



Mathematical modeling of software reliability testing with imperfect debugging

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ABSTRACT

Software reliability testing is concerned with the quantitative relationship between software testing and software reliability. Our previous work develops a mathematically rigorous modeling framework for software reliability testing. However the modeling framework is confined to the case of perfect debugging, where detected defects are removed without introducing new defects. In this paper the modeling framework is extended to the case of imperfect debugging and two models are proposed. In the first model it is assumed that debugging is imperfect and may make the number of remaining defects reduce by one, remain intact, or increase by one. In the second model it is assumed that when the number of remaining defects reaches the upper bound, the probability that the number of remaining defects is increased by one by debugging is zero. The expected behaviors of the cumulative number of observed failures and the number of remaining defects in the first model show that the software testing process may induce a linear or nonlinear dynamic system, depending on the relationship between the probability of debugging introducing a new defect and that of debugging removing a detected defect. The second-order behaviors of the first model also show that in the case of imperfect debugging, although there may be unbiased estimator for the initial number of defects remaining in the software under test, the cumulative number of observed failures and the current number of remaining defects are not sufficient for precisely estimating the initial number of remaining defects. This is because the variance of the unbiased estimator approaches a non-zero constant as the software testing process proceeds. This may be treated as an intrinsic principle of uncertainty for software testing. The expected behaviors of the cumulative number of observed failures and the number of remaining defects in the second model show that the software testing process may induce a nonlinear dynamic system. However theoretical analysis and simulation results show that, if defects are more often removed from than introduced into the software under test, the expected behaviors of the two models tend to coincide with each other as the upper bound of the number of remaining defects approaches infinity.

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1. Introduction

Software testing has long served as a major apparatus for software quality assurance [1,2]. There are many forms of software testing, including functional testing, data flow testing, boundary value testing, random testing, and so on. During

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software testing, test cases are selected and applied to the software under test. Defects are detected and removed one by one and thus the reliability of the software under test is improved. However the quantitative effects of conventional forms of software testing on the delivered software reliability are obscure. This is partially due to the fact that software testing can only prove the presence of software defects and is not capable of demonstrating the absence of software defects. The number of defects remaining in the software under test is unknown in principle. Another reason is that software reliability is a function of the operational profile and the discrepancy between the test profile and the operational profile is ambiguous.

On the other hand, since the software testing process is complex in the sense that it involves many factors including test case selection, test case execution, defect debugging, tester's knowledge and experience, and so on, it is not easy to formulate the software testing process in a mathematically rigorous manner. Various methods of software reliability modeling often ignore the applied software testing techniques and, for the sake of mathematical tractability, adopt simplifying assumptions such as the independence assumption (the times between successive software failures are independent), the exponentiality assumption (the times between successive software failures are exponentially distributed), and the NHPP assumption (the cumulative numbers of observed software failures follow a non-homogeneous Poisson process) [3–5]. However these assumptions are often unrealistic and this makes the validity of these methods highly questionable [6]. It is widely agreed in the software reliability community that no single method or model is universally valid for software reliability assessment [7,8]. Worse, it is not clear which existing method or model best fits a given software system. At the current stage it is a luxury to treat the quantitative goal of software reliability as a basis for the test case selection and execution in the software testing process.

In response to the undesirable status described above, in recent years, software reliability testing emerged as a new form of software testing that features the quantitative concern of the delivered software reliability. It formulates the quantitative relationships between software testing and software reliability, and addresses the effects of software testing on the delivered software reliability quantitatively. It also addresses how to achieve a given quantitative goal of software reliability via software testing. This can be justified in various related works. For example, in the so-called software reliability engineered testing [9], test cases are selected from the input domain of the software under testing in accordance with the expected operational profile that is described by a probability distribution, with the hope that the operational software reliability can accurately be estimated. Another form of random testing is Markov usage model based testing [10,11], by which test cases are executed in accordance with a Markov chain to reflect the interactive nature of the software under test. The testing data are then used to quantify the reliability improvement process and the delivered reliability. On the other hand, the CMC (controlled Markov chain) approach to software testing [12,13] can be treated as an approach for software reliability testing, where the feedback mechanisms in software testing are formalized, quantified and optimized for reliability improvement and/or reliability assessment from the perspective of test case selection during testing. However it is reasonable to say that the research on software reliability testing is still at its early stage. The above works are far from being systematic and the quantitative relationship between software testing and software reliability is still poorly understood.

In order to make a systematic contribution to the research on software reliability testing, a desirable mathematical modeling framework that is practically realistic, mathematically rigorous, and quantitatively precise, should be developed. To this end, Reference [14] proposes a simplifying model and a generalized model to constitute a systematic modeling framework of software reliability testing. In both models the selection process of test cases and their quantitative effects on software reliability improvement are formulated in a mathematically rigorous manner. This leads to interesting observations that the independence assumption, the exponentiality assumption, and the NHPP assumption are theoretically false [14], and the dynamics of software testing processes may be described by linear dynamic systems in certain circumstances [15]. However an unrealistic assumption, adopted in both models, is that the corresponding debugging is perfect. That is, when test cases are executed and failures are revealed, the corresponding failure-causing defects will be certainly removed from the software under test and no new defects will be introduced. This is unrealistic in many circumstances, as software developers may commit errors while actions are taken to debug the software under test and remove failure-causing defects from it [3,16].

This paper is aimed at removing the above unrealistic assumption and extending the mathematical modeling framework developed in Reference [14] by considering the case of imperfect debugging, in which failure-causing defects may or may not be removed, and new defects may or may not be introduced into the software under test. More specifically, following the research style of References [14,15], this paper proposes two models for software reliability testing and presents a theoretical study for the two proposed models. As observed in Reference [15], an advantage of theoretical studies over empirical studies is that conclusions drawn from theoretical studies can be rigorously proved, as long as the corresponding assumptions are valid. However, this by no means suggests that empirical studies are less important. Actually, theoretical and empirical studies should be complementary rather than conflicting. Empirical studies for the proposed two models should be carried out in the future.

The rest of this paper is organized as follows. Section 2 presents the assumptions of the first model proposed in this paper. Section 3 studies the Markovian properties of several sequences of concern for the first model. The expected behaviors and the second-order behaviors of the cumulative numbers of observed failures and the number of defects remaining in the software under test are studied in Sections 4 and 5, respectively. In order to circumvent the shortcoming of one assumption taken in the first model, a second model is proposed by adopting a more realistic assumption in Section 6. The Markovian properties of the second model are also studied therein. The expected behaviors of the second model are studied in Section 7. The relationship between the two models proposed in this paper is analyzed from the theoretical and simulative perspectives, respectively, in Sections 8 and 9. Concluding remarks are contained in Section 10.

2. Assumptions of Model I

2.1. Imperfect debugging and mathematical notations

Model I is basically an extension of the simplifying model presented in Reference [14] to the case of imperfect debugging. Test cases comprise a test suite that is divided into a number of different classes, and are selected in accordance with a Markov chain. The selected test cases are applied to the software under test and may or may not trigger failures. The number of selected and executed test cases in a time interval follows a Poisson process. If a failure is observed, then at most one failure-causing defect is detected and removed from the software under test. However two additional possibilities are admissible. The first possibility is that the debugging action fails to detect and remove any failure-causing defect. In this way the failure-causing defect remains in the software under test. The second possibility is that while no failure-causing defect is detected or removed, the debugging action introduces a new defect into the software under test. In this way the number of remaining software defects is increased by 1.

In order to formulate the proposed model, the following mathematical notations are adopted.

C : the input domain or the given test suite of the software under test.

C_j : the j th class of C , $j = 1, 2, \dots, m$; a class comprises a number of distinct test cases,¹ and it holds $C = \bigcup_{j=1}^m C_j$.

A_i : the i th testing action taken since the beginning of software testing, $i = 1, 2, \dots; \forall i, A_i \in \{1, 2, \dots, m\}$, and $A_i = j$ means that the i th testing action picks up a test case from C_j .

Z_i : indicator of failure revealed by A_i .

M_i : total number of failures revealed by A_1, A_2, \dots, A_i .

N_i : the number of defects remaining in the software under test after the action A_i is finished.

$A(t)$: the testing action or the test case that is executed at time instant t , $t \in [0, \infty)$; $A(t) \in \{1, 2, \dots, m\}$, and $A(t) = j$ means that a test case picked up from C_j is executed at time instant t .

$M(t)$: total number of failures revealed during the time interval $[0, t]$.

$N(t)$: the number of defects remaining in the software under test at time instant t .

$H(t)$: total number of testing actions taken during the time interval $[0, t)$, excluding the first testing action that is taken at the beginning of testing.

N : total number of defects remaining in the software under test at the beginning of testing; it holds $N_0 = N$.

λ : testing intensity.

p_{ij} : transition probability from state i to state j in a Markov chain.

2.2. Model assumptions

The following assumptions are taken in Model I.

- (1) The input domain or the given test suite, C , of the software under test comprises m classes of test cases, C_1, C_2, \dots, C_m , which may or may not be disjoint; that is, $C = \bigcup_{j=1}^m C_j$; C_1, C_2, \dots, C_m do not change in the course of software testing.
- (2) The software under test contains N defects at the beginning of testing.
- (3) Each test case picked up by an action or from a class may or may not reveal a failure; let

$$Z_i = \begin{cases} 1 & \text{if the } i\text{th action } A_i \text{ reveals a failure,} \\ 0 & \text{otherwise.} \end{cases}$$

- (4) Upon a failure being revealed, the execution of the current test case terminates; at most one failure-causing defect is removed immediately from the software under test, and a new defect may or may be introduced; more specifically, it holds

$$N_k = \begin{cases} N_{k-1}, & \text{with probability } 1 - p - q \\ N_{k-1} + 1, & \text{with probability } p \\ N_{k-1}, & \text{with probability } q \end{cases}$$

$$0 \leq p, q \leq 1, 0 \leq p + q \leq 1.$$

- (5) A next test case is selected and executed after the current action is finished; the sequence $\{A_1, A_2, \dots, A_i, A_{i+1}, \dots\}$ forms a Markov chain with

$$\Pr\{A_{i+1} = l | A_i = k\} = p_{kl}.$$

- (6) During the time interval $[0, t)$ a total of $H(t) + 1$ test cases are selected; the first one is taken at the beginning of software testing ($t = 0$) and $H(t)$ forms a Poisson process with parameter λ , or

$$\Pr\{H(t) = k\} = \frac{(\lambda t)^k}{k!} e^{-\lambda t}, \quad k = 0, 1, 2, \dots$$

where λ is referred to as the testing intensity; each testing action, including the first one, takes an exponentially distributed length of time with parameter λ .

¹ An alternative term adopted in the literature is “subdomain”.

- (7) The first i actions or test cases detect M_i defects and the testing process during the time interval $[0, t]$ detects $M(t)$ defects; that is,

$$M_i = \sum_{k=1}^i Z_k, \quad \text{with } M_0 = 0;$$

$$M(t) = \sum_{k=1}^{H(t)} Z_k, \quad \text{with } M(0) = 0, Z_0 = 0.$$

The $(H(t) + 1)$ st action is being executed at time t , and not finished yet. Therefore, $Z_{H(t)+1}$ is not counted into $M(t)$.

- (8) Let N_k denote the number of defects remaining in the software after the k th test. The probability of a test case revealing a failure is proportional to the number of defects remaining in the software under test; that is,

$$\Pr\{Z_i = 1 | A_i = j, N_{i-1} = k\} = k\theta_j,$$

$$\Pr\{Z_i = 0 | A_i = j, N_{i-1} = k\} = 1 - k\theta_j.$$

- (9) $\{M_1, N_1, M_2, N_2, \dots\}$ and $\{A_1, A_2, \dots\}$ are conditionally independent of each other as follows,

$$\Pr\{M_1, N_1 | A_1; M_0, N_0\} = \Pr\{M_1, N_1 | A_1\}.$$

And for $i > 1$,

$$\Pr\{M_i, N_i | A_i; M_{i-1}, N_{i-1}, A_{i-1}; \dots; M_1, N_1, A_1; M_0, N_0\} = \Pr\{M_i, N_i | A_i; M_{i-1}, N_{i-1}\}, \quad (2.1)$$

$$\Pr\{A_i | M_{i-1}, N_{i-1}, A_{i-1}; \dots; M_1, N_1, A_1; M_0, N_0\} = \Pr\{A_i | A_{i-1}\}. \quad (2.2)$$

- (10) The process $\{M_i, N_i; i = 0, 1, \dots\}$ is independent of the Poisson process $\{H(t), t \geq 0\}$; more accurately, it holds

$$\begin{aligned} \Pr\{M_0 = 0, N_0 = N, M_1 = k_1, N_1 = n_1, \dots, M_i = k_i, N_i = n_i | H(t) = i\} \\ = \Pr\{M_0 = 0, N_0 = N, M_1 = k_1, N_1 = n_1, \dots, M_i = k_i, N_i = n_i\}. \end{aligned}$$

- (11) The process $\{A_i; i = 1, 2, \dots\}$ is independent of the Poisson process $\{H(t), t \geq 0\}$; more accurately, it holds

$$\Pr\{A_1 = j_1, A_2 = j_2, \dots, A_i = j_i | H(t) = i\} = \Pr\{A_1 = j_1, A_2 = j_2, \dots, A_i = j_i\}.$$

- (12) The first testing action is selected according to the probability distribution $\{p_1, p_2, \dots, p_m\}$, that is, $\Pr\{A_1 = j\} = p_j, j = 1, 2, \dots, m$.

2.3. Remarks

- (1) The different classes of test cases may or may not overlap. When a test case is selected from a given class, we usually suppose that it is selected from the given class at random. However no specific selection mechanism is further given in the above assumptions.
- (2) An action actually comprises several subactions including selection, initialization, execution, termination, check, and debugging, and the software testing process can be described as follows. At the beginning of software testing the first action or test case is selected. This is finished instantaneously. Then the required test initialization is conducted. In general, the test initialization for an action may include setting initial state for the software under test and so on. The test initialization is finished instantaneously. This is followed by execution of the selected action or test case that takes an exponentially distributed length of time with parameter λ . The execution of the action then terminates. The termination is finished instantaneously. After termination, a test oracle or the tester checks or decides if a failure is revealed. The check is finished instantaneously. If no failure is revealed, then the current testing action is finished. If a failure is revealed, then a debugging subaction tries to locate and remove the failure-causing defect. The debugging subaction is finished with the consequence that the failure-causing defect may or may not be removed, and a new defect may or may not be introduced into the software under test. This completes the first action. Then the next action starts with another test case being selected, followed by the required test initialization and so on.
- (3) $A(t) = j$ means two things. First, a test case was selected from C_j prior to time t ; second, the selected test case is being conducted at time t . Assumption (5) describes how various actions are taken.
- (4) Assumption (4) describes the behavior of imperfect debugging. In the case of perfect debugging, it holds $N_i = N - M_i$. Of course, this relationship no longer holds in the case of imperfect debugging. Model I reduces to the simplifying model of Reference [14] in the case of $p = 0, q = 1$.
- (5) Assumption (9) mathematically characterizes the conditional independence between $\{M_1, N_1, M_2, N_2, \dots\}$ and $\{A_1, A_2, \dots\}$. It actually implies that given the current action A_1 (say) being executed, $\{M_1, N_1, M_2, N_2, \dots\}$ behaves as a Markov chain. Further, the selection of a next action depends on the current action only.
- (6) Simply speaking, the software testing process is determined by $\{\theta_1, \theta_2, \dots, \theta_m\}, (p_{ij})_{m \times m}, \{p, q\}, \{\Pr\{A_1 = j\}, j = 1, 2, \dots, m\}$ and N . We can say that these parameters fully define a testing strategy mathematically.

3. Markovian properties of Model I

Obviously, the three sequences $\{A_1, A_2, \dots\}$, $\{M_1, M_2, \dots\}$ and $\{N_1, N_2, \dots\}$ interact with one and another. Note that $\{A_1, A_2, \dots\}$ describes various actions taken in the software testing process; $\{M_1, M_2, \dots\}$ describes the observed software reliability behavior, and $\{N_1, N_2, \dots\}$ describes the behavior of the number of remaining software defects. In order to characterize the quantitative relationship between the software testing process and the delivered software reliability, it is interesting and necessary to examine how these sequences interact from a mathematical perspective. This is summarized in Propositions 3.1–3.4.

Proposition 3.1. *With the assumptions presented in Section 2.2, it holds*

1. $\{(M_i, N_i, A_i); i = 1, 2, \dots\}$ is a Markov chain;
2. $\{(M_i, N_i, A_{i+1}); i = 1, 2, \dots\}$ is a Markov chain;
3. $\{(M_{i+1}, N_i, A_{i+1}); i = 1, 2, \dots\}$ is not a Markov chain,

where $M_0 = 0, N_0 = N, (M_0, N_0, A_0) = (M_0, N_0)$.

Note that there is not A_0 in practice. Using A_0 is just for the convenience of mathematical notation.

Proof. Proof of 1:

$$\begin{aligned}
 & \Pr\{M_i, N_i, A_i; M_{i-1}, N_{i-1}, A_{i-1}; \dots; M_1, N_1, A_1; M_0, N_0\} \\
 &= \Pr\{M_i, N_i, A_i; M_{i-1}, N_{i-1}, A_{i-1}; \dots; M_1, N_1, A_1; M_0, N_0\} \Pr\{A_i; M_{i-1}, N_{i-1}, A_{i-1}; \dots; M_1, N_1, A_1; M_0, A_0\} \\
 &= \Pr\{M_i, N_i, A_i; M_{i-1}, N_{i-1}, A_{i-1}\} \Pr\{A_i; M_{i-1}, N_{i-1}, A_{i-1}; \dots; M_1, N_1, A_1; M_0, A_0\} \\
 &= \Pr\{M_i, N_i, A_i; M_{i-1}, N_{i-1}, A_{i-1}\} \Pr\{A_i | M_{i-1}, N_{i-1}, A_{i-1}; \dots; M_1, N_1, A_1; M_0, N_0\} \\
 &\quad \times \Pr\{M_{i-1}, N_{i-1}, A_{i-1}; \dots; M_1, N_1, A_1; M_0, N_0\} \\
 &= \frac{\Pr\{M_i, N_i, A_i; M_{i-1}, N_{i-1}, A_{i-1}\}}{\Pr\{A_i; M_{i-1}, N_{i-1}, A_{i-1}\}} \Pr\{A_i | A_{i-1}\} \Pr\{M_{i-1}, N_{i-1}, A_{i-1}; \dots; M_1, N_1, A_1; M_0, N_0\} \\
 &= \Pr\{N_i | M_i, A_i; M_{i-1}, N_{i-1}, A_{i-1}\} \frac{\Pr\{M_i, A_i; M_{i-1}, N_{i-1}, A_{i-1}\}}{\Pr\{A_i; M_{i-1}, N_{i-1}, A_{i-1}\}} \Pr\{A_i | A_{i-1}\} \\
 &\quad \times \Pr\{M_{i-1}, N_{i-1}, A_{i-1}; \dots; M_1, N_1, A_1; M_0, N_0\} \\
 &= \Pr\{N_i | M_i, A_i; M_{i-1}, N_{i-1}, A_{i-1}\} \Pr\{M_i | A_i; M_{i-1}, N_{i-1}, A_{i-1}\} \Pr\{A_i | A_{i-1}\} \\
 &\quad \times \Pr\{M_{i-1}, N_{i-1}, A_{i-1}; \dots; M_1, N_1, A_1; M_0, N_0\}.
 \end{aligned}$$

We have

$$\begin{aligned}
 & \Pr\{M_i, N_i, A_i | M_{i-1}, N_{i-1}, A_{i-1}; \dots; M_1, N_1, A_1; M_0, N_0\} \\
 &= \Pr\{N_i | M_i, A_i; M_{i-1}, N_{i-1}, A_{i-1}\} \Pr\{M_i | A_i; M_{i-1}, N_{i-1}, A_{i-1}\} \Pr\{A_i | A_{i-1}\},
 \end{aligned}$$

thus $\{(M_i, N_i, A_i); i = 1, 2, \dots\}$ is a Markov chain.

Proof of 2: According to Eqs. (2.1) and (2.2) we have

$$\begin{aligned}
 & \Pr\{A_{i+2}, M_{i+1}, N_{i+1}; A_{i+1}, M_i, N_i; \dots; A_1, M_0, N_0; A_0\} \\
 &= \Pr\{A_{i+2} | M_{i+1}, N_{i+1}, A_{i+1}; M_i, N_i, A_i; \dots; M_0, N_0, A_0\} \Pr\{M_{i+1}, N_{i+1}, A_{i+1}; M_i, N_i, A_i; \dots; M_0, N_0, A_0\} \\
 &= \Pr\{A_{i+2} | A_{i+1}\} \Pr\{M_{i+1}, N_{i+1}, A_{i+1}; M_i, N_i, A_i; \dots; M_0, N_0, A_0\} \\
 &= \Pr\{A_{i+2} | A_{i+1}\} \Pr\{M_{i+1}, N_{i+1} | A_{i+1}; \dots; M_0, N_0, A_0\} \Pr\{A_{i+1}, M_i, N_i, A_i; \dots; M_0, N_0, A_0\} \\
 &= \Pr\{A_{i+2} | A_{i+1}\} \Pr\{M_{i+1}, N_{i+1} | A_{i+1}; M_i, N_i\} \Pr\{M_i, N_i, A_{i+1}; M_{i-1}, N_{i-1}, A_i; \dots; M_0, N_0, A_1; A_0\}.
 \end{aligned}$$

Therefore,

$$\Pr\{M_{i+1}, N_{i+1}, A_{i+2} | M_i, N_i, A_{i+1}; \dots; M_0, N_0, A_1; A_0\} = \Pr\{A_{i+2} | A_{i+1}\} \Pr\{M_{i+1}, N_{i+1} | A_{i+1}; M_i, N_i\},$$

thus $\{(M_i, N_i, A_{i+1}); i = 1, 2, \dots\}$ is a Markov chain.

Proof of 3: We give the following example to show that $\{(M_{i+1}, N_i, A_{i+1}); i = 1, 2, \dots\}$ is not necessarily a Markov chain.

Let $N = 10, m = 2, \theta_1 = \frac{1}{20}, \theta_2 = \frac{1}{30}, p = \frac{2}{7}, q = \frac{3}{7}, \Pr\{A_1 = 1\} = \frac{1}{3}$,

$$\Pr\{A_1 = 2\} = \frac{2}{3}, \quad P = \begin{pmatrix} \frac{3}{5} & \frac{2}{5} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

For the case of $M_1 = 0$, we have

$$\begin{aligned} & \Pr\{M_3 = 2, N_2 = 10, A_3 = 2 | M_2 = 1, N_1 = 10, A_2 = 1; M_1 = 0, N_0 = 10, A_1 = 1\} \\ &= \Pr\{M_3 = 2 | A_3 = 2; M_2 = 1, N_2 = 10\} p_{12} \Pr\{N_2 = 10 | M_2 = 1, A_2 = 1; M_1 = 0, N_1 = 10\} \\ &= N_2 \theta_2 p_{12} p \\ &= \frac{4}{105}. \end{aligned}$$

When $M_1 = 1$, we have

$$\begin{aligned} & \Pr\{M_3 = 2, N_2 = 10, A_3 = 2 | M_2 = 1, N_1 = 10, A_2 = 1; M_1 = 1, N_0 = 10, A_1 = 1\} \\ &= \Pr\{M_3 = 2 | A_3 = 2; M_2 = 1, N_2 = 10\} p_{12} \Pr\{N_2 = 10 | M_2 = 1, A_2 = 1; M_1 = 1, N_1 = 10\} \\ &= N_2 \theta_2 p_{12} \\ &= \frac{2}{15}. \end{aligned}$$

Obviously, the probability of the event $(M_3 = 2, N_2 = 10, A_3 = 2)$ depends on the initial state of the process, which indicates that the process is not Markovian. \square

Based on the example in Proposition 3.1, the process $\{(M_{i+1}, N_i, A_{i+1}); i = 1, 2, \dots\}$ is not a Markov chain in general. If the conditional probabilities satisfy that $\Pr\{N_i | M_i, A_i; M_{i-1}, N_{i-1}\} = \Pr\{N_i | M_i, A_i, N_{i-1}\}$ for any $i \geq 2$, then $\{(M_{i+1}, N_i, A_{i+1}); i = 1, 2, \dots\}$ is a Markov chain.

Proposition 3.2. $\{(M(t), N(t), A(t)); t \geq 0\}$ is a Markov process.

Proof. From Proposition 3.1 we know that $\{(M_i, N_i, A_{i+1}); i = 1, 2, \dots\}$ is a Markov chain. With the help of assumptions (10) and (11), we have $\{(M_i, N_i, A_{i+1}); i = 1, 2, \dots\}$ is independent of $\{H(t); t \geq 0\}$, and $(M(t), N(t), A(t)) = (M_{H(t)}, N_{H(t)}, A_{H(t)+1})$. Thus, we conclude that $\{(M(t), N(t), A(t)); t \geq 0\}$ is a Markov process based on the Reference [17]. \square

Proposition 3.3. $\{(N_i, A_{i+1}); i = 1, 2, \dots\}$ is a Markov chain.

