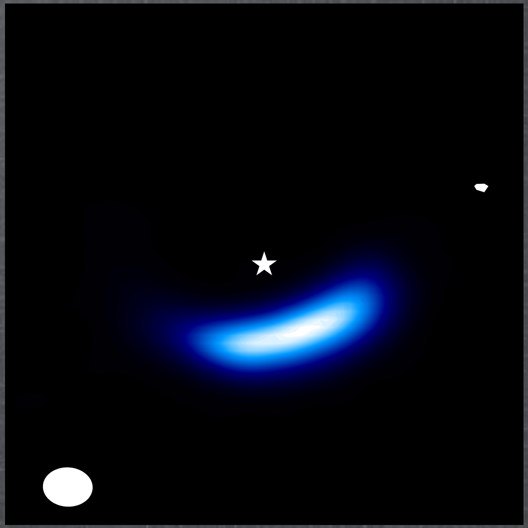
Dust and gas in SPH
(part I)

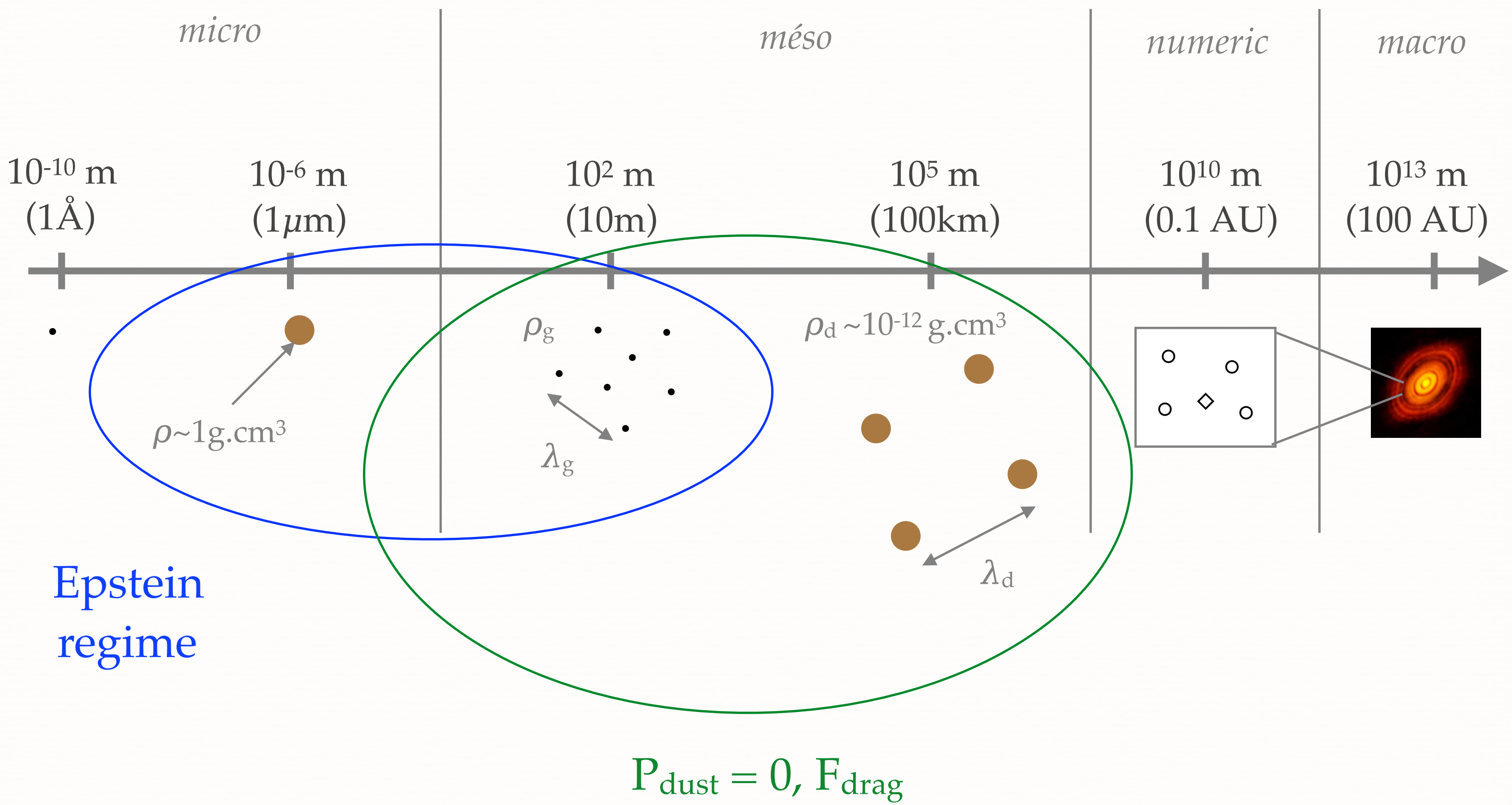


Guillaume Laibe

and the "*dustbusters*" consortium

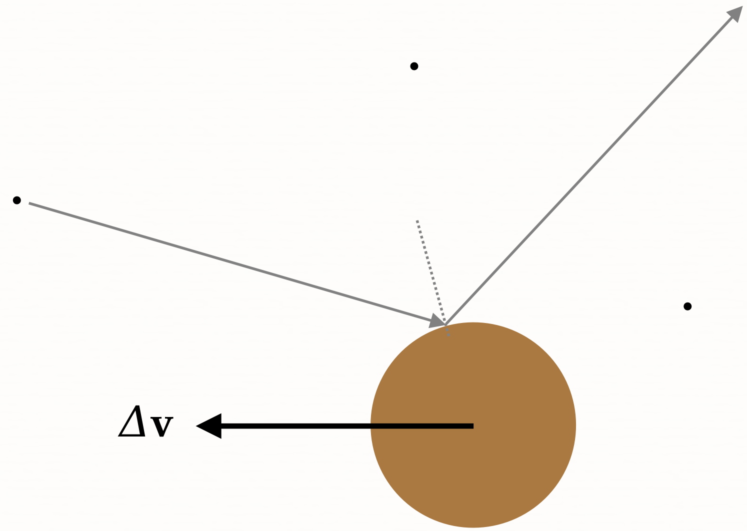
Scales

In a protoplanetary disc :



Force on a grain

Spherical homogeneous massive grain, specular reflexions :



Epstein drag force: $$\mathbf{F}_\mathrm{d} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \frac{\langle \mathbf{p}_{1\mathrm{c}} \rangle}{t_{1\mathrm{c}}}$$

$$\mathbf{p}_{1\mathrm{c}} = -\frac{4}{3} \times m_\mathrm{g} \Delta\mathbf{v}$$

$$t_{1\mathrm{c}} = \left(n_\mathrm{g} \underbrace{\left(\pi s^2\right)}_{\sigma_\mathrm{coll}} \underbrace{\sqrt{\frac{8}{\pi\gamma}} c_\mathrm{s}}_{v_\mathrm{th}} \right)^{-1}$$

$$\boxed{\mathbf{F}_\mathrm{d} = -\frac{4}{3}\pi \rho_\mathrm{g} s^2 \sqrt{\frac{8}{\pi\gamma}} c_\mathrm{s} \Delta\mathbf{v}}$$

correction if $\varDelta$v supersonic

Force balance on a grain:

$$m_\mathrm{d} \frac{\mathrm{d}\mathbf{v}_\mathrm{d}}{\mathrm{d}t} = \mathbf{F}_\mathrm{d} \longrightarrow \frac{\mathrm{d}\mathbf{v}_\mathrm{d}}{\mathrm{d}t} = -\frac{\Delta\mathbf{v}}{t_\mathrm{s,d}},$$

