

Appendix A: The proofs of Lemma 1 and Theorem 1

Derivation of the relationship between p_i^{n+1} and p_i^n is as follows:

$$\begin{aligned}
p_i^{n+1} &= p_i^n \theta_i (p_i^n + \varepsilon) + p_i^n (1 - \theta_i) (p_i^n - \varepsilon) \\
&\quad + \sum_{j \neq i} p_j^n \theta_j (p_i^n - \frac{\varepsilon}{m-1}) \\
&\quad + \sum_{j \neq i} p_j^n (1 - \theta_j) (p_i^n + \frac{\varepsilon}{m-1}) \\
&= (p_i^n)^2 \theta_i + p_i^n \theta_i \varepsilon + (p_i^n)^2 - p_i^n \varepsilon - (p_i^n)^2 \theta_i + p_i^n \theta_i \varepsilon \\
&\quad + (p_i^n - \frac{\varepsilon}{m-1}) \sum_{j \neq i} p_j^n \theta_j + (p_i^n + \frac{\varepsilon}{m-1}) \sum_{j \neq i} p_j^n \\
&\quad - (p_i^n + \frac{\varepsilon}{m-1}) \sum_{j \neq i} p_j^n \theta_j \\
&= (p_i^n)^2 + 2p_i^n \theta_i \varepsilon - p_i^n \varepsilon + (p_i^n - \frac{\varepsilon}{m-1} - p_i^n \\
&\quad - \frac{\varepsilon}{m-1}) \sum_{j \neq i} p_j^n \theta_j + (p_i^n + \frac{\varepsilon}{m-1}) (1 - p_i^n) \\
&= p_i^n + (p_i^n)^2 - (p_i^n)^2 + 2p_i^n \theta_i \varepsilon - p_i^n \varepsilon + \frac{\varepsilon}{m-1} - \\
&\quad \frac{\varepsilon}{m-1} p_i^n - \frac{2\varepsilon}{m-1} \sum_{j \neq i} p_j^n \theta_j \\
&= p_i^n + \frac{\varepsilon}{m-1} (2p_i^n \theta_i m - p_i^n m - 2p_i^n \theta_i + 1) \\
&\quad - \frac{2\varepsilon}{m-1} \sum_{j \neq i} p_j^n \theta_j \\
&= p_i^n + Y_i^n,
\end{aligned} \tag{1}$$

where

$$\begin{aligned}
Y_i^n &= \frac{\varepsilon}{m-1} (2p_i^n \theta_i m - p_i^n m - 2p_i^n \theta_i + 1) \\
&\quad - \frac{2\varepsilon}{m-1} \sum_{j \neq i} p_j^n \theta_j.
\end{aligned} \tag{2}$$

Derivation of the relationship between Y_M^n and Y_i^n is as follows:

$$\begin{aligned}
Y_M^n - Y_i^n &= \frac{\varepsilon}{m-1} (2p_M^n \theta_M m - p_M^n m - 2p_M^n \theta_M + 1) \\
&\quad - \frac{2\varepsilon}{m-1} \sum_{j \neq M} p_j^n \theta_j - \frac{\varepsilon}{m-1} (2p_i^n \theta_i m - p_i^n m \\
&\quad - 2p_i^n \theta_i + 1) + \frac{2\varepsilon}{m-1} \sum_{j \neq i} p_j^n \theta_j \\
&= \frac{\varepsilon}{m-1} (2m(p_M^n \theta_M - p_i^n \theta_i) - m(p_M^n - p_i^n) - \\
&\quad 2(p_M^n \theta_M - p_i^n \theta_i)) - \sum_{j \neq M} p_j^n \theta_j + \sum_{j \neq i} p_j^n \theta_j \\
&= \frac{2\varepsilon}{m-1} (m(p_M^n \theta_M - p_i^n \theta_i) - \frac{m(p_M^n - p_i^n)}{2} -
\end{aligned}$$

$$\begin{aligned}
& (p_M^n \theta_M - p_i^n \theta_i) + \frac{2\varepsilon}{m-1} (p_M^n \theta_M - p_i^n \theta_i) \\
&= \frac{2\varepsilon}{m-1} (m(p_M^n \theta_M - p_i^n \theta_i) - \frac{m(p_M^n - p_i^n)}{2}).
\end{aligned} \tag{3}$$

Lemma 1. If $p_i^n - p_M^n > 2(p_i^n \theta_i - p_M^n \theta_M)$, then $p_M^{n+1} > p_M^n$.