Proof.

$$\begin{aligned} \Pr\{N_i, A_{i+1} | N_{i-1}, A_i; \dots; N_1, A_2; N_0, A_1\} &= \Pr\{N_i | N_{i-1}, A_i; \dots; N_1, A_2; N_0, A_1\} \\ &\times \Pr\{A_{i+1} | N_i; N_{i-1}, A_i; \dots; N_1, A_1; A_0\}. \end{aligned} \quad (3.3)$$

From (2.1) we conclude

$$\frac{\Pr\{M_i, N_i, A_i; M_{i-1}, N_{i-1}, A_{i-1}; \dots; M_0, N_0\}}{\Pr\{A_i; M_{i-1}, N_{i-1} \dots; M_0, N_0\}} = \frac{\Pr\{M_i, N_i, A_i; M_{i-1}, N_{i-1}\}}{\Pr\{A_i; M_{i-1}, N_{i-1}\}},$$

then we have

$$\frac{\sum_{M_{i-1}, \dots, M_0} \Pr\{M_i, N_i, A_i; M_{i-1}, N_{i-1}, A_{i-1}; \dots; M_0, N_0\}}{\sum_{M_{i-1}, \dots, M_0} \Pr\{A_i; M_{i-1}, N_{i-1} \dots; M_0, N_0\}} = \frac{\sum_{M_{i-1}, \dots, M_0} \Pr\{M_i, N_i, A_i; M_{i-1}, N_{i-1}\}}{\sum_{M_{i-1}, \dots, M_0} \Pr\{A_i; M_{i-1}, N_{i-1}\}}.$$

That is,

$$\frac{\Pr\{M_i, N_i, A_i; N_{i-1}, A_{i-1}; \dots; N_0\}}{\Pr\{A_i; N_{i-1} \dots; N_0\}} = \frac{\Pr\{M_i, N_i, A_i; N_{i-1}\}}{\Pr\{A_i, N_{i-1}\}}.$$

Taking summation over M_i in the above equation, we obtain

$$\frac{\Pr\{N_i, A_i; N_{i-1}, A_{i-1}; \dots; N_0\}}{\Pr\{A_i; N_{i-1} \dots; N_0\}} = \frac{\Pr\{N_i, A_i; N_{i-1}\}}{\Pr\{A_i, N_{i-1}\}},$$

$$\Pr\{N_i | N_{i-1}, A_i; \dots; N_1, A_2; N_0, A_1\} = \Pr\{N_i | N_{i-1}, A_i\}. \quad (3.4)$$

From (2.2), by the same argument, we derive that

$$\Pr\{A_{i+1} | N_i, A_i; \dots; N_1, A_1; A_0\} = \Pr\{A_{i+1} | A_i\}. \quad (3.5)$$

From the Eqs. (3.3), (3.4) and (3.5), we obtain

$$\begin{aligned}\Pr\{N_i, A_{i+1}|N_{i-1}, A_i; \dots; N_1, A_2; N_0, A_1\} &= \Pr\{N_i|N_{i-1}, A_i; \dots; N_1, A_2; N_0, A_1\} \Pr\{A_{i+1}|N_i; N_{i-1}, A_i; \dots; N_1, A_1; A_0\} \\ &= \Pr\{N_i|N_{i-1}, A_i\} \Pr\{A_{i+1}|A_i\}.\end{aligned}$$

Therefore, $\{(N_i, A_{i+1}); i = 1, 2, \dots\}$ is a Markov chain. \square

Similarly, we can obtain the following result.

Proposition 3.4. $\{(N_i, A_i); i = 1, 2, \dots\}$ is a Markov chain.

4. Expected behaviors of Model I

In the case of perfect debugging [14], it is shown that the software state defined in terms of the initial number of software defects and the expected cumulative number of observed failures demonstrates the dynamics of a linear system. In this section we show this observation may or may not be valid in the case of imperfect debugging. The expected behavior of the dynamics of the system depends on the value of p and q .

4.1. Recursive formulae

Let

$$\begin{aligned}\alpha_k(i) &= E[M_k, A_k = i] = E[M_k|A_k = i] \Pr\{A_k = i\}, \quad \alpha_k = (\alpha_k(1), \alpha_k(2), \dots, \alpha_k(m)), \\ \beta_k(i) &= E[N_k, A_k = i] = E[N_k|A_k = i] \Pr\{A_k = i\}, \quad \beta_k = (\beta_k(1), \beta_k(2), \dots, \beta_k(m)), \\ \Theta &= \text{diag}(\theta_1, \theta_2, \dots, \theta_m), \quad P = (p_{ij})_{i,j=1,2,\dots,m}, \quad \mathbf{1} = (1, 1, \dots, 1),\end{aligned}$$

where P is a probability transition matrix satisfying $P \cdot \mathbf{1}^T = \mathbf{1}^T$, and p_{ij} is defined in Section 2.2. From the definition of the conditional probability and Markovian properties of the software testing process

$$\begin{aligned}\Pr\{M_i, N_i, A_i; M_{i-1}, N_{i-1}, A_{i-1}; \dots; M_1, N_1, A_1; M_0, N_0\} \\ &= \Pr\{M_i, N_i, A_i|M_{i-1}, N_{i-1}, A_{i-1}\} \\ &= \Pr\{N_i|M_i, A_i; M_{i-1}, N_{i-1}, A_{i-1}\} \Pr\{M_i|A_i; M_{i-1}, N_{i-1}, A_{i-1}\} \Pr\{A_i|A_{i-1}\},\end{aligned}$$

we have the following precise expression with given parameters:

$$\begin{aligned}\Pr\{M_i = m_i, N_i = n_i, A_i = a_i|M_{i-1} = m_{i-1}, N_{i-1} = n_{i-1}, A_{i-1} = a_{i-1}\} \\ &= \Pr\{N_i = n_i|M_i = m_i, A_i = a_i; M_{i-1} = m_{i-1}, N_{i-1} = n_{i-1}, A_{i-1} = a_{i-1}\} \\ &\quad \times \Pr\{M_i = m_i|A_i = a_i; M_{i-1} = m_{i-1}, N_{i-1} = n_{i-1}, A_{i-1} = a_{i-1}\} \Pr\{A_i = a_i|A_{i-1} = a_{i-1}\} \\ &= \begin{cases} (1 - n_{i-1}\theta_{a_i})p_{a_{i-1}a_i}, & n_i = n_{i-1}, m_i = m_{i-1}; \\ (1 - p - q) \cdot n_{i-1}\theta_{a_i} \cdot p_{a_{i-1}a_i}, & n_i = n_{i-1}, m_i = m_{i-1} + 1; \\ q \cdot n_{i-1}\theta_{a_i} \cdot p_{a_{i-1}a_i}, & n_i = n_{i-1} - 1, m_i = m_{i-1} + 1; \\ p \cdot n_{i-1}\theta_{a_i} \cdot p_{a_{i-1}a_i}, & n_i = n_{i-1} + 1, m_i = m_{i-1} + 1. \end{cases}\end{aligned}$$

With the above notations, we have the following recursive formulae.

Proposition 4.1. Under the assumptions of Section 2.2, it holds

$$\begin{aligned}\alpha_k &= \alpha_{k-1}P + \beta_{k-1}P\Theta, \\ \beta_k &= \beta_{k-1}P(I + (p - q)\Theta).\end{aligned}$$

Proof.

$$\begin{aligned}E[M_k, A_k = i, N_k = j] &= \sum_{\ell=0}^{\infty} \ell \Pr\{M_k = \ell, A_k = i, N_k = j\} \\ &= \sum_{\ell=1}^{\infty} \ell \sum_{i'=1}^m \sum_{j'=j-1}^{j+1} \Pr\{M_k = \ell, A_k = i, N_k = j; M_{k-1} = \ell, A_{k-1} = i', N_{k-1} = j'\} \\ &\quad + \sum_{\ell=1}^{\infty} \ell \sum_{i'=1}^m \sum_{j'=j-1}^{j+1} \Pr\{M_k = \ell, A_k = i, N_k = j; M_{k-1} = \ell - 1, A_{k-1} = i', N_{k-1} = j'\} \\ &= \sum_{\ell=1}^{\infty} \ell \sum_{i'=1}^m \sum_{j'=j-1}^{j+1} \Pr\{M_k = \ell, A_k = i, N_k = j|M_{k-1} = \ell, A_{k-1} = i', N_{k-1} = j'\}\end{aligned}$$

$$\begin{aligned}
& \times \Pr\{M_{k-1} = \ell, A_{k-1} = i', N_{k-1} = j'\} \\
& + \sum_{\ell=1}^{\infty} \ell \sum_{i'=1}^m \sum_{j'=j-1}^{j+1} \Pr\{M_k = \ell, A_k = i, N_k = j | M_{k-1} = \ell - 1, A_{k-1} = i', N_{k-1} = j'\} \\
& \times \Pr\{M_{k-1} = \ell - 1, A_{k-1} = i', N_{k-1} = j'\} \\
& = \sum_{\ell=1}^{\infty} \ell \sum_{i'=1}^m (1 - j\theta_i) p_{i'i} \Pr\{M_{k-1} = \ell, A_{k-1} = i', N_{k-1} = j\} \\
& + \sum_{\ell=1}^{\infty} \ell \sum_{i'=1}^m (1 - p - q) j\theta_i p_{i'i} \Pr\{M_{k-1} = \ell - 1, A_{k-1} = i', N_{k-1} = j\} \\
& + \sum_{\ell=1}^{\infty} \ell \sum_{i'=1}^m p(j-1)\theta_i p_{i'i} \Pr\{M_{k-1} = \ell - 1, A_{k-1} = i', N_{k-1} = j-1\} \\
& + \sum_{\ell=1}^{\infty} \ell \sum_{i'=1}^m q(j+1)\theta_i p_{i'i} \Pr\{M_{k-1} = \ell - 1, A_{k-1} = i', N_{k-1} = j+1\}. \\
\\
E[M_k, A_k = i] &= \sum_{j=0}^{\infty} \sum_{\ell=1}^{\infty} \ell \sum_{i'=1}^m (1 - j\theta_i) p_{i'i} \Pr\{M_{k-1} = \ell, A_{k-1} = i', N_{k-1} = j\} \\
& + \sum_{j=0}^{\infty} \sum_{\ell=1}^{\infty} \ell \sum_{i'=1}^m (1 - p - q) j\theta_i p_{i'i} \Pr\{M_{k-1} = \ell - 1, A_{k-1} = i', N_{k-1} = j\} \\
& + \sum_{j=1}^{\infty} \sum_{\ell=1}^{\infty} \ell \sum_{i'=1}^m p(j-1)\theta_i p_{i'i} \Pr\{M_{k-1} = \ell - 1, A_{k-1} = i', N_{k-1} = j-1\} \\
& + \sum_{j=0}^{\infty} \sum_{\ell=1}^{\infty} \ell \sum_{i'=1}^m q(j+1)\theta_i p_{i'i} \Pr\{M_{k-1} = \ell - 1, A_{k-1} = i', N_{k-1} = j+1\} \\
& = \sum_{j=0}^{\infty} \sum_{\ell=1}^{\infty} \ell \sum_{i'=1}^m (1 - j\theta_i) p_{i'i} \Pr\{M_{k-1} = \ell, A_{k-1} = i', N_{k-1} = j\} \\
& + \sum_{j=0}^{\infty} \sum_{\ell=1}^{\infty} \ell \sum_{i'=1}^m (1 - p - q) j\theta_i p_{i'i} \Pr\{M_{k-1} = \ell - 1, A_{k-1} = i', N_{k-1} = j\} \\
& + \sum_{j=0}^{\infty} \sum_{\ell=1}^{\infty} \ell \sum_{i'=1}^m p j\theta_i p_{i'i} \Pr\{M_{k-1} = \ell - 1, A_{k-1} = i', N_{k-1} = j\} \\
& + \sum_{j=0}^{\infty} \sum_{\ell=1}^{\infty} \ell \sum_{i'=1}^m q j\theta_i p_{i'i} \Pr\{M_{k-1} = \ell - 1, A_{k-1} = i', N_{k-1} = j\} \\
& = \sum_{j=0}^{\infty} \sum_{\ell=1}^{\infty} \ell \sum_{i'=1}^m (1 - j\theta_i) p_{i'i} \Pr\{M_{k-1} = \ell, A_{k-1} = i', N_{k-1} = j\} \\
& + \sum_{j=0}^{\infty} \sum_{\ell=1}^{\infty} \ell \sum_{i'=1}^m j\theta_i p_{i'i} \Pr\{M_{k-1} = \ell - 1, A_{k-1} = i', N_{k-1} = j\} \\
& = \sum_{j=0}^{\infty} \sum_{\ell=1}^{\infty} \ell \sum_{i'=1}^m (1 - j\theta_i) p_{i'i} \Pr\{M_{k-1} = \ell, A_{k-1} = i', N_{k-1} = j\} \\
& + \sum_{j=0}^{\infty} \sum_{\ell=0}^{\infty} (\ell + 1) \sum_{i'=1}^m j\theta_i p_{i'i} \Pr\{M_{k-1} = \ell, A_{k-1} = i', N_{k-1} = j\} \\
& = \sum_{j=0}^{\infty} \sum_{\ell=0}^{\infty} \ell \sum_{i'=1}^m (1 - j\theta_i) p_{i'i} \Pr\{M_{k-1} = \ell, A_{k-1} = i', N_{k-1} = j\} \\
& + \sum_{j=0}^{\infty} \sum_{\ell=0}^{\infty} \ell \sum_{i'=1}^m j\theta_i p_{i'i} \Pr\{M_{k-1} = \ell, A_{k-1} = i', N_{k-1} = j\}
\end{aligned}$$

$$\begin{aligned}
& + \sum_{j=0}^{\infty} \sum_{\ell=0}^{\infty} \sum_{i'=1}^m j \theta_i p_{i'} \Pr\{M_{k-1} = \ell, A_{k-1} = i', N_{k-1} = j\} \\
& = \sum_{j=0}^{\infty} \sum_{\ell=0}^{\infty} \sum_{i'=1}^m \ell p_{i'} \Pr\{M_{k-1} = \ell, A_{k-1} = i', N_{k-1} = j\} \\
& \quad + \sum_{j=0}^{\infty} \sum_{\ell=0}^{\infty} \sum_{i'=1}^m j \theta_i p_{i'} \Pr\{M_{k-1} = \ell, A_{k-1} = i', N_{k-1} = j\} \\
& = \sum_{i'=1}^m p_{i'} E[M_{k-1}, A_{k-1} = i'] + \sum_{j=0}^{\infty} \sum_{\ell=0}^{\infty} \sum_{i'=1}^m j \theta_i p_{i'} \Pr\{M_{k-1} = \ell, A_{k-1} = i', N_{k-1} = j\} \\
& = \sum_{i'=1}^m p_{i'} E[M_{k-1}, A_{k-1} = i'] + \sum_{i'=1}^m \sum_{j=0}^{\infty} j \theta_i p_{i'} \Pr\{A_{k-1} = i', N_{k-1} = j\} \\
& = \sum_{i'=1}^m p_{i'} E[M_{k-1}, A_{k-1} = i'] + \theta_i \sum_{i'=1}^m p_{i'} E[N_{k-1}, A_{k-1} = i'].
\end{aligned}$$

That is,

$$\alpha_k(i) = \sum_{i'=1}^m p_{i'} \alpha_{k-1}(i') + \theta_i \sum_{i'=1}^m p_{i'} \beta_{k-1}(i'). \quad (4.6)$$

$$\begin{aligned}
E[N_k, A_k = i, M_k = \ell] & = \sum_{j=0}^{\infty} j \Pr\{M_k = \ell, A_k = i, N_k = j\} \\
& = \sum_{j=0}^{\infty} j \sum_{i'=1}^m \sum_{j'=j-1}^{j+1} \Pr\{M_k = \ell, A_k = i, N_k = j; M_{k-1} = \ell, A_{k-1} = i', N_{k-1} = j'\} \\
& \quad + \sum_{j=0}^{\infty} j \sum_{i'=1}^m \sum_{j'=j-1}^{j+1} \Pr\{M_k = \ell, A_k = i, N_k = j; M_{k-1} = \ell - 1, A_{k-1} = i', N_{k-1} = j'\} \\
& = \sum_{j=0}^{\infty} j \sum_{i'=1}^m \sum_{j'=j-1}^{j+1} \Pr\{M_k = \ell, A_k = i, N_k = j; M_{k-1} = \ell, A_{k-1} = i', N_{k-1} = j'\} \\
& \quad + \sum_{j=0}^{\infty} j \sum_{i'=1}^m \sum_{j'=j-1}^{j+1} \Pr\{M_k = \ell, A_k = i, N_k = j; M_{k-1} = \ell - 1, A_{k-1} = i', N_{k-1} = j'\} \\
& = \sum_{j=0}^{\infty} j \sum_{i'=1}^m (1 - j \theta_i) p_{i'} \Pr\{M_{k-1} = \ell, A_{k-1} = i', N_{k-1} = j\} \\
& \quad + \sum_{j=0}^{\infty} j \sum_{i'=1}^m (1 - p - q) j \theta_i p_{i'} \Pr\{M_{k-1} = \ell - 1, A_{k-1} = i', N_{k-1} = j\} \\
& \quad + \sum_{j=0}^{\infty} j \sum_{i'=1}^m p(j - 1) \theta_i p_{i'} \Pr\{M_{k-1} = \ell - 1, A_{k-1} = i', N_{k-1} = j - 1\} \\
& \quad + \sum_{j=0}^{\infty} j \sum_{i'=1}^m q(j + 1) \theta_i p_{i'} \Pr\{M_{k-1} = \ell - 1, A_{k-1} = i', N_{k-1} = j + 1\}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
E[N_k, A_k = i] & = \sum_{\ell=0}^{\infty} \sum_{j=0}^{\infty} \sum_{i'=1}^m j (1 - j \theta_i) p_{i'} \Pr\{M_{k-1} = \ell, A_{k-1} = i', N_{k-1} = j\} \\
& \quad + \sum_{\ell=0}^{\infty} \sum_{j=0}^{\infty} \sum_{i'=1}^m j (1 - p - q) j \theta_i p_{i'} \Pr\{M_{k-1} = \ell - 1, A_{k-1} = i', N_{k-1} = j\} \\
& \quad + \sum_{\ell=0}^{\infty} \sum_{j=0}^{\infty} (j + 1) \sum_{i'=1}^m p j \theta_i p_{i'} \Pr\{M_{k-1} = \ell - 1, A_{k-1} = i', N_{k-1} = j\}
\end{aligned}$$

$$\begin{aligned}
& + \sum_{\ell=0}^{\infty} \sum_{j=0}^{\infty} (j-1) \sum_{i'=1}^m qj\theta_i p_{i'i} \Pr\{M_{k-1} = \ell - 1, A_{k-1} = i', N_{k-1} = j\} \\
= & \sum_{\ell=0}^{\infty} \sum_{j=1}^{\infty} j \sum_{i'=1}^m (1-j\theta_i) p_{i'i} \Pr\{M_{k-1} = \ell, A_{k-1} = i', N_{k-1} = j\} \\
& + \sum_{\ell=0}^{\infty} \sum_{j=1}^{\infty} j \sum_{i'=1}^m (1-p-q)j\theta_i p_{i'i} \Pr\{M_{k-1} = \ell - 1, A_{k-1} = i', N_{k-1} = j\} \\
& + \sum_{\ell=0}^{\infty} \sum_{j=1}^{\infty} j \sum_{i'=1}^m pj\theta_i p_{i'i} \Pr\{M_{k-1} = \ell - 1, A_{k-1} = i', N_{k-1} = j\} \\
& + \sum_{\ell=0}^{\infty} \sum_{j=1}^{\infty} \sum_{i'=1}^m pj\theta_i p_{i'i} \Pr\{M_{k-1} = \ell - 1, A_{k-1} = i', N_{k-1} = j\} \\
& + \sum_{\ell=0}^{\infty} \sum_{j=1}^{\infty} j \sum_{i'=1}^m qj\theta_i p_{i'i} \Pr\{M_{k-1} = \ell - 1, A_{k-1} = i', N_{k-1} = j\} \\
& - \sum_{\ell=0}^{\infty} \sum_{j=1}^{\infty} \sum_{i'=1}^m qj\theta_i p_{i'i} \Pr\{M_{k-1} = \ell - 1, A_{k-1} = i', N_{k-1} = j\} \\
= & \sum_{\ell=0}^{\infty} \sum_{j=1}^{\infty} j \sum_{i'=1}^m (1-j\theta_i) p_{i'i} \Pr\{M_{k-1} = \ell, A_{k-1} = i', N_{k-1} = j\} \\
& + \sum_{\ell=0}^{\infty} \sum_{j=1}^{\infty} j \sum_{i'=1}^m j\theta_i p_{i'i} \Pr\{M_{k-1} = \ell - 1, A_{k-1} = i', N_{k-1} = j\} \\
& + \sum_{\ell=0}^{\infty} \sum_{j=1}^{\infty} \sum_{i'=1}^m (p-q)j\theta_i p_{i'i} \Pr\{M_{k-1} = \ell - 1, A_{k-1} = i', N_{k-1} = j\} \\
= & \sum_{\ell=0}^{\infty} \sum_{j=1}^{\infty} j \sum_{i'=1}^m (1-j\theta_i) p_{i'i} \Pr\{M_{k-1} = \ell, A_{k-1} = i', N_{k-1} = j\} \\
& + \sum_{\ell=0}^{\infty} \sum_{j=1}^{\infty} j \sum_{i'=1}^m j\theta_i p_{i'i} \Pr\{M_{k-1} = \ell - 1, A_{k-1} = i', N_{k-1} = j\} \\
& + \sum_{\ell=0}^{\infty} \sum_{j=1}^{\infty} \sum_{i'=1}^m (p-q)j\theta_i p_{i'i} \Pr\{M_{k-1} = \ell, A_{k-1} = i', N_{k-1} = j\} \\
= & \sum_{\ell=0}^{\infty} \sum_{j=1}^{\infty} j \sum_{i'=1}^m p_{i'i} \Pr\{M_{k-1} = \ell, A_{k-1} = i', N_{k-1} = j\} \\
& + \sum_{\ell=0}^{\infty} \sum_{j=1}^{\infty} \sum_{i'=1}^m (p-q)j\theta_i p_{i'i} \Pr\{M_{k-1} = \ell, A_{k-1} = i', N_{k-1} = j\} \\
= & [1 + (p-q)\theta_i] \sum_{i'=1}^m p_{i'i} E[N_{k-1}, A_{k-1} = i'].
\end{aligned}$$

That is,

$$\beta_k(i) = [1 + (p-q)\theta_i] \sum_{i'=1}^m p_{i'i} \beta_{k-1}(i'). \quad (4.7)$$

Rewrite (4.6) and (4.7) in vector form, we have

$$\begin{aligned}
\alpha_k &= \alpha_{k-1}P + \beta_{k-1}P\Theta, \\
\beta_k &= \beta_{k-1}P(I + (p-q)\Theta). \quad \square
\end{aligned}$$

Now let us calculate the initial value of the parameters in the recursive formulae.

$$\alpha_1(i) = E[M_1, A_1 = i] = \Pr\{M_1 = 1, A_1 = i\}$$

$$\begin{aligned}
&= \Pr\{M_1 = 1|A_1 = i\} \Pr\{A_1 = i\} \\
&= N_0\theta_i p_i = N\theta_i p_i,
\end{aligned}$$

where $N_0 = N$, $p_i = \Pr\{A_1 = i\}$, $i = 1, 2, \dots, m$.

