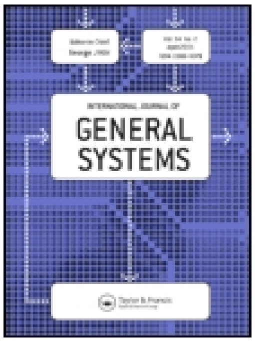

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Variable-structure coherent systems†

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The notion of coherent systems plays an essential role in conventional reliability theory. A system is said to be coherent if all of its components are relevant and the system reliability is improved as the component reliabilities are improved. However, in many complex systems or networks, not all the components are unconditionally relevant. As a result, in this paper we introduce the notion of variable-structure coherent systems to describe those systems that extensively exist and demonstrate essentially distinct features not observed in conventional coherent systems. A variable-structure coherent system consists of a number of substructures that are each a coherent system in conventional sense themselves. We then analyze the structural properties of variable-structure coherent systems; define the system operational profile, the system reliability, and the system structural profile. We study the system life distribution, the substructure importance, and the component importance. Finally, we deal with phase-cyclic systems in the context of variable-structure coherent systems.

Keywords: System reliability; Coherent system; Variable-structure coherent system; System operational profile; System structural profile; System life distribution

1. Introduction

Reliability research can be traced back to machine maintenance problems in 1930s and street-lighting lamp replacement problems in early 1940s [2]. With the painful experience with poor performance and reliability of military equipment suffered during World War II, reliability research drew a lot of attention thereafter. Up to the 1960s, the properties of various life distributions of components had been explored in-depth and the fundamental role of exponential distributions had been observed and established in reliability theory [3]. This might mark a mature status of "component reliability theory". From the 1960 to 1970s, the notion of coherent systems was introduced as a general system structure and the relationships between component reliabilities and system reliability were extensively investigated [4]. We may say that a mature theory of system reliability was thus developed. Although multi-state systems drew considerable attention [29], whether in conventional

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component reliability or system reliability theory, two fundamental assumptions were implicitly made: probability assumption and binary-state assumption. The probability assumption suggests that the reliability behavior can be completely characterized in the probability context, whereas the binary-state assumption suggests that the systems and components demonstrate only two classes of states: fully functioning or fully failed. These two assumptions had not been explicitly formulated until 1990 [11]. By relaxing the binary-state assumption to the fuzzy-state assumption and/or replacing the probability assumption by the possibility assumption, various fuzzy reliability theories including profust reliability theory, posbist reliability theory and posfust reliability theory can be developed [8]. To distinguish from fuzzy reliability theories, we call conventional reliability theory based on the probability assumption and the binary-state assumption, *probist* reliability theory.

In probist reliability theory, a system comprising n components is referred to as a coherent system if each component is relevant to the system and the system reliability is uniquely determined by an increasing function of all the component reliabilities. Let R be the system reliability and $p_1, p_2, p_3, \ldots, p_n$ the component reliabilities, respectively. The coherent system assumes that R is uniquely determined by exactly n variables $p_1, p_2, \ldots, p_n$ via an increasing function or coherent structure G

$$R = G(p_1, p_2, \ldots, p_n).$$

This is essentially a static or time-invariant notion. It assumes G is time-invariant and the state (functioning or failed) of each component always makes a contribution to the system reliability at any time.

Needless to say, the conventional notion of coherent systems mentioned above describes a large variety of real systems. This can be justified by the great success of probist reliability theory witnessed in the past few decades. However, we need to notice that many other systems demonstrate a different scenario. When a computer system runs a program on its CPU, the state of the printer interface would not make a contribution to the execution of the program if the program is not engaged in printing. In this way, the printer interface (or printer) is not relevant to the computer system with respect to the execution of the program. Similarly, when the computer system reads and prints a file from a hard disk, the floppy disk drive becomes an irrelevant component.

In a modern computer network, a huge number of nodes and links may be connected. Suppose a user needs to transmit a piece of message from node j to k. Then, only the nodes and links along the paths from node j to k are relevant to the network reliability with respect to the user, and other nodes and links become irrelevant. In a modern production line, each item may be required to go through several stations. When an item is processed at a station, other stations become irrelevant if they are not processing other items.

Multiple-phased systems (MPSs), such as on-board systems for the aided-guide of aircraft, whose mission consists of take-off, ascent, cruise, approach and landing phases, also constitute a large class of systems that the conventional notion of coherent systems does not fit. An MPS performs tasks to accomplish a given mission in a sequence of non-overlapping phases. System configuration, success criteria and component behavior may vary from phase to phase. MPSs have been extensively investigated in the literature [6,21] and three major reliability modeling approaches have been proposed for them, including representing all the phases of an MPS with a single model (e.g. Markovian model) [15], a separate modeling of each phase [28], and a hierarchical modeling of inter-phase and intra-phase behavior [24].

In comparison with various kinds of systems and system reliability problems that have been investigated [1,5,7,12,13,17–20,22,23,25–27], we have to say that the problem of conditionally relevant components in a system or MPS has not been tackled in-depth from a theoretical perspective. Existing works on PMSs are most concerned with how to model the systems of concern so that the mission reliability or phased system reliability can be defined and calculated in a convenient manner. They were carried out largely on the basis of case by case and were not aimed to develop a new reliability theory that might resemble the conventional theory of coherent systems. The structural properties of the systems of concern such as path, cut and component importance were seldom examined. The possible extension of the concept of coherent structure from single-phased- to multiple-phased systems was not discussed. The general properties among component, phase transition and system behavior were not investigated in a systematic manner.

Consequently, a logical question is how to model conditionally relevant components and their impacts on systems reliability in a rigorous manner. Alternatively, we are concerned with how to extend conventional concept of coherent structure to a multiple-phased or conditionally relevant components counterpart, just noting that the concept of coherent systems plays an essential role in conventional (probist) reliability theory. As a result, in this paper, within the context of probist reliability theory, we propose a new notion, namely variable-structure coherent system to describe a class of systems that are often encountered but do not fit the conventional notion of coherent systems. A variable-structure coherent system has two states: functioning or failed; and so does each component of the system. However a component, no matter whether it is functioning or failed, can be said to be in an idle state if the component becomes irrelevant to the system functioning or reliability, although at any time the system is coherent with respect to the corresponding relevant components. In other words, a variable-structure coherent system is a system whose structure can vary from one coherent structure to another according to some specified scheme. The notion of variable-structure coherent systems is a dynamic one in the sense that the system structure may change from time to time in system operation. It reflects the fact that in modern complex systems and networks performing multiple functions, it is seldom true that all the components are required to be consistently relevant.

With the new notion of variable-structure coherent systems, we may expect that many rebates can be brought about:

(1) Various different classes of hardware systems, including multiple-phased systems and other systems, can be modeled and investigated in a single framework.
(2) Researches on software reliability may benefit from those on hardware reliability by treating software systems as a variable-structure coherent system [10]. On the other hand, researches on software reliability may shed new light on hardware reliability theory. This can be exemplified in Section 3 by introducing the notion of system operational profile which is inspired from that of software operational profile. There is potential for the notion of variable-structure coherent systems to provide a unified reliability modeling framework for hardware as well as software systems.
(3) A new and systematic reliability theory can be developed to significantly generalize the conventional mature reliability theory of coherent systems. Many new notions such as unconditional cut, structural profile and substructure importance can be introduced and investigated as will be shown in the subsequent sections.

Here we briefly address the following questions that one may raise.

(1) Why Markov chains are not adequate models to model paths of a system with conditionally relevant components? The answer is that the paper does not reject Markov chains. Markov chains are adequate to model conditionally relevant components in some situations as discussed in Section 6. However, Markov chains are not a universal model. Other models such as semi-Markov chains should be adopted to model conditionally relevant components or system operational profile in other situations. Note that in modeling of software operational profile [9], no single model can be universal. This paper models conditionally relevant components in a more general context.
(2) Suppose the relationships between the system reliability and the reliabilities of conditionally relevant components are modeled by conditional probabilities. Why is a new theory needed to model the relationships? The answer is that the new theory proposed in the paper may bring out many rebates as discussed in previous paragraphs.
(3) If one uses conditional probabilities for conditionally relevant components, then each conditionally relevant component becomes relevant to the system and the system reliability is uniquely determined by those conditional probabilities. Where is there a contradiction to the definition of coherent system? The answer is that the existing definition of a coherent system assumes that components are relevant. It does not address conditionally relevant components in an explicit manner. Not every system with conditionally relevant components can be treated as a coherent system in the conventional sense as demonstrated in Example 2.5.

In the sequel, Section 2 introduces the notion of variable-structure coherent systems and discusses various structural properties of variable-structure coherent systems. Section 3 defines system operational profile, system reliability and system structural profile. Section 4 discusses life distributions of variable-structure coherent systems. Section 5 proposes a number of measures of substructure importance and component importance. Section 6 treats phase-cyclic systems as a variable-structure coherent system. In a companion paper [10] we will show that component-based software systems, under specific assumptions, can be treated as a variable-structure coherent system and discuss various related properties. Concluding remarks are contained in Section 7.

2. Definitions and structural properties

Consider a system comprising n components. Let

$$x_i = \begin{cases} 1 & \text{if component } i \text{ is functioning} \\ 0 & \text{if component } i \text{ is failed} \end{cases}$$

That is, x_i serves as a binary component variable. Here the idle state of component i is not defined explicitly. Component i is said to be idle if it is irrelevant (see Definition 2.1 below). A component in its idle state can be physically functioning or failed.

Let

$$\Phi = \begin{cases} 1 & \text{if the system is functioning} \\ 0 & \text{if the system is failed} \end{cases}$$

Φ serves as a binary system variable. If the system states are uniquely determined by the component states, then we have

$$\Phi = \Phi(X)$$

where

$$X = (x_1, x_2, \ldots, x_n)$$

The function $\Phi(X)$ is called the structure function of the system.

Denote

$$\begin{aligned}(1_i, X) &= (x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n)\\ (0_i, X) &= (x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n)\\ (\bullet_i, X) &= (x_1, \ldots, x_{i-1}, \bullet, x_{i+1}, \ldots, x_n)\end{aligned}$$

In probist (conventional) reliability theory, a coherent system is defined as follow [4].

DEFINITION 2.1. The ith component is irrelevant to the structure Φ if Φ is constant in x_i; i.e. $\Phi(1_i, X) = \Phi(0_i, X)$. Otherwise the ith component is relevant to the structure.

DEFINITION 2.2. A system of components is coherent if: (a) the system states are uniquely determined by the component states, (b) its structure function Φ is increasing and (c) each component is relevant.

LEMMA 2.3. (The pivotal decomposition theorem) For any function (coherent or non-coherent) Φ of order n, there holds

$$\Phi(X) = x_i\Phi(1_i, X) + (1 - x_i)\Phi(0_i, X) \quad \text{for all} \quad X (i = 1, 2, \ldots, n)$$

Proof. Trivial. Q.E.D.