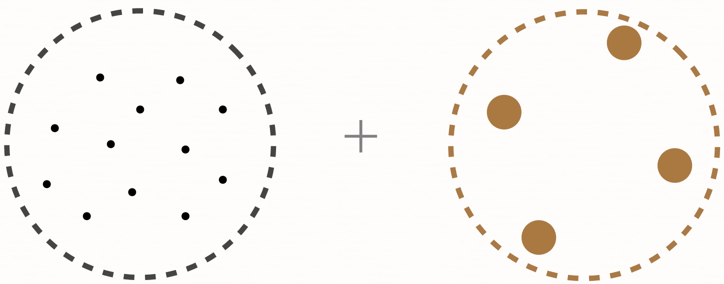
$$t_\mathrm{s,d} \equiv \sqrt{\frac{\pi\gamma}{8}} \frac{\rho s}{\rho_\mathrm{g} c_\mathrm{s}} \longrightarrow \propto \frac{\sigma_\mathrm{d}}{m_\mathrm{d}}$$

Two fluids description

Mass conservation:

$$\frac{\partial \rho_{\rm g}}{\partial t} + \nabla \cdot (\rho_{\rm g} \mathbf{v}_{\rm g}) = 0,$$

$$\frac{\partial \rho_{\rm d}}{\partial t} + \nabla \cdot (\rho_{\rm d} \mathbf{v}_{\rm d}) = 0,$$



$\rho_{\rm g}, \mathrm{v}_{\rm g}$ $\rho_{\rm d}, \mathrm{v}_{\rm d}$

Momentum conservation:

$$\rho_{\rm g}\left(\frac{\partial \mathbf{v}_{\rm g}}{\partial t} + \mathbf{v}_{\rm g} \cdot \nabla \mathbf{v}_{\rm g}\right) = \nabla P + \rho_{\rm g}\mathbf{g} + \{...\} + K\left(\mathbf{v}_{\rm d} - \mathbf{v}_{\rm g}\right),$$

$$\rho_{\rm d}\left(\frac{\partial \mathbf{v}_{\rm d}}{\partial t} + \mathbf{v}_{\rm d} \cdot \nabla \mathbf{v}_{\rm d}\right) = \underbrace{\cancel{\nabla P_{\rm d}}}_{0} + \rho_{\rm d}\mathbf{g} + \{...\} - K\left(\mathbf{v}_{\rm d} - \mathbf{v}_{\rm g}\right),$$

∇P: differential force

drag: exchange of momentum

differential advection

Energy conservation:

$$\rho_{\rm g}\frac{{\rm d}u_{\rm g}}{{\rm d}t} = -P\nabla \cdot \mathbf{v}_{\rm g} + \rho_{\rm g} K \Delta \mathbf{v}^2.$$

Two fluids SPH equations

bell-shaped kernel

$$\rho_a = \sum_b m_b W_{ab}(h_a); \qquad h_a = h_{\rm fact}\left(\frac{m_a}{\rho_a}\right)^{1/3}$$

$$\rho_i = \sum_j m_j W_{ij}(h_i); \qquad h_i = h_{\rm fact}\left(\frac{m_i}{\rho_i}\right)^{1/3}$$