Let $\mathbf{p}_0 = (p_1, p_2, \dots, p_m)$, we have $\mathbf{p}_0 \cdot \mathbf{1}^T = 1$, so $\alpha_1 = N\mathbf{p}_0\Theta$.

$$\begin{aligned}
\beta_1(i) &= E[N_1, A_1 = i] \\
&= N \Pr\{N_1 = N, A_1 = i\} + (N+1) \Pr\{N_1 = N+1, A_1 = i\} + (N-1) \Pr\{N_1 = N-1, A_1 = i\} \\
&= N[\Pr\{M_1 = 0, N_1 = N, A_1 = i\} + \Pr\{M_1 = 1, N_1 = N, A_1 = i\}] + (N+1) \Pr\{M_1 = 1, N_1 = N+1, A_1 = i\} \\
&\quad + (N-1) \Pr\{M_1 = 1, N_1 = N-1, A_1 = i\} \\
&= N[\Pr\{M_1 = 0|A_1 = i\} \Pr\{A_1 = i\} + \Pr\{M_1 = 1, N_1 = N|A_1 = i\}] \Pr\{A_1 = i\} \\
&\quad + (N+1) \Pr\{M_1 = 1, N_1 = N+1|A_1 = i\} \Pr\{A_1 = i\} \\
&\quad + (N-1) \Pr\{M_1 = 1, N_1 = N-1|A_1 = i\} \Pr\{A_1 = i\} \\
&= N[(1-N\theta_i) + (1-p-q)N\theta_i]p_i + (N+1)pN\theta_i p_i + (N-1)qN\theta_i p_i \\
&= Np_i(1 + (p-q)\theta_i).
\end{aligned}$$

Then

$$\beta_1 = N\mathbf{p}_0(I + (p-q)\Theta).$$

Therefore,

$$\begin{aligned}
\beta_k &= \beta_{k-1}P(I + (p-q)\Theta) \\
&= \beta_1[P(I + (p-q)\Theta)]^{k-1} \\
&= N\mathbf{p}_0(I + (p-q)\Theta)[P(I + (p-q)\Theta)]^{k-1}; \\
\alpha_k &= \alpha_{k-1}P + \beta_{k-1}P\Theta \\
&= \alpha_1 P^{k-1} + \sum_{j=0}^{k-2} N\mathbf{p}_0(I + (p-q)\Theta)[P(I + (p-q)\Theta)]^j P\Theta P^{k-2-j} \\
&= N\mathbf{p}_0\Theta + \sum_{j=0}^{k-2} N\mathbf{p}_0(I + (p-q)\Theta)[P(I + (p-q)\Theta)]^j P\Theta P^{k-2-j}.
\end{aligned}$$

4.2. The case of $p \neq q$

Theorem 4.1. Under the assumptions of Section 2.2, suppose $p \neq q$, it holds

$$\begin{aligned}
E[M(t)] &= -\frac{N}{p-q} + \frac{N\mathbf{p}_0}{p-q} e^{-\lambda t[I - (I + (p-q)\Theta)P]} \cdot \mathbf{1}^T, \\
E[N(t)] &= (p-q)E[M(t)] + N = N\mathbf{p}_0 e^{-\lambda t[I - (I + (p-q)\Theta)P]} \cdot \mathbf{1}^T.
\end{aligned}$$

Proof. The expectation of M_k is

$$\begin{aligned}
E[M_k] &= \sum_{i=1}^m E[M_k, A_k = i] = \sum_{i=1}^m \alpha_k(i) = \alpha_k \cdot \mathbf{1}^T \\
&= N\mathbf{p}_0\Theta \cdot \mathbf{1}^T + \sum_{j=0}^{k-2} N\mathbf{p}_0(I + (p-q)\Theta)[P(I + (p-q)\Theta)]^j P\Theta P^{k-2-j} \cdot \mathbf{1}^T \\
&= N\mathbf{p}_0\Theta \cdot \mathbf{1}^T + \sum_{j=0}^{k-2} N\mathbf{p}_0(I + (p-q)\Theta)[P(I + (p-q)\Theta)]^j P\Theta \cdot \mathbf{1}^T \\
&= N\mathbf{p}_0\Theta \cdot \mathbf{1}^T + \sum_{j=0}^{k-2} N\mathbf{p}_0(I + (p-q)\Theta)[P(I + (p-q)\Theta)]^j \frac{P - P(I + (p-q)\Theta)}{q-p} \cdot \mathbf{1}^T \\
&= N\mathbf{p}_0\Theta \cdot \mathbf{1}^T + N\mathbf{p}_0 \frac{(I + (p-q)\Theta)}{q-p} \sum_{j=0}^{k-2} [P(I + (p-q)\Theta)]^j - [P(I + (p-q)\Theta)]^{j+1} \cdot \mathbf{1}^T \\
&= N\mathbf{p}_0\Theta \cdot \mathbf{1}^T + N\mathbf{p}_0 \frac{(I + (p-q)\Theta)}{q-p} [I - [P(I + (p-q)\Theta)]^{k-1}] \cdot \mathbf{1}^T
\end{aligned}$$

$$\begin{aligned}
&= N\mathbf{p}_0 \left[-\frac{1}{p-q} + \frac{(I + (p-q)\Theta)}{p-q} [P(I + (p-q)\Theta)]^{k-1} \right] \cdot \mathbf{1}^T \\
&= N\mathbf{p}_0 \left[-\frac{1}{p-q} + \frac{1}{p-q} [(I + (p-q)\Theta)P]^k \right] \cdot \mathbf{1}^T.
\end{aligned} \tag{4.8}$$

Assumptions (6), (7) and (11) yield

$$\begin{aligned}
E[M(t)] &= E \left[\sum_{k=1}^{H(t)} Z_k \right] = \sum_{j=1}^{\infty} E \left[\sum_{k=1}^j Z_k \right] \Pr\{H(t) = j\} = \sum_{j=1}^{\infty} E[M_j] \frac{(\lambda t)^j}{j!} e^{-\lambda t} \\
&= N\mathbf{p}_0 \sum_{j=1}^{\infty} \left[-\frac{1}{p-q} + \frac{1}{p-q} [(I + (p-q)\Theta)P]^j \right] \cdot \mathbf{1}^T \frac{(\lambda t)^j}{j!} e^{-\lambda t} \\
&= -\frac{N}{p-q} (1 - e^{-\lambda t}) + \frac{N\mathbf{p}_0}{p-q} \sum_{j=1}^{\infty} \frac{(\lambda t (I + (p-q)\Theta)P)^j}{j!} e^{-\lambda t} \cdot \mathbf{1}^T \\
&= -\frac{N}{p-q} (1 - e^{-\lambda t}) + \frac{N\mathbf{p}_0}{p-q} [e^{-\lambda t + \lambda t (I + (p-q)\Theta)P} \cdot \mathbf{1}^T - \mathbf{1}^T e^{-\lambda t}] \\
&= -\frac{N}{p-q} + \frac{N\mathbf{p}_0}{p-q} e^{-\lambda t [I - (I + (p-q)\Theta)P]} \cdot \mathbf{1}^T.
\end{aligned}$$

The expectation of N_k is

$$\begin{aligned}
E[N_k] &= \sum_{i=1}^m E[N_k, A_k = i] = \sum_{i=1}^m \beta_k(i) = \beta_k \cdot \mathbf{1}^T \\
&= N\mathbf{p}_0 (I + (p-q)\Theta) [P(I + (p-q)\Theta)]^{k-1} \cdot \mathbf{1}^T \\
&= N\mathbf{p}_0 [(I + (p-q)\Theta)P]^k \cdot \mathbf{1}^T.
\end{aligned} \tag{4.9}$$

Combine Eqs. (4.8) and (4.9), we have $E[(q-p)M_k + N_k] = N\mathbf{p}_0 \mathbf{1}^T = N$. Furthermore,

$$\begin{aligned}
E[(q-p)M(t) + N(t)] &= \sum_{k=0}^{\infty} E[(q-p)M_k + N_k] \frac{(\lambda t)^k}{k!} e^{-\lambda t} \\
&= \sum_{k=0}^{\infty} N \frac{(\lambda t)^k}{k!} e^{-\lambda t} = N.
\end{aligned}$$

Thus, $E[N(t)] = (p-q)E[M(t)] + N = N\mathbf{p}_0 e^{-\lambda t [I - (I + (p-q)\Theta)P]} \cdot \mathbf{1}^T$. \square

A linear dynamic system can be induced from Theorem 4.1 as follows. Let

$$K = -\lambda [I - (I + (p-q)\Theta)P],$$

$$X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_m(t) \end{pmatrix}.$$

Then $e^{-\lambda t [I - (I + (p-q)\Theta)P]} \cdot \mathbf{1}^T$ is the solution to the following system of linear differential equations with the initial state $X(0) = \mathbf{1}^T$,

$$\dot{X}(t) = KX(t).$$

From Theorem 4.1 we have

$$\frac{\frac{N}{q-p} - E[M(t)]}{\frac{N}{q-p}} = (p_1, p_2, \dots, p_m) \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_m(t) \end{pmatrix},$$

$$\frac{E[N(t)]}{N} = (p_1, p_2, \dots, p_m) \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_m(t) \end{pmatrix}.$$

Especially, in the case of $p = 0$ and $q = 1$, we have $M(t) + N(t) = N$. So $N - E[M(t)] = E[N(t)]$, and

$$\frac{N - E[M(t)]}{N} = (p_1, p_2, \dots, p_m) \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_m(t) \end{pmatrix}.$$

This is consistent with the result in Reference [15].

Note that $\frac{E[N(t)]}{N}$ represents the expected ratio of defects that have not been removed by time t . $x_i(t)$ should be interpreted as the expected ratio of defects that have not been removed by an arbitrary sequence of actions starting with action i by time t . There are two kinds of probabilistic uncertainty associated with a sequence of actions. First, it is not certain how many actions are applied during the time interval $(0, t)$. Second, it is not certain what actions the sequence comprises and in what order these actions are applied.

Overall, we obtain a linear system for the software testing process defined in Section 2.2,

$$\begin{aligned} \dot{X}(t) &= KX(t), \\ v(t) &= p_0 X(t), \end{aligned}$$

where $v(t) = \frac{E[N(t)]}{N}$.

Accordingly, the software testing process defined in Section 2.2 can be classified as linear, and it is reasonable to expect there are nonlinear software testing processes too (this will be shown in the next section). A notable feature of the linear dynamic system is that the corresponding states are not defined a priori. However, they are interpreted in a reasonable manner as explained above. There are unexpected and intrinsic links between software testing and system dynamics, which have not been revealed in the literature. The theory of linear or non-linear control [18,19] may be adopted to guide or improve the software testing processes by adjusting the matrix K on-line. Without extra control, the testing process defined in Section 2.2 follows $\dot{X}(t) = KX(t)$ to evolve. The matrix K determines the nature of the dynamic behavior of the software testing process. A noticeable feature of K is that it is independent of the parameter N . This implies that the initial number of software defects is not an intrinsic factor for characterizing the dynamics of the software testing process. The extra control effect can take place as K is adjusted on-line by updating parameters λ , Θ and/or P . The corresponding control problem may be formulated in the setting of model predictive control [20]. We leave this problem to future investigations.

Next we examine the asymptotic behavior of $E[M(t)]$ and $E[N(t)]$. First we present an upper bound and a lower bound of the maximal eigenvalue of a non-negative matrix from [21] without proof.

Lemma 4.1. For any nonnegative matrix $L = (l_{ij})_{i,j=1,2,\dots,m}$, $l_i = \sum_{j=1}^m l_{ij} > 0$, $i = 1, 2, \dots, m$, we have

$$\min_{1 \leq i \leq m} \left(\frac{1}{l_i} \sum_{j=1}^m l_{ij} l_j \right) \leq \mu^{(L)} \leq \max_{1 \leq i \leq m} \left(\frac{1}{l_i} \sum_{j=1}^m l_{ij} l_j \right),$$

where $\mu^{(L)}$ denotes the maximal eigenvalue of matrix L .

Let $L = (I + (p - q)\Theta)P$, then $l_i = 1 + (p - q)\theta_i$, and we have

$$\min_{1 \leq i \leq m} \sum_{j=1}^m p_{ij} (1 + (p - q)\theta_j) \leq \mu^{(L)} \leq \max_{1 \leq i \leq m} \sum_{j=1}^m p_{ij} (1 + (p - q)\theta_j).$$

So we conclude that

$$1 + \min_{1 \leq i \leq m} \sum_{j=1}^m (p - q) p_{ij} \theta_j \leq \mu^{(L)} \leq 1 + \max_{1 \leq i \leq m} \sum_{j=1}^m (p - q) p_{ij} \theta_j.$$

Therefore, when $p > q$, the maximal eigenvalue of L is greater than 1, and the maximal eigenvalue of $-\lambda[I - (I + (p - q)\Theta)P]$ is greater than 0. Thus, $E[M(t)]$ converges exponentially to ∞ as $t \rightarrow \infty$. In other words, if defects are more often introduced than removed, then the number of failures revealed will increase and this process tends to continue forever.

When $p < q$, the maximal eigenvalue of L is less than 1, and the maximal eigenvalue of $-\lambda[I - (I + (p - q)\Theta)P]$ is less than 0. Therefore, $E[M(t)]$ converges exponentially to $\frac{N}{q-p}$ as $t \rightarrow \infty$. Since defects are more often removed than introduced, the software under test will eventually be defect-free and the expected total number of observed failures tends to be $\frac{N}{q-p}$.

4.3. The case of $p = q$

Based on the above analysis, we cannot directly extend the result from the case $p \neq q$ to $p = q$. When $p = q$, it is intuitively understandable that the expected number of remaining defects should keep invariant during software testing

and the expected total number of observed failures tends to infinity. However, this should be formulated and verified in a mathematically rigorous manner. On the other hand, we know that in a software testing process when $p \neq q$, $E[M(t)]$ and $E[N(t)]$ correspond to a linear system; in the case of $p = q$, it is more interesting to see how $E[M(t)]$ behaves as software testing proceeds and if it may demonstrate certain linear or nonlinear dynamic phenomenon. Moreover, $p = q$ is the critical case that separates the case of $p > q$ from the case of $p < q$.

Theorem 4.2. Under the assumptions of Section 2.2, suppose $p = q$ and P is irreducible, we have

$$E[M(t)] = N\mathbf{p}_0(I - e^{-\lambda t(I-P+P^*)})(I - P + P^*)^{-1}\Theta \cdot \mathbf{1}^T + N \sum_{i=1}^m \pi_i \theta_i (\lambda t - 1 + e^{-\lambda t}),$$

$$E[N(t)] = N,$$

where $P^* = \lim_{N \rightarrow \infty} \frac{1}{N+1} \sum_{i=0}^N P^i$.

Note that the limit of $\frac{1}{N+1} \sum_{i=0}^N P^i$ exists because the transition probability matrix P has finite dimension and is irreducible, therefore P^* can be written as $P^* = \mathbf{1} \cdot \pi = \mathbf{1} \cdot (\pi_1, \pi_2, \dots, \pi_m)$. The proof of Theorem 4.2 needs the following lemma.

Lemma 4.2. All eigenvalues of $I - P + P^*$ are greater than 0. (This implies $I - P + P^*$ is reversible.)

Proof. Proof by contradiction. If $I - P + P^*$ has an eigenvalue $-c$ ($c \geq 0$), then there exists a nonzero vector v such that $(I - P + P^*)v = -cv$, and $P^*(I - P + P^*)v = -cP^*v$. That is, $(1+c)P^*v = 0$, $P^*v = 0$. Therefore, $(I - P)v = -cv$, $Pv = (1+c)v$, $0 = P^*v = \lim_{n \rightarrow \infty} P^n v = \lim_{n \rightarrow \infty} P^{n-1}(1+c)v = \dots = \lim_{n \rightarrow \infty} (1+c)^n v$. We conclude $v = 0$, which contradicts with our assumption. Thus all eigenvalues of $I - P + P^*$ are greater than 0. \square

Proof of Theorem 4.2. The expectation of M_k is

$$\begin{aligned} E[M_k] &= \sum_{i=1}^m E[M_k, A_k = i] = \sum_{i=1}^m \alpha_k(i) = \alpha_k \cdot \mathbf{1}^T \\ &= N\mathbf{p}_0\Theta \cdot \mathbf{1}^T + \sum_{j=0}^{k-2} N\mathbf{p}_0(I + (p-q)\Theta)[P(I + (p-q)\Theta)]^j P\Theta P^{k-2-j} \cdot \mathbf{1}^T \\ &= N\mathbf{p}_0\Theta \cdot \mathbf{1}^T + \sum_{j=0}^{k-2} N\mathbf{p}_0 P^{j+1} \Theta \cdot \mathbf{1}^T \\ &= N\mathbf{p}_0\Theta \cdot \mathbf{1}^T + N\mathbf{p}_0 \sum_{j=1}^{k-1} P^j \Theta \cdot \mathbf{1}^T. \end{aligned}$$

Note that

$$\begin{aligned} \mathbf{p}_0 P^* &= \mathbf{p}_0 \mathbf{1} \cdot (\pi_1, \pi_2, \dots, \pi_m) = (\pi_1, \pi_2, \dots, \pi_m), \\ PP^* &= P^*P = P^*P^* = P^*. \end{aligned}$$

Therefore,

$$\begin{aligned} (P - P^*)^j &= (P - P^*)(P - P^*)(P - P^*)^{j-2} \\ &= (P^2 - PP^* - P^*P + P^*P^*)(P - P^*)^{j-2} \\ &= (P^2 - P^*)(P - P^*)^{j-2} \\ &= \dots = P^j - P^*. \end{aligned}$$

$$\begin{aligned} E[M_k] &= N\mathbf{p}_0\Theta \cdot \mathbf{1}^T + N\mathbf{p}_0 \sum_{j=1}^{k-1} P^j \Theta \cdot \mathbf{1}^T \\ &= N\mathbf{p}_0\Theta \cdot \mathbf{1}^T + N\mathbf{p}_0 \sum_{j=1}^{k-1} [(P - P^*)^j + P^*] \Theta \cdot \mathbf{1}^T \\ &= N\mathbf{p}_0\Theta \cdot \mathbf{1}^T + N\mathbf{p}_0 \sum_{j=1}^{k-1} (P - P^*)^j \Theta \cdot \mathbf{1}^T + N\mathbf{p}_0 \sum_{j=1}^{k-1} P^* \Theta \cdot \mathbf{1}^T \\ &= N\mathbf{p}_0\Theta \cdot \mathbf{1}^T + N\mathbf{p}_0(P - P^*) \frac{I - (P - P^*)^{k-1}}{I - P + P^*} \Theta \cdot \mathbf{1}^T + N(k-1)\mathbf{p}_0 P^* \Theta \cdot \mathbf{1}^T \\ &= N\mathbf{p}_0\Theta \cdot \mathbf{1}^T + N\mathbf{p}_0(P - P^*)[I - (P - P^*)^{k-1}](I - P + P^*)^{-1} \Theta \cdot \mathbf{1}^T + N(k-1) \sum_{i=1}^m \pi_i \theta_i. \end{aligned} \tag{4.10}$$

$$\begin{aligned}
E[M(t)] &= \sum_{k=0}^{\infty} E[M_k] \frac{(\lambda t)^k}{k!} e^{-\lambda t} = \sum_{k=1}^{\infty} E[M_k] \frac{(\lambda t)^k}{k!} e^{-\lambda t} \\
&= \sum_{k=1}^{\infty} N \mathbf{p}_0 \Theta \cdot \mathbf{1}^T + N \mathbf{p}_0 (P - P^*) [I - (P - P^*)^{k-1}] (I - P + P^*)^{-1} \Theta \cdot \mathbf{1}^T + N(k-1) \sum_{i=1}^m \pi_i \theta_i \frac{(\lambda t)^k}{k!} e^{-\lambda t} \\
&= N \mathbf{p}_0 \Theta \cdot \mathbf{1}^T (1 - e^{-\lambda t}) + N \mathbf{p}_0 (P - P^*) \sum_{k=1}^{\infty} [I - (P - P^*)^{k-1}] (I - P + P^*)^{-1} \frac{(\lambda t)^k}{k!} e^{-\lambda t} \Theta \cdot \mathbf{1}^T \\
&\quad + N \sum_{i=1}^m \pi_i \theta_i \sum_{k=1}^{\infty} (k-1) \frac{(\lambda t)^k}{k!} e^{-\lambda t} \\
&= N \mathbf{p}_0 \Theta \cdot \mathbf{1}^T (1 - e^{-\lambda t}) + N \mathbf{p}_0 (P - P^*) (I - P + P^*)^{-1} (1 - e^{-\lambda t}) \Theta \cdot \mathbf{1}^T \\
&\quad - N \mathbf{p}_0 (P - P^*) \sum_{k=1}^{\infty} (P - P^*)^{k-1} (I - P + P^*)^{-1} \frac{(\lambda t)^k}{k!} e^{-\lambda t} \Theta \cdot \mathbf{1}^T \\
&\quad + N \sum_{i=1}^m \pi_i \theta_i \left[\lambda t \sum_{k=1}^{\infty} \frac{(\lambda t)^{k-1}}{(k-1)!} e^{-\lambda t} - \sum_{k=1}^{\infty} \frac{(\lambda t)^k}{k!} e^{-\lambda t} \right] \\
&= N \mathbf{p}_0 (I - P + P^*)^{-1} (1 - e^{-\lambda t}) \Theta \cdot \mathbf{1}^T - N \mathbf{p}_0 \sum_{k=1}^{\infty} (P - P^*)^k \frac{(\lambda t)^k}{k!} e^{-\lambda t} (I - P + P^*)^{-1} \Theta \cdot \mathbf{1}^T \\
&\quad + N \sum_{i=1}^m \pi_i \theta_i (\lambda t - 1 + e^{-\lambda t}) \\
&= N \mathbf{p}_0 (I - P + P^*)^{-1} (1 - e^{-\lambda t}) \Theta \cdot \mathbf{1}^T - N \mathbf{p}_0 e^{-\lambda t} (e^{\lambda t(P-P^*)} - I) (I - P + P^*)^{-1} \Theta \cdot \mathbf{1}^T \\
&\quad + N \sum_{i=1}^m \pi_i \theta_i (\lambda t - 1 + e^{-\lambda t}) \\
&= N \mathbf{p}_0 (I - e^{-\lambda t(I-P+P^*)}) (I - P + P^*)^{-1} \Theta \cdot \mathbf{1}^T + N \sum_{i=1}^m \pi_i \theta_i (\lambda t - 1 + e^{-\lambda t}). \tag{4.11}
\end{aligned}$$

The expectation of N_k is $E[N_k] = \beta_k \cdot \mathbf{1}^T = N \mathbf{p}_0 P^{k-1} \cdot \mathbf{1}^T = N$.

Therefore, $E[N(t)] = N$. \square

A nonlinear dynamic system can be induced from Theorem 4.2 as follows. Note that $N(\lambda t - 1 + e^{-\lambda t}) \pi \Theta \cdot \mathbf{1}^T = N \mathbf{p}_0 (\lambda t - 1 + e^{-\lambda t}) P^* \Theta \cdot \mathbf{1}^T$, so if we let

$$K = I - P + P^*,$$

$$X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_m(t) \end{pmatrix} = (I - e^{-\lambda t K}) K^{-1} \Theta \cdot \mathbf{1}^T + (\lambda t - 1 + e^{-\lambda t}) P^* \Theta \cdot \mathbf{1}^T,$$

$$\text{then } \frac{E[M(t)]}{N} = \mathbf{p}_0 X(t) = (p_1, p_2, \dots, p_m) \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_m(t) \end{pmatrix}.$$

Similar to the last subsection, $\frac{E[M(t)]}{N}$ represents the expected ratio of observed failures by time t . $x_i(t)$ can be interpreted as the expected ratio of observed failures by an arbitrary sequence of actions starting with action i by time t .

From Lemma 4.2 we know all eigenvalues of K are larger than 0, so $E[M(t)] \rightarrow \infty$ linearly as $t \rightarrow \infty$.

$Y(t)$ is the unique solution satisfying

$$\dot{X}(t) = \lambda e^{-\lambda t K} \Theta \cdot \mathbf{1}^T + \lambda (1 - e^{-\lambda t}) P^* \Theta \cdot \mathbf{1}^T \tag{4.12}$$

and the initial condition $X(0) = \mathbf{0}^T$.

From (4.12) it holds $\lim_{t \rightarrow \infty} \dot{X}(t) = P^* \Theta \cdot \mathbf{1}^T$, so the rate of finding errors is $N \mathbf{p}_0 P^* \Theta \cdot \mathbf{1}^T = \lambda N \pi \Theta$.

Furthermore, $X(t)$ is a solution of a nonlinear ordinary differential equation. It tells us the expected cumulative number of observed failures $E[M(t)] = N \mathbf{p}_0 X(t)$ obeys a nonlinear dynamic behavior. It is different from the case of $p \neq q$.

5. Second-order behaviors of Model I

The variances of $M(t)$ and $N(t)$ can be treated as a measure of the stability of the software testing process. If the variances of $M(t)$ and $N(t)$ are small enough, the testing process can be thought to be stable, and $M(t)$ and $N(t)$ can be estimated from

$E[M(t)]$ and $E[N(t)]$ in a reasonable manner. It is highly desirable for a software testing process to be stable since a stable software testing process may be treated as repeatable in a heuristic sense and deserves more trust than non-stable software testing processes. Therefore it is worthwhile to study the variances of $M(t)$ and $N(t)$.

5.1. Recursive formulae

Let

$$\begin{aligned}\gamma_k(i) &= E[M_k^2, A_k = i], & \gamma_k &= (\gamma_k(1), \gamma_k(2), \dots, \gamma_k(m)), \\ \varphi_k(i) &= E[N_k^2, A_k = i], & \varphi_k &= (\varphi_k(1), \varphi_k(2), \dots, \varphi_k(m)), \\ \chi_k(i) &= E[M_k N_k, A_k = i], & \chi_k &= (\chi_k(1), \chi_k(2), \dots, \chi_k(m)).\end{aligned}$$

We have the following recursive formulae.

