Appendix A: The process of proofing the Theorem 1

The derivation process of the relationship between p_i^{n+1} and p_i^n is shown below:

$$\begin{aligned}
p_i^{n+1} =& p_i^n\theta_i(p_i^n+\varepsilon)+p_i^n(1-\theta_i)(p_i^n-\varepsilon)\\
&+\sum_{j\neq i}p_j^n\theta_j(p_i^n-\frac{\varepsilon}{m-1})\\
&+\sum_{j\neq i}p_j^n(1-\theta_j)(p_i^n+\frac{\varepsilon}{m-1})\\
=&(p_i^n)^2\theta_i+p_i^n\theta_i\varepsilon+(p_i^n)^2-p_i^n\varepsilon-(p_i^n)^2\theta_i+p_i^n\theta_i\varepsilon\\
&+(p_i^n-\frac{\varepsilon}{m-1})\sum_{j\neq i}p_j^n\theta_j+(p_i^n+\frac{\varepsilon}{m-1})\sum_{j\neq i}p_j^n\\
&-(p_i^n+\frac{\varepsilon}{m-1})\sum_{j\neq i}p_j^n\theta_j\\
=&(p_i^n)^2+2p_i^n\theta_i\varepsilon-p_i^n\varepsilon+(p_i^n-\frac{\varepsilon}{m-1}-p_i^n\\
&-\frac{\varepsilon}{m-1})\sum_{j\neq i}p_j^n\theta_j+(p_i^n+\frac{\varepsilon}{m-1})(1-p_i^n)\\
=&p_i^n+(p_i^n)^2-(p_i^n)^2+2p_i^n\theta_i\varepsilon-p_i^n\varepsilon+\frac{\varepsilon}{m-1}-\\
&\frac{\varepsilon}{m-1}p_i^n-\frac{2\varepsilon}{m-1}\sum_{j\neq i}p_j^n\theta_j\\
=&p_i^n+\frac{\varepsilon}{m-1}(2p_i^n\theta_i m-p_i^n m-2p_i^n\theta_i+1)\\
&-\frac{2\varepsilon}{m-1}\sum_{j\neq i}p_j^n\theta_j\\
=&p_i^n+Y_i^n,
\end{aligned} \tag{1}$$

where

$$\begin{aligned}
Y_i^n =&\frac{\varepsilon}{m-1}(2p_i^n\theta_i m-p_i^n m-2p_i^n\theta_i+1)\\
&-\frac{2\varepsilon}{m-1}\sum_{j\neq i}p_j^n\theta_j.
\end{aligned} \tag{2}$$

The derivation process of the relationship between Y_M^n and Y_i^n is shown below:

$$\begin{aligned}
Y_M^n-Y_i^n =&\frac{\varepsilon}{m-1}(2p_M^n\theta_M m-p_M^n m-2p_M^n\theta_M+1)\\
&-\frac{2\varepsilon}{m-1}\sum_{j\neq M}p_j^n\theta_j-\frac{\varepsilon}{m-1}(2p_i^n\theta_i m-p_i^n m\\
&-2p_i^n\theta_i+1)+\frac{2\varepsilon}{m-1}\sum_{j\neq i}p_j^n\theta_j\\
=&\frac{\varepsilon}{m-1}(2m(p_M^n\theta_M-p_i^n\theta_i)-m(p_M^n-p_i^n)-\\
&2(p_M^n\theta_M-p_i^n\theta_i))-\sum_{j\neq M}p_j^n\theta_j+\sum_{j\neq i}p_j^n\theta_j\\
=&\frac{2\varepsilon}{m-1}(m(p_M^n\theta_M-p_i^n\theta_i)-\frac{m(p_M^n-p_i^n)}{2}-\\
&(p_M^n\theta_M-p_i^n\theta_i))+\frac{2\varepsilon}{m-1}(p_M^n\theta_M-p_i^n\theta_i)
\end{aligned}$$

$$=\frac{2\varepsilon}{m-1}(m(p_M^n\theta_M - p_i^n\theta_i) - \frac{m(p_M^n - p_i^n)}{2}). \tag{3}$$

Lemma 1. If $p_i^n - p_M^n > 2(p_i^n\theta_i - p_M^n\theta_M)$, then $p_M^{n+1} > p_M^n$.