Now let us introduce the notion of variable-structure coherent systems. Following the previous notation except replacing $\Phi(x_1, x_2, \ldots, x_n)$ by $\Phi(x_1, x_2, \ldots, x_n; y)$, where y is a supplementary variable called *control or switching variable*, $y \in \{1, 2, \ldots, m\}$. For a system of n components, suppose

$$\Phi(x_1, x_2, \ldots, x_n; y) = \begin{cases} \Phi_1(x_{i(1,1)}, x_{i(1,2)}, \ldots, x_{i(1,k_1)}) & \text{if} \quad y = 1\\ \Phi_2(x_{i(2,1)}, x_{i(2,2)}, \ldots, x_{i(2,k_2)}) & \text{if} \quad y = 2\\ \ldots & \\ \Phi_m(x_{i(m,1)}, x_{i(m,2)}, \ldots, x_{i(m,k_m)}) & \text{if} \quad y = m \end{cases} \tag{2.1}$$

where Φ is the (binary) system structure function whose values are uniquely determined by the n (binary) component variables $x_1, x_2, \ldots, x_n$ and the switching variable y; $\Phi_1, \Phi_2, \ldots, \Phi_m$

are the (binary) substructure functions with Φ_i being uniquely determined by the k_j component variables $x_{i(j,1)}, x_{i(j,2)}, \ldots, x_{i(j,k_j)}$ out of the n component variables[†] x_1, x_2,…,x_n.

DEFINITION 2.4. Φ is a variable-structure coherent structure (system) if

(1) $\cup_{y=1}^{m} \{x_{i(y,1)}, x_{i(y,2)}, \ldots, x_{i(y,k_y)}\} = \{x_1, x_2, \ldots, x_n\}$; and
(2) The m substructures $\Phi_1, \Phi_2, \ldots, \Phi_m$ are distinct and each coherent in the sense of Definition 2.2. □

From the above definition we can immediately conclude:

(1) Each x_i must appear in at least one of the m substructures $\Phi_1, \Phi_2, \ldots, \Phi_m$, or no components can be unconditionally irrelevant to Φ;
(2) Φ is increasing in each component variable (argument) x_i;
(3) In general, Φ cannot be uniquely determined by the (binary) component states x_1, x_2,…,x_n; it also depends on the switching variable y; A special case is $\Phi_1 \equiv \Phi_2 \equiv \cdots \equiv \Phi_m$.
(4) Each coherent system in the sense of Definition 2.2 can be treated as a variable-structure coherent system as a result of the pivotal decomposition theorem and
(5) In general, a variable-structure coherent system Φ is not a coherent system in the sense of Definition 2.2, even if virtual (binary) component variables y_1, y_2,…,y_d, $2^{d-1} < m \leq 2^d$, are introduced to replace the switching variable y and included in the structure function Φ. This can be justified by the following example.

Example 2.5. Suppose

$$\Phi = \begin{cases} \Phi_1(x_1, x_2, x_3) & \text{if} \quad y = 1 \\ \Phi_2(x_2, x_3, x_4) & \text{if} \quad y = 2 \end{cases}$$

where

$$\Phi_1(x_1, x_2, x_3) = \begin{cases} 1 & \text{if} \quad x_1 + x_2 + x_3 \geq 2 \\ 0 & \text{if} \quad x_1 + x_2 + x_3 < 2 \end{cases}$$

$$\Phi_2(x_2, x_3, x_4) = \begin{cases} 1 & \text{if} \quad x_2 + x_3 + x_4 \geq 2 \\ 0 & \text{if} \quad x_2 + x_3 + x_4 < 2 \end{cases}$$

[†]We can say that χ_1 is in an idle state, no matter it is functioning or failed, if $y = j$ and $x_l \notin \{x_{i(j,1)}, x_{i(j,2)}, \ldots, x_{i(j,k_j)}\}$.

Obviously, this constitutes a variable-structure coherent system. However, it cannot be treated as a coherent system in the sense of Definition 2.2. Let

$$y_d = \begin{cases} 1 & \text{if} \quad y = 1 \\ 0 & \text{if} \quad y = 2 \end{cases}$$

$$\Phi^*(x_1, x_2, x_3, x_4, y_d) = \Phi(x_1, x_2, x_3, x_4; y)$$

If $\Phi*$ is a coherent system, then it must be increasing in argument y_d. This means that there should always hold

$$\Phi_1(x_1, x_2, x_3) \geq \Phi_2(x_2, x_3, x_4)$$

But the state $x_1 = 0, x_2 = 0, x_3 = 1, x_4 = 1$ refutes the conclusion. So Φ^* is not increasing in y_d and thus not coherent. Similarly, we can easily show that Φ^* is not coherent even if we let

$$y_d = \begin{cases} 0 & \text{if} \quad y = 1 \\ 1 & \text{if} \quad y = 2 \end{cases}$$

In some special cases, on the other hand, a variable-structure coherent system can be treated as a coherent system in the sense of Definition 2.2. Consider the system defined in Definition 2.4 again. Suppose $\forall(x_1, x_2, \ldots, x_n)$, there holds $\Phi_1 \leq \Phi_2 \leq \ldots \leq \Phi_m$. Then we can introduce virtual binary component variables $y_1, y_2, \ldots, y_m$ such that

$$\begin{array}{llllll}
y_1 = 1, & y_2 = 0, & y_3 = 0, \ldots, y_{m-1} = 0, & y_m = 0, & \text{if} & y = 1 \\
y_1 = 1, & y_2 = 1, & y_3 = 0, \ldots, y_{m-1} = 0, & y_m = 0, & \text{if} & y = 2 \\
 & & \ldots & & & \\
y_1 = 1, & y_2 = 1, & y_3 = 1, \ldots, y_{m-1} = 1, & y_m = 0, & \text{if} & y = m-1 \\
y_1 = 1, & y_2 = 1, & y_3 = 1, \ldots, y_{m-1} = 1, & y_m = 1, & \text{if} & y = m
\end{array}$$

Let

$$\Phi^*(x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_m) =$$

$$\begin{cases}
\Phi_1(x_{i(1,1)}, x_{i(1,2)}, \ldots, x_{i(1,k_1)}) & \text{if } y_1 = 1,\ y_2 = 0,\ y_3 = 0, \ldots, y_{m-1} = 0,\ y_m = 0 \\
\Phi_2(x_{i(2,1)}, x_{i(2,2)}, \ldots, x_{i(2,k_2)}) & \text{if } y_1 = 1,\ y_2 = 1,\ y_3 = 0, \ldots, y_{m-1} = 0,\ y_m = 0 \\
\ldots & \\
\Phi_{m-1}(x_{i(m-1,1)}, x_{i(m-1,2)}, \ldots, x_{i(m-1,k_{m-1})}) & \text{if } y_1 = 1,\ y_2 = 1,\ y_3 = 1, \ldots,\ y_{m-1} = 1,\ y_m = 0 \\
\Phi_m(x_{i(m,1)}, x_{i(m,2)}, \ldots, x_{i(m,k_m)}) & \text{if } y_1 = 1,\ y_2 = 1,\ y_3 = 1, \ldots,\ y_{m-1} = 1,\ y_m = 1 \\
0 & \text{otherwise}
\end{cases}$$

Then each virtual component variable is relevant to Φ^*. It is easy to show that Φ^* is a coherent structure in the sense of Definition 2.2. An interesting open problem is whether $\Phi_1 \leq \Phi_2 \leq \cdots \leq \Phi_m$ is a necessary condition for a variable-structure coherent system to be a coherent system in the sense of Definition 2.2.

Definition 2.6. The dual of a structure Φ is defined as

$$\Phi^{\mathrm{D}}(X;y) = 1 - \Phi(1 - X;y)$$

where

$$1 - X = (1 - x_1, 1 - x_2, \ldots, 1 - x_n)$$

Example 2.7. Consider the structure $\Phi(X;y)$ defined in Example 2.5. The dual system is

$$\Phi^{\mathrm{D}}(x_1, x_2, x_3, x_4; y) = \begin{cases} \Phi_1^{\mathrm{D}}(x_1, x_2, x_3) & \text{if} \quad y = 1 \\ \Phi_2^{\mathrm{D}}(x_2, x_3, x_4) & \text{if} \quad y = 2 \end{cases}$$

where

$$\Phi_1^{\mathrm{D}}(x_1, x_2, x_3) = \begin{cases} 0 & \text{if} \quad x_1 + x_2 + x_3 \leq 1 \\ 1 & \text{if} \quad x_1 + x_2 + x_3 > 1 \end{cases}$$

$$\Phi_2^{\mathrm{D}}(x_2, x_3, x_4) = \begin{cases} 0 & \text{if} \quad x_2 + x_3 + x_4 \leq 1 \\ 1 & \text{if} \quad x_2 + x_3 + x_4 > 1 \end{cases}$$

We see that $\Phi^{\mathrm{D}}(X;y) = \Phi(X;y)$.

Proposition 2.8. The dual of a variable-structure coherent system is a variable-structure coherent system.

Proof. Trivial. Q.E.D.

Since not all components are consistently relevant, we can distinguish between unconditionally relevant components and conditionally relevant components. Also, we can distinguish between unconditionally functioning/failed states and conditionally functioning/failed states.

Definition 2.9. Consider a variable-structure system of order n denoted in equation (2.1).

(1) x_i is referred to as being unconditionally relevant if it is relevant in each substructure of $\Phi_1, \Phi_2, \ldots, \Phi_m$;
(2) x_i is referred to as being conditionally relevant if it is not unconditionally relevant. □

We see that in Example 2.5, x_2 and x_3 are unconditionally relevant and x_1 and x_4 are conditionally relevant. Obviously, no unconditionally irrelevant components can appear in a variable-structure coherent system.

Definition 2.10. Consider a variable-structure system of order n denoted in equation (2.1).

(1) Φ is unconditionally functioning at state $(x_1, x_2, \ldots, x_n)$ if $\Phi_1 = \Phi_2 = \ldots = \Phi_m = 1$, or $\Phi\big(x_1, x_2, \ldots, x_n; y\big) \equiv 1$;

(2) Φ is unconditionally failed at state $(x_1, x_2, \ldots x_n)$ if $\Phi_1 = \Phi_2 = \cdots = \Phi_m = 0$, or $\Phi\big(x_1, x_2, \ldots, x_n; y\big) \equiv 0$;

(3) Otherwise Φ is said to be conditionally functioning or failed at state $(x_1, x_2, \ldots x_n)$.

We note that there must be an unconditionally functioning state and unconditionally failed state in a variable-structure coherent system Φ, since $\Phi(1, 1, \ldots, 1; y) \equiv 1$ and $\Phi(0, 0, \ldots, 0; y) \equiv 0$. Normally, there must be some conditionally functioning/failed states in a variable-structure (coherent) system denoted in equation (2.1). Otherwise there holds $\Phi_1 \equiv \Phi_2 \equiv \ldots \Phi_m$ and thus $\Phi \equiv \Phi_1$ which is irrelevant to the switching variable y, and this reduces to a trivial case with $m = 1$.