Theorem 5.1. Under the assumptions of Section 2.2, it holds

$$\gamma_k = \gamma_{k-1}P + 2\chi_{k-1}P\Theta + \beta_{k-1}P\Theta, \quad (5.13)$$

$$\varphi_k = \varphi_{k-1}P(I + 2(p - q)\Theta) + (p + q)\beta_{k-1}P\Theta, \quad (5.14)$$

$$\chi_k = \chi_{k-1}P(I + (p - q)\Theta) + \varphi_{k-1}P\Theta + (p - q)\beta_{k-1}P\Theta. \quad (5.15)$$

Proof.

$$\begin{aligned}E[M_k^2, A_k = i] &= \sum_{j=0}^{\infty} \sum_{\ell=1}^{\infty} \ell^2 \sum_{i'=1}^m (1 - j\theta_i) p_{i'i} \Pr\{M_{k-1} = \ell, A_{k-1} = i', N_{k-1} = j\} \\ &\quad + \sum_{j=0}^{\infty} \sum_{\ell=1}^{\infty} \ell^2 \sum_{i'=1}^m (1 - p - q) j \theta_i p_{i'i} \Pr\{M_{k-1} = \ell - 1, A_{k-1} = i', N_{k-1} = j\} \\ &\quad + \sum_{j=1}^{\infty} \sum_{\ell=1}^{\infty} \ell^2 \sum_{i'=1}^m p \cdot (j - 1) \theta_i p_{i'i} \Pr\{M_{k-1} = \ell - 1, A_{k-1} = i', N_{k-1} = j - 1\} \\ &\quad + \sum_{j=0}^{\infty} \sum_{\ell=1}^{\infty} \ell^2 \sum_{i'=1}^m p \cdot (j + 1) \theta_i p_{i'i} \Pr\{M_{k-1} = \ell - 1, A_{k-1} = i', N_{k-1} = j + 1\} \\ &= \sum_{j=0}^{\infty} \sum_{\ell=1}^{\infty} \ell^2 \sum_{i'=1}^m (1 - j\theta_i) p_{i'i} \Pr\{M_{k-1} = \ell, A_{k-1} = i', N_{k-1} = j\} \\ &\quad + \sum_{j=0}^{\infty} \sum_{\ell=1}^{\infty} \ell^2 \sum_{i'=1}^m (1 - p - q) j \theta_i p_{i'i} \Pr\{M_{k-1} = \ell - 1, A_{k-1} = i', N_{k-1} = j\} \\ &\quad + \sum_{j=0}^{\infty} \sum_{\ell=1}^{\infty} \ell^2 \sum_{i'=1}^m p j \theta_i p_{i'i} \Pr\{M_{k-1} = \ell - 1, A_{k-1} = i', N_{k-1} = j\} \\ &\quad + \sum_{j=0}^{\infty} \sum_{\ell=1}^{\infty} \ell^2 \sum_{i'=1}^m p j \theta_i p_{i'i} \Pr\{M_{k-1} = \ell - 1, A_{k-1} = i', N_{k-1} = j\} \\ &= \sum_{j=0}^{\infty} \sum_{\ell=1}^{\infty} \ell^2 \sum_{i'=1}^m (1 - j\theta_i) p_{i'i} \Pr\{M_{k-1} = \ell, A_{k-1} = i', N_{k-1} = j\} \\ &\quad + \sum_{j=0}^{\infty} \sum_{\ell=1}^{\infty} \ell^2 \sum_{i'=1}^m j \theta_i p_{i'i} \Pr\{M_{k-1} = \ell - 1, A_{k-1} = i', N_{k-1} = j\} \\ &= \sum_{j=0}^{\infty} \sum_{\ell=1}^{\infty} \ell^2 \sum_{i'=1}^m (1 - j\theta_i) p_{i'i} \Pr\{M_{k-1} = \ell, A_{k-1} = i', N_{k-1} = j\} \\ &\quad + \sum_{j=0}^{\infty} \sum_{\ell=0}^{\infty} (\ell + 1)^2 \sum_{i'=1}^m j \theta_i p_{i'i} \Pr\{M_{k-1} = \ell, A_{k-1} = i', N_{k-1} = j\} \\ &= \sum_{j=0}^{\infty} \sum_{\ell=0}^{\infty} \ell^2 \sum_{i'=1}^m (1 - j\theta_i) p_{i'i} \Pr\{M_{k-1} = \ell, A_{k-1} = i', N_{k-1} = j\}\end{aligned}$$

$$\begin{aligned}
& + \sum_{j=0}^{\infty} \sum_{\ell=0}^{\infty} \ell^2 \sum_{i'=1}^m j \theta_i p_{i'i} \Pr\{M_{k-1} = \ell, A_{k-1} = i', N_{k-1} = j\} \\
& + 2 \sum_{j=0}^{\infty} \sum_{\ell=0}^{\infty} \ell \sum_{i'=1}^m j \theta_i p_{i'i} \Pr\{M_{k-1} = \ell, A_{k-1} = i', N_{k-1} = j\} \\
& + \sum_{j=0}^{\infty} \sum_{\ell=0}^{\infty} \sum_{i'=1}^m j \theta_i p_{i'i} \Pr\{M_{k-1} = \ell, A_{k-1} = i', N_{k-1} = j\} \\
& = \sum_{j=0}^{\infty} \sum_{\ell=0}^{\infty} \ell^2 \sum_{i'=1}^m p_{i'i} \Pr\{M_{k-1} = \ell, A_{k-1} = i', N_{k-1} = j\} \\
& + 2 \sum_{j=0}^{\infty} \sum_{\ell=0}^{\infty} \ell \sum_{i'=1}^m j \theta_i p_{i'i} \Pr\{M_{k-1} = \ell, A_{k-1} = i', N_{k-1} = j\} \\
& + \sum_{j=0}^{\infty} \sum_{\ell=0}^{\infty} \sum_{i'=1}^m j \theta_i p_{i'i} \Pr\{M_{k-1} = \ell - 1, A_{k-1} = i', N_{k-1} = j\} \\
& = \sum_{i'=1}^m p_{i'i} E[M_{k-1}^2, A_{k-1} = i'] + 2 \theta_i \sum_{i'=1}^m p_{i'i} E[M_{k-1} N_{k-1}, A_{k-1} = i'] \\
& + \theta_i \sum_{i'=1}^m p_{i'i} E[N_{k-1}, A_{k-1} = i']. \tag{5.16}
\end{aligned}$$

$$\begin{aligned}
E[N_k^2, A_k = i] & = \sum_{\ell=0}^{\infty} \sum_{j=0}^{\infty} j^2 \sum_{i'=1}^m (1 - j \theta_i) p_{i'i} \Pr\{M_{k-1} = \ell, A_{k-1} = i', N_{k-1} = j\} \\
& + \sum_{\ell=0}^{\infty} \sum_{j=0}^{\infty} j^2 \sum_{i'=1}^m (1 - p - q) j \theta_i p_{i'i} \Pr\{M_{k-1} = \ell - 1, A_{k-1} = i', N_{k-1} = j\} \\
& + \sum_{\ell=0}^{\infty} \sum_{j=0}^{\infty} (j + 1)^2 \sum_{i'=1}^m p j \theta_i p_{i'i} \Pr\{M_{k-1} = \ell - 1, A_{k-1} = i', N_{k-1} = j\} \\
& + \sum_{\ell=0}^{\infty} \sum_{j=0}^{\infty} (j - 1)^2 \sum_{i'=1}^m q j \theta_i p_{i'i} \Pr\{M_{k-1} = \ell - 1, A_{k-1} = i', N_{k-1} = j\} \\
& = \sum_{\ell=0}^{\infty} \sum_{j=1}^{\infty} j^2 \sum_{i'=1}^m (1 - j \theta_i) p_{i'i} \Pr\{M_{k-1} = \ell, A_{k-1} = i', N_{k-1} = j\} \\
& + \sum_{\ell=0}^{\infty} \sum_{j=1}^{\infty} j^2 \sum_{i'=1}^m (1 - p - q) j \theta_i p_{i'i} \Pr\{M_{k-1} = \ell - 1, A_{k-1} = i', N_{k-1} = j\} \\
& + \sum_{\ell=0}^{\infty} \sum_{j=1}^{\infty} j^2 \sum_{i'=1}^m p j \theta_i p_{i'i} \Pr\{M_{k-1} = \ell - 1, A_{k-1} = i', N_{k-1} = j\} \\
& + 2 \sum_{\ell=0}^{\infty} \sum_{j=1}^{\infty} j \sum_{i'=1}^m p j \theta_i p_{i'i} \Pr\{M_{k-1} = \ell - 1, A_{k-1} = i', N_{k-1} = j\} \\
& + \sum_{\ell=0}^{\infty} \sum_{j=1}^{\infty} \sum_{i'=1}^m p j \theta_i p_{i'i} \Pr\{M_{k-1} = \ell - 1, A_{k-1} = i', N_{k-1} = j\} \\
& + \sum_{\ell=0}^{\infty} \sum_{j=1}^{\infty} j^2 \sum_{i'=1}^m q j \theta_i p_{i'i} \Pr\{M_{k-1} = \ell - 1, A_{k-1} = i', N_{k-1} = j\} \\
& - 2 \sum_{\ell=0}^{\infty} \sum_{j=1}^{\infty} j \sum_{i'=1}^m q j \theta_i p_{i'i} \Pr\{M_{k-1} = \ell - 1, A_{k-1} = i', N_{k-1} = j\} \\
& + \sum_{\ell=0}^{\infty} \sum_{j=1}^{\infty} \sum_{i'=1}^m q j \theta_i p_{i'i} \Pr\{M_{k-1} = \ell - 1, A_{k-1} = i', N_{k-1} = j\}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{\ell=0}^{\infty} \sum_{j=1}^{\infty} j^2 \sum_{i'=1}^m (1 - j\theta_i) p_{i'i} \Pr\{M_{k-1} = \ell, A_{k-1} = i', N_{k-1} = j\} \\
&\quad + \sum_{\ell=0}^{\infty} \sum_{j=1}^{\infty} j^2 \sum_{i'=1}^m j\theta_i p_{i'i} \Pr\{M_{k-1} = \ell - 1, A_{k-1} = i', N_{k-1} = j\} \\
&\quad + 2(p - q) \sum_{\ell=0}^{\infty} \sum_{j=1}^{\infty} j \sum_{i'=1}^m j\theta_i p_{i'i} \Pr\{M_{k-1} = \ell - 1, A_{k-1} = i', N_{k-1} = j\} \\
&\quad + (p + q) \sum_{\ell=0}^{\infty} \sum_{j=1}^{\infty} \sum_{i'=1}^m j\theta_i p_{i'i} \Pr\{M_{k-1} = \ell - 1, A_{k-1} = i', N_{k-1} = j\} \\
&= \sum_{\ell=0}^{\infty} \sum_{j=1}^{\infty} j^2 \sum_{i'=1}^m (1 - j\theta_i) p_{i'i} \Pr\{M_{k-1} = \ell, A_{k-1} = i', N_{k-1} = j\} \\
&\quad + \sum_{\ell=0}^{\infty} \sum_{j=1}^{\infty} j^2 \sum_{i'=1}^m j\theta_i p_{i'i} \Pr\{M_{k-1} = \ell, A_{k-1} = i', N_{k-1} = j\} \\
&\quad + 2(p - q) \sum_{\ell=0}^{\infty} \sum_{j=1}^{\infty} j \sum_{i'=1}^m j\theta_i p_{i'i} \Pr\{M_{k-1} = \ell, A_{k-1} = i', N_{k-1} = j\} \\
&\quad + (p + q) \sum_{\ell=0}^{\infty} \sum_{j=1}^{\infty} \sum_{i'=1}^m j\theta_i p_{i'i} \Pr\{M_{k-1} = \ell, A_{k-1} = i', N_{k-1} = j\} \\
&= \sum_{\ell=0}^{\infty} \sum_{j=1}^{\infty} j^2 \sum_{i'=1}^m p_{i'i} \Pr\{M_{k-1} = \ell, A_{k-1} = i', N_{k-1} = j\} \\
&\quad + 2(p - q) \sum_{\ell=0}^{\infty} \sum_{j=1}^{\infty} j \sum_{i'=1}^m j\theta_i p_{i'i} \Pr\{M_{k-1} = \ell, A_{k-1} = i', N_{k-1} = j\} \\
&\quad + (p + q) \sum_{\ell=0}^{\infty} \sum_{j=1}^{\infty} \sum_{i'=1}^m j\theta_i p_{i'i} \Pr\{M_{k-1} = \ell, A_{k-1} = i', N_{k-1} = j\} \\
&= [1 + 2(p - q)\theta_i] \sum_{i'=1}^m p_{i'i} E[N_{k-1}^2, A_{k-1} = i'] + (p + q)\theta_i \sum_{i'=1}^m p_{i'i} E[N_{k-1}, A_{k-1} = i']. \tag{5.17}
\end{aligned}$$

$$\begin{aligned}
&\sum_{j=0}^{\infty} \sum_{\ell=0}^{\infty} \ell j \Pr\{M_k = \ell, A_k = i, N_k = j\} \\
&= \sum_{j=1}^{\infty} \sum_{\ell=0}^{\infty} \ell j \sum_{i'=1}^m (1 - j\theta_i) p_{i'i} \Pr\{M_{k-1} = \ell, A_{k-1} = i', N_{k-1} = j\} \\
&\quad + \sum_{j=1}^{\infty} \sum_{\ell=0}^{\infty} \ell j \sum_{i'=1}^m (1 - p - q) j\theta_i p_{i'i} \Pr\{M_{k-1} = \ell - 1, A_{k-1} = i', N_{k-1} = j\} \\
&\quad + \sum_{j=1}^{\infty} \sum_{\ell=0}^{\infty} \ell j \sum_{i'=1}^m p(j - 1) \theta_i p_{i'i} \Pr\{M_{k-1} = \ell - 1, A_{k-1} = i', N_{k-1} = j - 1\} \\
&\quad + \sum_{j=1}^{\infty} \sum_{\ell=0}^{\infty} \ell j \sum_{i'=1}^m q(j + 1) \theta_i p_{i'i} \Pr\{M_{k-1} = \ell - 1, A_{k-1} = i', N_{k-1} = j + 1\} \\
&= \sum_{j=1}^{\infty} \sum_{\ell=0}^{\infty} \ell j \sum_{i'=1}^m (1 - j\theta_i) p_{i'i} \Pr\{M_{k-1} = \ell, A_{k-1} = i', N_{k-1} = j\} \\
&\quad + \sum_{j=1}^{\infty} \sum_{\ell=0}^{\infty} (\ell + 1) j \sum_{i'=1}^m (1 - p - q) j\theta_i p_{i'i} \Pr\{M_{k-1} = \ell, A_{k-1} = i', N_{k-1} = j\} \\
&\quad + \sum_{j=1}^{\infty} \sum_{\ell=0}^{\infty} (\ell + 1) (j + 1) \sum_{i'=1}^m p j \theta_i p_{i'i} \Pr\{M_{k-1} = \ell, A_{k-1} = i', N_{k-1} = j\}
\end{aligned}$$

$$\begin{aligned}
& + \sum_{j=1}^{\infty} \sum_{\ell=0}^{\infty} (\ell+1)(j-1) \sum_{i'=1}^m qj\theta_i p_{i'} \Pr\{M_{k-1} = \ell, A_{k-1} = i', N_{k-1} = j\} \\
& = \sum_{j=1}^{\infty} \sum_{\ell=0}^{\infty} \sum_{i'=1}^m p_{i'} [\ell j(1-j\theta_i) + (1+\ell)j(1-p-q)j\theta_i + (\ell+1)(j+1)pj\theta_i \\
& \quad + (\ell+1)(j-1)qj\theta_i] \Pr\{M_{k-1} = \ell, A_{k-1} = i', N_{k-1} = j\} \\
& = \sum_{j=1}^{\infty} \sum_{\ell=0}^{\infty} \sum_{i'=1}^m p_{i'} [\ell j + (p-q)j\theta_i + (p-q)\ell j\theta_i + j^2\theta_i] \Pr\{M_{k-1} = \ell, A_{k-1} = i', N_{k-1} = j\} \\
& = [1 + (p-q)\theta_i] \sum_{i'=1}^m p_{i'} E[M_{k-1}N_{k-1}, A_{k-1} = i'] \\
& \quad + (p-q)\theta_i \sum_{i'=1}^m p_{i'} E[N_{k-1}, A_{k-1} = i'] + \theta_i \sum_{i'=1}^m p_{i'} E[N_{k-1}^2, A_{k-1} = i']. \tag{5.18}
\end{aligned}$$

Rewriting (5.16)–(5.18) in vector form, we have

$$\begin{aligned}
\gamma_k &= \gamma_{k-1}P + 2\chi_{k-1}P\Theta + \beta_{k-1}P\Theta, \\
\varphi_k &= \varphi_{k-1}P(I + 2(p-q)\Theta) + (p+q)\beta_{k-1}P\Theta, \\
\chi_k &= \chi_{k-1}P(I + (p-q)\Theta) + \varphi_{k-1}P\Theta + (p-q)\beta_{k-1}P\Theta. \quad \square
\end{aligned}$$

Next let us calculate the initial value of the parameters in the recursive formulae. From the definition of γ_1 and $i = 1, 2, \dots, m$, we have

$$\begin{aligned}
\gamma_1(i) &= E[M_1^2, A_1 = i] = \Pr\{M_1 = 1, A_1 = i\} \\
&= p_i \Pr\{M_1 = 1 | A_1 = i\} = Np_i\theta_i.
\end{aligned}$$

In vector form, $\gamma_1 = N\mathbf{p}_0\Theta$. For φ_1 , we have

$$\begin{aligned}
\varphi_1(i) &= E[N_1^2, A_1 = i] \\
&= N^2 \Pr\{N_1 = N, A_1 = i\} + (N+1)^2 \Pr\{N_1 = N+1, A_1 = i\} + (N-1)^2 \Pr\{N_1 = N-1, A_1 = i\} \\
&= N^2 [\Pr\{M_1 = 0, N_1 = N, A_1 = i\} + \Pr\{M_1 = 1, N_1 = N, A_1 = i\}] \\
& \quad + (N+1)^2 \Pr\{M_1 = 1, N_1 = N+1, A_1 = i\} + (N-1)^2 \Pr\{M_1 = 1, N_1 = N-1, A_1 = i\} \\
&= N^2 [\Pr\{M_1 = 0 | A_1 = i\} \Pr\{A_1 = i\} + \Pr\{M_1 = 1, N_1 = N | A_1 = i\} \Pr\{A_1 = i\} \\
& \quad + (N+1)^2 \Pr\{M_1 = 1, N_1 = N+1 | A_1 = i\} \Pr\{A_1 = i\} \\
& \quad + (N-1)^2 \Pr\{M_1 = 1, N_1 = N-1 | A_1 = i\} \Pr\{A_1 = i\}] \\
&= N^2 [(1-N\theta_i) + (1-p-q)N\theta_i] p_i + (N+1)^2 pN\theta_i p_i + (N-1)^2 qN\theta_i p_i \\
&= p_i [N^2 + 2N^2(p-q)\theta_i + (p+q)N\theta_i].
\end{aligned}$$

In vector form, $\varphi_1 = N^2\mathbf{p}_0 + 2(p-q)N^2\mathbf{p}_0\Theta + (p+q)N\mathbf{p}_0\Theta$. For χ_1 , we have

$$\begin{aligned}
\chi_1(i) &= \sum_{j=0}^{\infty} \sum_{\ell=0}^{\infty} \ell j \Pr\{M_1 = \ell, A_1 = i, N_1 = j\} \\
&= \sum_{j=0}^{\infty} j \Pr\{M_1 = 1, A_1 = i, N_1 = j\} \\
&= N \Pr\{M_1 = 1, A_1 = i, N_1 = N\} + (N+1) \Pr\{M_1 = 1, A_1 = i, N_1 = N+1\} \\
& \quad + (N-1) \Pr\{M_1 = 1, A_1 = i, N_1 = N-1\} \\
&= Np_i(1-p-q)N\theta_i + (N+1)p_i pN\theta_i + (N-1)p_i qN\theta_i \\
&= [N^2 + (p-q)N] p_i \theta_i.
\end{aligned}$$

In vector form, $\chi_1 = [N^2 + (p-q)N]\mathbf{p}_0\Theta$.

5.2. Dynamic behavior of $(q-p)M(t) + N(t)$

Although it is hard to obtain the expression of γ_k , φ_k and χ_k by the recursive formulae (5.13)–(5.15) directly, with the help of the recursive formulae (5.13)–(5.15) we have following equation.

$$(p-q)^2\gamma_k + \varphi_k - 2(p-q)\chi_k = [(p-q)^2\gamma_{k-1} + \varphi_{k-1} - 2(p-q)\chi_{k-1}]P + [(p+q) - (p-q)^2]\beta_{k-1}P\Theta.$$

The above equation yields

$$\begin{aligned}
 (p-q)^2\gamma_k + \varphi_k - 2(p-q)\chi_k &= [(p-q)^2\gamma_{k-1} + \varphi_{k-1} - 2(p-q)\chi_{k-1}]P + [(p+q) - (p-q)^2] \\
 &\quad \times N\mathbf{p}_0(I + (p-q)\Theta)[P(I + (p-q)\Theta)]^{k-2}P\Theta \\
 &= [(p-q)^2\gamma_{k-1} + \varphi_{k-1} - 2(p-q)\chi_{k-1}]P + [(p+q) - (p-q)^2]N\mathbf{p}_0[(I + (p-q)\Theta)P]^{k-1}\Theta \\
 &= [(p-q)^2\gamma_1 + \varphi_1 - 2(p-q)\chi_1]P^{k-1} + [(p+q) - (p-q)^2]N\mathbf{p}_0 \sum_{j=1}^{k-1} [(I + (p-q)\Theta)P]^j \Theta P^{k-1-j} \\
 &= [(p-q)^2N\mathbf{p}_0\Theta + N^2\mathbf{p}_0 + 2(p-q)N^2\mathbf{p}_0\Theta + (p+q)N\mathbf{p}_0\Theta - 2(p-q)[N^2 + (p-q)N]\mathbf{p}_0\Theta]P^{k-1} \\
 &\quad + [(p+q) - (p-q)^2]N\mathbf{p}_0 \sum_{j=1}^{k-1} [(I + (p-q)\Theta)P]^j \Theta P^{k-1-j} \\
 &= [(p+q) - (p-q)^2]N\mathbf{p}_0\Theta + N^2\mathbf{p}_0]P^{k-1} + [(p+q) - (p-q)^2]N\mathbf{p}_0 \sum_{j=1}^{k-1} [(I + (p-q)\Theta)P]^j \Theta P^{k-1-j} \\
 &= N^2\mathbf{p}_0P^{k-1} + [(p+q) - (p-q)^2]N\mathbf{p}_0 \sum_{j=0}^{k-1} [(I + (p-q)\Theta)P]^j \Theta P^{k-1-j}. \tag{5.19}
 \end{aligned}$$

5.2.1. The case of $p \neq q$

In the case of $p \neq q$, we have

$$\begin{aligned}
 E[(q-p)M_k + N_k]^2 &= E[(q-p)^2M_k^2 + N_k^2 + 2(q-p)M_kN_k] \\
 &= \sum_{i=1}^m E[(q-p)^2M_k^2 + N_k^2 + 2(q-p)M_kN_k, A_k = i] \\
 &= \sum_{i=1}^m [(q-p)^2\gamma_k(i) + \varphi_k(i) + 2(q-p)\chi_k(i)] \\
 &= [(q-p)^2\gamma_k + \varphi_k + 2(q-p)\chi_k] \cdot \mathbf{1}^T \\
 &= N^2 + [(p+q) - (q-p)^2]N\mathbf{p}_0 \sum_{j=0}^{k-1} [(I + (p-q)\Theta)P]^j \Theta \cdot \mathbf{1}^T \\
 &= N^2 + [(p+q) - (q-p)^2]N\mathbf{p}_0 \sum_{j=0}^{k-1} [(I + (p-q)\Theta)P]^j \Theta P \cdot \mathbf{1}^T \\
 &= N^2 + [(p+q) - (q-p)^2]N\mathbf{p}_0 \sum_{j=0}^{k-1} [(I + (p-q)\Theta)P]^j \frac{P - (I + (p-q)\Theta)P}{q-p} \cdot \mathbf{1}^T \\
 &= N^2 + \frac{(p+q) - (q-p)^2}{q-p} N\mathbf{p}_0 \sum_{j=0}^{k-1} [(I + (p-q)\Theta)P]^j - [(I + (p-q)\Theta)P]^{j+1} \cdot \mathbf{1}^T \\
 &= N^2 + \frac{(p+q) - (q-p)^2}{q-p} N\mathbf{p}_0 [I - (I + (p-q)\Theta)P]^k \cdot \mathbf{1}^T \\
 &= N^2 + \frac{(p+q) - (q-p)^2}{q-p} N - \frac{(p+q) - (q-p)^2}{q-p} N\mathbf{p}_0 [(I + (p-q)\Theta)P]^k \cdot \mathbf{1}^T.
 \end{aligned}$$

Consequently,

$$\begin{aligned}
 E[(q-p)M(t) + N(t)]^2 &= \sum_{k=0}^{\infty} E[(q-p)M_k + N_k]^2 \frac{(\lambda t)^k}{k!} e^{-\lambda t} \\
 &= \sum_{k=0}^{\infty} \left[N^2 + \frac{(p+q) - (q-p)^2}{q-p} N - \frac{(p+q) - (q-p)^2}{q-p} N\mathbf{p}_0 [(I + (p-q)\Theta)P]^k \cdot \mathbf{1}^T \frac{(\lambda t)^k}{k!} e^{-\lambda t} \right] \\
 &= N^2 + \frac{(p+q) - (q-p)^2}{q-p} N - \frac{(p+q) - (q-p)^2}{q-p} N\mathbf{p}_0 e^{\lambda t(I + (p-q)\Theta)P - \lambda t} \cdot \mathbf{1}^T \\
 &= N^2 + \frac{(p+q) - (q-p)^2}{q-p} N - \frac{(p+q) - (q-p)^2}{q-p} N\mathbf{p}_0 e^{-\lambda t[I - (I + (p-q)\Theta)P]} \cdot \mathbf{1}^T.
 \end{aligned}$$

Therefore,

$$\begin{aligned}\text{Var}[(q-p)M(t) + N(t)] &= E[(q-p)M(t) + N(t)]^2 - [E[(q-p)M(t) + N(t)]]^2 \\ &= N^2 + \frac{(p+q) - (q-p)^2}{q-p} N - \frac{(p+q) - (q-p)^2}{q-p} N \mathbf{p}_0 e^{-\lambda t [I - (I + (p-q)\Theta)P]} \cdot \mathbf{1}^T - (-N)^2 \\ &= \frac{(p+q) - (q-p)^2}{q-p} N - \frac{(p+q) - (q-p)^2}{q-p} N \mathbf{p}_0 e^{-\lambda t [I - (I + (p-q)\Theta)P]} \cdot \mathbf{1}^T.\end{aligned}$$

In the case of $p < q$, $\text{Var}[(q-p)M(t) + N(t)] \rightarrow \frac{(p+q) - (q-p)^2}{q-p} N$ exponentially as $t \rightarrow \infty$. Note that $E[(q-p)M(t) + N(t)] = N$. This implies that in the case that defects are more often removed than introduced, although $(q-p)M(t) + N(t)$ can serve as an unbiased estimator for the initial number of remaining defects N , there is intrinsic uncertainty associated with the estimator. The amount of uncertainty is measured in terms of $\frac{(p+q) - (q-p)^2}{q-p} N$. The cumulative number of observed failures up to the current time t , $M(t)$, and the current number of remaining defects, $N(t)$, are not sufficient for precisely estimating the initial number of defects, N . This may be treated as a principle of uncertainty for software defect estimation in the case of imperfect debugging that does not emerge in the case of perfect debugging.

