Breaking the Curse of Kareken and Wallace with Private Information*

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Abstract

We study the endogenous choice to accept fiat objects as media of exchange and the implications for nominal exchange rate determination. We consider an economy with two currencies which can be used to settle any transactions. However, currencies can be counterfeited at a fixed cost and the decision to counterfeit is private information. This induces equilibrium liquidity constraints on the currencies in circulation. We show that the threat of counterfeiting can pin down the nominal exchange rate even when the currencies are perfect substitutes, thus breaking the Kareken-Wallace indeterminacy result. We also find that with appropriate fiscal policies we can enlarge the set of monetary equilibria with determinate nominal exchange rates. Finally, we show that the threat of counterfeiting can also help determine nominal exchange rates in a variety of different trading environments, including a more familiar two-country setup with nontradable-goods decentralized markets and an alternative timing of money injections.

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1 Introduction

When agents have unrestricted access to currency markets and are free to use any currency as means of payment, Kareken and Wallace (1981) showed that the rate of return on the two currencies must be identical for both of them to circulate, which means that these currencies are perfect substitutes.[1] However, in this case the nominal exchange rate between these currencies is indeterminate. In the three decades since Kareken and Wallace, it has been extremely difficult to generate nominal exchange rate determinacy without imposing *ad hoc* frictions that inhibit trade using one or more of the currencies. We refer to this as 'the curse of Kareken and Wallace'. The frictions include currencies in the utility function, imposing restrictions on the use of currency for certain transactions, assuming differential transaction costs and having differential terms of trade depending on the currency. By assuming distinct liquidity properties for each currency, nominal exchange rate determinacy is effectively imposed on the model by making these currencies imperfect substitutes. A more desirable approach is to have the liquidity properties of currencies determined endogenously implying that the determinacy, or indeterminacy, of the nominal exchange rate is an equilibrium outcome.[2] This is the approach we take in this paper with our main contribution being that we break the curse of Kareken and Wallace without imposing ad hoc and differential restrictions on the liquidity properties of the currencies. We also show how inflation rates and the severity of the private information problem affect the properties of the nominal exchange rate.

In this paper we study the endogenous choice to accept fiat objects as media of exchange, the fundamentals that drive their acceptance, and the implications for their bilateral nominal exchange rate. To this end, we consider an economy where a medium of exchange is essential in the tradition of Rocheteau and Wright (2005) or Lagos and Wright (2005). Agents have no restrictions on what divisible fiat currency can be used to settle transactions. These currencies face a private information problem regarding their quality. We build on the insights of Li, Rocheteau and Weill (2012) and allow both fiat currencies to be counterfeited at a fixed cost. Since sellers cannot recognize counterfeited currency, in equilibrium they put a limit on how much of each currency they are willing to accept. These upper bounds in turn are endogenous and depend on the relative inflation rates associated with each currency and the severity of the private information problem.

In this environment currency substitution occurs as a response to the endogenous liquidity constraints arising from the private information problem. These constraints are such that increased matching efficiency or fixed costs of counterfeiting tighten the buyers' upper bound on payment

[1]With perfect currency substitution, there is only one single world market clearing condition determining the supplies and demands of all currencies jointly. Thus an indeterminate monetary equilibrium can only be pinned down by an exogenous selection of the nominal exchange rate. This exogenous information is often interpreted as arbitrary speculation.

[2]Consequently, we are trying to adhere to Wallace's dictum (1998). We interpret Wallace's dictum to mean that 1) monetary economists should explain why fiat money is essential, not assume that it is; 2) the value of fiat money should be determined without resorting to ad hoc restrictions; and 3) any "good" model of money should have a non-monetary equilibrium as a possibility.

As a corollary, we argue that 1) monetary economists should explain why the nominal exchange rate is determinate, not assume that it is; 2) determine its value without resorting to ad hoc restrictions on currencies and 3) a "good" model of fiat currencies should have nominal exchange rate indeterminacy as a possible equilibrium outcome.

offers. A critical feature of these liquidity constraints is that the marginal liquidity value of an additional unit of currency beyond the upper bound is zero.