Proposition 2.11. For any structure function (coherent or non-coherent) $\Phi(x_1, x_2, \ldots, x_n; y)$ of order n denoted in equation (2.1), there holds

$$\Phi(X; y) = x_i \Phi(1_i, X; y) + (1 - x_i)\Phi(0_i, X; y) \quad \text{for all} \quad X(i = 1, 2, \ldots, n)$$

Proof. Consider the case $y = 1$.

$$\Phi(X; y = 1) = \Phi_1(x_{i(1,1)}, x_{i(1,2)}, \ldots, x_{i(1,k_1)})$$

If $x_i \in \{x_{i(1,1)}, x_{i(1,2)}, \ldots, x_{i(1,k_1)}\}$, then

$$\begin{aligned} x_i\Phi(1_i, X; y = 1) + (1 - x_i)\Phi(0_i, X; y = 1) &= x_i\Phi_1(1_i, X^*) + (1 - x_i)\Phi_1(0_i, X^*) \\ &= \Phi_1(X^*) = \Phi(X; y = 1) \end{aligned}$$

where

$$X^* = \{x_{i(1,1)}, x_{i(1,2)}, \ldots, x_{i(1,k_1)}\} - \{x_i\}$$

If $x_i \notin \{x_{i(1,1)}, x_{i(1,2)}, \ldots, x_{i(1,k_1)}\}$, then

$$\begin{aligned} & x_i\Phi(1_i, X; y = 1) + (1 - x_i)\Phi(0_i, X; y = 1) \\ & \quad = x_i\Phi_1(x_{i(1,1)}, x_{i(1,2)}, \ldots, x_{i(1,k_1)}) + (1 - x_i)\Phi_1(x_{i(1,1)}, x_{i(1,2)}, \ldots, x_{i(1,k_1)}) \\ & \quad = \Phi_1(x_{i(1,1)}, x_{i(1,2)}, \ldots, x_{i(1,k_1)}) = \Phi(X : y = 1) \end{aligned}$$

Obviously, the conclusion can be proved for other cases of y. Q.E.D. □

Proposition 2.12. For a variable-structure coherent system $\Phi(x_1, x_2, \ldots, x_n; y)$ of order n, there holds†

$$\prod_{i=1}^{n} x_i \leq \Phi(x_1, x_2, \ldots, x_n; y) \leq \coprod_{i=1}^{n} x_i$$

† $\coprod_{i=1}^{n} x_i = \max(x_1, x_2, \ldots, x_n) = 1 - \prod_{i=1}^{n} (1 - x_i)$.

Proof. If $x_1 = x_2 = \ldots = x_n = 1$, then $\prod_{i=1}^{n} x_i = 1 = \Phi(1, 1, \ldots, 1; y)$. In other cases, $0 = \prod_{i=1}^{n} x_i \leq \Phi(x_1, x_2, \ldots, x_n; y)$.
If $x_1 = x_2 = \ldots = x_n = 0$, then $\coprod_{i=1}^{n} x_i = 0 = \Phi(x_1, x_2, \ldots, x_n; y)$. In other cases, $1 = \coprod_{i=1}^{n} x_i \geq \Phi(x_1, x_2, \ldots, x_n; y)$. Q.E.D. □

As in probist (conventional) reliability theory [4], Proposition 2.13 tells us that the performance of a variable-structure coherent system is bounded below by the performance of a series system and above by the performance of a parallel system.

PROPOSITION 2.13. Given $X = (x_1, x_2, \ldots, x_n)$, $Z = (z_1, z_2, \ldots, z_n)$. Denote

$$X \coprod Z = (x_1 \coprod z_1, x_2 \coprod z_2, \ldots, x_n \coprod z_n)$$

$$X \cdot Z = (x_1 z_1, x_2 z_2, \ldots, x_n z_n)$$

Then for a variable-structure coherent system, there holds

(1) $\Phi(X \coprod Z; y) \geq \Phi(X; y) \coprod \Phi(Z; y)$
(2) $\Phi(X \cdot Z) \leq \Phi(X)\Phi(Z) = \min(\Phi(X), \Phi(Z))$

Proof. Trivial, since $\Phi(X; y)$ is increasing in each argument of X. Q.E.D.

Also, as in probist (conventional) reliability theory [4], the first inequality of Proposition 2.13 states that the redundancy at the component level is more effective that reduancy at the system level.

Consider a conventional structure $\Phi(x_1, x_2, \ldots x_n)$ of order n in probist reliability theory. A subset of $\{x_1, x_2, \ldots, x_n\}$, say, $\{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\}$, is said to constitute a path (set) if $x_{i_1} = x_{i_2} = \cdots = x_{i_k} = 1$ means $\Phi(x_1, x_2, \ldots, x_n) = 1$. The path is said to be minimal if any proper subset of $\{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\}$ does not constitute a path (set) of the system. On the other hand, $\{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\}$ is said to constitute a cut (set) of the system if $x_{i_1} = x_{i_2} = \ldots = x_{i_k} = 0$ means $\Phi(x_1, x_2, \ldots, x_n) = 0$. The cut is said to be minimal if any proper subset of $\{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\}$ does not constitute a cut of the system. All these definitions can be extended to variable-structure (coherent) systems.

DEFINITION 2.14. Consider a variable-structure system $\Phi(x_1, x_2, \ldots, x_n; y)$ of order n denoted in equation (2.1). A subset of $\{x_1, x_2, \ldots x_n\}$, say, $\{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\}$, is said to constitute an unconditional path (set) of the system if $x_{i_1} = x_{i_2} = \ldots = x_{i_k} = 1$ means $\Phi(X; y) \equiv 1$. The path is said to be minimal if any proper subset of $\{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\}$ does not constitute an unconditional path of the system. $\{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\}$ is said to be a multiple-conditional path of the system if it is a path of more than one substructure. $\{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\}$ is said to be single-conditional path of the system if it is a path of one and only one substructure. □

Obviously, unconditional paths can be obtained from unconditionally functioning states of the system. Suppose the system is unconditionally functioning at state $\{x_1, x_2, \ldots x_n\}$, that is, $\Phi(x_1, x_2, \ldots, x_n; y) \equiv 1$, then $\{x_j : x_j = 1\}$ constitutes an unconditional path of the system.

Another way to obtain unconditional paths of a variable-structure system is to use the paths of its substructures. Suppose $\mathrm{P}_i = \{x_{j(i,1)}, x_{j(i,2)}, \ldots, x_{j(i,k_i)}\}$ is a path of substructure Φ_i,

$i = 1, 2, \ldots, m$, then $\mathrm{P} = \cup_{i=1}^{m} \mathrm{P}_i$ is certainly an unconditional path of the system. Now a problem is how to find all minimal unconditional paths in a convenient manner.

Example 2.15. Consider Example 2.5 again. Φ_1 has minimal paths $\{x_1, x_2\}$, $\{x_2, x_3\}$, $\{x_1, x_3\}$; Φ_2 has minimal paths $\{x_2, x_3\}$, $\{x_3, x_4\}$, $\{x_2, x_4\}$. We see that $\{x_1, x_2\} \cup \{x_3, x_4\} = \{x_1, x_2, x_3, x_4\}$ is an unconditional path of the system, but not a minimal unconditional path, since $\{x_2, x_3\}$ constitutes an unconditional path of the system. This suggests that the union of minimal paths of substructures is not necessarily a minimal unconditional path of the system. However, we have the following proposition.

PROPOSITION 2.16. Consider a variable-structure system $\Phi(x_1, x_2, \ldots, x_n; y)$ of order n denoted in equation (2.1). A minimal unconditional path of the system must include a minimal path for each structure.

Proof. Trivial. Q.E.D.

Since a union of minimal paths one from each substructure must be an unconditional path of the system, the above proposition suggests that, in order to find a minimal unconditional path of a variable-structure system, we can begin with finding all the minimal paths of each substructure. Then we construct the unions of these minimal paths in a combinatorial manner and reduce these unions. An interesting question is how many minimal unconditional paths are available in total for $\Phi(x_1, x_2, \ldots, x_n; y)$. Suppose Φ_i has h_i minimal paths in total, $i = 1, 2, \ldots, m$, and $\Phi(x_1, x_2, \ldots, x_n; y)$ has h minimal unconditional paths in total. It is easy to show that $h \leq \prod_{i=1}^{m} h_i$.

Definitions and properties can be developed for cut (sets) of a variable-structure system in a similar way. Here, we only present the following definition.

DEFINITION 2.17. Consider a variable-structure system $\Phi(x_1, x_2, \ldots, x_n; y)$ of order n denoted in equation (2.1). A subset of $\{x_1, x_2, \ldots, x_n\}$, say, $\{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\}$, is said to constitute an unconditional cut (set) of the system if $x_{i_1} = x_{i_2} = \cdots = x_{i_k} = 0$ means $\Phi(X; y) \equiv 0$. The cut is said to be minimal if any proper subset of $\{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\}$ does not constitute an unconditional cut of the system. $\{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\}$ is said to be multiple-conditional cut of the system if it is a cut of more than one substructure. $\{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\}$ is said to be single-conditional cut of the system if it is a cut of one and only one substructure $\square$.

From the above definition we immediately conclude that a minimal unconditional path of $\Phi(X; y)$ must be a minimal unconditional cut of $\Phi^D(X; y) = 1 - \Phi(1 - X; y)$.

Suppose $\{x_{j_1}, x_{j_2}, \ldots, x_{j_k}\}$ constitutes a minimal unconditional cut of the variable-structure system of order n denoted in equation (2.1), and τ_i is the time to failure of component i. Then the time to an unconditional failure (i.e. the time to an unconditional failed state of the system) is given by

$$\min_{\text{All minimal unconditional cuts}} \left(\max_{\{i \mid x_i \in \{x_{j_1}, x_{j_2}, \ldots, x_{j_k}\}\}} \tau_i \right).$$

Suppose $\{x_{v_1}, x_{v_2}, \ldots, x_{v_l}\}$ constitutes a minimal unconditional path of the system. Then the time to a conditional failure, i.e., the time to a conditionally failed state of the system, is given by

$$\max_{\text{All minimal unconditional paths}} \left(\min_{\{i | x_i \in \{x_{v_1}, x_{v_2}, \ldots, x_{v_l}\}\}} \tau_i \right).$$

Let $\tau(\Phi)$ denote the time to a failure of the system $\Phi(X; y)$. We arrive at the following proposition.

PROPOSITION 2.18. There holds

$$\min_{\text{All minimal unconditional cuts}} \left(\max_{\{i | x_i \in \{x_{j_1}, x_{j_2}, \ldots, x_{j_k}\}\}} \tau_i \right) \leq \tau(\Phi)$$

$$\leq \max_{\text{All minimal unconditional paths}} \left(\min_{\{i | x_i \in \{x_{v_1}, x_{v_2}, \ldots, x_{v_l}\}\}} \tau_i \right)$$

Proof. Obvious from the above formulation, as long as we note that a system failure really occurs when the system at a conditionally failed state switches to a failed substructure. Q.E.D.

In probist (conventional) reliability theory, for a given coherent system $\Phi(X)$ in the sense of Definition 2.2,

$$n_\Phi(i) = \sum_{\{X | x_i = 1\}} [\Phi(1_i, X) - \Phi(0_i, X)]$$

is actually the number of times component i appears in all the minimal paths of the system. The underlying idea can also apply to a variable-structure system $\Phi(x_1, x_2, \ldots, x_n; y)$ of order n denoted in equation (2.1).

Let

$$n_{\text{up}}(x_i) = \text{times of } x_i \text{ appearing in all the minimal unconditional paths}$$

$$n_{\text{sp}}(x_i) = \text{times of } x_i \text{ appearing in all the minimal paths of substructures}$$

Then

$$n_{\text{sp}}(x_i) = \sum_{y, \{X | x_i = 1\}} [\Phi(1_i, X; y) - \Phi(0_i, X; y)]$$

$$n_{\text{up}}(x_i) = \sum_{\{X | x_i = 1\}} \prod_y [\Phi(1_i, X; y) - \Phi(0_i, X; y)]$$

We will discuss in Section 5 whether $n_{\text{up}}(x_i)$ and $n_{\text{sp}}(x_i)$ can be used to define structural importance of component i.