On the other hand, if $p = 1$, $q = 0$, then $\text{Var}[(q-p)M(t) + N(t)] = 0$, and it always holds $N(t) - M(t) = N$. In other cases with $p > q$, it holds $\text{Var}[(q-p)M(t) + N(t)] \rightarrow \infty$ exponentially as $t \rightarrow \infty$. This makes the estimator $(q-p)M(t) + N(t)$ meaningless for estimating N .

5.2.2. The case of $p = q$

Now we consider the case that $p = q$ and P is irreducible below.

From (5.19) we know $\varphi_k = N^2 \mathbf{p}_0 P^{k-1} + 2qN \mathbf{p}_0 \sum_{j=0}^{k-1} P^j \Theta P^{k-1-j}$, then

$$\begin{aligned}E[N_k^2] &= \varphi_k \cdot \mathbf{1}^T = N^2 + 2qN \mathbf{p}_0 \sum_{j=0}^{k-1} P^j \Theta \cdot \mathbf{1}^T \\ &= N^2 + 2qN \mathbf{p}_0 \sum_{j=0}^{k-1} [(P - P^*)^j + P^*] \Theta \cdot \mathbf{1}^T \\ &= N^2 + 2qN \mathbf{p}_0 \sum_{j=0}^{k-1} (P - P^*)^j \Theta \cdot \mathbf{1}^T + 2qN \mathbf{p}_0 \sum_{j=0}^{k-1} P^* \Theta \cdot \mathbf{1}^T \\ &= N^2 + 2qN \mathbf{p}_0 \frac{I - (P - P^*)^k}{I - P + P^*} \Theta \cdot \mathbf{1}^T + 2qNk \mathbf{p}_0 P^* \Theta \cdot \mathbf{1}^T \\ &= N^2 + 2qN \mathbf{p}_0 [I - (P - P^*)^k] (I - P + P^*)^{-1} \Theta \cdot \mathbf{1}^T + 2qNk \sum_{i=1}^m \pi_i \theta_i. \\ E[N(t)^2] &= \sum_{k=0}^{\infty} E[N_k^2] \frac{(\lambda t)^k}{k!} e^{-\lambda t} \\ &= \sum_{k=0}^{\infty} \left[N^2 + 2qN \mathbf{p}_0 [I - (P - P^*)^k] (I - P + P^*)^{-1} \Theta \cdot \mathbf{1}^T + 2qNk \sum_{i=1}^m \pi_i \theta_i \right] \frac{(\lambda t)^k}{k!} e^{-\lambda t} \\ &= N^2 + 2qN \mathbf{p}_0 (I - P + P^*)^{-1} \Theta \cdot \mathbf{1}^T - 2qN \mathbf{p}_0 e^{\lambda t (P - P^*)} \cdot \mathbf{1}^T e^{-\lambda t} + 2qN \sum_{i=1}^m \pi_i \theta_i \sum_{k=0}^{\infty} k \frac{(\lambda t)^k}{k!} e^{-\lambda t} \\ &= N^2 + 2qN \mathbf{p}_0 (I - P + P^*)^{-1} \Theta \cdot \mathbf{1}^T - 2qN \mathbf{p}_0 e^{-\lambda t (I - P + P^*)} \cdot \mathbf{1}^T + 2qN \lambda t \sum_{i=1}^m \pi_i \theta_i.\end{aligned}$$

Therefore,

$$\begin{aligned}\text{Var}[N(t)] &= E[N(t)^2] - [E[N(t)]]^2 \\ &= N^2 + 2qN \mathbf{p}_0 (I - P + P^*)^{-1} \Theta \cdot \mathbf{1}^T - 2qN \mathbf{p}_0 e^{-\lambda t (I - P + P^*)} \cdot \mathbf{1}^T + 2qN \lambda t \sum_{i=1}^m \pi_i \theta_i - N^2 \\ &= 2qN \mathbf{p}_0 (I - P + P^*)^{-1} \Theta \cdot \mathbf{1}^T - 2qN \mathbf{p}_0 e^{-\lambda t (I - P + P^*)} \cdot \mathbf{1}^T + 2qN \lambda t \sum_{i=1}^m \pi_i \theta_i.\end{aligned}$$

This implies that $\text{Var}[(q-p)M(t) + N(t)] = \text{Var}[N(t)] \rightarrow \infty$ linearly as $t \rightarrow \infty$. Similar to the case of $p > q$, the estimator $(q-p)M(t) + N(t)$ does not make sense for N if $p = q$.

5.3. The martingale property and asymptotic behavior of $M(t)$ and $N(t)$

5.3.1. Martingale property

We denote σ -field $\mathcal{F}_k = \sigma(M_0, N_0, A_1, M_1, N_1, \dots, A_k, M_k, N_k, A_{k+1})$, then we have

$$\begin{aligned} E[M_{k+1}|\mathcal{F}_k] &= M_k \Pr[M_{k+1} = M_k|\mathcal{F}_k] + (M_k + 1) \Pr[M_{k+1} = M_k + 1|\mathcal{F}_k] \\ &= M_k + \Pr[M_{k+1} = M_k + 1|\mathcal{F}_k] \\ &= M_k + N_k \theta_{A_{k+1}} \\ &\geq M_k, \end{aligned} \quad (5.20)$$

which means that $\{M_k, \mathcal{F}_k, k \geq 0\}$ is a submartingale. And let us consider

$$\begin{aligned} E[N_{k+1}|\mathcal{F}_k] &= N_k \Pr[M_{k+1} = M_k|\mathcal{F}_k] + N_k \Pr[M_{k+1} = M_k + 1, N_{k+1} = N_k|\mathcal{F}_k] \\ &\quad + (N_k + 1) \Pr[M_{k+1} = M_k + 1, N_{k+1} = N_k + 1|\mathcal{F}_k] + (N_k - 1) \Pr[M_{k+1} = M_k + 1, N_{k+1} = N_k - 1|\mathcal{F}_k] \\ &= N_k(1 - N_k \theta_{A_{k+1}}) + N_k N_k \theta_{A_{k+1}}(1 - p - q) + (N_k + 1) N_k \theta_{A_{k+1}} p + (N_k - 1) N_k \theta_{A_{k+1}} q \\ &= N_k + (p - q) N_k \theta_{A_{k+1}}. \end{aligned} \quad (5.21)$$

Combine (5.20) and (5.21), we have $E[(q - p)M_{k+1} + N_{k+1}|\mathcal{F}_k] = (q - p)M_k + N_k$, therefore, $\{(q - p)M_k + N_k, \mathcal{F}_k, k \geq 0\}$ is a martingale.

Notice that $M(t) = M_{H(t)}$, $N(t) = N_{H(t)}$, and $A(t) = A_{H(t)+1}$, so $\{(q - p)M(t) + N(t), \mathcal{F}_t, t \geq 0\}$ is a martingale, and $\{M(t), \mathcal{F}_t, t \geq 0\}$ is a submartingale.

5.3.2. Asymptotic behavior

In the case of $p < q$, $E[(q - p)M_k + N_k]^2 \leq E[(q - p)M_k + N_k]^2$ is bounded. According to martingale convergence theorem, there exists a random variable W , such that $\lim_{k \rightarrow \infty} (q - p)M_k + N_k = W$, almost surely (a.s. for short).

Note $E[M_k]^2 \leq E[(q - p)M_k + N_k]^2 / (q - p)^2$ is bounded. Then martingale convergence theorem implies there exists a random variable U , such that $\lim_{k \rightarrow \infty} M_k = U$, a.s. Therefore, there exists a random variable $V = W - (q - p)U$, such that $\lim_{k \rightarrow \infty} N_k = V$, a.s. While $M_{k+1} \geq M_k$, by monotone convergence theorem, $EU = E[\lim_{k \rightarrow \infty} M_k] = \lim_{k \rightarrow \infty} E[M_k] = \frac{N}{q - p}$.

Since $\lim_{k \rightarrow \infty} E[N_k] = 0$, by Fatou Lemma, we have $EV = E[\liminf_{k \rightarrow \infty} N_k] \leq \liminf_{k \rightarrow \infty} E[N_k] = 0$. Obviously $V \geq 0$, then we have $EV = 0$, $V = 0$, a.s. Thus $W = (q - p)U$, a.s.

Lemma 5.1. In the case of $p < q$ we have

$$\lim_{k \rightarrow \infty} E[N_k^2] = 0.$$

Proof. For any $k \geq 1$, we have

$$\begin{aligned} \varphi_k &= \varphi_{k-1} P(I + 2(p - q)\Theta) + (p + q)\beta_{k-1} P\Theta \\ &= \varphi_{k-1} P(I + 2(p - q)\Theta) + (p + q)N\mathbf{p}_0[(I + (p - q)\Theta)P]^{k-1}\Theta \\ &= \varphi_1 [P(I + 2(p - q)\Theta)]^{k-1} + (p + q)N\mathbf{p}_0 \sum_{j=0}^{k-2} [(I + (p - q)\Theta)P]^{k-1-j}\Theta [P(I + 2(p - q)\Theta)]^j \\ &= N^2 \mathbf{p}_0 (I + 2(p - q)\Theta) [P(I + 2(p - q)\Theta)]^{k-1} + (p + q)N\mathbf{p}_0 \sum_{j=0}^{k-1} [(I + (p - q)\Theta)P]^{k-1-j}\Theta [P(I + 2(p - q)\Theta)]^j. \end{aligned}$$

Therefore,

$$\begin{aligned} E[N_k^2] &= \varphi_k \cdot \mathbf{1}^T \\ &= N^2 \mathbf{p}_0 [(I + 2(p - q)\Theta)P]^k \cdot \mathbf{1}^T + (p + q)N\mathbf{p}_0 \sum_{j=0}^{k-1} [(I + (p - q)\Theta)P]^{k-1-j}\Theta [P(I + 2(p - q)\Theta)]^j \cdot \mathbf{1}^T. \end{aligned}$$

In the case of $p < q$ both the maximal eigenvalues of $(I + 2(p - q)\Theta)P$ and $(I + 2(p - q)\Theta)P$ are less than 1, so $E[N_k^2] \rightarrow 0, k \rightarrow \infty$. \square

From Lemma 5.1 we obtain

$$\begin{aligned} |E[(q - p)M_k + N_k]^2 - (q - p)^2 E[M_k^2]|^2 &= |E[N_k(2(q - p)M_k + N_k)]|^2 \leq 4|E[N_k((q - p)M_k + N_k)]|^2 \\ &\leq 4E[N_k^2]E[(q - p)M_k + N_k]^2 \rightarrow 0, \end{aligned}$$

therefore, $\lim_{k \rightarrow \infty} (q-p)^2 E[M_k^2] = \lim_{k \rightarrow \infty} E[(q-p)M_k + N_k]^2 = N^2 + \frac{(p+q)-(q-p)^2}{q-p} N$. While $M_{k+1} \geq M_k$, by monotone convergence theorem, we have $EW^2 = (q-p)^2 EU^2 = (q-p)^2 E[\lim_{k \rightarrow \infty} M_k^2] = (q-p)^2 \lim_{k \rightarrow \infty} E[M_k^2] = N^2 + \frac{(p+q)-(q-p)^2}{q-p} N$, and $\text{Var } W = \frac{(p+q)-(q-p)^2}{q-p} N$. Thus $\text{Var } U = \text{Var} \left(\frac{W}{q-p} \right) = \frac{(p+q)-(q-p)^2}{(q-p)^3} N$.

The above analysis yields the following result.

Theorem 5.2. *In the case of $p < q$, M_k has a limit U , and N_k has a limit 0. The random variable U satisfies $EU = \frac{N}{q-p}$ and $\text{Var } U = \frac{(p+q)-(q-p)^2}{(q-p)^3} N$.*

Based on Theorem 5.2, we have the following result.

Proposition 5.1. *In the case of $p < q$, random variable U (the limit of M_k) is a constant if and only if $p = 0$ and $q = 1$.*

Proof. U is a constant if and only if $\text{Var } U = 0$, that is $(p+q) - (q-p)^2 = 0$, which is equivalent to $p = 0$ and $q = 1$. \square

5.3.3. Central limit theorem

Now let us investigate the convergence in distribution for the series of random variables $\{(q-p)M_k + N_k, \mathcal{F}_k, k \geq 0\}$.

We denote $Y_k = (q-p)M_k + N_k - (q-p)M_{k-1} - N_{k-1}$, $k \geq 1$; $Y_{nk} = \frac{1}{n^{1+\epsilon}} Y_k$, and $\mathcal{F}_{nk} = \mathcal{F}_k$, where $\epsilon > 0$. Therefore, $\{Y_{nk}, \mathcal{F}_{nk}, 1 \leq k \leq n, n \geq 1\}$ is a sequence of difference of square integrable martingales with zero mean, and $|Y_k| = |(q-p)(M_k - M_{k-1}) + (N_k - N_{k-1})| \leq |q-p| + 1$.

Based on $\max_{1 \leq j \leq n} |Y_{nj}| = \frac{1}{n^{1+\epsilon}} \max_{1 \leq j \leq n} |Y_j| \leq \frac{1}{n^{1+\epsilon}} \max_{1 \leq j \leq n} (|q-p| + 1) = \frac{|q-p|+1}{n^{1+\epsilon}} \rightarrow 0$, a.s., we have $\max_{1 \leq j \leq n} |Y_{nj}| \xrightarrow{P} 0$.

Let $U_n = \sum_{j=1}^n Y_{nj}^2$, then $U_n = \frac{1}{n^{1+\epsilon}} \sum_{j=1}^n Y_j^2 \leq \frac{1}{n^{1+\epsilon}} n(|q-p| + 1)^2 = \frac{(|q-p|+1)^2}{n^\epsilon} \rightarrow 0$, when $n \rightarrow \infty$. Therefore, $U_n \rightarrow 0$, a.s. Thus, we have $U_n \xrightarrow{P} 0$.

Because $E[\max_{1 \leq j \leq n} Y_{nj}^2] = \frac{1}{n^{1+\epsilon}} E[\max_{1 \leq j \leq n} Y_j^2] \leq \frac{(|q-p|+1)^2}{n^{1+\epsilon}} \leq (|q-p| + 1)^2$, then $E[\max_{1 \leq j \leq n} Y_{nj}^2]$ is uniformly bounded in n , and for $1 \leq j \leq n$, $\mathcal{F}_{nj} = \mathcal{F}_j = \mathcal{F}_{n+1j}$. By martingale central theorem [22], we have

$$\sum_{j=1}^n Y_{nj} = \frac{(q-p)M_n + N_n - N}{n^{1+\epsilon}} \xrightarrow{d} W,$$

where W is a random variable with characteristic function 1, which means that $W = 0$, a.s.

In this way, we have

$$\frac{(q-p)M_n + N_n}{n^{1+\epsilon}} \xrightarrow{d} 0.$$

6. Assumptions and Markovian properties of Model II

6.1. Notations and assumptions

Note that assumption (8) in Section 2.2 does not fit the reality. This is because for sufficiently large N_{i-1} , it may emerge that $\Pr\{Z_i = 1 | A_i = j, N_{i-1} = k\} = k\theta_j > 1$. Thus the assumption should be replaced by a more realistic one or some constraint should be imposed on the value of N_{i-1} . Before doing so, we introduce the following mathematical notations that are different from those adopted in Model I:

\hat{M}_i : total number of failures revealed by A_1, A_2, \dots, A_i .

\hat{N}_i : the number of defects remaining in the software under test after the action A_i is finished.

$\hat{M}(t)$: total number of failures revealed during the time interval $[0, t]$.

$\hat{N}(t)$: the number of defects remaining in the software under test at the time instant t .

N_b : the largest possible number of defects remaining in the software under test.

\tilde{P} : the transition matrix of the Markov chain $(\hat{N}_k, A_{k+1})_{k=0,1,\dots}$.

\tilde{P}^* : the limit matrix of \tilde{P} .

O : the matrix of dimension $m \times m$, with all entries being 0.

$\tau(n, i)$: the number of testing steps taken when no defect is remaining in the software, given that there are n defects starting with action i .

Recall that assumption (4) in Model I implies that no matter how many defects are remaining in the software under test, it is always possible that new defects are introduced. This is unrealistic for competent programmers. Normally, if there are sufficiently large number of defects remaining in the software under test, some of them should be easily located and

removed. Debugging activities should not increase the number of remaining defects, if they do not reduce it. Therefore, the following assumption can be adopted to replace assumption (4) of Model I. This is the major difference between Model I and Model II introduced in this section. The remaining assumptions of Model I are kept for Model II. The new assumption is as follows:

$$\hat{N}_k = \begin{cases} \hat{N}_{k-1}, & \text{with probability } 1 - p^{(\hat{N}_{k-1})} - q, \\ \hat{N}_{k-1} + 1, & \text{with probability } p^{(\hat{N}_{k-1})}, \\ \hat{N}_{k-1} - 1, & \text{with probability } q \end{cases}$$

$$p^{(i)} = \begin{cases} p, & \text{if } i < N_b, \\ 0, & \text{else} \end{cases} \quad (6.22)$$

$$0 \leq p, q \leq 1, \quad \text{and} \quad 0 \leq p + q \leq 1.$$

Eq. (6.22) implies that there is an upper bound for the number of remaining defects throughout the software testing process. Since assumption (8) of Model I remains valid and $k \leq N_b$, an additional assumption is required for the new model: $\theta_j N_b \leq 1, j = 1, 2, \dots, m$.

Note that assumptions (1)–(12) are kept for Model II except assumption (4), so it is easy to verify that Propositions 3.1–3.4 still hold.

6.2. Transition probability matrix of Markov chain

Consider the bivariate Markov chain $\{(\hat{N}_k, A_{k+1}); k = 0, 1, \dots\}$ in state space $S = \{(n, i) | 0 \leq n \leq N_b, 1 \leq i \leq m\}$. Its one-step transition probability is

$$p_{(i,j),(i',j')} = \begin{cases} (1 - (p + q)\theta_j)p_{jj'}, & i' = i, & 1 \leq i < N_b, \\ p\theta_j p_{jj'}, & i' = i + 1, & 1 \leq i < N_b, \\ q\theta_j p_{jj'}, & i' = i - 1, & 1 \leq i < N_b, \\ (1 - q\theta_j)p_{jj'}, & i' = i, & i = N_b, \\ q\theta_j p_{jj'}, & i' = i - 1, & i = N_b, \\ 1, & i' = i, & i = 0, j' = j, \\ 0, & \text{other.} \end{cases}$$

Then the transition matrix of state space S is

$$\tilde{P} = (p_{(i,j),(i',j')})_{i,i' \in \{0,1,\dots,N_b\}; j,j' \in \{1,2,\dots,m\}}$$

$$= \begin{pmatrix} I & & & & & & & \\ q\Theta P & (I - (p + q)\Theta)P & p\Theta P & & & & & \\ & 2q\Theta P & (I - 2(p + q)\Theta)P & 2p\Theta P & & & & \\ & & & \ddots & \ddots & \ddots & & \\ & & & & (N_b - 1)q\Theta P & (I - (N_b - 1)(p + q)\Theta)P & (N_b - 1)p\Theta P & \\ & & & & & N_b q\Theta P & (I - N_b q\Theta)P \end{pmatrix},$$

where \tilde{P} is a $(N_b + 1)m \times (N_b + 1)m$ -dimension matrix.

6.3. The stationary transition probability matrix \tilde{P}

Let $\mathbf{y} = (\mathbf{y}_0, \mathbf{y}_1, \dots, \mathbf{y}_{N_b})$ be a row vector, where \mathbf{y}_i is a nonnegative row vector of $1 \times m, i = 0, 1, \dots, N_b$. We solve $\mathbf{y}\tilde{P} = \mathbf{y}$, and $\sum_{k=0}^{N_b} \mathbf{y}_k \cdot \mathbf{1} = 1$ below.

Based on the different values of q , we investigate two cases.

Case 1. If $q \neq 0, \mathbf{y}_i = 0, i = 1, 2, \dots, N_b, \mathbf{y}_0 \cdot \mathbf{1} = 1$. Obviously, $\{(0, i), 1 \leq i \leq m\}$ are all absorbing states, and the states $D = \{(n, i), 1 \leq n \leq N_b, 1 \leq i \leq m\}$ are all transient. Thus we have

$$\lim_{k \rightarrow \infty} \hat{N}_k = 0, \quad \text{a.s.} \quad (6.23)$$

First we consider the limit matrix \tilde{P}^* of Markov chain $\{(\hat{N}_k, A_{k+1}); k = 0, 1, \dots\}$. Based on the general theory of finite state Markov chains, we know that $\tilde{P}^* = \lim_{N \rightarrow \infty} \frac{1}{N+1} \sum_{i=0}^N \tilde{P}^i$.

Let $\tilde{P}^* = (q_{(i,j),(i',j')}^*)$, then

$$q_{(i,j),(i',j')}^* = \begin{cases} 1, & i = i' = 0, j = j', \\ \alpha_{(i,j),j'}, & i > 0, i' = 0, \\ 0, & \text{other.} \end{cases} \quad (6.24)$$

The parameter set $\{\alpha_{(n,i),j}\}$ satisfies the linear system

$$\alpha_{(n,i),j} = p_{(n,i),(0,j)} + \sum_{(n',i') \in D} p_{(n,i),(n',i')} \alpha_{(n',i'),j}. \quad (6.25)$$

Furthermore, according to (6.24) and (6.25), $\{\alpha_{(n,i),j}\}$ is the unique solution of the following equations.

$$\begin{aligned} \alpha_{(1,i),j} &= qp_{ij}\theta_i + \sum_{k=1}^m p_{ik}(1 - (p+q)\theta_k)\alpha_{(1,k),j} + \sum_{k=1}^m pp_{ik}\theta_k\alpha_{(2,k),j}, \\ \alpha_{(n,i),j} &= \sum_{k=1}^m qnp_{ik}\theta_k\alpha_{(n-1,k),j} + \sum_{k=1}^m p_{ik}(1 - (p+q)n\theta_k)\alpha_{(n,k),j} + \sum_{k=1}^m pnp_{ik}\theta_k\alpha_{(n+1,k),j}, \quad (2 \leq n \leq N_b - 1) \\ \alpha_{(N_b,i),j} &= \sum_{k=1}^m qN_bp_{ik}\theta_k\alpha_{(N_b-1,k),j} + \sum_{k=1}^m p_{ik}(1 - qN_b\theta_k)\alpha_{(N_b,k),j}. \end{aligned}$$

Let $\alpha_{n,j} = (\alpha_{(n,1),j}, \alpha_{(n,2),j}, \dots, \alpha_{(n,m),j})^T$, we have

$$\begin{aligned} \alpha_{1,j} &= q\Theta Pe_j + (I - (p+q)\Theta)P\alpha_{1,j} + p\Theta P\alpha_{2,j}, \\ \alpha_{n,j} &= qn\Theta P\alpha_{n-1,j} + (I - (p+q)n\Theta)P\alpha_{n,j} + p\Theta P\alpha_{n+1,j}, \quad (2 \leq n \leq N_b - 1) \\ \alpha_{N_b,j} &= qN_b\Theta P\alpha_{N_b-1,j} + (I - qN_b\Theta)P\alpha_{N_b,j}, \end{aligned} \quad (6.26)$$

where e_j is a $m \times 1$ column vector in which the j th entry is 1 and other entries are 0.