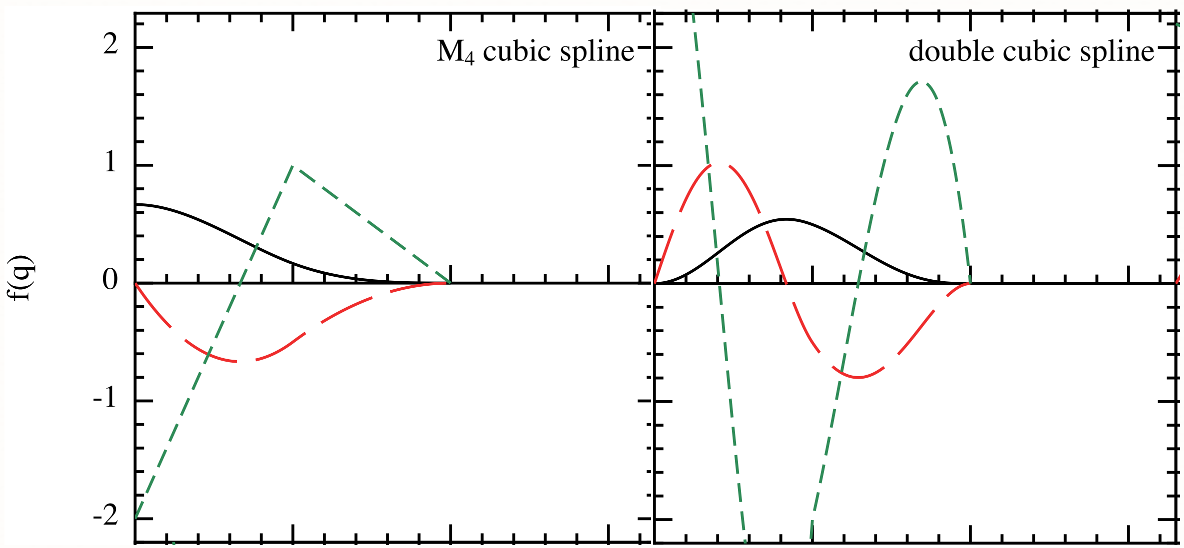
neighbours of same type

own smoothing length

$$\left(\frac{\mathrm{d}\boldsymbol{v}_a}{\mathrm{d}t}\right)_{\rm drag} = -3\sum_j m_j \frac{\boldsymbol{v}_{aj}\cdot\hat{\boldsymbol{r}}_{aj}}{(\rho_a+\rho_j)t^{\rm s}_{aj}}\hat{\boldsymbol{r}}_{aj} D_{aj}(h_a),$$

$$\left(\frac{\mathrm{d}\boldsymbol{v}_i}{\mathrm{d}t}\right)_{\rm drag} = -3\sum_b m_b \frac{\boldsymbol{v}_{ib}\cdot\hat{\boldsymbol{r}}_{ib}}{(\rho_i+\rho_b)t^{\rm s}_{ib}}\hat{\boldsymbol{r}}_{ib} D_{ib}(h_b)$$

double hump drag kernel



$$D(r,h) = \frac{\sigma}{h^3} g(q),$$

$$g(q) = q^2 f(q)$$

Angular momentum conservation

Conservation of angular momentum:

$$
\begin{aligned}
\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_a m_a \mathbf{r}_a \times \mathbf{v}_a + \sum_i m_i \mathbf{r}_i \times \mathbf{v}_i\right) = \sum_a m_a \mathbf{r}_a \times \frac{\mathrm{d}\mathbf{v}_a}{\mathrm{d}t} + \sum_i m_i \mathbf{r}_i \times \frac{\mathrm{d}\mathbf{v}_i}{\mathrm{d}t} \\
&= -3 \sum_a m_a \mathbf{r}_a \times \left\{ \sum_j \frac{m_j}{(\rho_a + \rho_j)\, t_{\mathrm{s},aj}} (\mathbf{v}_{aj} \cdot \hat{\mathbf{r}}_{aj})\, \hat{\mathbf{r}}_{aj} D_{aj}(h_a) \right\} + \leftrightarrow \\
&= -3 \sum_a \sum_j \underbrace{\frac{m_a m_j}{(\rho_a + \rho_j)\, t_{\mathrm{s},aj}}}_{\mathcal{S}} \Big(\underbrace{\mathbf{v}_{aj}}_{\mathcal{A}} \cdot \underbrace{\hat{\mathbf{r}}_{aj}}_{\mathcal{A}} \Big) \underbrace{\frac{\mathbf{r}_a \times \mathbf{r}_j}{r_{aj}}}_{\mathcal{A}} D_{aj}(h_a) + \leftrightarrow, \\
&= 0.
\end{aligned}
$$