Similarly, we can also calculate the number of times component i appears in a given multiple-conditional path

$$n_{\mathrm{mp}}(x_i) = \sum_{\{X|x_i=1\}} \prod_{y \in \text{a given multiple conditional path}} [\Phi(1_i, X; y) - \Phi(0_i, X; y)]$$

Modules and modular decompositions of a coherent system in the sense of Definition 2.2 are important structural aspects in probist (conventional) reliability theory. Following the manner of defining paths and cuts for variable-structure systems, modules of a variable-structure coherent system can also be defined. However, finding the existence conditions of appropriate modules of a variable-structure coherent system is an open problem.

3. System operational profile, reliability, and structural profile

Section 2 discusses the deterministic behavior of variable-structure systems. From this section we discuss the uncertain aspects of variable-structure systems in the probability context. Actually, the probability assumption and the binary-state assumption are not violated and thus the ongoing discussion fits the framework of probist reliability theory.

3.1 System operational profile

It is well known that software reliability is a function of software operational profile. This is because software operational profile describes the behavior of input values of the software system, whereas different input values may invoke different subsets of software modules/components and thus lead to different reliabilities [9]. System operational profile is a generization or counterpart of software operational profile for variable-structure systems.

Definition 3.1. The operational profile of a variable-structure system $\Phi(X; y)$ denoted in equation (2.1) characterizes the behavior of the control or switching variable y.

Typically, we may have

$$\Pr\{y = i\} = p_i \sum_{i=1}^{m} p_i = 1$$

That is, the system operational profile is determined by a static probability distribution. Symbolically,

$$\mathrm{SOP} = \{y = i, p_i; i = 1, 2, \ldots, m\}$$

where SOP designates system operational profile. From the probability perspective, it states that the switching variable switches among the m options in accordance with a time-invariant probability distribution.

An alternative general case is that the system operational profile is described as a continuous-time Markov chain. Specifically, let y_t denote the value of the switching variable y at time t. Then, symbolically, the Markov chain can be described as follows

$$p_{ij}(s,t) = \Pr\{y_t = j | y_s = i\} = p_{ij}(t-s) \quad 0 \le s \le t$$
$$p_{ij}(t) = \Pr\{y_t = j | y_0 = i\}$$

Let $P(t)$ be the probability transition matrix, or

$$P(t) = [p_{ij}(t)]_{m\times m} \sum_{j=1}^{m} p_{ij}(t) \equiv 1; \quad i = 1,2,\ldots,m$$
$$P(0) = \lim_{t\downarrow 0} P(t) = I$$

where I denotes the identity matrix of dimensions $m \times m$. Further,

$$q_{ij} = p'_{ij}(0) = \lim_{t\downarrow 0} \frac{p_{ij}(t) - p_{ij}(0)}{t} = \lim_{t\downarrow 0} \frac{p_{ij}(t) - \theta_{ij}}{t}$$

where

$$\theta_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{otherwise} \end{cases}$$

Obviously, there hold

$$\sum_{j=1}^{m} q_{ij} = 0, \quad i = 1,2,\ldots,m$$

$$q_{ij} \ge 0 \text{ if } i \ne j; q_{ii} \le 0, i,j = 1,2,\ldots,m$$

The corresponding infinitesimal generator of the continuous-time Markov chain is defined as a $m \times m$ matrix

$$Q = [q_{ij}]_{m\times m}$$

Q and the initial state y_0 completely characterize the behavior of the Markov chain. There hold

$$P'(t) = QP(t) \quad \text{Kolmogorov's backward equations}$$
$$P'(t) = P(t)Q \quad \text{Kolmogorov's forward equations}$$

$$P(t) = e^{tQ} = \sum_{i=0}^{\infty} \frac{(tQ)^i}{i!} = I + \sum_{i=1}^{\infty} \frac{t^i}{i!} Q^i$$

The limiting probabilities are determined as follows

$$\pi_j = \lim_{t\rightarrow\infty} p_{ij}(t); \quad i = 1,2,\ldots,m$$
$$\sum_{i=1}^{m} q_{ij}\pi_i = 0; \quad j = 1,2,\ldots,m$$
$$\sum_{i=1}^{m} \pi_i = 1$$

Of course, the system operational profile may be different from one application to another, and other SOP models can be developed. This is particularly true for software systems [9].

3.2 System reliability

Since we follow the probability assumption and the binary-state assumption, the reliability of a variable-structure system at time t is defined as the probability that the system keeps functioning over the time interval $[0, t]$. Note that keeping the system functioning over the time interval $[0, t]$ is equivalent to ensuring that no failures of substructures (if any) cause a system failure over the time interval $[0, t]$. A failure of any substructure can lead to a system failure when the system switches to or is just in the substructure. This means that the system operational profile plays an essential role in determining the system reliability.

The above definition is actually consistent with the conventional definition of system reliability in probist reliability theory, which is the probability that the given system is free of system failure in a given time interval under a given environment. Here the given system is a variable-structure coherent, not a coherent system in general. The structure of the given variable-structure coherent system varies from time to time, depending on the given environment that describes the behavior of the corresponding switching variable. An important feature of the given variable-structure coherent system is that a substructure failure will not take effect immediately if the substructure is not in operation at the time of the substructure failure.

Consider a variable-structure system $\Phi(x_1, x_2, \ldots, x_n; y)$ of order n denoted in equation (2.1). Let

$$R_i(t) = \Pr\{\text{no failure occurs to substructure } \Phi_i \text{ over the time interval } [0, t]\}$$

$$p_i(t) = \Pr\{y_t = i, \text{ or the system is in substructure } \Phi_i \text{ at time } t\}$$

we have the following proposition.

Proposition 3.2. For a variable-structure system $\Phi(x_1, x_2, \ldots, x_n; y)$ of order n denoted in equation (2.1), suppose no maintenance is conducted even if failures of substructures occur. Further, the system fails to function when it switches to or is in a failed substructure, but the system failures or substructure failures make no effects on the underlying system operational profile. The system can still switch to other substructures in accordance with the underlying operational profile even in the presence of system failures. Let $R(t)$ denote the system reliability at time t, that is, the probability that no system failures occur over the time interval $[0, t]$. Then

$$\prod_{i=1}^{m} R_i(t) \le R(t) \le \sum_{i=1}^{m} R_i(t) p_i(t) \le 1 - \prod_{i=1}^{m} (1 - R_i(t))$$

Proof. The first and third inequalities are obvious. For the second inequalities, we note

$$R(t) = \Pr\{\text{the system keeps functioning over the time interval } [0, \mathrm{t}]\}$$

$$= \sum_{i=1}^{m} \left(\begin{array}{l} \Pr\left\{ \begin{array}{l} \text{the system keeps functioning over the time interval } [0, t] \\ \left| \begin{array}{l} \text{the system is in substructure} \\ \Phi_i \text{ at time } t \end{array} \right. \end{array} \right\} \times \\ \times \Pr\left\{ \begin{array}{l} \text{the system is in substructure} \\ \Phi_i \text{ at time } t \end{array} \right\} \end{array} \right)$$

If the system is in substructure Φ_i at time t, then a necessary condition for the system to keep functioning over the time interval [0,t] is that Φ_i must be functioning over the time interval [0,t]. However, that Φ_i is functioning over the time interval [0,t] is not sufficient for the system to keep functioning over the time interval [0,t]. It should also be required that the system did not switch to or was not in a failed substructure other than Φ_i. Therefore

$$\Pr\{\text{the system keeps functioning over the time interval } [0,t] \mid y_t = i\} \le R_i(t)$$

$$= \Pr\{\Phi_i \text{ keeps functioning over the time interval } [0,t]\}$$

or

$$R(t) \le \sum_{i=1}^{m} R_i(t) p_i(t)$$

Q.E.D. □

Note the above proposition does not assume that the substructure reliabilities are independent; the substructure reliabilities may not be independent if two substructures contain an identical component. The proposition states that the reliability of a variable-structure system is bounded below by that of a series system and above by that of a parallel system. More accurately, the reliability of a variable-structure system is bounded above by the availability of the variable-structure system. However, how to calculate the exact reliability of a variable-structure system is an open problem.

3.3 System structural profile

Since in a variable-structure system not all components are consistently relevant, it is reasonable to believe that some components are more frequently selected or activated than other components. Intuitively, we may imagine that those components that are more frequently selected should be more important or more relevant to the system. Then how frequently may one component be selected? The notion of system structural profile addresses this question as follows.

Definition 3.3. For a variable-structure system of order n, the system structural profile describes the relative frequencies of all the n components being selected.

Here we note that the notion of system structural profile is concerned with how frequently each component is selected or is relevant, not with how or whether it is functioning or failed. Although multiple components (within a substructure) may be simultaneously selected to make the system function in actuality, the structural profile tries to address such a question: suppose a single component is to be selected each time, what is the probability of a particular component (out of the n components) being selected. If we treat system operational profile as an external profile of the system, then we may treat system structural profile as an "internal" profile, or simply, as a "component operational profile". Obviously, the system structural profile is dependent on the substructures as well as on the system operational profile.

3.3.1 Case 1. Consider the first system operational profile model mentioned in Section 3.1. That is, for a variable-structure system denoted in equation (2.1), there are m substructures

$\Phi_1, \Phi_2, \ldots, \Phi_m$ and the behavior of the switching variable y is described by a static probability distribution $\{p_1, p_2, \ldots, p_m\}$. Since the system contains n components, or simply, $x_1, x_2, \ldots, x_n$, and each substructure contains some of them, let

$$\delta_{ij} = \begin{cases} 1 & \text{if } \Phi_i \text{ contains } x_j \\ 0 & \text{otherwise} \end{cases}$$

In this way, we obtain a matrix of dimensions $m \times n$, called *system structural matrix*, as follows

$$A = [\delta_{ij}]_{m \times n} = \begin{bmatrix} \delta_{11}\,\delta_{12}\cdots\delta_{1n} \\ \delta_{21}\,\delta_{22}\cdots\delta_{2n} \\ \vdots \\ \delta_{m1}\,\delta_{m2}\cdots\delta_{mn} \end{bmatrix}$$

Suppose there are $l-1$ transitions among the substructures $\Phi_1, \Phi_2, \ldots, \Phi_m$. Then the system has l visits ($l-1$ transitions plus the first (starting) visit to one substructure of the system) to $\{\Phi_1, \Phi_2, \ldots, \Phi_m\}$ in total. In average, there are lp_i visits to Φ_i. This means that x_j is selected $lp_i \cdot \delta_{ij}$ times as a result of Φ_i being visited (selected). In this way, there are $\sum_{i=1}^{m} lp_i \cdot \delta_{ij}$ visits to x_j in total. Therefore, the relative frequency of component j being selected is

$$p(x_j) = \frac{\sum_{i=1}^{m} lp_i \cdot \delta_{ij}}{\sum_{j=1}^{n} \sum_{i=1}^{m} lp_i \cdot \delta_{ij}} = \frac{\sum_{i=1}^{m} p_i \cdot \delta_{ij}}{\sum_{i=1}^{m} p_i \left(\sum_{j=1}^{n} \delta_{ij} \right)}$$

Obviously

$$\sum_{j=1}^{n} p(x_j) = 1$$

The system structural profile is described by the relative frequencies $\{p(x_j); j = 1, 2, \ldots, n\}$.

