 3, 0);$$

- Optimal value:

$$f^* = 2.$$

The efficiency of the simplex method

- One weakness of simplex method is that, in theory, it has exponential computation time. This is shown by Klee-Minty with the following example:

$$\sum_{j=1}^{n} 10^{n-j} x_j \to \max,$$

$$2\sum_{j=1}^{i-1} 10^{i-j} x_j + x_i \le 100^{i-1}, \quad i = 1, 2, \ldots, n,$$

$$x_j \ge 0, \quad j = 1, 2, \ldots, n$$

- To solve this problem, simplex method requires 2^n -1 iterations.

The efficiency of the simplex method

- Example Klee-Minty with n=3:

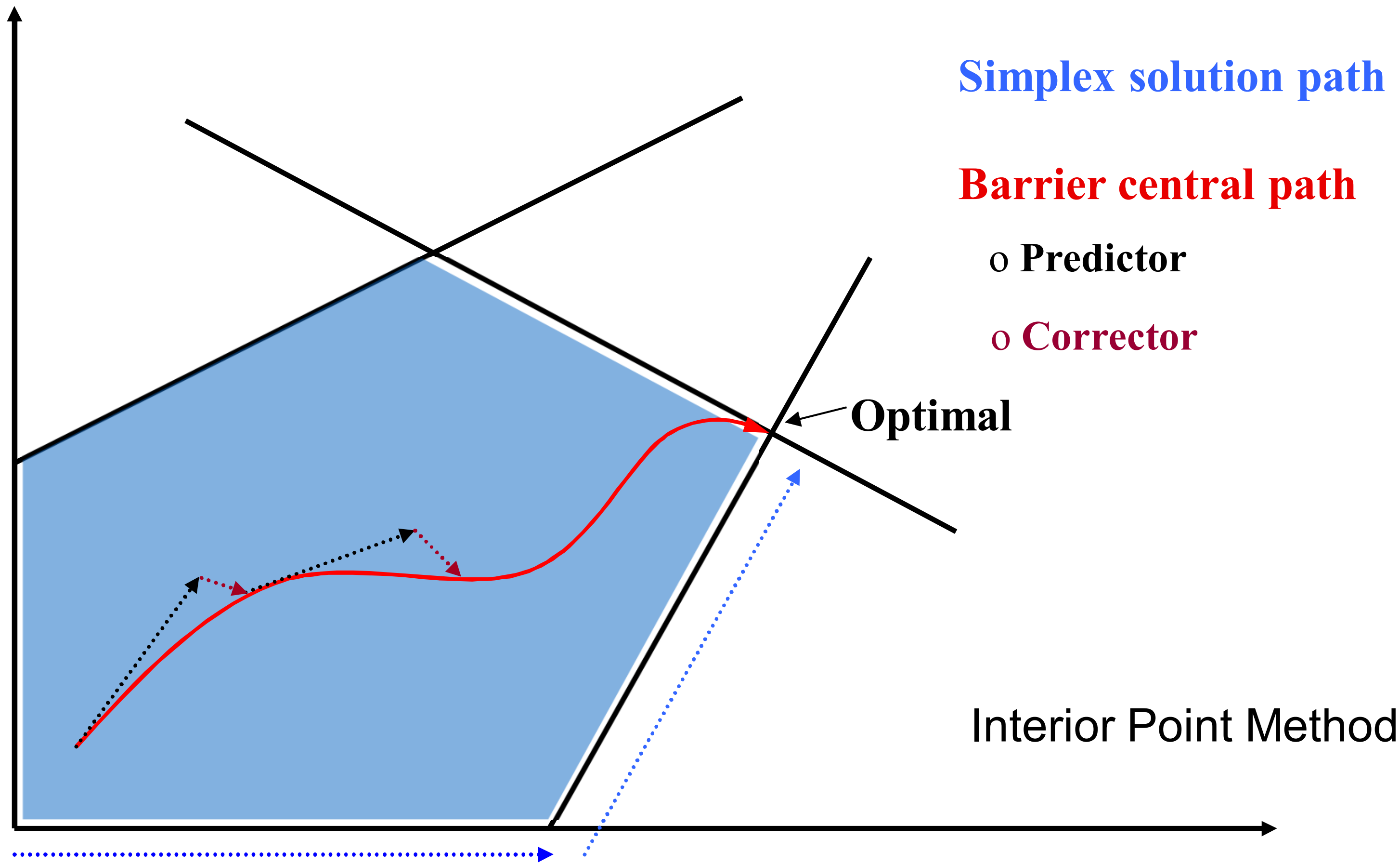
$$
\begin{array}{ccccccccc}
100x_1 & + & 10x_2 & + & x_3 & \rightarrow & \max \\
x_1 & & & & & \leq & 1 \\
20x_1 & + & x_2 & & & \leq & 100 \\
200x_1 & + & 20x_2 & + & x_3 & \leq & 10000 \\
 & & & & & x_1, x_2, x_3 \geq 0 &
\end{array}
$$