Proof: The condition $p_i^n - p_M^n > 2(p_i^n \theta_i - p_M^n \theta_M)$ can be equivalently expressed as:

$$\frac{p_M^n - p_i^n}{2} < p_M^n \theta_M - p_i^n \theta_i. \tag{4}$$

From Formula 4, $(p_M^n \theta_M - p_i^n \theta_i) - \frac{p_M^n - p_i^n}{2} > 0$, and because $0 < \varepsilon < 1$, and $m > 1$, therefore:

$$\frac{2m\varepsilon}{m-1} ((p_M^n \theta_M - p_i^n \theta_i) - \frac{p_M^n - p_i^n}{2}) > 0. \tag{5}$$

Furthermore:

$$\frac{2\varepsilon}{m-1} (m(p_M^n \theta_M - p_i^n \theta_i) - \frac{m(p_M^n - p_i^n)}{2}) > 0. \tag{6}$$

According to Formulas 6 and 3, if $p_i^n - p_M^n > 2(p_i^n \theta_i - p_M^n \theta_M)$, then $Y_M^n - Y_i^n > 0$.

Also, because $\sum_{i=1}^m p_i^{n+1} = 1$, and $\sum_{i=1}^m p_i^n = 1$, therefore $Y_M^n > 0$, and thus $\sum_{i=1}^m Y_i^n = 0$.

According to Formula 1, $p_M^{n+1} = p_M^n + Y_M^n$.

Because $Y_M^n > 0$, therefore $p_M^{n+1} > p_M^n$. □

Theorem 1. For failure rate $\theta_{\min} = \min\{\theta_1, \dots, \theta_m\}$, $\theta_M > \theta_{\min}$, if $0 < \theta_{\min} < \frac{1}{2}$, the following condition is sufficient to guarantee that $p_M^{n+1} > p_M^n$:

$$\frac{2m\theta_{\min}^2}{1 - 2\theta_{\min}} < \varepsilon < \frac{(m-1)m\theta_{\min}}{2(m+1)}. \tag{7}$$

Proof: In order to guarantee $p_M^{n+1} > p_M^n$, we consider the following three situations (where $i \in \{1, 2, \dots, m\}$ and $i \neq M$).

Situation 1 ($p_i^n = p_M^n$): Because $\theta_i < \theta_M$, therefore $(p_i^n \theta_i - p_M^n \theta_M) < 0$.

Therefore, $(p_i^n - p_M^n) > 2(p_i^n \theta_i - p_M^n \theta_M)$.

According to Lemma 1, we have $p_M^{n+1} > p_M^n$.

Situation 2 ($p_i^n > p_M^n$): According to Formula 7, we have the following:

$$\varepsilon > \frac{2m\theta_{\min}^2}{1 - 2\theta_{\min}}.$$

Because

$$\frac{2m\theta_{\min}^2}{1 - 2\theta_{\min}} = \frac{\theta_{\min}}{1/2m\theta_{\min} - 1/m},$$

we have the following:

$$\varepsilon > \frac{\theta_{\min}}{1/2m\theta_{\min} - 1/m}.$$

Because $\theta_{\min} < 1/2$, therefore $1/2m\theta_{\min} - 1/m > 0$ and $\varepsilon(1/2m\theta_{\min} - 1/m) > \theta_{\min}$, which gives $\varepsilon/2m\theta_{\min} > \theta_{\min} + \varepsilon/m$.