Example 3.4. Consider Example 2.5 again. We have

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

Suppose $p_1 = 0.2$, $p_2 = 0.8$.Then

$$p(x_1) = \frac{0.2}{3} \quad p(x_2) = \frac{1}{3} \quad p(x_3) = \frac{1}{3} \quad p(x_4) = \frac{0.8}{3}$$

□

Note that the system structural profile can be determined from another viewpoint. Let $N_i(t)$ denote the number of visits to Φ_i by time t and

$$N(t) = \sum_{i=1}^{m} N_i(t)$$

Let $p_i(x_j)$ be the relative frequency of x_j being selected at time t. Then

$$p_t(x_j) = \frac{\sum_{i=1}^{m} N_i(t)\delta_{ij}}{\sum_{j=1}^{n}\sum_{i=1}^{m} N_i(t)\delta_{ij}} = \frac{\sum_{i=1}^{m} \frac{N_i(t)}{N(t)}\delta_{ij}}{\sum_{j=1}^{n}\sum_{i=1}^{m} \frac{N_i(t)}{N(t)}\delta_{ij}}$$

Since the system operational profile is determined by a static probability distribution $\{p_1, p_2, \ldots, p_m\}$, we have

$$\lim_{t\to\infty} \frac{N_i(t)}{N(t)} = p_i$$

In this way

$$\lim_{t\to\infty} p_t(x_j) = \frac{\sum_{i=1}^{m} p_i \cdot \delta_{ij}}{\sum_{i=1}^{m} p_i\left(\sum_{j=1}^{n} \delta_{ij}\right)}$$

3.3.2 Case 2. Suppose a variable-structure system denoted in equation (2.1) is subjected to an operational profile described by a continuous-time Markov chain as formulated in Section 3.1. In order to determine the system structural profile, we need to determine the numbers of visits to substructures $\Phi_1, \Phi_2, \ldots, \Phi_m$. Let $N_i(t)$ denote the number of visits of the system to Φ_i by time t. Let $p_i(x_j)$ be the relative frequency of x_j being selected at time t. As in Case 1, we have

$$p_t(x_j) = \frac{\sum_{i=1}^{m} N_i(t)\delta_{ij}}{\sum_{j=1}^{n}\sum_{i=1}^{m} N_i(t)\delta_{ij}}$$

The problem now is how to determine $N_i(t)$. Note the continuous-time Markov chain can be treated as a standard or delayed renewal process [16]: the chain stays in state $y = i$ for a time distributed by $F_i(t) = 1 - e^{q_{ii}t}$ and then leaves state $y = i$. It stays in other states ($y \neq i$) for a time distributed by $G_i(t)$ and then returns to state $y = i$. Since in general it is not easy to determine $G_i(t)$, calculating the distribution of $N_i(t)$ is rather difficult.

As far as the limiting system structural profile is concerned, we have

$$\omega_j = \lim_{t\to\infty} p_t(x_j) = \lim_{t\to\infty} \frac{\sum_{i=1}^{m} N_i(t)\delta_{ij}}{\sum_{j=1}^{n}\sum_{i=1}^{m} N_i(t)\delta_{ij}} = \lim_{t\to\infty} \frac{\sum_{i=1}^{m} \frac{N_i(t)}{N(t)}\delta_{ij}}{\sum_{j=1}^{n}\sum_{i=1}^{m} \frac{N_i(t)}{N(t)}\delta_{ij}} = \frac{\sum_{i=1}^{m} \pi_i\delta_{ij}}{\sum_{i=1}^{m} \pi_i\left(\sum_{j=1}^{n} \delta_{ij}\right)}$$

where $N(t) = N_1(t) + N_2(t) + \cdots + N_m(t)$, and $\{\pi_1, \pi_2, \ldots, \pi_m\}$ is the limiting distribution of the Markov chain.

Example 3.5. Suppose the variable-structure system of concern consists of three substructures and five components. The system structural matrix is given by

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

Further, the system operational profile is described by a continuous-time Markov chain governed by the infinitesimal generator

$$Q = \begin{bmatrix} -10 & 5 & 5 \\ 1.6 & -8 & 6.4 \\ 2.8 & 4.2 & -7 \end{bmatrix}$$

Then the limiting distribution of the Markov chain is given by

$$\pi_1 = \frac{26}{111} \quad \pi_2 = \frac{40}{111} \quad \pi_3 = \frac{45}{111}$$

The limiting distribution of the system structural profile turns to be

$$\omega_1 = \frac{\pi_1 + \pi_2}{\pi_1 \times 2 + \pi_2 \times 3 + \pi_3 \times 3} = \frac{66}{307}$$
$$\omega_2 = \frac{\pi_3}{\pi_1 \times 2 + \pi_2 \times 3 + \pi_3 \times 3} = \frac{45}{307}$$
$$\omega_3 = \frac{\pi_1 + \pi_3}{\pi_1 \times 2 + \pi_2 \times 3 + \pi_3 \times 3} = \frac{71}{307}$$
$$\omega_4 = \frac{\pi_2}{\pi_1 \times 2 + \pi_2 \times 3 + \pi_3 \times 3} = \frac{40}{307}$$
$$\omega_5 = \frac{\pi_2 + \pi_3}{\pi_1 \times 2 + \pi_2 \times 3 + \pi_3 \times 3} = \frac{85}{307}$$

A special case of accurately determining $\{p_t(x_j)\}$ comes up if the system consists of only two substructures Φ_1 and Φ_2. Suppose the corresponding infinitesimal generator of the Markov chain is

$$Q = \begin{bmatrix} -\lambda_1 & \lambda_2 \\ \lambda_2 & -\lambda_2 \end{bmatrix}$$

Then the system stays in state $y = 1$ for a time, say, ξ_1, distributed by $F_1(t) = 1 - e^{-\lambda_1 t}$ and stays in state $y = 2$ for a time, say, ξ_2, distributed by $F_2(t) = 1 - e^{-\lambda_2 t}$. In this way the renewal theory can apply here to determine $N_1(t)$ and $N_2(t)$ in a straightward manner.

4. Life distribution

From Proposition 3.2 we see that

$$\prod_{i=1}^{m} R_i(t) \le R(t) \le \sum_{i=1}^{m} R_i(t) p_i(t)$$

Since it is not easy to calculate $R(t)$ accurately, direct research into the properties of life distribution of a variable-structure coherent system looks unfeasible for the time being. Instead, let

$$F_i(t) = 1 - R_i(t)$$

and

$$L(t) = 1 - \sum_{i=1}^{m} R_i(t)p_i(t) = \sum_{i=1}^{m} F_i(t)p_i(t) \quad \text{with} \quad \sum_{i=1}^{m} p_i(t) \equiv 1$$

If $p_1(t), p_2(t), \ldots, p_m(t)$ are time-homogeneous or time-invariant, then $L(t)$ is a mixture distribution of $F_1(t), F_2(t), \ldots, F_m(t)$ in conventional sense. In this section, we show that some properties of a mixture distribution in conventional sense preserve for $L(t)$ even if $p_1(t), p_2(t), \ldots, p_m(t)$ are time-dependent.

Definition 4.1. A non-discrete distribution F is DFR (Decreasing Failure Rate) if and only if

$$\frac{F(t+x) - F(t)}{1 - F(t)}$$

is increasing in t for $x > 0$, $t \geq 0$ such that $F(t) < 1$. F has decreasing failure rate average (F is DFRA) if $-(1/t)\log(1 - F(t))$ s is decreasing in $t \geq 0$.

It follows that F is DFR if and only if $\log(1 - F(t))$ is convex for t in $\{t|F(t) < 1,\ t \geq 0\}$ [3, p. 25].

Definition 4.2. The hazard transform of the (time-dependent) mixture

$$L(t) = \sum_{i=1}^{m} F_i(t)p_i(t)$$

is

$$\eta(\mathbf{U}) = -\log \sum_{i=1}^{m} e^{-u_i p_i(t)}$$

where

$$\mathbf{U} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} \quad 0 \leq u_i \leq \infty, \quad i = 1, 2, \ldots, m$$

Let

$$\bar{L}(t) = 1 - L(t) = e^{-\int_0^t r(x)\mathrm{d}x} = e^{-\beta(t)}$$

$$R_i(t) = 1 - F_i(t) = e^{-\int_0^t r_i(x)\mathrm{d}x} = e^{-\beta_i(t)}$$

Then we have

$$\log \bar{L}(t) = \log \sum_{i=1}^{m} e^{-\beta_i(t)} p_i(t)$$

$$\beta(t) = -\log \sum_{i=1}^{m} e^{-\beta_i(t)} p_i(t) = \eta\big(\beta(t)\big)$$

where we denote

$$\beta(t) = \begin{bmatrix} \beta_1(t) \\ \beta_2(t) \\ \vdots \\ \beta_m(t) \end{bmatrix}$$

PROPOSITION 4.3. The hazard transform of a (time-dependent) mixture as defined in Definition 4.2 is concave, that is,

$$\eta(a\mathbf{U} + (1-a)\mathbf{V}) \geq a\eta(\mathbf{U}) + (1-a)\eta(\mathbf{V})$$

$$0 \leq a \leq 1, \quad 0 \leq u_i \leq \infty, \quad 0 \leq v_i \leq \infty, i = 1, 2, \ldots, m$$

Proof. The Holder inequality suggests

$$\sum_{i=1}^{m} |f_i g_i| p_i(t) \leq \left[\sum_{i=1}^{m} |f_i|^{\alpha} p_i(t)\right]^{1/\alpha} \left[\sum_{i=1}^{m} |g_i|^{v} p_i(t)\right]^{1/v} \quad \alpha > 1, \quad \frac{1}{\alpha} + \frac{1}{v} = 1$$

we have

$$\sum_{i=1}^{m} e^{-au_i} \cdot e^{-(1-a)v_i} p_i(t) \leq \left[\sum_{i=1}^{m} e^{-u_i} p_i(t)\right]^{a} \left[\sum_{i=1}^{m} e^{-v_i} p_i(t)\right]^{1-a}, \quad 0 \leq a \leq 1$$

Thus

$$-\log \sum_{i=1}^{m} e^{-[au_i + (1-a)v_i]} p_i(t) \geq -a \log \sum_{i=1}^{m} e^{-u_i} p_i(t) - (1-a) \log \sum_{i=1}^{m} e^{-v_i} p_i(t)$$

Q.E.D.

LEMMA 4.4. Let $h(\mathbf{U})$ be concave and increasing in each argument. Let each $u_i(t)$ be concave. Then $g_U(t) = h(U(t))$ is concave in t.

Proof. Refer to [4, p. 103]. Q.E.D.

PROPOSITION 4.5. For the (time-dependent) mixture

$$L(t) = \sum_{i=1}^{m} F_i(t)p_i(t)$$

(1) If each F_i is DFR, then F is DFR;
(2) If each F_i is DFRA, then F is DFRA.

Proof. Note that whether $L(t)$ is DFR (DFRA) or not can be judged with the help of $\log \bar{L}(t)$, then the proof can be completed by use of Proposition 4.3 and Lemma 4.4. Q.E.D.