- In fact: The simplex method has computation time of $O(m^3)$

Polynomial Algorithms to solve the LP

- Ellipsoid. (Khachian 1979, 1980)
  - Computation time: $O(n^4 L)$.
    - n = number of variables
    - L = number of bits to represent the input data
  - This is a theoretical breakthrough.
  - No actual effect.
- Karmarkar's algorithm. (Karmarkar 1984)
  - Computation time: $O(n^{3.5} L)$.
  - Polynomial time and can implement efficiently
- Interior point algorithms.
  - Computation time: $O(n^3 L)$.
  - Comparable to simplex method!
    - Superior to simplex method when solving large size problems.
  - This method is extended to solve more general problems.

Barrier Function Algorithms



Interior Point Methods

DUAL THEORY

The dual theory of LP is a research field in which the LP is investigated through an auxiliary LP closely related to it called dual problem.

Content

1. The dual problem of the general LP.
2. Duality theorem.
3. Solve LP on MATLAB

The dual problem

The general LP

- Consider the general LP

$$f(x_1, x_2, \ldots, x_n) = \sum_{j=1}^{n} c_j x_j \to \min,$$

$$\sum_{j=1}^{n} a_{ij} x_j = b_i, i = 1, 2, \ldots, p \quad (p \le m)$$

$$\sum_{j=1}^{n} a_{ij} x_j \ge b_i, i = p+1, p+2, \ldots, m$$

$$x_j \ge 0, j = 1, 2, \ldots, n_1 \quad (n_1 \le n)$$

$$x_j <> 0, j = n_1 + 1, n_1 + 2, \ldots, n$$

The general LP

- Give the notations:

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

- Constraint matrix,

General LP

$a_i = (a_{i1}, a_{i2}, ..., a_{in})$ - i^{th} row of matrix A

$A_j = (a_{1j}, a_{2j}, ..., a_{mj})'$ - j^{th} column of matrix A

$M = \{1, 2, ..., p\}, \ \bar{M} = \{p+1, p+2, ..., m\}$ - constraint index set

$N = \{1, 2, ..., n_1\}, \ \bar{N} = \{n_1+1, n_1+2, ..., n\}$ - variable index sets

$b = (b_1, b_2, ..., b_m)'$ - right hand side vector

$x = (x_1, x_2, ..., x_n)'$ - variable vector

$c = (c_1, c_2, ..., c_n)'$ - Objective function coefficient vector

Build the dual problem

- Then the general LP could be rewritten as following form

$$
\begin{aligned}
& f(x) = c'x \to \min, \\
& a_i x = b_i,\ i \in M, \\
& a_i x \geq b_i,\ i \in \overline{M}, \\
& x_j \geq 0,\ j \in N, \\
& x_j <> 0,\ j \in \overline{N}.
\end{aligned}
\qquad (2.1)
$$

The general LP

- Transform the general LP into canonical form by:
  - using slack variables x_i^s to convert the inequality constraint to the equality form,
  - replace each unsigned variable by two signed conditional variables: $x_j = x_j^+ - x_j^-$, then each column A_j will be replaced by two columns A_j^+ and $-A_j^-$,

We get the following canonical LP:

Build the dual problem

- Problem (2.3):

$$\widehat{c}\hat{x} \to \min,$$
$$\widehat{A}\widehat{x} = b, \qquad (2.3)$$
$$\widehat{x} \geq 0,$$

where
$$\hat{c} = (c_j : j \in N;\ (c_j, -c_j) \ : \ j \in \bar{N};\ 0 : i \in \bar{M})',$$
$$\hat{x} = (x_j : j \in N;\ (x_j^+, x_j^-) \ : \ j \in \bar{N};\ 0 : x_i^s \in \bar{M})',$$
$$\hat{A} = [A_j : j \in N;\ (A_j, -A_j) : j \in \bar{N};\ e_i : i \in \bar{M}],$$

$e_i = (0,0,\ldots,0,-1,0,\ldots,0)', (i \in \bar{M})$ – vector has the i^{th} element equals to -1 and the remaining elements equal to 0.