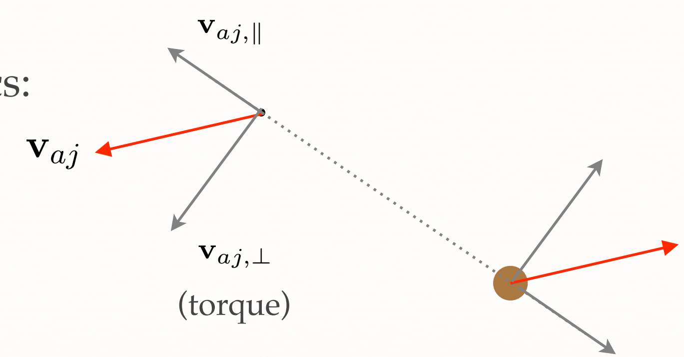
Why not the simpler $\displaystyle \frac{\mathrm{d}\mathbf{v}_a}{\mathrm{d}t} = -\sum_j \frac{m_j}{(\rho_a + \rho_j)\, t_{\mathrm{s},aj}} \mathbf{v}_{aj} D_{aj}(h_a)$

maths:

$$
\mathbf{v}_{aj} = \underbrace{(\mathbf{v}_{aj} \cdot \hat{\mathbf{r}}_{aj})\, \hat{\mathbf{r}}_{aj}}_{\mathbf{v}_{aj,\parallel}} + \underbrace{\hat{\mathbf{r}}_{aj} \times (\mathbf{v}_{aj} \times \hat{\mathbf{r}}_{aj})}_{\mathbf{v}_{aj,\perp}}
$$

gives symetric terms

physics:



Consistency - the factor 3 and approximations

Continuous limit:

$$
\begin{aligned}
I &= \sum_a \frac{m_a}{\rho_a} \left(\mathbf{v}_{aj} \cdot \hat{\mathbf{r}}_{aj}\right) \hat{\mathbf{r}}_{aj} D_{aj}\left(h_a\right) \simeq \int \mathrm{d}\mathbf{r}' \left(\mathbf{v}_{\mathrm{g}}\left(\mathbf{r}\right) - \mathbf{v}_{\mathrm{d}}\left(\mathbf{r}'\right)\right) \cdot \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} D\left(\mathbf{r}'-\mathbf{r}\right) \\
&\simeq \iiint \mathrm{d}x'\mathrm{d}y'\mathrm{d}z' \{ \frac{\Delta v_x\left(\mathbf{r}\right)\left(x-x'\right)^2 + \Delta v_y\left(\mathbf{r}\right)\left(y-y'\right)\left(x-x'\right) + \Delta v_z\left(\mathbf{r}\right)\left(z-z'\right)\left(x-x'\right)}{\left(x-x'\right)^2 + \left(y-y'\right)^2 + \left(z-z'\right)^2} \} D\left(\mathbf{r}'-\mathbf{r}\right) \hat{\mathbf{x}} \\
&\quad + ()\, \hat{\mathbf{y}} + ()\, \hat{\mathbf{z}} - \iiint \mathrm{d}\mathbf{r}' \left[\left(\mathbf{r}'-\mathbf{r}\right) \cdot \nabla_{\mathbf{r}'=\mathbf{r}} \mathbf{v}_{\mathrm{d}}\right] \cdot \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} D\left(\mathbf{r}'-\mathbf{r}\right)
\end{aligned}
$$