By rewriting (6.26) in the matrix form we obtain

$$\begin{pmatrix} \alpha_{1,j} \\ \alpha_{2,j} \\ \vdots \\ \alpha_{N_b,j} \end{pmatrix} = \begin{pmatrix} (I - (p+q)\Theta)P & p\Theta P & & \\ 2q\Theta P & (I - 2(p+q)\Theta)P & \ddots & \\ & \ddots & \ddots & \\ qN_b\Theta P & (I - qN_b\Theta)P & & \end{pmatrix} \begin{pmatrix} \alpha_{1,j} \\ \alpha_{2,j} \\ \vdots \\ \alpha_{N_b,j} \end{pmatrix} + \begin{pmatrix} qP\Theta e_j \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Thus

$$\begin{aligned} \begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} & \cdots & \alpha_{1,m} \\ \alpha_{2,1} & \alpha_{2,2} & \cdots & \alpha_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{N_b,1} & \alpha_{N_b,2} & \cdots & \alpha_{N_b,m} \end{pmatrix} &= \begin{pmatrix} (I - (p+q)\Theta)P & p\Theta P & & \\ 2q\Theta P & (I - 2(p+q)\Theta)P & \ddots & \\ & \ddots & \ddots & \\ qN_b\Theta P & (I - qN_b\Theta)P & & \end{pmatrix} \\ &\times \begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} & \cdots & \alpha_{1,m} \\ \alpha_{2,1} & \alpha_{2,2} & \cdots & \alpha_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{N_b,1} & \alpha_{N_b,2} & \cdots & \alpha_{N_b,m} \end{pmatrix} + \begin{pmatrix} q\Theta P \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \end{aligned} \quad (6.27)$$

Let $A = \begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} & \cdots & \alpha_{1,m} \\ \alpha_{2,1} & \alpha_{2,2} & \cdots & \alpha_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{N_b,1} & \alpha_{N_b,2} & \cdots & \alpha_{N_b,m} \end{pmatrix}$, the above equation can be rewritten as

$$\begin{pmatrix} I \\ A \end{pmatrix} = \tilde{P} \begin{pmatrix} I \\ A \end{pmatrix}.$$

Then we have following result.

Lemma 6.1. If P is irreducible and $q \neq 0$, the maximal eigenvalue of

$$B = \begin{pmatrix} (I - (p+q)\Theta)P & p\Theta P & & \\ 2q\Theta P & (I - 2(p+q)\Theta)P & \ddots & \\ & \ddots & \ddots & \\ qN_b\Theta P & (I - qN_b\Theta)P & & \end{pmatrix} \quad (6.28)$$

is less than 1.

Proof. In fact the maximal row sum of the nonnegative matrix B is less or equal to 1, thus the maximal eigenvalue of B is less or equal to 1. If 1 is B 's eigenvalue, from Frobenius–Perron theorem we know that there exists nonnegative nonzero vectors $(\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_{N_b})$ such that $(\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_{N_b}) = (\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_{N_b})B$. That is,

$$\begin{aligned}\mathbf{y}_1 &= \mathbf{y}_1(I - (p + q)\Theta)P + 2\mathbf{y}_2q\Theta P, \\ \mathbf{y}_i &= (i - 1)\mathbf{y}_{i-1}p\Theta P + \mathbf{y}_i(I - i(p + q)\Theta)P + \mathbf{y}_{i+1}(i + 1)q\Theta P, \quad (2 \leq i \leq N_b - 1) \\ \mathbf{y}_{N_b} &= (N_b - 1)\mathbf{y}_{N_b-1}p\Theta P + \mathbf{y}_{N_b}(I - N_bq\Theta)P.\end{aligned}$$

Multiplying P^* from the right side of the above N_b equations, we obtain

$$\mathbf{y}_1(p + q)\Theta P^* = 2\mathbf{y}_2q\Theta P^*, \quad (6.29)$$

$$i\mathbf{y}_i(p + q)\Theta P^* = (i - 1)\mathbf{y}_{i-1}p\Theta P^* + \mathbf{y}_{i+1}(i + 1)q\Theta P^*, \quad (2 \leq i \leq N_b - 1) \quad (6.30)$$

$$N_b\mathbf{y}_{N_b}q\Theta P^* = (N_b - 1)\mathbf{y}_{N_b-1}p\Theta P^*. \quad (6.31)$$

From (6.30) and (6.31) we know

$$i\mathbf{y}_iq\Theta P^* = (i - 1)\mathbf{y}_{i-1}p\Theta P^*, \quad 2 \leq i \leq N_b. \quad (6.32)$$

Especially for $i = 2$, we have $2\mathbf{y}_2q\Theta P^* = \mathbf{y}_1p\Theta P^*$. And substituting above equation to (6.29) and considering $q \neq 0$, we obtain $\mathbf{y}_1\Theta P^* = \mathbf{0}$. Thus, from (6.32) we obtain that $\mathbf{y}_i\Theta P^* = \mathbf{0}$, $1 \leq i \leq N_b$. Because P is irreducible, $\mathbf{y}_i = \mathbf{0}$ for all $i = 1, 2, \dots, m$, which contradicts with the assumption of $(\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_{N_b}) \neq \mathbf{0}$. \square

From Lemma 6.1, the inverse matrix of $I - B$ exists. From (6.27) we obtain

$$A = \begin{pmatrix} I - (I - (p + q)\Theta)P & -p\Theta P & & & \\ -2q\Theta P & I - (I - 2(p + q)\Theta)P & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & -qN_b\Theta P & I - (I - qN_b\Theta)P \end{pmatrix}^{-1} \begin{pmatrix} q\Theta P \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Thus, \tilde{P}^* is

$$\begin{pmatrix} I & 0 \\ 0 & A \end{pmatrix}.$$

Case 2. If $q = 0$, then

$$\mathbf{y}_1 = \mathbf{y}_1(I - p\Theta)P, \quad (6.33)$$

$$\mathbf{y}_k = \mathbf{y}_{k-1}(k - 1)p\Theta P + \mathbf{y}_k(I - kp\Theta)P, \quad (2 \leq k \leq N_b - 1) \quad (6.34)$$

$$\mathbf{y}_{N_b} = \mathbf{y}_{N_b-1}(N_b - 1)p\Theta P + \mathbf{y}_{N_b}P. \quad (6.35)$$

The maximal eigenvalue of $(I - kp\Theta)P$ ($1 \leq k \leq N_b$) is less than 1, then $\mathbf{y}_1 = \mathbf{0}$ by (6.33). Recursively, we have $\mathbf{y}_k = \mathbf{0}$, $1 \leq k \leq N_b - 1$ by (6.34), and $\mathbf{y}_{N_b} = \mathbf{y}_{N_b}P$ by (6.35).

In this case, $\{(0, i)\}$, $1 \leq i \leq m$ and $\{(N_b, i)\}$, $1 \leq i \leq m\}$ are all absorbing states, and $D = \{(n, i), 1 \leq n \leq N_b - 1, 1 \leq i \leq m\}$ are all transient states. Starting from any state in D , the process will enter the state set $\{(N_b, i), 1 \leq i \leq m\}$ with probability one.

Then, we obtain \tilde{P}^* as follows:

$$\tilde{P}^* = \begin{pmatrix} I & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & P^* \\ 0 & 0 & \cdots & 0 & P^* \\ & \vdots & & & \\ 0 & 0 & \cdots & 0 & P^* \end{pmatrix},$$

where P^* is the limit of $\frac{1}{N+1} \sum_{i=0}^N P^i$.

6.4. Testing steps analysis for $\hat{N}_k = 0$

In order to study the dynamic behavior and terminal action of Markov chain $\{(\hat{N}_k, A_{k+1}), k = 1, 2, \dots\}$, we define $\tau(n, i) = \min\{k \geq 1 | \hat{N}_k = 0, \hat{N}_0 = n, A_1 = i\}$, and $\tau(n) = (\tau(n, 1), \tau(n, 2), \dots, \tau(n, m))^T, 0 \leq n \leq N_b$.

Proposition 6.1.

$$\begin{aligned} E\tau(0) &= \mathbf{0}, \\ E\tau(n) &= \mathbf{1} + (1 - n(p + q)\Theta)PE\tau(n) + nq\Theta PE\tau(n - 1) + np\Theta PE\tau(n + 1), \quad (1 \leq n \leq N_b - 1) \\ E\tau(N_b) &= \mathbf{1} + (1 - N_b q)\Theta PE\tau(N_b) + nq\Theta PE\tau(N_b - 1). \end{aligned} \quad (6.36)$$

Proof. Because $\tau(0) = \mathbf{0}$, we have $E\tau(0) = \mathbf{0}$.

By the first-step-analysis, we obtain

$$\begin{aligned} E\tau(n, i) &= 1 + (1 - n(p + q)\theta_i) \sum_{j=1}^m E\tau(n, j) + nq\theta_i \sum_{j=1}^m E\tau(n - 1, j) + np\theta_i \sum_{j=1}^m E\tau(n + 1, j), \quad (1 \leq n \leq N_b - 1) \\ E\tau(N_b, i) &= 1 + (1 - N_b q\theta_i) \sum_{j=1}^m E\tau(N_b, j) + N_b q\theta_i \sum_{j=1}^m E\tau(N_b - 1, j). \end{aligned}$$

Writing the above equations in the vector form, we complete the proof of Proposition 6.1. \square

We rewrite (6.36) as:

$$\begin{pmatrix} E\tau(1) \\ E\tau(2) \\ \vdots \\ E\tau(N_b) \end{pmatrix} = \begin{pmatrix} (I - (p + q)\Theta)P & p\Theta P & & \\ 2q\Theta P & (I - 2(p + q)\Theta)P & \ddots & \\ & \ddots & \ddots & \\ N_b q\Theta P & & (I - N_b q\Theta)P & \end{pmatrix} \begin{pmatrix} E\tau(1) \\ E\tau(2) \\ \vdots \\ E\tau(N_b) \end{pmatrix} + \begin{pmatrix} \mathbf{1} \\ \mathbf{1} \\ \vdots \\ \mathbf{1} \end{pmatrix}. \quad (6.37)$$

When $q > 0$, since the matrix $I - B$ is non-singular, (6.37) yields

$$\begin{pmatrix} E\tau(1) \\ E\tau(2) \\ \vdots \\ E\tau(N_b) \end{pmatrix} = \begin{pmatrix} I - (I - (p + q)\Theta)P & -p\Theta P & & \\ -2q\Theta P & I - (I - 2(p + q)\Theta)P & \ddots & \\ & \ddots & \ddots & \\ -N_b q\Theta P & & I - (I - N_b q\Theta)P & \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{1} \\ \mathbf{1} \\ \vdots \\ \mathbf{1} \end{pmatrix}.$$

The following proposition can be obtained in a way that is similar to that for Proposition 6.1, and the proof is omitted.

Proposition 6.2.

$$\begin{aligned} E[\tau(0)]^2 &= \mathbf{0}, \\ E[\tau(n)]^2 &= \mathbf{1} + 2E\tau(n) + (1 - n(p + q)\Theta)PE\tau(n)^2 + nq\Theta PE\tau(n - 1)^2 + np\Theta PE\tau(n + 1)^2, \\ &\quad (1 \leq n \leq N_b - 1) \\ E[\tau(N_b)]^2 &= \mathbf{1} + 2E\tau(N_b) + (1 - N_b q)\Theta PE\tau(N_b)^2 + nq\Theta PE\tau(N_b - 1)^2. \end{aligned}$$

In the vector form, it holds

$$\begin{aligned} \begin{pmatrix} E[\tau(1)]^2 \\ E[\tau(2)]^2 \\ \vdots \\ E[\tau(N_b)]^2 \end{pmatrix} &= \begin{pmatrix} I - (I - (p + q)\Theta)P & -p\Theta P & & \\ -2q\Theta P & I - (I - 2(p + q)\Theta)P & \ddots & \\ & \ddots & \ddots & \\ -N_b q\Theta P & & I - (I - N_b q\Theta)P & \end{pmatrix}^{-1} \\ &\quad \times \left(\begin{pmatrix} \mathbf{1} \\ \mathbf{1} \\ \vdots \\ \mathbf{1} \end{pmatrix} + \begin{pmatrix} E\tau(1) \\ E\tau(2) \\ \vdots \\ E\tau(N_b) \end{pmatrix} \right). \end{aligned}$$

7. Expected behaviors of Model II

Let

$$\begin{aligned}\hat{\alpha}_k(i) &= E[\hat{M}_k, A_k = i], \quad 1 \leq i \leq m, & \hat{\alpha}_k &= (\hat{\alpha}_k(1), \hat{\alpha}_k(2), \dots, \hat{\alpha}_k(m)); \\ \hat{\beta}_k(i) &= E[\hat{N}_k, A_k = i], \quad 1 \leq i \leq m, & \hat{\beta}_k &= (\hat{\beta}_k(1), \hat{\beta}_k(2), \dots, \hat{\beta}_k(m));\end{aligned}$$

and

$$\hat{\gamma}_k = (\hat{\gamma}_k(0), \hat{\gamma}_k(1), \dots, \hat{\gamma}_k(N_b)),$$

where

$$\hat{\gamma}_k(i) = (\Pr\{\hat{N}_k = i, A_{k+1} = 1\}, \Pr\{\hat{N}_k = i, A_{k+1} = 2\}, \dots, \Pr\{\hat{N}_k = i, A_{k+1} = m\}), \quad 0 \leq i \leq N_b, 1 \leq j \leq m.$$

Then we have following result.

Theorem 7.1.

$$\begin{aligned}\hat{\alpha}_k &= \hat{\alpha}_{k-1}P + \hat{\beta}_{k-1}P\Theta, \\ \hat{\beta}_k &= \hat{\beta}_{k-1}P(I + (p - q)\Theta) - pN_b\hat{\gamma}_{k-1}(N_b)\Theta, \\ \hat{\gamma}_k &= \hat{\gamma}_{k-1}\tilde{P} = \dots = \hat{\gamma}_0\tilde{P}^k,\end{aligned}$$

where $\hat{\gamma}_k(N_b) = \hat{\gamma}_k e_{N_b}$, and e_{N_b} is a $(N_b + 1)m \times m$ matrix defined as:

$$e_{N_b} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ I \end{pmatrix}.$$

Proof. We first calculate $\hat{\alpha}_k(i)$.

$$\begin{aligned}E[\hat{M}_k, A_k = i] &= \sum_{j=0}^{N_b} \sum_{\ell=1}^{\infty} \ell \sum_{i'=1}^m (1 - j\theta_i) p_{i'i} \Pr\{\hat{M}_{k-1} = \ell, A_{k-1} = i', \hat{N}_{k-1} = j\} \\ &\quad + \sum_{j=0}^{N_b} \sum_{\ell=1}^{\infty} \ell \sum_{i'=1}^m (1 - p^{(j)} - q) j\theta_i p_{i'i} \Pr\{\hat{M}_{k-1} = \ell - 1, A_{k-1} = i', \hat{N}_{k-1} = j\} \\ &\quad + \sum_{j=1}^{N_b} \sum_{\ell=1}^{\infty} \ell \sum_{i'=1}^m p^{(j-1)} \cdot (j - 1)\theta_i p_{i'i} \Pr\{\hat{M}_{k-1} = \ell - 1, A_{k-1} = i', \hat{N}_{k-1} = j - 1\} \\ &\quad + \sum_{j=0}^{N_b-1} \sum_{\ell=1}^{\infty} \ell \sum_{i'=1}^m q(j + 1)\theta_i p_{i'i} \Pr\{\hat{M}_{k-1} = \ell - 1, A_{k-1} = i', \hat{N}_{k-1} = j + 1\} \\ &= \sum_{j=0}^{N_b} \sum_{\ell=1}^{\infty} \ell \sum_{i'=1}^m (1 - j\theta_i) p_{i'i} \Pr\{\hat{M}_{k-1} = \ell, A_{k-1} = i', \hat{N}_{k-1} = j\} \\ &\quad + \sum_{j=0}^{N_b} \sum_{\ell=1}^{\infty} \ell \sum_{i'=1}^m (1 - p^{(j)} - q) j\theta_i p_{i'i} \Pr\{\hat{M}_{k-1} = \ell - 1, A_{k-1} = i', \hat{N}_{k-1} = j\} \\ &\quad + \sum_{j=0}^{N_b} \sum_{\ell=1}^{\infty} \ell \sum_{i'=1}^m p^{(j)} \cdot j\theta_i p_{i'i} \Pr\{\hat{M}_{k-1} = \ell - 1, A_{k-1} = i', \hat{N}_{k-1} = j\} \\ &\quad + \sum_{j=0}^{N_b} \sum_{\ell=1}^{\infty} \ell \sum_{i'=1}^m qj\theta_i p_{i'i} \Pr\{\hat{M}_{k-1} = \ell - 1, A_{k-1} = i', \hat{N}_{k-1} = j\} \\ &= \sum_{j=0}^{N_b} \sum_{\ell=1}^{\infty} \ell \sum_{i'=1}^m (1 - j\theta_i) p_{i'i} \Pr\{\hat{M}_{k-1} = \ell, A_{k-1} = i', \hat{N}_{k-1} = j\}\end{aligned}$$

$$\begin{aligned}
& + \sum_{j=0}^{N_b} \sum_{\ell=1}^{\infty} \ell \sum_{i'=1}^m j \theta_i p_{i'i} \Pr\{\hat{M}_{k-1} = \ell - 1, A_{k-1} = i', \hat{N}_{k-1} = j\} \\
& = \sum_{j=0}^{N_b} \sum_{\ell=1}^{\infty} \ell \sum_{i'=1}^m (1 - j \theta_i) p_{i'i} \Pr\{\hat{M}_{k-1} = \ell, A_{k-1} = i', \hat{N}_{k-1} = j\} \\
& \quad + \sum_{j=0}^{N_b} \sum_{\ell=0}^{\infty} (\ell + 1) \sum_{i'=1}^m j \theta_i p_{i'i} \Pr\{\hat{M}_{k-1} = \ell, A_{k-1} = i', \hat{N}_{k-1} = j\} \\
& = \sum_{j=0}^{N_b} \sum_{\ell=0}^{\infty} \ell \sum_{i'=1}^m (1 - j \theta_i) p_{i'i} \Pr\{\hat{M}_{k-1} = \ell, A_{k-1} = i', \hat{N}_{k-1} = j\} \\
& \quad + \sum_{j=0}^{N_b} \sum_{\ell=0}^{\infty} \ell \sum_{i'=1}^m j \theta_i p_{i'i} \Pr\{\hat{M}_{k-1} = \ell, A_{k-1} = i', \hat{N}_{k-1} = j\} \\
& \quad + \sum_{j=0}^{N_b} \sum_{\ell=0}^{\infty} \sum_{i'=1}^m j \theta_i p_{i'i} \Pr\{\hat{M}_{k-1} = \ell, A_{k-1} = i', \hat{N}_{k-1} = j\} \\
& = \sum_{j=0}^{N_b} \sum_{\ell=0}^{\infty} \ell \sum_{i'=1}^m p_{i'i} \Pr\{\hat{M}_{k-1} = \ell, A_{k-1} = i', \hat{N}_{k-1} = j\} \\
& \quad + \sum_{j=0}^{N_b} \sum_{\ell=0}^{\infty} \sum_{i'=1}^m j \theta_i p_{i'i} \Pr\{\hat{M}_{k-1} = \ell, A_{k-1} = i', \hat{N}_{k-1} = j\} \\
& = \sum_{i'=1}^m p_{i'i} E[\hat{M}_{k-1}, A_{k-1} = i'] + \theta_i \sum_{i'=1}^m p_{i'i} E[\hat{N}_{k-1}, A_{k-1} = i']. \tag{7.38}
\end{aligned}$$

Then we calculate $\hat{\beta}_k(i)$.

$$\begin{aligned}
E[\hat{N}_k, A_k = i] & = \sum_{j=0}^{N_b} j \sum_{\ell=0}^{\infty} \sum_{i'=1}^m (1 - j \theta_i) p_{i'i} \Pr\{\hat{M}_{k-1} = \ell, A_{k-1} = i', \hat{N}_{k-1} = j\} \\
& \quad + \sum_{j=0}^{N_b} j \sum_{\ell=0}^{\infty} \sum_{i'=1}^m (1 - p^{(j)} - q) j \theta_i p_{i'i} \Pr\{\hat{M}_{k-1} = \ell - 1, A_{k-1} = i', \hat{N}_{k-1} = j\} \\
& \quad + \sum_{j=0}^{N_b-1} (j + 1) \sum_{\ell=0}^{\infty} \sum_{i'=1}^m p^{(j)} \cdot j \theta_i p_{i'i} \Pr\{\hat{M}_{k-1} = \ell - 1, A_{k-1} = i', \hat{N}_{k-1} = j\} \\
& \quad + \sum_{j=1}^{N_b} (j - 1) \sum_{\ell=0}^{\infty} \sum_{i'=1}^m q j \theta_i p_{i'i} \Pr\{\hat{M}_{k-1} = \ell - 1, A_{k-1} = i', \hat{N}_{k-1} = j\} \\
& = \sum_{j=0}^{N_b} j \sum_{\ell=0}^{\infty} \sum_{i'=1}^m (1 - j \theta_i) p_{i'i} \Pr\{\hat{M}_{k-1} = \ell, A_{k-1} = i', \hat{N}_{k-1} = j\} \\
& \quad + \sum_{j=0}^{N_b} j \sum_{\ell=0}^{\infty} \sum_{i'=1}^m (1 - p^{(j)} - q) j \theta_i p_{i'i} \Pr\{\hat{M}_{k-1} = \ell, A_{k-1} = i', \hat{N}_{k-1} = j\} \\
& \quad + \sum_{j=0}^{N_b-1} j \sum_{\ell=0}^{\infty} \sum_{i'=1}^m p^{(j)} \cdot j \theta_i p_{i'i} \Pr\{\hat{M}_{k-1} = \ell, A_{k-1} = i', \hat{N}_{k-1} = j\} \\
& \quad + \sum_{j=0}^{N_b-1} \sum_{\ell=0}^{\infty} \sum_{i'=1}^m p^{(j)} \cdot j \theta_i p_{i'i} \Pr\{\hat{M}_{k-1} = \ell, A_{k-1} = i', \hat{N}_{k-1} = j\} \\
& \quad + \sum_{j=1}^{N_b} j \sum_{\ell=0}^{\infty} \sum_{i'=1}^m q j \theta_i p_{i'i} \Pr\{\hat{M}_{k-1} = \ell, A_{k-1} = i', \hat{N}_{k-1} = j\}
\end{aligned}$$

$$\begin{aligned}
& - \sum_{j=1}^{N_b} \sum_{\ell=0}^{\infty} \sum_{i'=1}^m qj\theta_i p_{i'} \Pr\{\hat{M}_{k-1} = \ell, A_{k-1} = i', \hat{N}_{k-1} = j\} \\
& = \sum_{j=0}^{N_b} j \sum_{\ell=0}^{\infty} \sum_{i'=1}^m \Pr\{\hat{M}_{k-1} = \ell, A_{k-1} = i', \hat{N}_{k-1} = j\} \\
& \quad + \sum_{j=0}^{N_b-1} \sum_{\ell=0}^{\infty} \sum_{i'=1}^m p^{(j)} \cdot j\theta_i p_{i'} \Pr\{\hat{M}_{k-1} = \ell, A_{k-1} = i', \hat{N}_{k-1} = j\} \\
& \quad - \sum_{j=1}^{N_b} \sum_{\ell=0}^{\infty} \sum_{i'=1}^m qj\theta_i p_{i'} \Pr\{\hat{M}_{k-1} = \ell, A_{k-1} = i', \hat{N}_{k-1} = j\} \\
& = \sum_{j=0}^{N_b} j \sum_{\ell=0}^{\infty} \sum_{i'=1}^m p_{i'} \Pr\{\hat{M}_{k-1} = \ell, A_{k-1} = i', \hat{N}_{k-1} = j\} \\
& \quad + \sum_{j=0}^{N_b} \sum_{\ell=0}^{\infty} \sum_{i'=1}^m (p^{(j)} - q)j\theta_i p_{i'} \Pr\{\hat{M}_{k-1} = \ell, A_{k-1} = i', \hat{N}_{k-1} = j\} \\
& = \sum_{j=0}^{N_b} j \sum_{\ell=0}^{\infty} \sum_{i'=1}^m p_{i'} (1 + (p - q)\theta_i) \Pr\{\hat{M}_{k-1} = \ell, A_{k-1} = i', \hat{N}_{k-1} = j\} \\
& \quad - pN_b\theta_i \sum_{i'=1}^m p_{i'} \Pr\{\hat{N}_{k-1} = N_b, A_{k-1} = i'\} \\
& = \sum_{i'=1}^m p_{i'} (1 + (p - q)\theta_i) E[\hat{N}_{k-1}, A_{k-1} = i'] - pN_b\theta_i \Pr\{\hat{N}_{k-1} = N_b, A_k = i\}. \tag{7.39}
\end{aligned}$$

Rewrite (7.38) and (7.39) in the vector form and this completes the proof. \square

Now we calculate the initial values $\hat{\alpha}_1$, $\hat{\beta}_1$, and $\hat{\gamma}_0$, respectively. For any $\hat{N}_0 = N \leq N_b - 1$, and let $p_i = \Pr\{A_1 = i\}$, $i = 1, 2, \dots, m$, we have

$$\begin{aligned}
\hat{\alpha}_1(i) &= E[\hat{M}_1, A_1 = i] = \Pr\{\hat{M}_1 = 1, A_1 = i\} = \Pr\{\hat{M}_1 = 1|A_1 = i\} \Pr\{A_1 = i\} \\
&= N_0\theta_i p_i = N\theta_i p_i,
\end{aligned}$$

thus $\hat{\alpha}_1 = N\mathbf{p}_0\Theta$; and

$$\begin{aligned}
\hat{\beta}_1(i) &= E[\hat{N}_1, A_1 = i] \\
&= N \Pr\{\hat{N}_1 = N, A_1 = i\} + (N + 1) \Pr\{\hat{N}_1 = N + 1, A_1 = i\} + (N - 1) \Pr\{\hat{N}_1 = N - 1, A_1 = i\} \\
&= N[\Pr\{\hat{M}_1 = 0, \hat{N}_1 = N, A_1 = i\} + \Pr\{\hat{M}_1 = 1, \hat{N}_1 = N, A_1 = i\}] \\
&\quad + (N + 1) \Pr\{\hat{M}_1 = 1, \hat{N}_1 = N + 1, A_1 = i\} + (N - 1) \Pr\{\hat{M}_1 = 1, \hat{N}_1 = N - 1, A_1 = i\} \\
&= N[\Pr\{\hat{M}_1 = 0|A_1 = i\} \Pr\{A_1 = i\} + \Pr\{\hat{M}_1 = 1, \hat{N}_1 = N|A_1 = i\} \Pr\{A_1 = i\} \\
&\quad + (N + 1) \Pr\{\hat{M}_1 = 1, \hat{N}_1 = N + 1|A_1 = i\} \Pr\{A_1 = i\} + (N - 1) \Pr\{\hat{M}_1 = 1, \hat{N}_1 = N - 1|A_1 = i\} \Pr\{A_1 = i\}] \\
&= N[(1 - N\theta_i) + (1 - p - q)N\theta_i]p_i + (N + 1)pN\theta_i p_i + (N - 1)qN\theta_i p_i \\
&= Np_i(1 + (p - q)\theta_i),
\end{aligned}$$

that is $\hat{\beta}_1 = N\mathbf{p}_0(I + (p - q)\Theta)$.