Build the dual problem

- From the optimality criteria of simplex method, we see that if problem (2.3) has optimal solution $\hat{x}^0$ and basis corresponding to it is $\hat{B}$. then we must have:

$$(\hat{c}'_B \hat{B}^{-1})' \hat{A} - \hat{c} \leq 0.$$

- Therefore, vector $y^0 = \hat{c}'\hat{B}^{-1}$ is feasible solution of following linear inequalities:

$$y'\hat{A} \leq \hat{c}, \qquad (2.4)$$

where $y \in R^m$: m-vector. Linear inequalities in (2.4) could be divided into three groups depending on which column vector of matrix A appears in it

Build the dual problem

- The first constraint group is:

$$y'A_j \le c_j,\ j \in N. \tag{2.5}$$

- The second group corresponds to unsigned variables x_j, $j \in \bar{N}$, constraints will go together one by one

$$y'A_j \le c_j,\ \ -y'A_j \le\ -c_j,\ j \in \bar{N}$$

or they corresponds to the linear equalities

$$y'A_j = c_j,\ \ j \in \bar{N}. \tag{2.6}$$

Build the dual problem

- The last constraint group corresponding to slack variables $x_i^s, i \in \bar{M}$

$$-y_i \leq 0,\ i \in \bar{M}$$

or

$$y_i \geq 0,\ i \in \bar{M}. \tag{2.7}$$

Conditions (2.5)-(2.7) define the feasible region of the new LP which is called dual problem of the original LP (2.1). Then, the original LP (2.1) is called as primal problem.

Build the dual problem

- Then vector

$$y^0 = \hat{c}_B B^{-1}$$

is feasible solution of dual problem. If we define the objective function of dual problem as:

$$y'b \to \max,$$

then y^0 is not only feasible solution but also the optimal solution to dual problem.

- The above results will be stated exactly in the definitions and theorems below.

Dual problem

- **Definition.** *The general LP (called as primal problem) has the dual problem which is the following LP:*

Primal Problem		Dual Problem
$f(x) = c'x \to \min,$		$g(y) = y'b \to \max,$
$a_i x = b_i,$	$i \in M,$	$y_i <> 0,$
$a_i x \geq b_i,$	$i \in \overline{M},$	$y_i \geq 0,$
$x_j \geq 0,$	$j \in N,$	$yA_j \leq c_j,$
$x_j <> 0,$	$j \in \overline{N}$	$yA_j = c_j,$

Duality theorem

Duality theorem

First, we prove the following lemma:

- **Lemma 2.1** (Weak duality theorem). Assume $\{x, y\}$ is a pair of feasible solutions of primal and dual problems. Then

$$f(x) = c^T x \geq y^T b = g(y)$$

Primal problem		Dual problem
$f(x) = c^T x \to \min,$		$g(y) = y^T b \to \max,$
$a_i x = b_i,$	$i \in M,$	$y_i <> 0,$
$a_i x \geq b_i,$	$i \in \overline{M},$	$y_i \geq 0,$
$x_j \geq 0,$	$j \in N,$	$yA_j \leq c_j,$
$x_j <> 0,$	$j \in \overline{N}$	$yA_j = c_j,$

Prove: As y is feasible solution of dual problem, so:

$$c_j \geq yA_j, j \in N, c_j = yA_j, j \in \bar{N}$$

On the other hand, we have $x_j \geq 0, j \in N$ (as x is feasible solution of primal problem), so:

$$c_j x_j \geq yA_j x_j, j \in N,$$
$$c_j x_j = yA_j x_j, j \in \bar{N},$$

Similarly, we have: $yAx \geq y^T b$

Therefore: $c^T x \geq yAx \geq y^T b$

Duality theorem

- **Lemma 2.1** (Weak duality theorem). Assume $\{x, y\}$ is a pair of feasible solutions of primal and dual problems. Then

$$f(x) = c^T x \geq y^T b = g(y)$$

Primal problem		Dual problem
$f(x) = c^T x \to \min,$		$g(y) = y^T b \to \max,$
$a_i x = b_i,$	$i \in M,$	$y_i <> 0,$
$a_i x \geq b_i,$	$i \in \overline{M},$	$y_i \geq 0,$
$x_j \geq 0,$	$j \in N,$	$yA_j \leq c_j,$
$x_j <> 0,$	$j \in \overline{N}$	$yA_j = c_j,$

Corollary 2.1. Assume $\{x^*, y^*\}$ is a pair of feasible solutions of primal and dual problems that satisfy $c^T x^* = (y^*)^T b.$

Then $\{x^*, y^*\}$ is a pair of optimal solutions of primal and dual problems.