Proof: The condition $p_i^n - p_M^n > 2(p_i^n\theta_i - p_M^n\theta_M)$ can be equivalently expressed as:

$$\frac{p_M^n - p_i^n}{2} < p_M^n\theta_M - p_i^n\theta_i. \tag{4}$$

From Formula 4, $(p_M^n\theta_M - p_i^n\theta_i) - \frac{p_M^n - p_i^n}{2} > 0$, and because $0 < \varepsilon < 1$, and $m > 1$, therefore:

$$\frac{2m\varepsilon}{m-1}((p_M^n\theta_M - p_i^n\theta_i) - \frac{p_M^n - p_i^n}{2}) > 0. \tag{5}$$

Furthermore:

$$\frac{2\varepsilon}{m-1}(m(p_M^n\theta_M - p_i^n\theta_i) - \frac{m(p_M^n - p_i^n)}{2}) > 0. \tag{6}$$

According to Formulas 6 and 3, if $p_i^n - p_M^n > 2(p_in\theta_i - p_M^n\theta_M)$, then $Y_M^n - Y_i^n > 0$.
Also, because $\sum_{i=1}^m p_i^{n+1} = 1$, and $\sum_{i=1}^m p_i^n = 1$, therefore $Y_M^n > 0$, and thus $\sum_{i=1}^m Y_i^n = 0$.
According to Formula 1, $p_M^{n+1} = p_M^n + Y_M^n$. Because $Y_M^n > 0$, therefore $p_M^{n+1} > p_M^n$. □

Theorem 1. For failure rate $\theta_{min} = min\{\theta_1, \ldots, \theta_m\}$, $\theta_M > \theta_{min}$, if $0 < \theta_{min} < \frac{1}{2}$, the following condition is sufficient to guarantee that $p_M^{n+1} > p_M^n$:

$$\frac{2m\theta_{min}^2}{1-2\theta_{min}} < \varepsilon < \frac{(m-1)m\theta_{min}}{2(m+1)}. \tag{7}$$

Proof: In order to guarantee $p_M^{n+1} > p_M^n$, we consider the following three situations (where $i \in \{1, 2, \ldots, m\}$ and $i \neq M$).

Situation 1 ($p_i^n = p_M^n$): Because $\theta_i < \theta_M$, therefore $(p_i^n\theta_i - p_M^n\theta_M) < 0$.
Therefore, $(p_i^n - p_M^n) > 2(p_i^n\theta_i - p_M^n\theta_M)$.
According to Lemma 1, we have $p_M^{n+1} > p_M^n$.

Situation 2 ($p_i^n > p_M^n$): According to Formula 7, we have the following:

$$\varepsilon > \frac{2m\theta_{min}^2}{1-2\theta_{min}}.$$

Because

$$\frac{2m\theta_{min}^2}{1-2\theta_{min}} = \frac{\theta_{min}}{1/2m\theta_{min} - 1/m},$$

we have the following:

$$\varepsilon > \frac{\theta_{min}}{1/2m\theta_{min} - 1/m}.$$

Because $\theta_{min} < 1/2$, therefore $1/2m\theta_{min} - 1/m > 0$ and $\varepsilon(1/2m\theta - 1/m) > \theta_{min}$, which gives $\varepsilon/2m\theta_{min} > \theta_{min} + \varepsilon/m$.