5. Substructure importance and component importance

A variable-structure system comprises substructures and components. Intuitively, it is unlikely that all substructures or components are equally important. Ranking substructures and components and quantifying their importance become necessary when reliability improvement list and maintenance checklist are concerned. The most important substructure or component should be improved first for the system reliability improvement and checked first for possible failure causes. This is particularly true when some resources, e.g., time, financial support, are tightly limited.

Obviously, different criteria may be applied to quantify substructure and component importance and different quantitative measures can be defined. Recall that in a non-variable-structure system, the importance of a component depends on two factors [13]

- The location of the component in the system; and
- The reliability of the component in question.

Similarly, these two factors should also make sense for a variable-structure system. Moreover, since the reliability of a variable-structure system is largely subject to the corresponding system operational profile, system operational profile should be one of candidate contributing factors when we define the importance of a substructure or component.

5.1 Substructure importance

When the importance of a substructure is concerned, we discard the internal structure of the substructure and take it as a single component. In this way, from a structural or static viewpoint, all substructures of a variable-structure system are equally important since the system switches among the substructures and none of the substructures takes structural priority. Thus, we can forgo such a trivial system structural or static viewpoint when substructure importance is concerned. In order to distinguish the importance of various substructures, we can adopt other viewpoints. Here we introduce the operational profile viewpoint and the reliability viewpoint.

DEFINITION 5.1. Consider a variable-structure system $\Phi(x_1, x_2, \ldots, x_n; y)$ of order n denoted in equation (2.1). Let

$$\begin{aligned} R_i(t) &= \Pr\{\text{no failures occur to substructure } \Phi_i \text{ over the time interval } [0, t]\} \\ R(t) &= \Pr\{\text{the system keeps functioning over the time interval } [0, t]\} \\ p_i(t) &= \Pr\{y_t = i, \text{or the system is in substructure } \Phi_i \text{ at time } t\} \end{aligned}$$

From the system operational profile viewpoint, the importance of substructure Φ_i at time t is defined as

$$I_{\mathrm{SOP}}(i, t) = p_i(t)$$

From the system reliability viewpoint, the importance of substructure Φ_i at time t is defined as

$$I_{\mathrm{sr}}(i, t) = \frac{\partial R(t)}{\partial R_i(t)}$$

It is not difficult to calculate $I_{\mathrm{SOP}}(i, t)$ since $p_i(t)$ can be obtained from the system operational profile directly. For example, if the system operational profile is described by a static probability distribution $\{p_1, p_2, \ldots, p_m\}$, then $p_i(t) \equiv p_i$. If the system operational profile is described by a continuous-time Markov chain with infinitesimal generator $Q = [q_{ij}]_{m\times m}$, then

$$\begin{bmatrix} p_1(t) \\ p_2(t) \\ \vdots \\ p_m(t) \end{bmatrix} = \left(\sum_{k=0}^{\infty} \frac{(Qt)^k}{k!} \right)^{\mathrm{T}} \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_m \end{bmatrix}$$

where T denotes the transpose of a matrix, and

$$\gamma_i = \Pr\{\text{the system is in substructure } \Phi_i \text{ at time } t = 0\} = p_i(0)$$

Things are quite different for $I_{\mathrm{sr}}(i, t)$. As shown in Proposition 3.2, there holds

$$\prod_{i=1}^{m} R_i(t) \le R(t) \le \sum_{i=1}^{m} R_i(t) p_i(t)$$

but in general we observe

$$R(t) \ne \sum_{i=1}^{m} R_i(t) p_i(t)$$

Also, it is an open problem how to determine $R(t)$ accurately. Thus, it is not easy to determine $(\partial R(t)/\partial R_i(t))$. If

$$R(t) = \sum_{i=1}^{m} R_i(t) p_i(t)$$

holds in some special case, then

$$I_{\mathrm{sr}}(i,t) = I_{\mathrm{SOP}}(i,t)$$

5.2 Component importance

As in probist (conventional) reliability theory, various viewpoints can apply to define the importance of a component in a variable-structure system. Among them, the structural or static viewpoint tries to use structural or static properties of a component to quantify the importance of the component. Two cases can be distinguished further as follows.

DEFINITION 5.2. Consider a variable-structure system $\Phi(x_1, x_2, \ldots, x_n; y)$ of order n denoted in equation (2.1). Suppose the system structural matrix is $A = [\delta_{ij}]_{m\times n}$. Then the importance of x_j using the times of x_j appearing in all the substructures is defined as

$$I_{\mathrm{ts}}(j) = \frac{\sum_{i=1}^{m} \delta_{ij}}{\sum_{i=1}^{m}\sum_{j=1}^{n} \delta_{ij}}$$

Let $n_{\mathrm{up}}(x_j)$ be the times of x_j appearing in all the minimal paths of substructures as shown in Section 2. Then the importance of x_j using $n_{\mathrm{up}}(x_j)$ is defined as

$$I_{\mathrm{up}}(j) = \frac{n_{\mathrm{up}}(x_j)}{\sum_{j=1}^{n} n_{\mathrm{up}}(x_j)}$$

Example 5.3. Consider Example 2.5 again. The system structural matrix is

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

Thus

$$I_{\mathrm{ts}}(1) = \frac{1}{6} \quad I_{\mathrm{ts}}(2) = \frac{2}{6} \quad I_{\mathrm{ts}}(3) = \frac{2}{6} \quad I_{\mathrm{ts}}(1) = \frac{1}{6}$$

Further note $\Phi_1(x_1, x_2, x_3)$ has three minimal paths: $\{x_1, x_2\}$, $\{x_2, x_3\}$, $\{x_1, x_3\}$; $\Phi_2(x_2, x_3, x_4)$ has three minimal paths: $\{x_2, x_3\}$, $\{x_3, x_4\}$, x_2, x_4. In this way

$$n_{\mathrm{up}}(x_1) = 2 \quad n_{\mathrm{up}}(x_2) = 4 \quad n_{\mathrm{up}}(x_3) = 4 \quad n_{\mathrm{up}}(x_4) = 2$$

Hence

$$I_{\mathrm{up}}(1) = \frac{1}{6} \quad I_{\mathrm{up}}(2) = \frac{2}{6} \quad I_{\mathrm{up}}(3) = \frac{2}{6} \quad I_{\mathrm{up}}(1) = \frac{1}{6}$$

In the above special example, we have

$$I_{\mathrm{ts}}(j) = I_{\mathrm{up}}(j)$$

However, in general this equation should not hold. The conditions for the equation to hold remain being sought. Also from the above special example we see that $n_{\mathrm{sp}}(x_j)$, the times of x_j appearing in all the minimal unconditional paths (refer to Section 2), is questionable if it is used for defining the importance of x_j. This is because there is no guarantee that each component can appear in at least one minimal path. The above special example has only one minimal unconditional path: $\{x_2, x_3\}$. A quantitative measure of component importance should not take a misleading value of zero.

Definition 5.3. Consider a variable-structure system $\Phi(x_1, x_2, \ldots, x_n; y)$ of order n denoted in equation (2.1). Let

$$r_i(t) = \Pr\{\text{no failures occurs to component } x_i \text{ over the time interval } [0, t]\}$$

$$R(t) = \Pr\{\text{the system keeps functioning over the time interval } [0, t]\}$$

Let $\{p_t(x_1), p_t(x_2), \ldots, p_t(x_n)\}$ be the system structural profile at time t. Then from the system structural profile viewpoint, the importance of x_j at time t is defined as

$$I_{\mathrm{ssp}}(j, t) = p_t(x_j)$$

From the system reliability viewpoint, the importance of x_j at time t is defined as

$$I_{\mathrm{src}}(j, t) = \frac{\partial R(t)}{\partial r_j(t)}$$

From the system failure viewpoint, the importance of x_j at time t is defined as

$$I_{\mathrm{sfc}}\left(j, t\right) = \Pr\left\{x_j \text{ is failed } | \text{the system is failed}\right\}$$

Although Definition 5.3 follows a similar way of defining component importance in probist (conventional) reliability theory [13], it is rather difficult to obtain the various measures for practical variable-structure systems. This is because the system structural profile and system reliability can hardly be determined accurately for a variable-structure system. For example, suppose the system operational profile is described by a continuous-time Markov chain, what we can accurately determine is the limiting system structural profile, as shown in Section 3.3.2, whereas the transient system structural profile is not easy to be determined. The system reliability $R(t)$ is bounded lower by $\prod_{i=1}^{m} R_i(t)$ and upper by $\sum_{i=1}^{m} R_i(t) p_i(t)$, but its accurate expression is not available. In order to determine the probability that x_j is failed under a given system failure, we need to determine if x_j is selected. This is not easier than determining the system structural profile. In summary, Definition 5.3 only presents typical viewpoints of defining component importance. The practical methods of determining the corresponding measures need further investigation.

Of course, other viewpoints can also be applied to define component importance. The improvement potential can be of concern, that is, how much the system reliability would be improved if component x_j is replaced by a perfect component, i.e., by a component with $r_j(t) \equiv 1$. However, the practical methods of determining or approximating the resulting measures them are, as yet, unavailable.

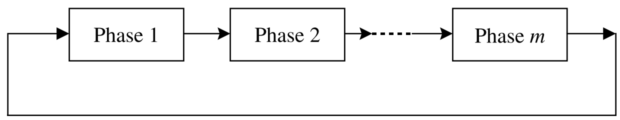

Figure 1. Flow chart of a phase-cyclic system.

6. Phase-cyclic systems

By phase-cyclic systems we mean those that perform functions phase by phase and then repeat the phases cyclically. This can be shown by use of figure 1. Suppose the system begins with phase 1. Then it turns to phase 2, 3,. . .., and to phase m consecutively. After phase m, the system returns to phase 1 again and repeats the previous procedure. In each phase the system activates a number of components and the other components become irrelevant to the system. Figure 2 shows an example† [15]. The system consists of four components and 3 phases in total. In phase 1 all the components are relevant. In phase 2, x_4 becomes irrelevant. In phase 3 x_2 becomes irrelevant. If we treat the system structure in each phase as a substructure, then the system is certainly a variable-structure system. Specifically, let Φ_i be the system structure in phase i, then equation (2.1) is still valid for representing a phase-cyclic system. Figure 2 represents a variable-structure coherent system since each substructure is coherent in a conventional sense.

6.1 System operational profile

6.1.1 Case 1 Deterministic system operational profile. Suppose the phase-cyclic system is initialized with phase 1, and the holding time of phase i is certainly a deterministic value of τ_i, $i = 1, 2, \ldots, m$. Then a deterministic system operational profile comes up, and no uncertainty is associated with it. Let $y = i$ if the system is in phase i, i.e., y serves as the switching variable of the phase-cyclic system. In this way y_t, the value of y at time t, is a deterministic function of t. Denote $\tau_0 = 0$. We see that $y_t = i$ if and only if there exists a non-negative integer k such that

$$k\sum_{j=0}^{m}\tau_j + \tau_0 + \tau_1 + \cdots + \tau_{i-1} \le t < k\sum_{j=0}^{m}\tau_j + \tau_0 + \tau_1 + \cdots + \tau_i; \quad i = 1, 2, \ldots, m$$

Let

$$k = \operatorname{int}\left[\frac{t}{\sum_{j=0}^{m}\tau_j}\right]$$

denote the number of system cycles by time t. Then

$$y_t = i \Leftrightarrow \tau_0 + \tau_1 + \cdots\tau_{i-1} \le t - k\sum_{j=0}^{m}\tau_j < \tau_0 + \tau_1 + \cdots\tau_{i-1} + \tau_i$$

†Kim & Park discussed mission reliability. They did not introduce the notion of phase-cyclic systems.