Duality theorem

Corollary 2.1. Assume $\{x^*,\ y^*\}$ is a pair of feasible solutions of primal and dual problems that satisfy

$$c^T x^* = (y^*)^T b. \quad 2$$

Then $\{x^*,\ y^*\}$ is a pair of optimal solutions of primal and dual problems.

Prove. From the lemma, we have:

$c^T x \geq y^T b$ $\forall x$ *is feasible solution of primal problem and* $\forall y$ *is feasible solution of dual problem* $\rightarrow c^T x^* \geq (y^*)^T b$ 1

1 2 $\rightarrow$ $c^T x \geq (y^*)^T b = c^T x^* \geq y^T b$

For every pair of feasible solutions {x, y} of the primal and dual problems. The last inequality proves the optimality of the pair of solutions $\{x^*, y^*\}$.

Duality theorem

Theorem 2.1. If the primal problem (2.1) has optimal solution then its dual problem also has optimal solution, and their optimal values are equal.

Prove. Assume the primal problem has optimal solution. Therefore, its corresponding canonical LP (2.3) also has optimal basic feasible solution $\hat{x}^0$ respect to basis $\hat{B}$. Then, as seen above, vector $y^0 = \hat{c}_B^T \hat{B}^{-1}$ is feasible solution of dual problem. Let x^0 be the optimal solution of the primal problem obtained from $\hat{x}^0$, we have

$$(y^0)^T b = \hat{c}_B^T \hat{B}^{-1} b = \hat{c}_B^T \hat{x}_B^0 = c^T x^0 \quad (2.9)$$

From Corollary 2.1, it follows that y^0 is the optimal solution of the dual problem, and from (2.9) deduces that the optimal value of the primal and dual problems is equal..

The theorem is proven

Primal problem		Dual problem
$f(x) = c^T x \to \min,$		$g(y) = y^T b \to \max,$
$a_i x = b_i,$	$i \in M,$	$y_i <> 0,$
$a_i x \geq b_i,$	$i \in \overline{M},$	$y_i \geq 0,$
$x_j \geq 0,$	$j \in N,$	$yA_j \leq c_j,$
$x_j <> 0,$	$j \in \overline{N}$	$yA_j = c_j,$

Duality theorem

A feature of duality is the symmetry shown in the following theorem:

Theorem 2.2. The dual problem of the dual of primal problem is the same as the primal problem.

Prove. Rewrite the dual problem as

$$
\begin{aligned}
&-y^T b \to \min, \\
&-A_j y \geq -c_j, \quad j \in N, \\
&-A_j y = -c_i, \quad j \in \overline{N}, \\
&y_i \geq 0, \qquad i \in \overline{M}, \\
&y_i <> 0, \qquad i \in M
\end{aligned}
$$

and consider it as primal problem. By definition its dual problem has the form::

$$
\begin{aligned}
&-c^T x \to \max, \\
&-a_i x = -b_i, \quad i \in M, \\
&-a_i x \leq -b_i, \quad i \in \overline{M}, \\
&x_j \geq 0, \qquad j \in N, \\
&x_j <> 0, \qquad j \in \overline{N}\ .
\end{aligned}
$$

which conspicuously coincides with the primal problem.

Duality theorem

- For any LP, there are three possibilities:
  1) The problem has an optimal solution;
  2) The problem has objective function unbounded;
  3) The problem has no feasible solution.
- Therefore, for the pair of primal-dual problems, there can be 9 situations described in the following table:

Duality theorem

Dual Primal	Has optimal solution	Unbounded	No feasible solution
Has optimal solution	1)	√	√
Unbounded	√	√	3)
No feasible solution	√	3)	2)

Duality theorem

- **The duality theorem.** For the pair of primal-dual problems of the LP, only three of the following three situations can occur:

1) Both problems have optimal solutions and their optimal values are equal;
2) Both problems have no feasible solutions;
3) This problem has an objective function unbounded and the other problem has no feasible solution.