Since $\hat{\gamma}_0 = \mathbf{p}_0(\underbrace{O, O, \dots, O}_N, I, O, \dots, O)$, we have

$$\begin{aligned}
\hat{\beta}_k &= \hat{\beta}_{k-1}P(I + (p - q)\Theta) - pN_b\hat{\gamma}_{k-1}(N_b)\Theta \\
&= \hat{\beta}_{k-1}P(I + (p - q)\Theta) - pN_b\hat{\gamma}_0\tilde{P}^{k-1}e_{N_b}\Theta \\
&= \hat{\beta}_1[P(I + (p - q)\Theta)]^{k-1} - pN_b\hat{\gamma}_0 \sum_{j=1}^{k-1} \tilde{P}^j e_{N_b} \Theta [P(I + (p - q)\Theta)]^{k-1-j} \\
&= N\mathbf{p}_0(I + (p - q)\Theta)[P(I + (p - q)\Theta)]^{k-1} - pN_b\hat{\gamma}_0 \sum_{j=1}^{k-1} \tilde{P}^j e_{N_b} \Theta [P(I + (p - q)\Theta)]^{k-1-j}.
\end{aligned}$$

With the above analysis, we have the following result.

Theorem 7.2. For $\hat{N}(t)$ we have

$$E[\hat{N}(t)] = N\mathbf{p}_0 e^{-\lambda t[I-(I+(p-q)\Theta)P]} \cdot \mathbf{1}^T - pN_b\hat{\gamma}_0 \sum_{k=0}^{\infty} \sum_{j=1}^{k-1} \tilde{P}^j e_{N_b} \Theta [P(I+(p-q)\Theta)]^{k-1-j} \cdot \mathbf{1}^T \frac{(\lambda t)^k}{k!} e^{-\lambda t}.$$

For $\hat{M}(t)$, when $p \neq q$, we have

$$E[\hat{M}(t)] = -\frac{N}{p-q} + \frac{N}{p-q} \mathbf{p}_0 e^{-\lambda t[I-(I+(p-q)\Theta)P]} \cdot \mathbf{1}^T \\ - \frac{pN_b}{p-q} \hat{\gamma}_0 \sum_{k=0}^{\infty} \sum_{j=1}^{k-2} \tilde{P}^j e_{N_b} \Theta \{[P(I+(p-q)\Theta)]^{k-1-j} - I\} \frac{(\lambda t)^k}{k!} e^{-\lambda t} \cdot \mathbf{1}^T;$$

and when $p = q$ and P is irreducible, we have

$$E[\hat{M}(t)] = N\mathbf{p}_0(I - e^{-\lambda t(I-P+P^*)})(I - P + P^*)^{-1}\Theta \cdot \mathbf{1}^T + N \sum_{i=1}^m \pi_i \theta_i (\lambda t - 1 + e^{-\lambda t}) \\ - pN_b\hat{\gamma}_0 \sum_{k=0}^{\infty} \sum_{j=1}^{k-2} \tilde{P}^j e_{N_b} \Theta (P - P^*)[I - (P - P^*)^{k-j-1}](I - P + P^*)^{-1}\Theta \cdot \mathbf{1}^T \frac{(\lambda t)^k}{k!} e^{-\lambda t} \\ - pN_b\hat{\gamma}_0 \sum_{k=0}^{\infty} \sum_{j=1}^{k-2} (k-j-1) \tilde{P}^j e_{N_b} \Theta P^* \Theta \cdot \mathbf{1}^T \frac{(\lambda t)^k}{k!} e^{-\lambda t}.$$

Proof. In order to consider $\hat{N}(t)$, let us investigate

$$E[\hat{N}_k] = \hat{\beta}_k \cdot \mathbf{1}^T \\ = N\mathbf{p}_0(I + (p-q)\Theta)[P(I+(p-q)\Theta)]^{k-1} - pN_b\hat{\gamma}_0 \sum_{j=1}^{k-1} \tilde{P}^j e_{N_b} \Theta [P(I+(p-q)\Theta)]^{k-1-j} \cdot \mathbf{1}^T \\ = N\mathbf{p}_0[I + (p-q)\Theta]P^k \cdot \mathbf{1}^T - pN_b\hat{\gamma}_0 \sum_{j=1}^{k-1} \tilde{P}^j e_{N_b} \Theta [P(I+(p-q)\Theta)]^{k-1-j} \cdot \mathbf{1}^T.$$

And then

$$E[\hat{N}(t)] = \sum_{k=0}^{\infty} E[\hat{N}_k] \frac{(\lambda t)^k}{k!} e^{-\lambda t} \\ = \sum_{k=0}^{\infty} N\mathbf{p}_0[I + (p-q)\Theta]P^k \cdot \mathbf{1}^T \frac{(\lambda t)^k}{k!} e^{-\lambda t} - \sum_{k=0}^{\infty} pN_b\hat{\gamma}_0 \sum_{j=1}^{k-1} \tilde{P}^j e_{N_b} \Theta [P(I+(p-q)\Theta)]^{k-1-j} \cdot \mathbf{1}^T \frac{(\lambda t)^k}{k!} e^{-\lambda t} \\ = N\mathbf{p}_0 e^{-\lambda t[I-(I+(p-q)\Theta)P]} \cdot \mathbf{1}^T - pN_b\hat{\gamma}_0 \sum_{k=0}^{\infty} \sum_{j=1}^{k-1} \tilde{P}^j e_{N_b} \Theta [P(I+(p-q)\Theta)]^{k-1-j} \cdot \mathbf{1}^T \frac{(\lambda t)^k}{k!} e^{-\lambda t}.$$

When $p \neq q$, we have

$$\hat{\beta}_k P \Theta \cdot \mathbf{1}^T = N\mathbf{p}_0(I + (p-q)\Theta)[P(I+(p-q)\Theta)]^{k-1} P \Theta \cdot \mathbf{1}^T - pN_b\hat{\gamma}_0 \sum_{j=1}^{k-1} \tilde{P}^j e_{N_b} \Theta [P(I+(p-q)\Theta)]^{k-1-j} P \Theta \cdot \mathbf{1}^T \\ = N\mathbf{p}_0[I + (p-q)\Theta]P^k \Theta \cdot \mathbf{1}^T - pN_b\hat{\gamma}_0 \sum_{j=1}^{k-1} \tilde{P}^j e_{N_b} \Theta P[I + (p-q)\Theta]P^{k-1-j} \Theta \cdot \mathbf{1}^T \\ = N\mathbf{p}_0[I + (p-q)\Theta]P^k \frac{(I + (p-q)\Theta)P - P}{p-q} \cdot \mathbf{1}^T \\ - pN_b\hat{\gamma}_0 \sum_{j=1}^{k-1} \tilde{P}^j e_{N_b} \Theta P[I + (p-q)\Theta]P^{k-1-j} \frac{(I + (p-q)\Theta)P - P}{p-q} \cdot \mathbf{1}^T$$

$$\begin{aligned}
&= \frac{N}{p-q} \mathbf{p}_0 \{ [(I + (p-q)\Theta)P]^{k+1} - [(I + (p-q)\Theta)P]^k \} \cdot \mathbf{1}^T \\
&\quad - \frac{pN_b}{p-q} \hat{\gamma}_0 \sum_{j=1}^{k-1} \tilde{p}^j e_{N_b} \Theta P \{ [(I + (p-q)\Theta)P]^{k-j} - [(I + (p-q)\Theta)P]^{k-1-j} \} \cdot \mathbf{1}^T.
\end{aligned}$$

For \hat{M}_k , if $p \neq q$, we have

$$\begin{aligned}
E[\hat{M}_k] &= \hat{\alpha}_k \cdot \mathbf{1}^T = \hat{\alpha}_1 \cdot \mathbf{1}^T + \sum_{i=1}^{k-1} \hat{\beta}_i P \Theta \cdot \mathbf{1}^T \\
&= N \mathbf{p}_0 \Theta \cdot \mathbf{1}^T + \frac{N}{p-q} \mathbf{p}_0 \{ [(I + (p-q)\Theta)P]^k - [(I + (p-q)\Theta)P] \} \cdot \mathbf{1}^T \\
&\quad - \frac{pN_b}{p-q} \hat{\gamma}_0 \sum_{i=1}^{k-1} \sum_{j=1}^{i-1} \tilde{p}^j e_{N_b} \Theta P \{ [(I + (p-q)\Theta)P]^{i-j} - [(I + (p-q)\Theta)P]^{i-1-j} \} \cdot \mathbf{1}^T \\
&= -\frac{N}{p-q} \mathbf{p}_0 \cdot \mathbf{1}^T + \frac{N}{p-q} \mathbf{p}_0 [(I + (p-q)\Theta)P]^k \cdot \mathbf{1}^T \\
&\quad - \frac{pN_b}{p-q} \hat{\gamma}_0 \sum_{j=1}^{k-2} \sum_{i=j+1}^{k-1} \tilde{p}^j e_{N_b} \Theta P \{ [(I + (p-q)\Theta)P]^{i-j} - [(I + (p-q)\Theta)P]^{i-1-j} \} \cdot \mathbf{1}^T \\
&= -\frac{N}{p-q} + \frac{N}{p-q} \mathbf{p}_0 [(I + (p-q)\Theta)P]^k \cdot \mathbf{1}^T - \frac{pN_b}{p-q} \hat{\gamma}_0 \sum_{j=1}^{k-2} \tilde{p}^j e_{N_b} \Theta P \{ [(I + (p-q)\Theta)P]^{k-1-j} - I \} \cdot \mathbf{1}^T \\
&= -\frac{N}{p-q} + \frac{N}{p-q} \mathbf{p}_0 [(I + (p-q)\Theta)P]^k \cdot \mathbf{1}^T - \frac{pN_b}{p-q} \hat{\gamma}_0 \sum_{j=1}^{k-2} \tilde{p}^j e_{N_b} \Theta \{ [P(I + (p-q)\Theta)]^{k-1-j} - I \} \cdot \mathbf{1}^T.
\end{aligned}$$

Thus, in this case, we have

$$\begin{aligned}
E[\hat{M}(t)] &= \sum_{k=0}^{\infty} E[\hat{M}_k] \frac{(\lambda t)^k}{k!} e^{-\lambda t} \\
&= -\frac{N}{p-q} + \frac{N}{p-q} \mathbf{p}_0 \sum_{k=0}^{\infty} [(I + (p-q)\Theta)P]^k \frac{(\lambda t)^k}{k!} e^{-\lambda t} \cdot \mathbf{1}^T \\
&\quad - \frac{pN_b}{p-q} \hat{\gamma}_0 \sum_{k=0}^{\infty} \sum_{j=1}^{k-2} \tilde{p}^j e_{N_b} \Theta P \{ [(I + (p-q)\Theta)P]^{k-1-j} - I \} \frac{(\lambda t)^k}{k!} e^{-\lambda t} \cdot \mathbf{1}^T \\
&= -\frac{N}{p-q} + \frac{N}{p-q} \mathbf{p}_0 e^{-\lambda t [I - (I + (p-q)\Theta)P]} \cdot \mathbf{1}^T \\
&\quad - \frac{pN_b}{p-q} \hat{\gamma}_0 \sum_{k=0}^{\infty} \sum_{j=1}^{k-2} \tilde{p}^j e_{N_b} \Theta \{ [P(I + (p-q)\Theta)]^{k-1-j} - I \} \frac{(\lambda t)^k}{k!} e^{-\lambda t} \cdot \mathbf{1}^T.
\end{aligned}$$

When $p = q$, we have

$$\hat{\beta}_k P \Theta \cdot \mathbf{1}^T = N \mathbf{p}_0 P^k \Theta \cdot \mathbf{1}^T - pN_b \hat{\gamma}_0 \sum_{j=1}^{k-1} \tilde{p}^j e_{N_b} \Theta P^{k-j} \Theta \cdot \mathbf{1}^T.$$

According to Eq. (4.10), we have

$$\begin{aligned}
E[\hat{M}_k] &= \hat{\alpha}_k \cdot \mathbf{1}^T = \hat{\alpha}_1 \cdot \mathbf{1}^T + \sum_{i=1}^{k-1} \hat{\beta}_i P \Theta \cdot \mathbf{1}^T \\
&= N \mathbf{p}_0 \Theta \cdot \mathbf{1}^T + N \mathbf{p}_0 \sum_{i=1}^{k-1} P^i \Theta \cdot \mathbf{1}^T - pN_b \hat{\gamma}_0 \sum_{i=1}^{k-1} \sum_{j=1}^{i-1} \tilde{p}^j e_{N_b} \Theta P^{i-j} \Theta \cdot \mathbf{1}^T \\
&= N \mathbf{p}_0 \Theta \cdot \mathbf{1}^T + N \mathbf{p}_0 \sum_{i=1}^{k-1} P^i \Theta \cdot \mathbf{1}^T - pN_b \hat{\gamma}_0 \sum_{j=1}^{k-2} \tilde{p}^j e_{N_b} \Theta \sum_{i=j+1}^{k-1} P^{i-j} \Theta \cdot \mathbf{1}^T
\end{aligned}$$

$$\begin{aligned}
&= N\mathbf{p}_0\Theta \cdot \mathbf{1}^T + N\mathbf{p}_0 \sum_{i=1}^{k-1} P^i \Theta \cdot \mathbf{1}^T - pN_b\hat{\gamma}_0 \sum_{j=1}^{k-2} \tilde{P}^j e_{N_b} \Theta \sum_{i=1}^{k-j-1} P^i \Theta \cdot \mathbf{1}^T \\
&= N\mathbf{p}_0\Theta \cdot \mathbf{1}^T + N\mathbf{p}_0(P - P^*)[I - (P - P^*)^{k-1}](I - P + P^*)^{-1} \Theta \cdot \mathbf{1}^T + N(k-1) \sum_{i=1}^m \pi_i \theta_i \\
&\quad - pN_b\hat{\gamma}_0 \sum_{j=1}^{k-2} \tilde{P}^j e_{N_b} \Theta \sum_{i=1}^{k-j-1} [(P - P^*)^i + P^*] \Theta \cdot \mathbf{1}^T \\
&= N\mathbf{p}_0\Theta \cdot \mathbf{1}^T + N\mathbf{p}_0(P - P^*)[I - (P - P^*)^{k-1}](I - P + P^*)^{-1} \Theta \cdot \mathbf{1}^T + N(k-1) \sum_{i=1}^m \pi_i \theta_i \\
&\quad - pN_b\hat{\gamma}_0 \sum_{j=1}^{k-2} \tilde{P}^j e_{N_b} \Theta (P - P^*)[I - (P - P^*)^{k-j-1}](I - P + P^*)^{-1} \Theta \cdot \mathbf{1}^T \\
&\quad - pN_b\hat{\gamma}_0 \sum_{j=1}^{k-2} (k-j-1) \tilde{P}^j e_{N_b} \Theta P^* \Theta \cdot \mathbf{1}^T.
\end{aligned}$$

And with the help of Eq. (4.11) we obtain

$$\begin{aligned}
E[\hat{M}(t)] &= \sum_{k=0}^{\infty} E[\hat{M}_k] \frac{(\lambda t)^k}{k!} e^{-\lambda t} \\
&= N\mathbf{p}_0(I - e^{-\lambda t(I-P+P^*)})(I - P + P^*)^{-1} \Theta \cdot \mathbf{1}^T + N \sum_{i=1}^m \pi_i \theta_i (\lambda t - 1 + e^{-\lambda t}) \\
&\quad - pN_b\hat{\gamma}_0 \sum_{k=0}^{\infty} \sum_{j=1}^{k-2} \tilde{P}^j e_{N_b} \Theta (P - P^*)[I - (P - P^*)^{k-j-1}](I - P + P^*)^{-1} \Theta \cdot \mathbf{1}^T \frac{(\lambda t)^k}{k!} e^{-\lambda t} \\
&\quad - pN_b\hat{\gamma}_0 \sum_{k=0}^{\infty} \sum_{j=1}^{k-2} (k-j-1) \tilde{P}^j e_{N_b} \Theta P^* \Theta \cdot \mathbf{1}^T \frac{(\lambda t)^k}{k!} e^{-\lambda t}. \quad \square
\end{aligned}$$

Similar to the discussion in Section 4.3, a nonlinear dynamic system can be introduced as follows.

Let $Z(t) = N\mathbf{p}_0 e^{-\lambda t[I - (I + (p-q)\Theta)P]} - pN_b\hat{\gamma}_0 \sum_{k=0}^{\infty} \sum_{j=1}^{k-1} \tilde{P}^j e_{N_b} \Theta [P(I + (p-q)\Theta)]^{k-1-j} \frac{(\lambda t)^k}{k!} e^{-\lambda t}$, then $E[\hat{N}(t)] = Z(t) \cdot \mathbf{1}^T$.

Note that

$$\begin{aligned}
\dot{Z}(t) &= -N\mathbf{p}_0 e^{-\lambda t[I - (I + (p-q)\Theta)P]} \cdot \lambda[I - (I + (p-q)\Theta)P] \\
&\quad - pN_b\hat{\gamma}_0 \sum_{k=0}^{\infty} \sum_{j=1}^{k-1} \tilde{P}^j e_{N_b} \Theta [P(I + (p-q)\Theta)]^{k-1-j} \left[\lambda \frac{(\lambda t)^{k-1}}{(k-1)!} e^{-\lambda t} - \lambda \frac{(\lambda t)^k}{k!} e^{-\lambda t} \right] \\
&= -N\mathbf{p}_0 e^{-\lambda t[I - (I + (p-q)\Theta)P]} \cdot \lambda[I - (I + (p-q)\Theta)P] - pN_b\hat{\gamma}_0 \sum_{k=0}^{\infty} \sum_{j=1}^k \tilde{P}^j e_{N_b} \Theta [P(I + (p-q)\Theta)]^{k-j} \lambda \frac{(\lambda t)^k}{k!} e^{-\lambda t} \\
&\quad + pN_b\hat{\gamma}_0 \sum_{k=0}^{\infty} \sum_{j=1}^{k-1} \tilde{P}^j e_{N_b} \Theta [P(I + (p-q)\Theta)]^{k-1-j} \lambda \frac{(\lambda t)^k}{k!} e^{-\lambda t} \\
&= -N\mathbf{p}_0 e^{-\lambda t[I - (I + (p-q)\Theta)P]} \cdot \lambda[I - (I + (p-q)\Theta)P] \\
&\quad - pN_b\hat{\gamma}_0 \sum_{k=0}^{\infty} \sum_{j=1}^{k-1} \tilde{P}^j e_{N_b} \Theta [P(I + (p-q)\Theta)]^{k-j} \lambda \frac{(\lambda t)^k}{k!} e^{-\lambda t} - pN_b\hat{\gamma}_0 \sum_{k=0}^{\infty} \tilde{P}^k e_{N_b} \Theta \lambda \frac{(\lambda t)^k}{k!} e^{-\lambda t} \\
&\quad + pN_b\hat{\gamma}_0 \sum_{k=0}^{\infty} \sum_{j=1}^{k-1} \tilde{P}^j e_{N_b} \Theta [P(I + (p-q)\Theta)]^{k-1-j} \lambda \frac{(\lambda t)^k}{k!} e^{-\lambda t} \\
&= -N\mathbf{p}_0 e^{-\lambda t[I - (I + (p-q)\Theta)P]} \cdot \lambda[I - (I + (p-q)\Theta)P] \\
&\quad + pN_b\hat{\gamma}_0 \sum_{k=0}^{\infty} \sum_{j=1}^{k-1} \tilde{P}^j e_{N_b} \Theta [P(I + (p-q)\Theta)]^{k-j} \lambda \frac{(\lambda t)^k}{k!} e^{-\lambda t} [I - (I + (p-q)\Theta)P] \\
&\quad - pN_b\hat{\gamma}_0 \sum_{k=0}^{\infty} \tilde{P}^k e_{N_b} \Theta \lambda \frac{(\lambda t)^k}{k!} e^{-\lambda t}.
\end{aligned}$$

Therefore,

$$\dot{Z}(t) = -Z(t) \cdot \lambda[I - (I + (p - q)\Theta)P] - \lambda p N_b \hat{\gamma}_0 e^{\lambda t \tilde{P}} e_{N_b} \Theta e^{-\lambda t}.$$

Because $\lambda p N_b \hat{\gamma}_0 e^{\lambda t \tilde{P}} e_{N_b} \Theta e^{-\lambda t}$ is in $\dot{Z}(t)$, the system is not a linear system. We call that $E[\hat{N}(t)]$ obeys a nonlinear dynamic system.

By a similar discussion, $E[\hat{M}(t)]$ also obeys a nonlinear dynamic system.

8. Theoretical comparison of the two models

When $p < q$, we have $\hat{\beta}_k \leq N \mathbf{p}_0(I + (p - q)\Theta)[P(I + (p - q)\Theta)]^{k-1}$, thus

$$\begin{aligned} E[\hat{N}_k] &= \hat{\beta}_k \cdot \mathbf{1}^T \\ &\leq N \mathbf{p}_0(I + (p - q)\Theta)[P(I + (p - q)\Theta)]^{k-1} \cdot \mathbf{1}^T \\ &= N \mathbf{p}_0[(I + (p - q)\Theta)P]^k \cdot \mathbf{1}^T = E[N_k]. \end{aligned}$$

Recall $E[N_k]$ converges to 0 exponentially, therefore, $E[\hat{N}_k]$ converges to 0 exponentially.

Note that $(I + (p - q)\Theta)P \cdot \mathbf{1}^T = \mathbf{1}^T + (p - q)\Theta \cdot \mathbf{1}^T < \mathbf{1}^T$, then $\forall k \geq 1$, $[(I + (p - q)\Theta)P]^k \cdot \mathbf{1}^T < \mathbf{1}^T$. Furthermore, $\frac{p N_b}{p - q} \hat{\gamma}_0 \sum_{j=1}^{k-2} \tilde{P}^j e_{N_b} \Theta P \{[(I + (p - q)\Theta)P]^{k-1-j} - I\} \cdot \mathbf{1}^T \geq 0$, then $E[\hat{M}_k] \leq E[M_k]$. By the fact of $E[M_k]$ converging exponentially, we know that $E[\hat{M}_k]$ also converges exponentially.