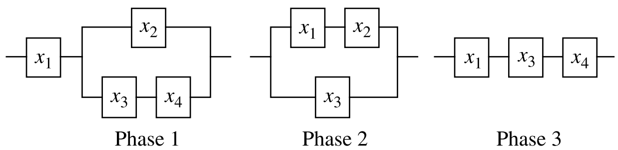

Figure 2. Example phase-cyclic system.

6.1.2 Case 2 Markov System Operational Profile. Suppose the holding time of phase i follows an exponential distribution with parameter λ_i, $i = 1, 2, \ldots, m$. Then the system operational profile is characterized by a continuous-time Markov chain with infinitesimal generator

$$Q = \begin{bmatrix} -\lambda_1 & \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & -\lambda_2 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & & & & & \\ \lambda_m & 0 & 0 & 0 & \cdots & -\lambda_m \end{bmatrix}$$

If the system consists of only two phases, or $m = 2$, then

$$Q = \begin{bmatrix} -\lambda_1 & \lambda_1 \\ \lambda_2 & -\lambda_2 \end{bmatrix}$$

Suppose the system is initialized with the first phase. It is easy to verify

$$p_1(t) = \tfrac{\lambda_2}{\lambda_1+\lambda_2} + \tfrac{\lambda_1}{\lambda_1+\lambda_2} e^{-(\lambda_1+\lambda_2)t}$$
$$p_2(t) = \tfrac{\lambda_1}{\lambda_1+\lambda_2}(1 - e^{-(\lambda_1+\lambda_2)t})$$

where $p_i(t)$ denotes the probability that the system is in phase i at time t, $i = 1, 2$.

6.1.3 Case 3 Semi-Markov system operational profile. In calculating mission reliability of phased systems, Kim & Park considered three cases† [15]:

(1) Phase durations are deterministic;

†However Kim & Park did not introduce or study the notion of system operational profile.

(2) Phase durations are random variables and there is a maximum mission time;
(3) Phase durations are random variables of general distributions, and there is no maximum mission time.

Obviously, the first case corresponds to a deterministic system operational profile, while the other two cases correspond to a semi-Markov chain. Let $p_i(t)$ be the probability that the system is in phase i at time t, $i = 1, 2, \ldots, m$. In theory there is no problem determining $p_1(t), p_2(t), \ldots, p_m(t)$ [14].

For the semi-Markov system operational profile, we note that the corresponding embedded Markov chain is determined by the state (phase) transition probability matrix

$$\mathrm{P} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

Observing this fact should be useful for determining system structural profile (refer to Section 6.3).

6.2 System reliability

System reliability is certainly subject to system operational profile. Basically, we distinguish two cases: deterministic and non-deterministic.

6.2.1 System reliability under deterministic operational profile. In order to calculate the reliability of a phase-cyclic system under deterministic operational profile, we assume:

(1) The system consists of m phases, as shown in figure 6.1;
(2) The system is initialized with phase 1;
(3) The holding time of phase i is τ_i, which is deterministic and irrelevant of whether the corresponding substructure is functioning or failed; denote $\tau_0 = 0$;
(4) The time to failure of substructure Φ_i is a random variable, denoted Z_i, measured from the time instant $t = 0$ when the system is initialized, and is irrelevant of the holding times of all phases;
(5) A system failure occurs when the system turns to a failed substructure, or the corresponding substructure fails to function while the system is in a phase;
(6) No maintenance applies to substructures or the system and
(7) The system can resume functioning when it turns to a functioning substructure from a failed substructure.

Let $R(t)$ denote the reliability of the system at time t. We note

$$0 \le t < \tau_1: \quad R(t) = \Pr\{Z_1 > t\}$$
$$\tau_1 \le t < \tau_1 + \tau_2: \quad R(t) = \Pr\{Z_1 > \tau_1, Z_2 > t\}$$
$$\tau_1 + \tau_2 \le t < \tau_1 + \tau_2 + \tau_3: \quad R(t) = \Pr\{Z_1 > \tau_1, Z_2 > \tau_1 + \tau_2, Z_3 > t\}$$
$$\vdots$$
$$\tau_1 + \tau_2 + \cdots + \tau_{m-1} \le t < \tau_1 + \tau_2 + \cdots + \tau_m:$$
$$R(t) = \Pr\{Z_1 > \tau_1, Z_2 > \tau_1 + \tau_2, \ldots, Z_{m-1} > \tau_1 + \tau_2 + \cdots + \tau_{m-1}, Z_m > t\}$$
$$\tau_1 + \tau_2 + \cdots + \tau_m \le t < \tau_1 + \tau_2 + \cdots + \tau_m + \tau_1:$$
$$R(t) = \Pr\left\{\begin{array}{l} Z_1 > t, Z_2 > \tau_1 + \tau_2, \ldots, Z_{m-1} > \tau_1 + \tau_2 + \cdots + \tau_{m-1}, \\ Z_m > \tau_1 + \tau_2 + \cdots + \tau_m \end{array}\right\}$$
$$\tau_1 + \tau_2 + \cdots + \tau_m + \tau_1 \le t < \tau_1 + \tau_2 + \cdots + \tau_m + \tau_1 + \tau_2:$$
$$R(t) = \Pr\left\{\begin{array}{c} Z_1 > \tau_1 + \tau_2 + \cdots + \tau_m + \tau_1, Z_2 > t, \ldots, Z_{m-1} > \tau_1 + \tau_2 + \cdots + \tau_{m-1}, \\ Z_m > \tau_1 + \tau_2 + \cdots + \tau_m \end{array}\right\}$$
$$\vdots$$

In general, as denoted in Case 1 of Section 6.1, let

$$k = \operatorname{int}\left[\frac{t}{\sum_{j=0}^{m} \tau_j}\right]$$

Then, if $y_t = i$, we have

$$R(t) = \Pr\left\{\begin{array}{l} Z_0 \ge \tau_0, \quad Z_1 > k\sum_{j=0}^{m} \tau_j + \tau_1, \ldots, \quad Z_{i-1} > k\sum_{j=0}^{m} \tau_j + \tau_1 + \ldots + \tau_{i-1}, \\ Z_i > t, Z_{i+1} > (k-1)\sum_{j=0}^{m} \tau_j + \tau_1 + \tau_2 + \ldots + \tau_i, \ldots, \\ Z_m > (k-1)\sum_{j=0}^{m} \tau_j + \tau_1 + \tau_2 + \ldots + \tau_m \end{array}\right\}$$

where we denote $Z_0 = 0$.

If we further assume that Z_1, Z_2,...,Z_m are independent† and let

$$R_i(t) = \Pr\{Z_i > t\}; \quad i = 0, 1, \ldots, m$$

†This assumption is over-strong in general, however it may hold if no single identical component is contained in two substructures and all components are distinct and independent.

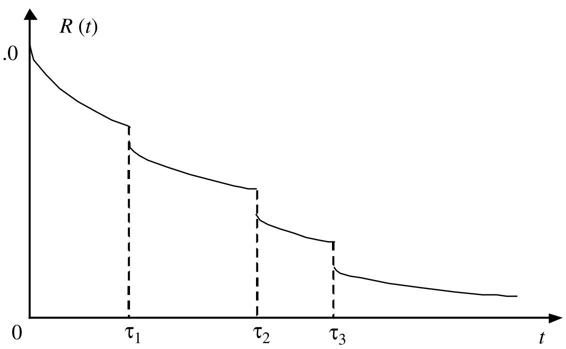

Figure 3. Example reliability function of a phase-cyclic system under deterministic operational profile.

Then

$$R(t) = \left[\prod_{l=0}^{i-1} R_l\left(k\sum_{j=0}^{m}\tau_j + \tau_1 + \ldots + \tau_l\right)\right] \cdot \left[\prod_{l=i+1}^{m} R_l\left((k-1)\sum_{j=0}^{m}\tau_j + \tau_1 + \cdots + \tau_l\right)\right] \cdot R_i(t)$$

$$\text{if} \quad y_t = i$$

Here we follow the convention that $\prod_{l=i+1}^{m}(\cdot) = 1$ if $i = m$.

From the above formula we see that $R(t)$ is a discontinuous function in general. A typical example is shown in figure 3. Discontinuity occurs when the system turns from one phase (substructure) to another phase (substructure). This may be a characteristic feature of variable-structure systems. We note that the reliability of a coherent system in the sense of Definition 2.2 is continuous in general.

6.2.2 System reliability under non-deterministic operational profile. Here we assume that the holding time of each phase is random instead of being deterministic and follow the other six assumptions of Section 6.2. Then the procedure for calculating system reliability presented there is still valid, provided we note that in each case the reliability must be treated as conditional reliability (probability) for calculating the overall reliability. For example,

$$0 \le t < \tau_1 : R(t) = \Pr\{Z_1 > t\}$$

Note τ_1 is a random variable and thus the condition $0 \le t < \tau_1$ follows a probability distribution. The overall reliability can be calculated, in theory, by use of the total probability formula

$$R(t) = \Pr\{Z_1 > t\}\Pr\{0 \le t < \tau_1\} + \Pr\{Z_1 > \tau_1, Z_2 > t\}\Pr\{\tau_1 \le t < \tau_1 + \tau_2\} + \ldots$$

Unfortunately, the possible cases (conditions) seem uncountable and thus a closed formula can hardly exist for the reliability function. It is a challenge even to seek an approximation to the reliability function.

6.3 System structural profile

6.3.1 System structural profile under deterministic operational profile. As in Section 3.3, we let $A = [\delta_{ij}]_{m\times n}$ be the system structural matrix. Following Case 1 of Section 6.1, let

$$k = \operatorname{int}\left[\frac{t}{\sum_{j=0}^{m} \tau_j}\right]$$

Let $N_i(t)$ be the number of visits to phase i by time t, i = 1, 2,. . .,m, then

$$N_1(t) = N_2(t) = \cdots = N_i(t) = k + 1$$
$$N_{i+1}(t) = N_{i+2}(t) = \cdots = N_m(t) = k$$

The number of visits to component x_v by time t is given by

$$M_v(t) = \sum_{i=1}^{m} N_i(t)\delta_{iv}; \quad v = 1, 2, \ldots, n$$

In this way the relative frequency of x_v being selected at time t is

$$p_t(x_v) = \frac{M_v(t)}{\sum_{v=1}^{n} M_v(t)} = \frac{\sum_{i=1}^{m} N_i(t)\delta_{iv}}{\sum_{i=1}^{m} N_i(t)\sum_{v=1}^{n} \delta_{iv}}$$

The limiting system structural profile becomes

$$\omega_v = \lim_{t\to\infty} p_t(x_v) = \lim_{t\to\infty} \frac{\sum_{i=1}^{m} \frac{N_i(t)}{t}\delta_{iv}}{\sum_{i=1}^{m} \frac{N_i(t)}{t}\sum_{v=1}^{n} \delta_{iv}}$$

$$= \lim_{t\to\infty} \frac{\sum_{i=1}^{m} \frac{\operatorname{int}\left[\frac{t}{\sum_{j=0}^{m} \tau_j}\right]}{t}\delta_{iv}}{\sum_{i=1}^{m} \frac{\operatorname{int}\left[\frac{t}{\sum_{j=0}^{m} \tau_j}\right]}{t}\sum_{v=1}^{n} \delta_{iv}} = \frac{\sum_{i=1}^{m} \frac{1}{\sum_{j=0}^{m} \tau_j}\delta_{iv}}{\sum_{i=1}^{m} \frac{1}{\sum_{j=0}^{m} \tau_j}\sum_{v=1}^{n} \delta_{iv}} = \frac{\sum_{i=1}^{m} \delta_{iv}}{\sum_{i=1}^{m} \sum_{v=1}^{n} \delta_{iv}}$$

We note that $\{\omega_1, \omega_2, \ldots, \omega_n\}$ depends only on the system structural matrix, and is irrelevant of the holding times of phases (states). In other words, from a long-term viewpoint, the system structural profile measures how many visits there are to a component, not how long a visit may consume. The limiting system structural profile coincides with the component importance $\{I_{\text{ts}}(j)\}$ of Definition 5.2 each other.