Duality theorem

Example 1. Both problems have no feasible solutions.

Primal problem

$$x_1 \rightarrow min,$$

$$x_1 + x_2 \geq 1,$$

$$-x_1 - x_2 \geq 1,$$

$$x_1 <> 0, \; x_2 <> 0.$$

Dual problem

$$y_1 + y_2 \rightarrow max,$$

$$y_1 - y_2 = 1,$$

$$y_1 - y_2 = 0,$$

$$y_1 \geq 0, \; y_2 \geq 0.$$

Duality theorem

Example 2. Primal problem has no feasible solution, and dual problem has objective function unbounded.

Primal problem

$$x_1 \rightarrow min,$$

$$x_1 + x_2 \geq 1,$$

$$-x_1 - x_2 \geq 1,$$

$$x_1 \geq 0, \; x_2 \geq 0.$$

Dual problem

$$y_1 + y_2 \rightarrow max,$$

$$y_1 - y_2 \leq 1,$$

$$y_1 - y_2 \leq 0,$$

$$y_1 \geq 0, \; y_2 \geq 0.$$

Theorem about complement

- If looking closely at the definition of the primal-dual problems, one could feel the antagonism between the primal problem and dual problem: The tighter constraints are in one problem (e.g. : $a_i x = b_i, i \in M$), then to rebalance, in the other problem, the corresponding constraint is loosened ($y_i <> 0, i \in M$).
- To express this equilibrium exactly, we state necessary and sufficient conditions (called the condition of complement) for the pair of feasible solutions x and y of the primal and dual problem to be optimal.

Theorem about complement

- **Theorem 2.4.** The pair of feasible solutions of primal-dual problems $\{x, y\}$ is optimal if and only if the following conditions are satisfied:

$$(a_i x - b_i)y_i = 0,\ i = 1,2,\ldots,m;$$
$$x_j(c_j - yA_j) = 0,\ j = 1,2,\ldots,n.$$

Primal problem		Dual problem
$f(x) = c^T x \to \min,$		$g(y) = y^T b \to \max,$
$a_i x = b_i,$	$i \in M,$	$y_i <> 0,$
$a_i x \geq b_i,$	$i \in \overline{M},$	$y_i \geq 0,$
$x_j \geq 0,$	$j \in N,$	$yA_j \leq c_j,$
$x_j <> 0,$	$j \in \overline{N}$	$yA_j = c_j,$

- **Prove.** Set

$$u_i = (a_i x - b_i)y_i\ ,\ i = 1,2,\ldots,m;$$
$$v_j = x_j(c_j - yA_j)\ ,\ j = 1,2,\ldots,n.$$

As $\{x, y\}$ is the pair of feasible solutions, so

$$u_i \geq 0,\ i = 1,2,\ldots,m;\quad v_j \geq 0\ ,\ j = 1,2,\ldots,n.$$

Theorem about complement

Set

$$\alpha + \beta = \sum_{i=1}^{m}(a_i x - b_i) y_i + \sum_{j=1}^{n}(c_j - yA_j) x_j$$
$$= -\sum_{i=1}^{m} b_i y_i + \sum_{j=1}^{n} c_j x_j$$
$$= c^T x - y^T b$$

So α+β=0 if and only if $c^T x = y^T b$

According to the Corollary 2.1. and Duality theorem, condition $c^T x = y^T b$ is the necessary and sufficient conditions for the pair of feasible solutions x and y of primal-dual problems are optimal. Theorem is proven

Theorem about complement

Example. Consider the LP

$$
\begin{array}{llllll}
x_1 & +x_2 & +x_3 & +x_4 & +x_5 & \longrightarrow min, \\
3x_1 & +2x_2 & +x_3 & & & = 1, \\
5x_1 & +x_2 & +x_3 & +x_4 & & = 3, \\
2x_1 & +5x_2 & +x_3 & & +x_5 & = 4,
\end{array}
$$

$$x_j \geq 0,\ j = 1, 2, \ldots, 5.$$

has the optimal solution $x^* = (0, ½, 0, 5/2, 3/2)$, the optimal value is $f^*=9/2$