Let $\rho(n, i, j) = \sum_{k=0}^{\infty} \Pr\{\hat{N}_k = N_b, A_{k+1} = j | \hat{N}_0 = n, A_1 = i\}$, and $\rho(n) = (\rho(n, i, j))_{i,j=1,2,\dots,m}$. Then we have $\hat{\gamma}_0 \tilde{P}^j e_{N_b} = (\Pr\{\hat{N}_{j+1} = N_b, A_{j+2} = 1 | \hat{N}_0 = N\}, \Pr\{\hat{N}_{j+1} = N_b, A_{j+2} = 2 | \hat{N}_0 = N\}, \dots, \Pr\{\hat{N}_{j+1} = N_b, A_{j+2} = m | \hat{N}_0 = N\})$. Therefore, we know that $\hat{\gamma}_0 \sum_{j=0}^{\infty} \tilde{P}^j e_{N_b} = \mathbf{p}_0 \rho(N)$.

We have the following result.

Proposition 8.1. If $p < q$ and every entry in P is greater than 0 (denoted by $P > 0$), for any fixed integer n , we have $N_b \rho(n) \Theta \rightarrow 0$, as $N_b \rightarrow \infty$.

Proof. Obviously, for $n = 0$ we have $\rho(0) = 0$.

When $1 \leq n \leq N_b - 1$, we have

$$\Pr\{\hat{N}_{k+1} = n', A_{k+2} = j | \hat{N}_k = n, A_{k+1} = i\} = \begin{cases} (1 - (p + q)n\theta_i)p_{ij}, & n' = n, \\ nq\theta_i p_{ij}, & n' = n - 1, \\ np\theta_i p_{ij}, & n' = n + 1. \end{cases}$$

Then we obtain

$$\rho(n) = (I - n(p + q)\Theta)P\rho(n) + nq\Theta P\rho(n - 1) + np\Theta P\rho(n + 1). \quad (8.40)$$

When $n = N_b$, we have

$$\Pr\{\hat{N}_{k+1} = n', A_{k+2} = j | \hat{N}_k = N_b, A_{k+1} = i\} = \begin{cases} (1 - qN_b\theta_i)p_{ij}, & n' = N_b, \\ N_b q\theta_i p_{ij}, & n' = N_b - 1. \end{cases}$$

Then we obtain

$$\rho(N_b) = I + (I - N_b q\Theta)P\rho(N_b) + N_b q\Theta P\rho(N_b - 1). \quad (8.41)$$

Multiplying P^* from left side to Eqs. (8.40) and (8.41), we obtain

$$\begin{aligned} 0 &= -(p + q)P^* \Theta P\rho(n) + qP^* \Theta P\rho(n - 1) + pP^* \Theta P\rho(n + 1), \quad 1 \leq n \leq N_b - 1, \\ 0 &= P^* - N_b qP^* \Theta P\rho(N_b) + N_b qP^* \Theta P\rho(N_b - 1). \end{aligned}$$

Considering $P^* \Theta P\rho(0) = 0$ and $p \neq q$, and solving above linear system we obtain the solution

$$N_b P^* \Theta P\rho(n) = \frac{p^{N_b}}{(q - p)q^{N_b}} \left(\left(\frac{q}{p} \right)^n - 1 \right) P^*.$$

Because $p < q$, for any fixed integer n ,

$$N_b P^* \Theta P\rho(n) \rightarrow 0, \quad N_b \rightarrow \infty. \quad (8.42)$$

By the fact that $P > 0$ implies $P^* > 0$ and from (8.42) we know that $N_b \Theta P\rho(n) \rightarrow 0$, as $N_b \rightarrow \infty$, thus, $N_b \rho(n) \Theta \rightarrow 0$, as $N_b \rightarrow \infty$. \square

Note that in Proposition 8.1 the extra condition $P > 0$ cannot be neglected. Next we present a new condition instead of the condition $P > 0$. If there exists $C > 0$, such that $\frac{\max_{1 \leq i \leq m} \theta_i}{\min_{1 \leq i \leq m} \theta_i} < C$, and P is irreducible, then from $N_b \Theta P \rho(n) \rightarrow 0$, as $N_b \rightarrow \infty$, we can derive $N_b \rho(n) \Theta \rightarrow 0$, as $N_b \rightarrow \infty$.

Theorem 8.1. When $p < q$, $P > 0$ and initial defect number is N , both $E[M(t)] - E[\hat{M}(t)]$ and $E[N(t)] - E[\hat{N}(t)]$ converge to 0 for any t as $N_b \rightarrow \infty$.

Proof. When $p \leq q$, $[P(I + (p - q)\Theta)]^k \cdot \mathbf{1}^T \leq \mathbf{1}^T$, then,

$$\begin{aligned} 0 &\leq E[N(t)] - E[\hat{N}(t)] \\ &= pN_b \hat{\gamma}_0 \sum_{k=0}^{\infty} \sum_{j=1}^{k-1} \tilde{p}^j e_{N_b} \Theta [P(I + (p - q)\Theta)]^{k-1-j} \cdot \mathbf{1}^T \frac{(\lambda t)^k}{k!} e^{-\lambda t} \\ &\leq pN_b \hat{\gamma}_0 \sum_{k=0}^{\infty} \sum_{j=1}^{k-1} \tilde{p}^j e_{N_b} \Theta \cdot \mathbf{1}^T \frac{(\lambda t)^k}{k!} e^{-\lambda t} \\ &\leq pN_b \hat{\gamma}_0 \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \tilde{p}^j e_{N_b} \Theta \cdot \mathbf{1}^T \frac{(\lambda t)^k}{k!} e^{-\lambda t} \\ &= pN_b \hat{\gamma}_0 \sum_{j=0}^{\infty} \tilde{p}^j e_{N_b} \Theta \cdot \mathbf{1}^T = pN_b \rho(N) \Theta \cdot \mathbf{1}^T. \end{aligned}$$

From Proposition 8.1 we know that $E[N(t)] - E[\hat{N}(t)]$ converges to 0 as $N_b \rightarrow \infty$.

$$\begin{aligned} N_b \hat{\gamma}_0 \sum_{k=0}^{\infty} \sum_{j=1}^{k-1} \tilde{p}^j e_{N_b} \Theta [P(I + (p - q)\Theta)]^{k-1-j} \frac{(\lambda t)^k}{k!} &= N_b \hat{\gamma}_0 \sum_{j=1}^{\infty} \sum_{k=j+1}^{\infty} \tilde{p}^j e_{N_b} \Theta [P(I + (p - q)\Theta)]^{k-1-j} \frac{(\lambda t)^k}{k!} \\ &= N_b \hat{\gamma}_0 \sum_{j=1}^{\infty} \sum_{k=0}^{\infty} \tilde{p}^j e_{N_b} \Theta [P(I + (p - q)\Theta)]^k \frac{(\lambda t)^{k+j+1}}{(k + j + 1)!} \\ &\leq N_b \hat{\gamma}_0 \sum_{j=1}^{\infty} \sum_{k=0}^{\infty} \tilde{p}^j e_{N_b} \Theta [P(I + (p - q)\Theta)]^k \frac{(\lambda t)^{k+j+1}}{k!(j + 1)!} \\ &= N_b \hat{\gamma}_0 \sum_{j=1}^{\infty} \frac{(\lambda t)^{j+1}}{(j + 1)!} \tilde{p}^j e_{N_b} \Theta \sum_{k=0}^{\infty} [P(I + (p - q)\Theta)]^k \frac{(\lambda t)^k}{k!} \\ &= N_b \hat{\gamma}_0 \sum_{j=1}^{\infty} \frac{(\lambda t)^{j+1}}{(j + 1)!} \tilde{p}^j e_{N_b} \Theta \sum_{k=0}^{\infty} [P(I + (p - q)\Theta)]^k \frac{(\lambda t)^k}{k!} \\ &\leq N_b \hat{\gamma}_0 \sum_{j=1}^{\infty} e^{\lambda t} \tilde{p}^j e_{N_b} \Theta e^{\lambda t [P(I + (p - q)\Theta)]}. \end{aligned}$$

When $p < q$, we have

$$\begin{aligned} 0 &\leq E[M(t)] - E[\hat{M}(t)] \\ &= \frac{pN_b}{p - q} \hat{\gamma}_0 \sum_{k=0}^{\infty} \sum_{j=1}^{k-2} \tilde{p}^j e_{N_b} \Theta \{ [P(I + (p - q)\Theta)]^{k-1-j} - I \} \frac{(\lambda t)^k}{k!} e^{-\lambda t} \cdot \mathbf{1}^T \\ &\leq -\frac{pN_b}{p - q} \hat{\gamma}_0 \sum_{k=0}^{\infty} \sum_{j=1}^{k-2} \tilde{p}^j e_{N_b} \Theta \frac{(\lambda t)^k}{k!} e^{-\lambda t} \cdot \mathbf{1}^T \\ &\leq \frac{pN_b}{q - p} \hat{\gamma}_0 \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \tilde{p}^j e_{N_b} \Theta \frac{(\lambda t)^k}{k!} e^{-\lambda t} \cdot \mathbf{1}^T \\ &= \frac{pN_b}{q - p} \hat{\gamma}_0 \sum_{j=0}^{\infty} \tilde{p}^j e_{N_b} \Theta \cdot \mathbf{1}^T = \frac{p}{q - p} N_b \rho(N) \Theta \cdot \mathbf{1}^T. \end{aligned}$$

Therefore, $E[M(t)] - E[\hat{M}(t)]$ converges to 0 as $N_b \rightarrow \infty$. \square

Theorem 8.1 tells that the values of $E[M(t)] - E[\hat{M}(t)]$ and $E[N(t)] - E[\hat{N}(t)]$ are very small when N_b is sufficiently large in the case of $p < q$. Actually, some numerical examples mentioned in Section 9 show that when N_b is not too large, the differences are sufficient small. Therefore, in this case Model I is a good approximation of Model II. However, this result does not hold in the case of $p \geq q$.

When $p > q$, $E[M(t)]$ and $E[N(t)]$ exponentially tend to infinity, but $E[\hat{N}(t)]$ and $E[\hat{M}(t)]$ increase bounded by a linear function. Therefore, $E[M(t)] - E[\hat{M}(t)]$ and $E[N(t)] - E[\hat{N}(t)]$ will tend to infinity as $t \rightarrow \infty$ for any N_b .

Now we consider the case of $p = q$.

Since $E[N(t)] = N$ is fixed, based on (6.23) we have $\lim_{k \rightarrow \infty} \hat{N}_k = 0$, a.s. (in the case of $q > 0$). Note that $\hat{N}_k \leq N_b$, so for any fixed N_b , we have $\lim_{k \rightarrow \infty} E[\hat{N}_k] = 0$ according to bounded convergence theorem. Then $\lim_{t \rightarrow \infty} E[\hat{N}(t)] = 0$. Therefore, $E[N(t)] - E[\hat{N}(t)]$ converges to N as $t \rightarrow \infty$.

On the other hand, we know that if no defect is remaining in the software under test, the total number of failures revealed will not increase anymore, so we have $\hat{M}_k \leq \min\{\ell \geq 0 | \hat{N}_\ell = 0, \hat{N}_0 = N\}$ for any k . Therefore, for any fixed N_b , $E[\hat{M}_k] \leq E[\min\{\ell \geq 0 | \hat{N}_\ell = 0, \hat{N}_0 = N\}] = \sum_{i=1}^m p_i E[\min\{\ell \geq 0 | \hat{N}_\ell = 0, \hat{N}_0 = N, A_1 = i\}] = \mathbf{p}_0 E\tau(N)$ (refer to Section 6.4), $E[\hat{M}(t)] \leq \mathbf{p}_0 E\tau(N)$. This implies $E[\hat{M}(t)]$ is bounded by the constant $\mathbf{p}_0 E\tau(N)$. While $E[M(t)]$ is increasing linearly in t , $E[M(t)] - E[\hat{M}(t)]$ converges to infinity as $t \rightarrow \infty$ for any N_b .

Theorem 8.1 implies that in the case that remaining defects are more often removed from than that new defects are introduced into the software under test, the discrepancy between Models I and II tends to diminish as the upper bound of the number of remaining defects increases. However the discrepancy between the two models cannot be reduced if new defects are more often introduced into than that remaining defects are removed from the software under test. In this case the number of remaining defects in Model I will approach infinity, whereas the number of remaining defects in Model II is always bounded by N_b . In the case that remaining defects are removed from as often as new defects are introduced into the software under test, since there is no upper bound for the number of remaining defects throughout the software testing process in Model I, the expected number of remaining defects tends to be the initial number of software defects. However in Model II the number of remaining defects will bounce back to a smaller one if it reaches N_b , and will be absorbed if it reaches zero. Therefore the expected number of remaining defects tends to be zero eventually. This implies that the discrepancy between the two models cannot be neglected.

9. Simulation results

Next we calculate the values of $E[M(t)]$ and $E[N(t)]$ of Model I by using analytic expressions derived, and get the values of $E[M(t)]$ and $E[N(t)]$ of Model II by simulating the software testing process.

Example 9.1. Suppose

$$\Theta = \begin{pmatrix} \theta_1 & 0 & 0 \\ 0 & \theta_2 & 0 \\ 0 & 0 & \theta_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{50} & 0 & 0 \\ 0 & \frac{1}{100} & 0 \\ 0 & 0 & \frac{1}{150} \end{pmatrix},$$

$$P = \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{3}{10} & \frac{1}{5} \\ \frac{2}{5} & \frac{2}{5} & \frac{1}{5} \\ \frac{3}{10} & \frac{1}{5} & \frac{1}{2} \end{pmatrix},$$

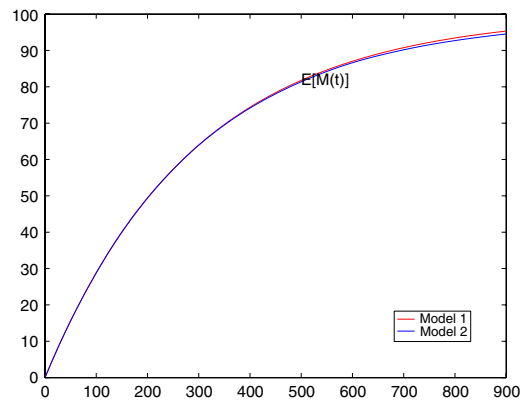
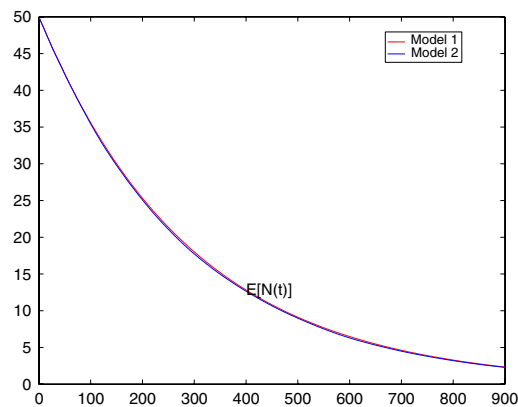
$$\mathbf{p}_0 = (p_1, p_2, p_3) = (3/10, 2/5, 3/10),$$

$$\lambda = 0.9, \quad p = 0.2, \quad q = 0.7, \quad N_b = 100, \quad N = 50.$$

Figs. 1 and 2 show that in the case of $p > q$ the differences of $E[M(t)]$ and $E[N(t)]$ between two models are small enough to be neglected. Therefore, Model I is a good approximation of Model II.

Example 9.2. Suppose

$$\Theta = \begin{pmatrix} \theta_1 & 0 & 0 \\ 0 & \theta_2 & 0 \\ 0 & 0 & \theta_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{50} & 0 & 0 \\ 0 & \frac{1}{100} & 0 \\ 0 & 0 & \frac{1}{150} \end{pmatrix},$$

Fig. 1. Behavior of $E[M(t)]$ for Example 9.1.Fig. 2. Behavior of $E[N(t)]$ for Example 9.1.

$$P = \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{3}{10} & \frac{1}{5} \\ \frac{2}{5} & \frac{2}{5} & \frac{1}{5} \\ \frac{3}{10} & \frac{1}{5} & \frac{1}{2} \end{pmatrix},$$

$$\mathbf{p}_0 = (p_1, p_2, p_3) = (3/10, 2/5, 3/10),$$

$$\lambda = 0.9, \quad p = 0.7, \quad q = 0.2, \quad N_b = 50, \quad N = 20.$$

Figs. 3 and 4 show that in the case of $p > q$ the two models make a great difference in the values of $E[M(t)]$ and $E[N(t)]$.

Example 9.3. Suppose

$$\Theta = \begin{pmatrix} \theta_1 & 0 & 0 \\ 0 & \theta_2 & 0 \\ 0 & 0 & \theta_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{50} & 0 & 0 \\ 0 & \frac{1}{100} & 0 \\ 0 & 0 & \frac{1}{150} \end{pmatrix},$$

$$P = \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{3}{10} & \frac{1}{5} \\ \frac{2}{5} & \frac{2}{5} & \frac{1}{5} \\ \frac{3}{10} & \frac{1}{5} & \frac{1}{2} \end{pmatrix},$$

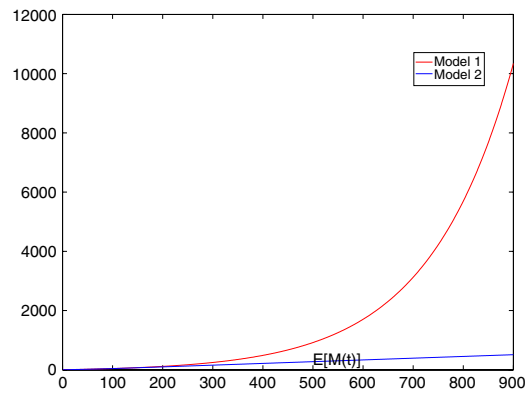


Fig. 3. Behavior of $E[M(t)]$ for Example 9.2.

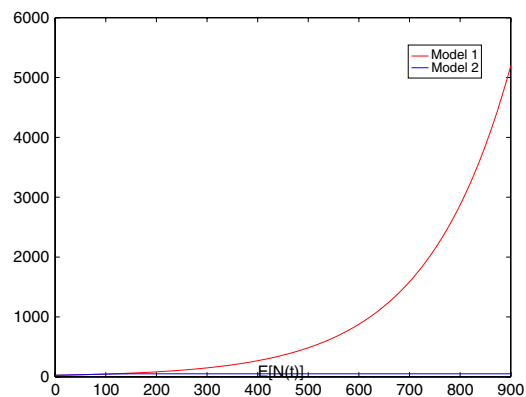


Fig. 4. Behavior of $E[N(t)]$ for Example 9.2.

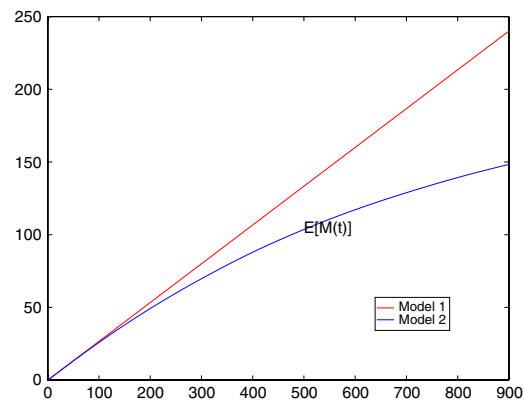
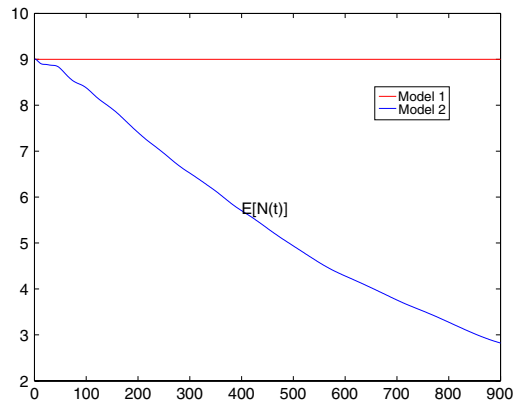
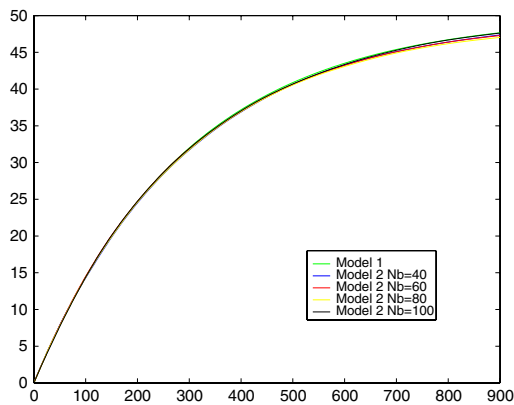


Fig. 5. Behavior of $E[M(t)]$ for Example 9.3.

$$\mathbf{p}_0 = (p_1, p_2, p_3) = (3/10, 2/5, 3/10),$$

$$\lambda = 0.9, \quad p = 0.45, \quad q = 0.45, \quad N_b = 15, \quad N = 9.$$

Fig. 5 shows that in the case of $p = q$ the difference of $E[M(t)]$ between two models is not negligible, and Fig. 6 shows that the difference of $E[N(t)]$ between two models is also not negligible.

Fig. 6. Behavior of $E[N(t)]$ for Example 9.3.Fig. 7. Behavior of $E[M(t)]$ for Example 9.4.

Example 9.4. Suppose

$$\Theta = \begin{pmatrix} \theta_1 & 0 & 0 \\ 0 & \theta_2 & 0 \\ 0 & 0 & \theta_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{50} & 0 & 0 \\ 0 & \frac{1}{100} & 0 \\ 0 & 0 & \frac{1}{150} \end{pmatrix},$$

$$P = \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{3}{10} & \frac{1}{5} \\ \frac{2}{5} & \frac{2}{5} & \frac{1}{5} \\ \frac{3}{10} & \frac{1}{5} & \frac{1}{2} \end{pmatrix},$$

$$\mathbf{p}_0 = (p_1, p_2, p_3) = (3/10, 2/5, 3/10),$$

$$\lambda = 0.9, \quad p = 0.2, \quad q = 0.7, \quad N = 25$$

and N_b varies from 40 to 100.

From Figs. 7 and 8 we know that N_b nearly has no effect on the values of $E[M(t)]$ and $E[N(t)]$, in the case of $p < q$.

10. Concluding remarks

The quantitative relationship between software testing and software reliability has been obscure and hard to be formulated in a mathematically rigorous manner. This motivates software reliability testing to emerge in recent years as a new form of software testing to take account of the quantitative aspects of software reliability in the process of software

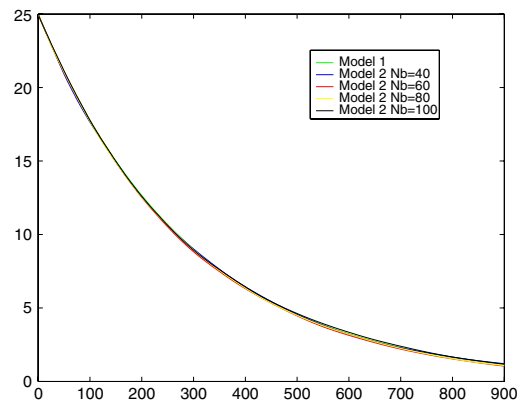


Fig. 8. Behavior of $E[N(t)]$ for Example 9.4.

testing. A modeling framework was developed in our previous work [7,9] for software reliability testing, which was intended to be practically realistic, mathematically rigorous, and quantitatively precise. However, a shortcoming of the modeling framework is that it is confined to the case of perfect debugging. In the preceding sections the modeling framework is extended to the case of imperfect debugging. Two models are proposed for the software testing process. In the first model it is assumed that the probability of a test case revealing a failure is proportional to the number of defects remaining in the software under test, and upon a failure being revealed, debugging may make the number of remaining defects reduce by one, remain intact, or increase by one. In the second model it is still assumed that the probability of a test case revealing a failure is proportional to the number of defects remaining in the software under test. However, there is an upper bound for the number of remaining defects. When the number of remaining defects reaches the upper bound, the probability that the number of remaining defects is increased by one by debugging is zero.

The expected behaviors of the cumulative number of observed failures and the number of remaining defects in the first model show that the software testing process may induce a linear or nonlinear dynamic system, depending on the relationship between the probability of debugging introducing a new defect and that of debugging removing a detected defect. The second-order behaviors of the first model also show that in the case of imperfect debugging, although there may be an unbiased estimator for the initial number of defects remaining in the software under test, the cumulative number of observed failures and the current number of remaining defects are not sufficient for precisely estimating the initial number of remaining defects. This is because the variance of the unbiased estimator approaches a non-zero constant as the software testing process proceeds. This may be treated as an intrinsic principle of uncertainty for software testing. The expected behaviors of the cumulative number of observed failures and the number of remaining defects in the second model show that, if remaining defects are more often removed from than new defects are introduced into the software under test, the software testing process may induce a nonlinear dynamic system. However the theoretical analysis and simulation results show that the expected behaviors of the two models tend to coincide with each other as the upper bound of the number of remaining defects approaches infinity.

Many research topics can be investigated in the future. For example, what are the properties of the induced dynamic systems of the software testing process? What is the implication of the intrinsic principle of uncertainty for software reliability? How can the software testing process be improved by estimating the current reliability of the software under test? All these topics look challenging.

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