We find that when neither endogenous liquidity constraint is binding then the nominal exchange rate is indeterminate as in Kareken and Wallace. However, if it binds for one or both currencies, then we have nominal exchange rate determinacy. When both liquidity constraints are binding and the currencies are identical in every respect, i.e., same counterfeiting costs and rate of return, we obtain the surprising result that the nominal exchange rate is the ratio of the two money stocks, which is the standard solution coming out of a two country cash in advance model. We also show that when there is nominal exchange rate indeterminacy, there exist fiscal policies that can restore determinacy of the nominal exchange rate. Another surprising result is that the first best may not be attainable even if the Friedman rule is implemented for both currencies. The reason is that the endogenous counterfeiting constraints may still bind such that the first best quantity of goods cannot be purchased.

An interesting feature of our results is that there is no counterfeiting in equilibrium. It is the threat of counterfeiting that pins down the nominal exchange rate and because of this both currencies can circulate even though one of them is dominated in rate of return. This is interesting because empirical evidence suggests that observed counterfeiting of currencies is not a big problem in practice as substantial resources and penalties are applied to those that counterfeit.[3] However, our results show that even if counterfeiting is not important quantitatively, its threat is nevertheless of first-order importance for nominal exchange rate determination.

We also show that ours results are robust to a variety of environments. For instance, relative to the benchmark economy, the introduction of credit enlarges the set of equilibria where nominal exchange rate is indeterminate. Finally, we generalize our results to a two country model.[4]

In what follows, Section 2 reviews the literature and Section 3 describes the model environment. In particular, the key private information friction giving rise to endogenous liquidity constraints is described and the equilibrium of an associated signalling game is characterized. The equilibrium characterization of the game is then embedded in a general monetary equilibrium in Section 4. In this section, we also consider the implications of the endogenous liquidity constraints for equilibrium and exchange rate determinacy. In Section 5, we explore the robustness of the proposed mechanism, the threat of counterfeiting, in helping determine nominal exchange rates by considering a variety of trading environments and alternative timing and composition assumptions regarding monetary transfers. We also discuss how cross-country international monetary policies, and, in conjunction

[3]Central Banks around the world spend resources to prevent counterfeiting by incorporating several security features on notes. Also, counterfeiting currencies is a punishable criminal offence. Several law enforcement entities like INTERPOL, the United States Secret Service and Europol as well as the European Anti-Fraud Office (OLAF), European Central Bank, the US Federal Reserve Bank, and the Central Bank Counterfeit Deterrence Group provide forensic support, operational assistance, and technical databases in order to assist countries in addressing counterfeit currency on a global scale. All these features and efforts substantially reduce the number of circulating counterfeited notes.

[4]This alternative setup results in an explicit international finance model with spatially separated decentralized trades in each country. This alternative model will be more familiar to the standard literature on international macroeconomics.

with domestic fiscal policy may further rescue the economy from the Kareken and Wallace indeterminacy result. Finally, Section 6 offers some concluding remarks. All proofs are given in the Supplementary Appendix.

2 Related literature

Models in mainstream international monetary economics typically pin down the value of a currency by imposing exogenous assumptions on what objects may be used as media of exchange. For instance, Stockman (1980) and Lucas (1982), among others, assume that in order to buy a good produced by a particular country, only that country's currency can be used. That is, in these environments, the demand for a specific fiat currency is solely driven by the demand for goods produced by that particular country. Devereaux and Shi (2013) study a trading post model under the assumption that there is only bilateral exchange at each trading post. Thus, by assumption, the ability to pay for goods with combinations of currencies is eliminated. Assumptions of this sort are *exogenous* currency constraints. By construction, they yield determinacy in agents' portfolio holdings of any two fiat currencies, and therefore determinacy in their nominal exchange rate.[5] Other researchers have introduced local currency in the utility function as in Obstfeld and Rogoff (1984) or have assumed differential trading cost advantages through network externalities as in Uribe (1997) or having different terms of trade depending on the currency used when purchasing goods as in Nosal and Rocheteau (2011) or different costly technologies to recognize currencies as in Zhang (2014).[6] In short, *endogenous* currency choice effectively is assumed away in this literature.