6.3.2 System structural profile under non-deterministic operational profile. Let τ_i, the holding time of phase i, be a random variable distributed by

$$F_i(t) = \Pr\{\tau_i \le t\}$$

In order to determine $N_i(t)$, the number of visits to phase i by time t, we note that the system operational profile can be formulated as a standard or delayed renewal process as follows. Suppose that the system is initialized with phase 1, then the system operates as depicted in figure 4, where $\xi_1, \xi_2, \ldots,$ are each distributed by $F_1(t)$, and $\zeta_1, \zeta_2, \ldots,$ are each

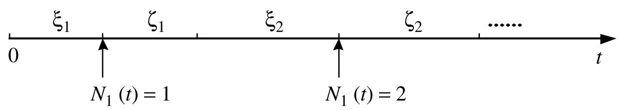

Figure 4. $N_1(t)$ as a delayed renewal process.

distributed by $G(t)$ given as

$$G(t) = \Pr\{\tau_2 + \tau_3 + \ldots + \tau_m \leq t\} = F_2(t)^* F_3(t)^* \ldots ^* F_m(t)$$

where $*$ denotes the convolution operator. In this way $N_1(t)$ follows a delayed renewal process given by $\{\xi_1, \zeta_1 + \xi_2, \zeta_2 + \xi_3, \ldots\}$. In theory there should be no problem with determining $N_1(t)$ by use of renewal process theory. Also, $N_2(t), N_3(t), \ldots, N_m(t)$ can be determined in a similar way.

With $N_1(t), N_2(t), \ldots, N_m(t)$, the system structural profile is given by

$$p_t(x_v) = \frac{\sum_{i=1}^m N_i(t)\delta_{iv}}{\sum_{i=1}^m N_i(t) \sum_{v=1}^n \delta_{iv}}$$

In order to determine the limiting system structural profile, we note that if the system is initialized with phase 1, then $N_1(t) \geq N_2(t) \geq \cdots \geq N_m(t)$, and

$$\left|N_i(t) - N_j(t)\right| \leq 1; \quad \forall i, j$$

Therefore, let $N(t) = N_1(t) + N_2(t) + \ldots + N_m(t)$,

$$\omega_v = \lim_{t \to \infty} p_t(x_v) = \lim_{t \to \infty} \frac{\sum_{i=1}^m \frac{N_i(t)}{N(t)} \delta_{iv}}{\sum_{i=1}^m \frac{N_i(t)}{N(t)} \sum_{v=1}^n \delta_{iv}} = \frac{\sum_{i=1}^m \frac{1}{m} \delta_{iv}}{\sum_{i=1}^m \frac{1}{m} \sum_{v=1}^n \delta_{iv}} = \frac{\sum_{i=1}^m \delta_{iv}}{\sum_{i=1}^m \sum_{v=1}^n \delta_{iv}}$$

Actually, this conclusion can also be drawn by use of the corresponding embedded Markov chain (refer to Case 3 of Section 6.1). So we do not need to distinguish deterministic and non-deterministic system operational profiles as far as the limiting system structural profile is concerned.

7. Concluding remarks

The notion of coherent systems plays an essential role in conventional reliability theory (referred to as probist reliability theory for distinguishing it from fuzzy reliability theories). A system is said to be coherent if the system states are uniquely determined by the component states, each of its components is relevant, and the system reliability is improved as the component reliabilities are improved. Although the notion of coherent systems describes a large variety of real systems, as practical systems and networks become more and more complex and perform more functions, however, it becomes evident that not all components of a complex system or network are always relevant. Instead, a component may be relevant at one time, and becomes irrelevant at another. This can often be observed in avionic systems, manufacturing systems, software systems and computer communication networks. Consequently, in this paper, we introduce the notion of variable-structure coherent systems. A variable-structure coherent system consists of a number of substructures, which are each coherent in the conventional sense themselves, nevertheless, not all the components of the system are contained in each substructure. The notion of variable-structure coherent systems is an extension of the conventional notion of coherent systems. However, a variable-structure coherent system is not a coherent system in the conventional sense in general, although a coherent system can be treated as a variable-structure coherent system. In the preceding sections, we analyzed structural properties of variable-structure coherent systems, define system operational profile, system reliability and system structural profile. Also, we study system life distributions, introduce substructure importance and component importance, and discuss phase-cyclic systems in the seating of variable-structure coherent systems. Various essentially different features that are not observed in conventional coherent systems emerge. For example, unconditional paths (cuts) and conditional paths (cuts) must be distinguished. The reliability of a variable-structure coherent system (e.g., a phase-cyclic system) may not be a continuous function of system operation time. It is a function of the system operational profile. All these features will play an important role in dealing with different problems of variable-structure coherent systems in the future. An essentially difficult problem is how to calculate accurate reliability of a variable-structure coherent system. This should not be surprising since it is also a problem even for a coherent system in conventional sense. How to design or optimize a variable-structure coherent system is another difficult problem. In short, in order to deal with conditionally relevant components of complex systems and networks, the notion of variable-structure coherent systems is a natural and inevitable extension of that of conventional coherent systems and many important topics deserve further in-depth investigation.

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References

[1] Y. Afek, H. Attiya, A. Fekete, M. Fischer, M. Lynch, Y. Mansour, D.W. Wang and L. Zuck, "Reliable communication over unreliable channels", *J. ACM*, 41(6), pp. 1267–1297, 1994.
[2] R.E. Barlow, "Mathematical theory of reliability: a historical perspective", *IEEE Trans. Reliability*, R-34(5), pp. 473–479, 1985.
[3] R.E. Barlow and F. Proschan, *Mathematical Theory of Reliability*, New York: Wiley, 1965.
[4] R.E. Barlow and F. Proschan, *Statistical Theory of Reliability and Life Testing: Probability Models*, Holt: Rinbart and Winston, 1975.
[5] R.A. Bazzi and G. Neihger, "Simplifying fault-tolerance: proving the abstraction of crash failures", *J. ACM*, 48(3), pp. 499–554, 2001.
[6] A. Bondavalli, S. Chiaradonna, F. Di Giandomenico and I. Mura, "Dependability modeling and evaluation of multiple-phased systems using DEEM", *IEEE Trans. Reliability*, 53(4), pp. 509–522, 2004.
[7] B. Brunelle and K. Kapur, "Continuous-state system-reliability: an interpolation approach", *IEEE Trans. Reliability*, 47(2), pp. 181–187, 1998.
[8] K.Y. Cai, *Introduction to Fuzzy Reliability*, Dordrecht: Kluwer Academic Publishers, 1996.
[9] K.Y. Cai, *Software Defect and Operational Profile Modeling*, Dordrecht: Kluwer Academic Publishers, 1998.
[10] K.Y. Cai, "Software systems as a variable-structure coherent systems", submitted.
[11] K.Y. Cai and C.Y. Wen, "Street-lighting lamps replacement: a fuzzy viewpoint", *Fuzzy Sets Systems*, 37, pp. 161–172, 1990.
[12] K. Goseva-Popstojanova and K.S. Trivedi, "Architecture-based approach to reliability assessment of software systems", *Performance Eval*, 45, pp. 179–204, 2001.
[13] M. Rausand A. Hoyland, *System Reliability Theory: Models and Statistical Methods*, New York: Wiley, 1994, Chapter 4.

[14] M. Kijima, *Markov Processes for Stochastic Modeling*, London: Chapman & Hall, 1997, Section 4.9, pp. 229–235.
[15] K. Kim and K.S. Park, "Phased-mission system reliability under markov environment", *IEEE Trans. Reliability*, 43(2), pp. 301–309, 1994.
[16] V.G. Kulkarni, *Modeling and Analysis of Stochastic Systems*, London: Chapman & Hall, 1995, Chapter 8.
[17] V.R. Lalli, "Space-system reliability: a historical perspective", *IEEE Trans. Reliability*, 47(3-SP), pp. SP-355–SP-360, 1998.
[18] J.C. Laprie and K. Kanoun, "X-ware reliability and availability modeling", *IEEE Trans. Software Eng.*, 18(2), pp. 159–168, 1992.
[19] B. Littlewood, "A reliability model for systems with Markov structures", *Appl. Stat.*, 24(2), pp. 172–177, 1975.
[20] H. Liu, "Reliability of a load-sharing *k*-out-of-*n*: G system: non-idd components with arbitrary distributions", *IEEE Trans. Reliability*, 47(3), pp. 279–284, 1998.
[21] Y. Ma and K.S. Trivedi, "An algorithm for reliability analysis of phased-mission systems", *Reliability Eng. Syst Safety*, 66, pp. 157–170, 1999.
[22] S. Mitra, N.R. Saxena and E.J. McCluskey, "A Design Diversity metric and analysis of redundant systems", *IEEE Trans. Comput*, 51(5), pp. 498–510, 2002.
[23] G. Mohan, S.R. Murthy and A.K. Somani, "Efficient algorithms for rounting dependable connections in WDM optical networks", *IEEE/ACM Trans. Networking*, 9(5), pp. 553–566, 2001.
[24] I. Mura and A. Bondavalli, "Hierarchical modeling & evaluation of phased-mission systems", *IEEE Trans. Reliability*, 48(4), pp. 360–368, 1999.
[25] A. Pasquini, G. Pistolesi and A. Rizzo, "Reliability analysis of systems based on software and human resources", *IEEE Trans. Reliability*, 50(4), pp. 337–345, 2001.
[26] A.P. Snow, "Network reliability: the concurrent challenges of innovation, competition, and complexity", *IEEE Trans. Reliability*, 5(4), pp. 38–40, 2001.
[27] J.M. Voas, "Certifying off-the-shelf software components", *IEEE Comput*, 31(6), pp. 53–59, 1998.
[28] L. Xing and J.B. Dugan, "Analysis of generalized phased mission system reliability, performance and sensitivity", *IEEE Trans. Reliability*, 51(2), pp. 199–211, 2002.
[29] K. Yu, I. Koren and Y. Guo, "Generalized multi-state monotone coherent systems", *IEEE Trans. Reliability*, 43(2), pp. 242–250, 1994.

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