Theorem about complement

- Its dual problem:

$$\begin{array}{llllll} x_1 & +x_2 & +x_3 & +x_4 & +x_5 & \longrightarrow min, \\ 3x_1 & +2x_2 & +x_3 & & & = 1, \\ 5x_1 & +x_2 & +x_3 & +x_4 & & = 3, \\ 2x_1 & +5x_2 & +x_3 & & +x_5 & = 4, \end{array}$$

$$x_j \geq 0, \ j = 1, 2, ..., 5.$$

$$\begin{array}{llll} y_1 & +3y_2 & +4y_3 & \longrightarrow max, \\ 3y_1 & +5y_2 & +2y_3 & \leq 1, \\ 2y_1 & +y_2 & +5y_3 & \leq 1, \\ y_1 & +y_2 & +y_3 & \leq 1, \\ & y_2 & & \leq 1, \\ & & y_3 & \leq 1, \end{array}$$

$$y_i <> 0, \ i = 1, 2, 3.$$

Primal problem		Dual problem
$f(x) = c^T x \to \min,$		$g(y) = y^T b \to \max,$
$a_i x = b_i,$	$i \in M,$	$y_i <> 0,$
$a_i x \geq b_i,$	$i \in \overline{M},$	$y_i \geq 0,$
$x_j \geq 0,$	$j \in N,$	$yA_j \leq c_j,$
$x_j <> 0,$	$j \in \overline{N}$	$yA_j = c_j,$

Theorem about complement

- Let $y^* = (y_1^*, y_2^*, y_3^*)$ be optimal solution to the dual problem. Since the primal problem has a canonical form, so according to the theorem about complement, condition $(a_i x - b_i)y_i = 0, i = 1,..,m$ is always fulfilled at all feasible solutions of the primal problem $a_i x = b_i, i \in M$. From condition $x_j(c_j - yA_j) = 0, j = 1,..,n$, do $x_2^*, x_4^*, x_5^* > 0$, we have

$$c_2 - yA_2 = 0$$
$$c_4 - yA_4 = 0$$
$$c_5 - yA_5 = 0$$

That means the 2nd, 4th, 5th conditions of the dual problem at its optimal solution y^* will have to be satisfied in terms of equality. So y* is the solution of the following system of linear equations. Therefore, y^* is solution to the following linear equations:

$$\begin{array}{llllll}
x_1 & +x_2 & +x_3 & +x_4 & +x_5 & \longrightarrow min, \\
3x_1 & +2x_2 & +x_3 & & & = 1, \\
5x_1 & +x_2 & +x_3 & +x_4 & & = 3, \\
2x_1 & +5x_2 & +x_3 & & +x_5 & = 4, \\
\multicolumn{6}{l}{x_j \geq 0,\ j = 1,2,...,5.}
\end{array}$$

Theorem about complement

$$\begin{aligned} 2y_1 \quad +y_2 \quad +5y_3 &= 1, \\ y_2 &= 1, \\ y_3 &= 1, \end{aligned}$$

- Thus we have the optimal solution of the dual problem:

$$y_1^* = -\frac{5}{2},\ y_2^* = 1,\ y_3^* = 1,$$

and the optimal values of the dual problem is 9/2

Corollary

- **Corollary.** Feasible solution x^* is optimal solution of the LP if and only if following linear equations inequalities have solutions:

$$(a_i x^* - b_i) y_i = 0, \forall i \in \overline{M},$$

$$(c_j - yA_j) x_j^* = 0, \forall j \in N,$$

$$y_i \geq 0, i \in \overline{M},$$

$$yA_j \leq c_j, \ j \in N,$$

$$yA_j = c_j, \ j \in \overline{N} \ .$$

Primal problem		Dual problem
$f(x) = c^T x \to \min,$		$g(y) = y^T b \to \max,$
$a_i x = b_i,$	$i \in M,$	$y_i <> 0,$
$a_i x \geq b_i,$	$i \in \overline{M},$	$y_i \geq 0,$
$x_j \geq 0,$	$j \in N,$	$yA_j \leq c_j,$
$x_j <> 0,$	$j \in \overline{N}$	$yA_j = c_j,$