In the early search theoretic models of money, agents are able to choose which currencies to accept and use for payment. This literature shows that multiple currencies can circulate even if one is dominated in rate of return and the nominal exchange rate is determinate [see Matsuyama, Kiyotaki and Matsui (1993), Zhou (1997) and Waller and Curtis (2003), Craig and Waller (2004), Camera, Craig and Waller (2004)]. In these models, currency exchange can occur in bilateral matches if agents' portfolios are overly weighted towards one currency or the other. In fact, this leads to a distribution of determinate exchange rates. However, these findings are driven solely by the decentralized nature of exchange since agents never have access to a centralized market for rebalancing their portfolios. Once agents have the ability to rebalance their currency holdings, be it by the large family assumption in Shi (1997) or the periodic centralized market structure in Lagos and Wright (2005), the curse of Kareken and Wallace rears its head. To get around the curse, Head and Shi (2003) consider an environment where the large household can hold a portfolio of currencies but individual buyers are constrained to hold only one currency. So although the household endogenously chooses a portfolio of currencies, bilateral exchange requires using one currency or the other but not both simultaneously. In another paper, Liu and Shi (2010) assume

[5]In another strand of literature coined as the "New Open Economy Macroeconomics", which is partially summarized in Obstfeld and Rogoff (1996) and used extensively for monetary policy prescriptions, similar assumptions are in place.

[6]We refer the reader to chapter 10 section 2.2 of Nosal and Rocheteau's book for more on this issue.

that buyers can offer any currency but sellers can only accept one currency. Nosal and Rocheteau (2011), instead, adopt a trading mechanism in decentralized markets whereby a buyer obtains better terms of trade in a country by using the domestic money rather than the foreign one.[7] The main contribution of our paper relative to this literature is that we have centralized exchange and no exogenous restrictions on currency exchange nor differential trading protocols, yet we can obtain nominal exchange rate determinacy, even when the currencies are perfect substitutes. All we require is a private information problem between buyers and sellers in decentralized exchanges.

The paper closest in spirit to ours is that of Zhang (2014), who considers an open economy search model with multiple competing currencies and governments that require transactions to be made in a local currency. Buyers can always costlessly produce counterfeit currencies while sellers face a recognizability problem, as in Lester, Postlewaite and Wright (2012). The recognizability problem is only in terms of foreign currencies. In order for sellers to detect counterfeits they have to purchase a counterfeit detection technology by incurring a fixed cost each period.[8] Here, trade in bilateral matches occur under full information—it is common knowledge in a match whether the seller has invested in the detection technology and that sellers do not accept currencies they do not recognize. This allows for strategic complementarities so that multiple equilibria exist.[9] Because producing a counterfeit when meeting an uninformed seller is a dominant strategy, unrecognizable fiat currency cannot be used as means of payment in a fraction of the matches where sellers do not have the relevant detection technology. When there are no government agents and both currencies have the same rate of return, Zhang (2014) shows that there exist equilibria where the nominal exchange rates is determinate. In other words, in Zhang (2014) currency-choice outcomes and currency coexistence emerge at the extensive margin as possible equilibrium phenomena.

In contrast to Zhang (2014), we propose an environment that has an explicit private information problem and study its implications for the determination of nominal exchange rates. As a result, we can examine the seller's decision to accept different fiat currencies at the intensive margin rather than the extensive margin as in Zhang (2014).[10] Monetary equilibria in our environment have the property that when the liquidity constraint of one currency binds, the marginal value of an additional unit of this currency is zero since the seller will not accept it. At the margin sellers will produce an extra unit of output only for the fiat currency that has a non-binding liquidity constraint. Thus, generally a buyer would offer—as payment for goods—the currency with the best rate of return. If the endogenous liquidity constraint is binding on the higher return currency, the buyer pays for additional units of the good with the lower return currency. This feature is critical in determining the properties of the nominal exchange rate. We can also allow for situations where the two currencies are perfect substitutes —i.e. when the cost of counterfeiting and the inflation rates of the two currencies are the same—and yet there is coexistence of the currencies and determinacy

[7]By doing so the authors are able to exploit all potential gains from trade as opposed to currency in advance models and determine the nominal exchange rate.