drag kernel

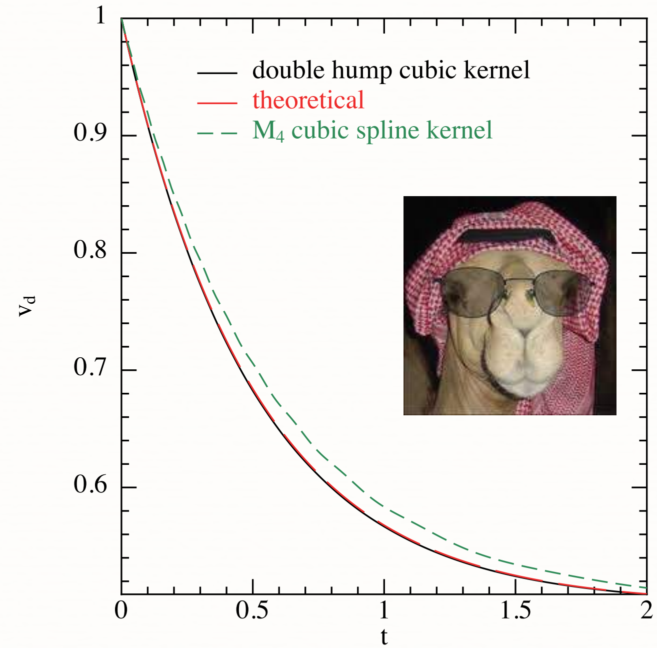
$$
\begin{aligned}
&\simeq \Delta v_x\left(\mathbf{r}\right) \iiint \mathrm{d}x'\mathrm{d}y'\mathrm{d}z' \{ \frac{\left(x-x'\right)^2}{\left(x-x'\right)^2 + \left(y-y'\right)^2 + \left(z-z'\right)^2} \} D\left(\mathbf{r}'-\mathbf{r}\right) \hat{\mathbf{x}} + ()\, \hat{\mathbf{y}} + ()\, \hat{\mathbf{z}} \\
&\simeq \frac{\Delta v_x\left(\mathbf{r}\right)}{3} \iiint \mathrm{d}x'\mathrm{d}y'\mathrm{d}z' \{ \frac{\left(x-x'\right)^2 + \left(y-y'\right)^2 + \left(z-z'\right)^2}{\left(x-x'\right)^2 + \left(y-y'\right)^2 + \left(z-z'\right)^2} \} D\left(\mathbf{r}'-\mathbf{r}\right) \hat{\mathbf{x}} + ()\, \hat{\mathbf{y}} + ()\, \hat{\mathbf{z}} \\
&\simeq \frac{\Delta \mathbf{v}\left(\mathbf{r}\right)}{3}.
\end{aligned}
$$

Note: Projection and velocity gradient issues avoided with reconstruction

Dustybox

Initially: homogeneous mixture $\rho_{\rm d} = \rho_{\rm d,0},\ \rho_{\rm g} = \rho_{\rm g,0}$

$$\frac{\partial v_{\rm g}}{\partial t} = \frac{K}{\rho_{\rm g,0}} \left(v_{\rm g} - v_{\rm d}\right),$$
$$\frac{\partial v_{\rm d}}{\partial t} = -\frac{K}{\rho_{\rm d,0}} \left(v_{\rm g} - v_{\rm d}\right).$$



reach barycentric velocity

barycentric stopping time

$$v_{\rm d} = v_0 - \frac{\rho_{\rm g,0}}{\rho_{\rm d,0} + \rho_{\rm d,0}} \Delta v_0 {\rm e}^{-t/t_{\rm s}},$$
$$v_{\rm g} = v_0 + \frac{\rho_{\rm d,0}}{\rho_{\rm d,0} + \rho_{\rm d,0}} \Delta v_0 {\rm e}^{-t/t_{\rm s}}.$$

damps differential velocity

$$t_{\rm s} \equiv \frac{K\left(\rho_{\rm d,0} + \rho_{\rm g,0}\right)}{\rho_{\rm d,0}\rho_{\rm g,0}} \qquad v_0 \equiv \frac{\rho_{g,0} v_{\rm g,0} + \rho_{d,0} v_{\rm d,0}}{\rho_{g,0} + \rho_{d,0}} \qquad \Delta v_0 \equiv v_{\rm d,0} - v_{\rm g,0}$$

Let diagonalise the system:

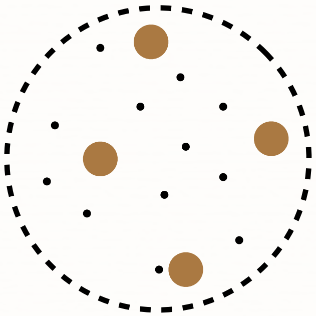
$$\frac{\partial}{\partial t}\begin{pmatrix} v \\ \Delta v \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1/t_{\rm s} \end{pmatrix}\begin{pmatrix} v \\ \Delta v \end{pmatrix} \longrightarrow \begin{array}{l} v = {\rm cste} \\ \Delta v = \Delta v_0 {\rm e}^{-t/t_{\rm s}} \end{array}$$

One fluid description

One fluid, two *phases*:

$$\rho \equiv \rho_{\mathrm{g}} + \rho_{\mathrm{d}} \qquad \epsilon \equiv \rho_{\mathrm{d}}/\rho \qquad \tilde{u} = u\left(1-\epsilon\right)$$

$$\mathbf{v} \equiv \frac{\rho_{\mathrm{g}} v_{\mathrm{g}} + \rho_{\mathrm{d}} v_{\mathrm{d}}}{\rho_{\mathrm{g}} + \rho_{\mathrm{d}}} \qquad \Delta\mathbf{v} \equiv \mathbf{v}_{\mathrm{d}} - \mathbf{v}_{\mathrm{g}}$$



ρ, v

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla$$

one advection velocity

Conservation equations:

$$\frac{\partial \rho}{\partial t} + \nabla\cdot\left(\rho\mathbf{v}\right) = 0$$

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right) = \nabla P + \rho\mathbf{g} - \nabla\left(\epsilon\left(1-\epsilon\right)\rho\Delta\mathbf{v}\Delta\mathbf{v}\right)$$

$$\rho\frac{\mathrm{d}\epsilon}{\mathrm{d}t} = -\nabla\left(\epsilon\left(1-\epsilon\right)\rho\Delta\mathbf{v}\right)$$

$$\frac{\mathrm{d}\Delta\mathbf{v}}{\mathrm{d}t} = -\frac{\Delta\mathbf{v}}{t_{\mathrm{s}}} + \frac{\nabla P}{\left(1-\epsilon\right)\rho_{\mathrm{g}}} - \left(\Delta\mathbf{v}\cdot\nabla\right)\mathbf{v} + \frac{1}{2}\nabla\left(\left(2\epsilon-1\right)\Delta\mathbf{v}\Delta\mathbf{v}\right)$$

$$\rho\frac{\mathrm{d}\tilde{u}}{\mathrm{d}t} = P\nabla\left(\mathbf{v}-\epsilon\Delta\mathbf{v}\right) + \epsilon\left(1-\epsilon\right)\frac{\Delta\mathbf{v}^{2}}{t_{\mathrm{s}}}$$

Infinite drag limit

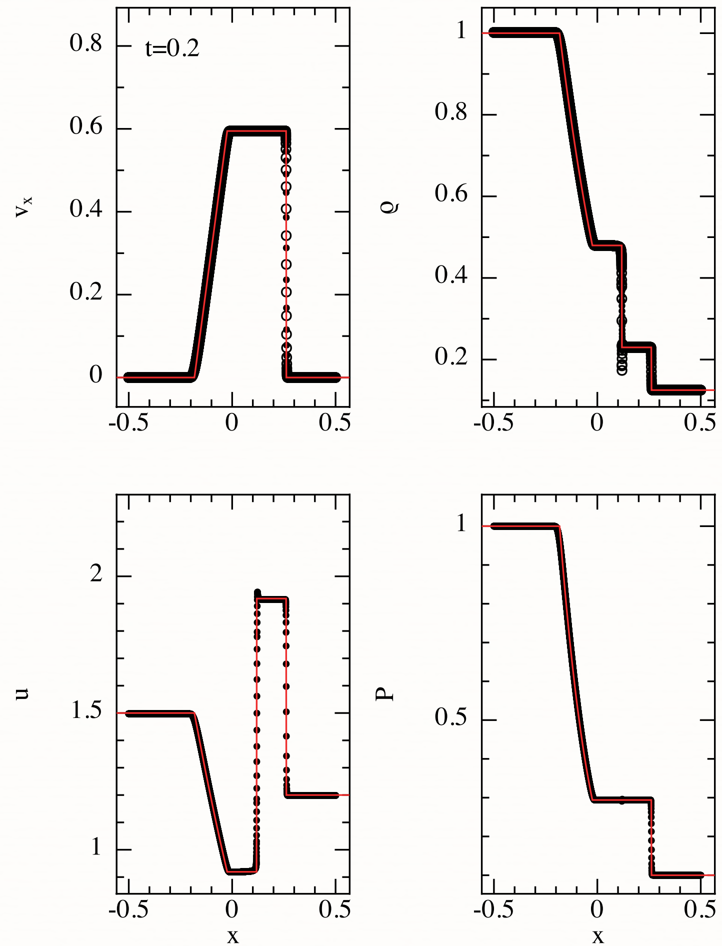
When ts = 0, Δv = 0.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = \nabla P + \rho \mathbf{g}$$