Example

- Consider the LP

$$
\begin{array}{cccccc}
-2x_1 & -6x_2 & +5x_3 & -x_4 & -4x_5 & \to max \\
x_1 & -4x_2 & +2x_3 & -5x_4 & +9x_5 & = 3 \\
 & x_2 & -3x_3 & +4x_4 & -5x_5 & = 6 \\
 & x_2 & -x_3 & +x_4 & -x_5 & = 1
\end{array}
$$

$$x_j \geq 0 \quad \forall j.$$

- Optimality test for vector

$$x^* = (0, 0, 16, 31, 14)$$

to the above LP

Example

- It is easy to check that x^* is the feasible solution to the given LP:

```
A=[1 -4 2 -5 9; 0 1 -3 4 -5; 0 1 -1 1 -1];
  x=[0;0;16;31;14]; A*x

ans =

     3

     6

     1
```

- Using the lemma, x^* is optimal if and only if the following equations and inequalities have solutions

Example

$$(y_1 + 2)\, x^*_1 = 0$$
$$(-4y_1 + y_2 + y_3 + 6)\, x^*_2 = 0$$
$$(2y_1 - 3y_2 - y_3 - 5)\, x^*_3 = 0$$
$$(-5y_1 + 4y_2 + y_3 + 1)\, x^*_4 = 0$$
$$(9y_1 - 5y_2 - y_3 + 4)\, x^*_5 = 0$$
$$y_1 \geq -2$$
$$-4y_1 + y_2 + y_3 \geq -6$$
$$2y_1 - 3y_2 - y_3 \geq 5$$
$$-5y_1 + 4y_2 + y_3 \geq -1$$
$$9y_1 - 5y_2 - y_3 \geq -4$$

$$x^*_1 = 0$$
$$x^*_2 = 0$$
$$x^*_3 = 16$$
$$x^*_4 = 31$$
$$x^*_5 = 14$$

Dual problem

$$3y_1 + 6y_2 + y_3 \to \min$$
$$y_1 \geq -2$$
$$-4y_1 + y_2 + y_3 \geq -6$$
$$2y_1 - 3y_2 - y_3 \geq 5$$
$$-5y_1 + 4y_2 + y_3 \geq -1$$
$$9y_1 - 5y_2 - y_3 \geq -4$$

Example

- It corresponds to following equations and inequalities:

$$
\begin{aligned}
y_1 \qquad\qquad\qquad &\geq -2 \\
-4y_1 + y_2 + y_3 &\geq -6 \\
2y_1 - 3y_2 - y_3 &= 5 \\
-5y_1 + 4y_2 + y_3 &= -1 \\
9y_1 - 5y_2 - y_3 &= -4
\end{aligned}
$$

- The last three equations has a unique solution $y^* = (-1, 1, -10)$.

```
(A=[2 -3 -1;-5 4 1; 9 -5 -1]; b=[5;-1;-4]; y=A\b)
```

- It is easy to check that y^* satisfies the first two inequalities. Hence y^* is the solution of the above of equations and inequations. Using the lemma, it proves that x^* is the optimal solution of the given LP.

Solve LP on MATLAB

Function LINPROG

- MATLAB provides function **`linprog`** to solve the LP.
- Here are some ways to use this function
  - **`X=LINPROG(f,A,b)`**
  - **`X=LINPROG(f,A,b,Aeq,beq)`**
  - **`X=LINPROG(f,A,b,Aeq,beq,LB,UB)`**
  - **`X=LINPROG(f,A,b,Aeq,beq,LB,UB,X0)`**
  - **`X=LINPROG(f,A,b,Aeq,beq,LB,UB,X0,OPTIONS)`**
  - **`[X,FVAL]=LINPROG(...)`**
  - **`[X,FVAL,EXITFLAG] = LINPROG(...)`**
  - **`[X,FVAL,EXITFLAG,OUTPUT] = LINPROG(...)`**
  - **`[X,FVAL,EXITFLAG,OUTPUT,LAMBDA]=LINPROG(...)`**

Function LINPROG

- Statement **`X=LINPROG(f,A,b)`** used to solve the LP:

  min { f 'x : A x <= b }

- Statement **`X=LINPROG(f,A,b,Aeq,beq)`** used to solve the LP with additional basic constraint in equality form **`Aeq*x = beq`**.
- Statement **`X=LINPROG(f,A,b,Aeq,beq,LB,UB)`** determines the lower and upper bounds for the variables **`LB <= X <= UB.`**
  - Assign **`Aeq=[]`** (**`A=[]`**) and **`beq=[]`** (**`b=[]`**) if without these constraints
  - Assign **`LB`** and **`UB`** as empty matrix ([]) if without using these bounds.
  - Assign **`LB(i)=-Inf`** if **`X(i)`** is not lower bounded and assign **`UB(i)=Inf`** if **`X(i)`** is not upper bounded.