Because $\varepsilon > 0$, and $m > 1$, therefore

$$\frac{1}{2\theta_{min}} > \frac{(\theta_{min} + \varepsilon/m)}{(\varepsilon/m)}.$$

$(1/2\theta_{min})(p_i^n - p_M^n) > (p_i^n - p_M^n)(\theta_{min} + \varepsilon/m)/(\varepsilon/m)$ as $p_i^n > p_M^n$, and

$$p_i^n - p_M^n > 2\theta_{min}(p_i^n - p_M^n)\frac{\theta_{min} + \varepsilon/m}{\varepsilon/m}.$$

Because $(\theta_{min} + \varepsilon/m)/(\varepsilon/m) > 1$, therefore

$$2\theta_{min}(p_i^n - p_M^n)\frac{\theta_{min} + \varepsilon/m}{\varepsilon/m} > 2\theta_{min}(p_i^n - p_M^n).$$

Because $\theta_{min} < \theta_M$, therefore

$$2\theta_{min}(p_i^n - p_M^n) > 2(p_i^n\theta_{min} - p_M^n\theta_M).$$

Thus,

$$p_i^n - p_M^n > 2(p_i^n\theta_{min} - p_M^n\theta_M).$$

According to Lemma 1, we have $p_M^{n+1} > p_M^n$.

Situation 3 ($p_i^n < p_M^n$): For this proof, we make the assumption that $\frac{1}{2} < \theta_M < 1$.

Because we have

$$\varepsilon < \frac{(m-1)m\theta_{min}}{2(m+1)}$$

and

$$\frac{(m-1)m\theta_{min}}{2(m+1)} = \frac{2m-(m+1)}{2(m+1)}m\theta_{min},$$

thus

$$\varepsilon < (\frac{m}{m+1} - \frac{1}{2})m\theta_{min}.$$

Obviously, $\varepsilon/m < (m/(m+1) - 1/2)\theta_{min}$ as $m > 1$.

Furthermore, we have

$$-\frac{\varepsilon}{m} > (\frac{1}{2} - \frac{m}{m+1})\theta_{min}$$

and

$$\frac{m\theta_{min}}{m+1} - \frac{\varepsilon}{m} + \frac{2\varepsilon}{m} > \frac{\theta_{min}}{2} + \frac{2\varepsilon}{m}$$

which means that

$$\frac{m\theta_{min}}{m+1} + \frac{\varepsilon}{m} > \frac{1}{2}(\theta_{min} + \frac{4\varepsilon}{m}).$$

It follows that

$$(m\theta_{min}/(m+1) + \varepsilon/m)/(4\varepsilon/m + \theta_{min}) > 1/2$$

for any $m > 1, \varepsilon > 0$, and $0 < \theta_{min} < 1$.

Because $\frac{1}{2} < \theta_M < 1$, therefore $(m\theta_{min}/(m+1) + \varepsilon/m)/(4\varepsilon/m + \theta_{min}) > 1/2\theta_M$.

Thus, we have

$$2(p_M^n - p_i^n)\theta_M \frac{\frac{\varepsilon}{m} + \frac{m\theta_{min}}{m+1}}{\frac{4\varepsilon}{m} + \theta_{min}} > p_M^n - p_i^n$$

as $p_M^n > p_i^n$.

Because $\varepsilon/m < 4\varepsilon/m$, and $m\theta_{min}/(m+1) < \theta_{min}$, therefore

$$\frac{\frac{\varepsilon}{m} + \frac{m\theta_{min}}{m+1}}{\frac{4\varepsilon}{m} + \theta_{min}} < 1$$

and

$$2(p_M^n - p_i^n)\theta_M > 2(p_M^n - p_i^n)\theta_M \frac{\frac{\varepsilon}{m} + \frac{m\theta_{min}}{m+1}}{\frac{4\varepsilon}{m} + \theta_{min}}$$

Hence we have

$$2(p_M^n - p_i^n)\theta_M > p_M^n - p_i^n,$$

which can be equivalently expressed as

$$p_i^n - p_M^n > 2(p_i^n - p_M^n)\theta_M.$$

Because $\theta_{min} < \theta_M$, therefore $2(p_i^n - p_M^n)\theta_M > 2(p_i^n\theta_{min} - p_M^n\theta_M)$, and thus

$$p_i^n - p_M^n > 2(p_i^n\theta_{min} - p_M^n\theta_M).$$

According to Lemma 1, we have $p_M^{n+1} > p_M^n$. $\square$