$$\longrightarrow \quad \tilde{c}_{\rm s} = c_{\rm s} \left(1 + \frac{\rho_{\rm d}}{\rho_{\rm g}} \right)^{-1/2}$$

DUSTYSHOCK



Physically:

$$\tilde{c}_{\rm s} = \sqrt{\frac{\text{stiffness}}{\text{inertia}}}$$

$$\rho_{\rm g} \rightarrow \rho = \rho_{\rm g} / \left(1 - \epsilon \right)$$

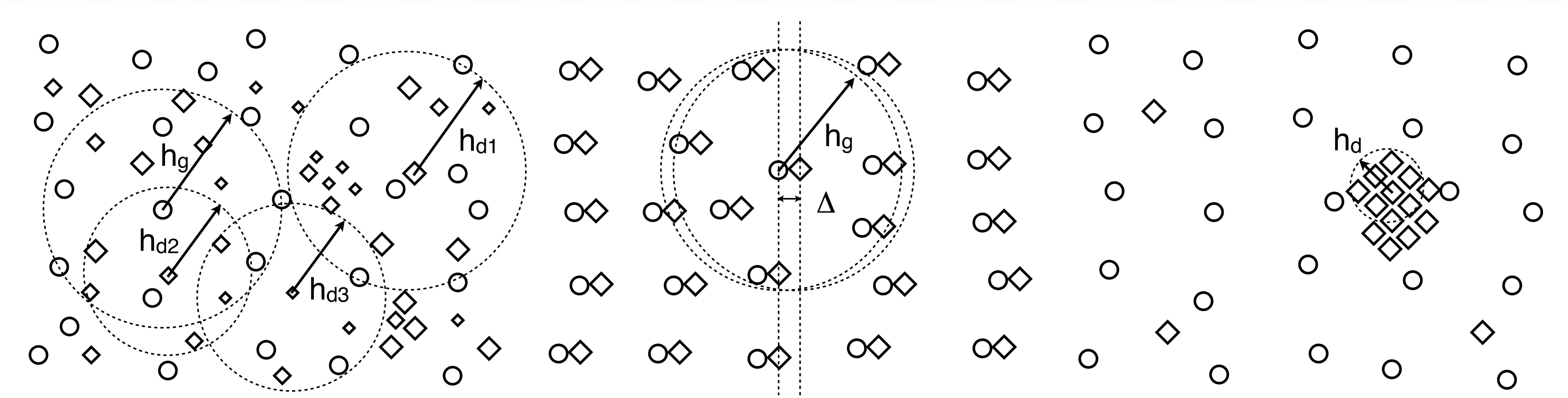
Gas accounts for the mass of the dust, transmitted instantaneously by the drag

Some comments

Main advantages of two fluid dust with SPH:

- Exact orbits in the zero-drag limit
- Track the history of the particle
- Allows dust interpenetration
- Sharp dust-gas discontinuities

But:



interpolation issues
for multiple sizes

h > cs ts ?

↳ numerical artefacts !