Function LINPROG

- Statement `X=LINPROG(f,A,b,Aeq,beq,LB,UB,X0)` determines the starting point `X0`.
  - **Note:** This choice is only accepted if *the positive set algorithm* is used. The default method for solving is that the *interior point algorithm* will not accept a starting point.
- Statement `X=LINPROG(f,A,b,Aeq,beq,LB,UB,X0,OPTIONS)` performs to solve the LP with optimal parameters defined by the structured variable **`OPTIONS`**, created by the function **`OPTIMSET`**.
  - Asign **`option=optimset('LargeScale','off', 'Simplex','on')`** to select simplex method to solve the problem.
  - Type **`help OPTIMSET`** to see more details.

Function LINPROG

- Statement `[X,FVAL]=LINPROG(...)` returns the value of objective function at solution **X**: **FVAL = f'*X.**
- Statement `[X,FVAL,EXITFLAG]=LINPROG(...)` **returns** **EXITFLAG** the description of the termination condition of **LINPROG.** Values of **EXITFLAG** have the following meanings
  - 1 LINPROG converges to the solution X.
  - 0 Reach the the number of iterations limit.
  - -2 Could not found the feasible solution.
  - -3 The problem has objective function unbounded.
  - -4 value **NaN** appears during the execution of the algorithm.
  - -5 Both the primal and dual problems are incompatible.
  - -7 The search direction is too small, can't be improved anymore.

Function LINPROG

- Statement **[X,FVAL,EXITFLAG,OUTPUT] = LINPROG(...)** returns the structured variable **OUTPUT** with
  - **OUTPUT.iterations** - number of iterations to perform
  - **OUTPUT.algorithm** - algorithm used
  - **OUTPUT.message** – announcement
- Statement
  [X,FVAL,EXITFLAG,OUTPUT,LAMBDA]=LINPROG(...)

returns the Lagrangian multiplier LAMBDA , corresponding to the optimal solution:

  - **LAMBDA.ineqlin** – corresponds to the inequality constraint A,
  - **LAMBDA.eqlin** corresponds to the equality constraint Aeq,
  - **LAMBDA.lower** – corresponds to the LB,
  - **LAMBDA.upper** – corresponds to the UB.

Example

- Solve the LP:

$$
\begin{aligned}
&2x_1 + x_2 + 3x_3 && \rightarrow \min \\
&x_1 + x_2 + x_3 + x_4 + x_5 && = 5 \\
&x_1 + x_2 + 2x_3 + 2x_4 + 2x_5 && = 8 \\
&x_1 + x_2 && = 2 \\
&x_3 + x_4 + x_5 && = 3 \\
&x_1,\ x_2,\ x_3,\ x_4,\ x_5 && \geq 0
\end{aligned}
$$

- `f=[2 1 3 0 0]; beq=[5; 8; 2; 3];`
- `Aeq=[1 1 1 1 1; 1 1 2 2 2;1 1 0 0 0;0 0 1 1 1];`
- `A=[]; b=[]; LB=[0 0 0 0 0]; UB=[];X0=[];`
- `[X,FVAL,EXITFLAG,OUTPUT,LAMBDA]=linprog(f,A,b,Aeq,beq,LB,UB,X0)`

Solution

- **X =**

  0.0000

  2.0000

  0.0000

  1.5000

  1.5000

- **FVAL =**

  2.0000

- **EXITFLAG =**

  1

- **OUTPUT =**

  iterations: 5

  algorithm: 'large-scale: interior point'

  cgiterations: 0

  message: 'Optimization terminated.'

- **LAMBDA =**

  ineqlin: [0x1 double]

  eqlin: [4x1 double]

  upper: [5x1 double]

  lower: [5x1 double]

Example

- **Using simplex method:**

```
opt=optimset('LargeScale','off','Simplex','on')
[X,FVAL,EXITFLAG,OUTPUT]=LINPROG(f,A,b,Aeq,beq,LB,UB,X0,opt)
```

we obtain the result:

- `X = [0 2 0 3 0]`
- `FVAL =  2`
- `EXITFLAG = 1`
- `OUTPUT =`

```
        iterations: 1
        algorithm: 'medium scale: simplex'
        cgiterations: []
        message: 'Optimization terminated.'
```