Because $\varepsilon > 0$, and $m > 1$, therefore

$$\frac{1}{2\theta_{\min}} > \frac{(\theta_{\min} + \varepsilon/m)}{(\varepsilon/m)}.$$

$(1/2\theta_{\min})(p_i^n - p_M^n) > (p_i^n - p_M^n)(\theta_{\min} + \varepsilon/m)/(\varepsilon/m)$ as $p_i^n > p_M^n$, and

$$p_i^n - p_M^n > 2\theta_{\min}(p_i^n - p_M^n) \frac{\theta_{\min} + \varepsilon/m}{\varepsilon/m}.$$

Because $(\theta_{\min} + \varepsilon/m)/(\varepsilon/m) > 1$, therefore

$$2\theta_{\min}(p_i^n - p_M^n) \frac{\theta_{\min} + \varepsilon/m}{\varepsilon/m} > 2\theta_{\min}(p_i^n - p_M^n).$$

Because $\theta_{\min} < \theta_M$, therefore

$$2\theta_{\min}(p_i^n - p_M^n) > 2(p_i^n \theta_{\min} - p_M^n \theta_M).$$

Thus,

$$p_i^n - p_M^n > 2(p_i^n \theta_{\min} - p_M^n \theta_M).$$

According to Lemma 1, we have $p_M^{n+1} > p_M^n$.

Situation 3 ($p_i^n < p_M^n$): For this proof, we make the assumption that $\frac{1}{2} < \theta_M < 1$.

Because we have

$$\varepsilon < \frac{(m-1)m\theta_{\min}}{2(m+1)}$$

and

$$\frac{(m-1)m\theta_{\min}}{2(m+1)} = \frac{2m - (m+1)}{2(m+1)} m\theta_{\min},$$

thus

$$\varepsilon < \left(\frac{m}{m+1} - \frac{1}{2}\right)m\theta_{\min}.$$

Obviously, $\varepsilon/m < (m/(m+1) - 1/2)\theta_{\min}$ as $m > 1$.

Furthermore, we have

$$-\frac{\varepsilon}{m} > \left(\frac{1}{2} - \frac{m}{m+1}\right)\theta_{\min}$$

and

$$\frac{m\theta_{\min}}{m+1} - \frac{\varepsilon}{m} + \frac{2\varepsilon}{m} > \frac{\theta_{\min}}{2} + \frac{2\varepsilon}{m}$$

which means that

$$\frac{m\theta_{\min}}{m+1} + \frac{\varepsilon}{m} > \frac{1}{2}(\theta_{\min} + \frac{4\varepsilon}{m}).$$

It follows that

$$(m\theta_{\min}/(m+1) + \varepsilon/m)/(4\varepsilon/m + \theta_{\min}) > 1/2$$

for any $m > 1, \varepsilon > 0$, and $0 < \theta_{\min} < 1$.

Because $\frac{1}{2} < \theta_M < 1$, therefore $(m\theta_{\min}/(m+1) + \varepsilon/m)/(4\varepsilon/m + \theta_{\min}) > 1/2\theta_M$.

Thus, we have

$$2(p_M^n - p_i^n)\theta_M \frac{\frac{\varepsilon}{m} + \frac{m\theta_{\min}}{m+1}}{\frac{4\varepsilon}{m} + \theta_{\min}} > p_M^n - p_i^n$$

as $p_M^n > p_i^n$.

Because $\varepsilon/m < 4\varepsilon/m$, and $m\theta_{\min}/(m+1) < \theta_{\min}$, therefore

$$\frac{\frac{\varepsilon}{m} + \frac{m\theta_{\min}}{m+1}}{\frac{4\varepsilon}{m} + \theta_{\min}} < 1$$

and

$$2(p_M^n - p_i^n)\theta_M > 2(p_M^n - p_i^n)\theta_M \frac{\frac{\varepsilon}{m} + \frac{m\theta_{\min}}{m+1}}{\frac{4\varepsilon}{m} + \theta_{\min}}$$

Hence we have

$$2(p_M^n - p_i^n)\theta_M > p_M^n - p_i^n,$$

which can be equivalently expressed as

$$p_i^n - p_M^n > 2(p_i^n - p_M^n)\theta_M.$$

Because $\theta_{\min} < \theta_M$, therefore $2(p_i^n - p_M^n)\theta_M > 2(p_i^n\theta_{\min} - p_M^n\theta_M)$, and thus

$$p_i^n - p_M^n > 2(p_i^n\theta_{\min} - p_M^n\theta_M).$$

According to Lemma 1, we have $p_M^{n+1} > p_M^n$. □