[8]These information costs try to reflect the costly nature of dealing with multiple currencies.

[9]This multiplicity is in the spirit of Rocheteau and Wright (2005) where there exist complementarities between sellers' decision to enter and buyers' choice of real balances.

[10]We thank Randy Wright for helping us fine-tune this point.

of their nominal exchange rate.

3 Model

We propose an environment that has an explicit private information problem and studies its implications for the determination of nominal exchange rates. Agents can trade with two fiat currencies. In this environment a medium of exchange is essential and agents face private information in some markets. We assume a per-period sequential decentralized-then-centralized market structure and anonymous trading in decentralized markets as in Lagos and Wright (2005) and sellers face a private information problem with both currencies as in Li, Rocheteau and Weill (2012).

The Kareken and Wallace indeterminacy result is present in a closed or in a multi-country economy with two independently issued fiat currencies. To ease exposition and to highlight the role that private information plays in determining nominal exchange rates, we first consider the closed economy setting. In a later section we analyze the two-country version and illustrate different types of equilibria that can emerge.

General Description The economy has a continuum of agents of measure 2. Time is discrete and indexed by $t \in \mathbb{N} := \{0, 1, 2, ...\}$. Each period is divided into two sub-periods with different trading protocols and informational frictions. In the first sub-period, anonymous agents meet pairwise and at random in a decentralized market (DM).[11] Sellers in this market face informational asymmetry regarding the quality of the fiat currencies to be exchanged for goods. These currencies are perfectly divisible and are the only assets in the economy which can grow potentially at different constant rates. In the second sub-period, all activity occurs in a full information and frictionless centralized market (CM).

DM production is specialized and agents take on fixed trader types so that they are either buyers (consumers) or sellers (producers) as in Rocheteau and Wright (2005).[12] In CM all agents can produce and consume an homogenous perishable good. Agents, in this market can trade the CM good and rebalance their currency portfolio.

Preferences Agents derive utility from DM and CM consumption and some disutility from effort. A common discount factor $\beta \in (0, 1)$ applies to utility flows one period ahead. Given the

[11] The search literature uses the term anonymity to encompass these three frictions: (i) no record-keeping over individual trading histories ("memory"), (ii) no public communication of histories and (iii) insufficient enforcement (or punishment). An environment with any of these frictions imply that credit between buyer and seller is not incentive compatible.

[12] The justification for this assumption is twofold. First, it allows for a simple description of production specialization as in Alchian (1977). Second, it allows us to abstract away from the additional role of money as a medium of insurance against buyer/seller idiosyncratic shocks. An instance of the latter can be found in the Lagos and Wright (2005) sort of environment.

specialization structure in DM, the (discounted) total expected utility of a DM-buyer is given by

$$\mathbb{E}\left\{\sum_{t=0}^{\infty}\beta^t\left[u(q_t)+\mathcal{U}(C_t)-N_t\right]\right\},\tag{1}$$

where q_t represents DM goods, N_t is the CM labor supply and C_t denotes consumption of perishable CM good. Finally, $\mathbb{E}$ is a linear expectations operator with respect to an equilibrium distribution of idiosyncratic agent types.[13] The utility function $u:\mathbb{R}_+\to\mathbb{R}$ is such that $u(0)=0$, $u'(q)>0$ and $u''(q)<0$, for all $q\in\mathbb{R}_+$. Also, $\mathcal{U}:\mathbb{R}_+\to\mathbb{R}$ has the property that $\mathcal{U}(0)=0$, $\mathcal{U}'(C)>0$, and $\mathcal{U}''(C)<0$, for all $C\in\mathbb{R}_+$.

The (discounted) total expected utility of a DM-seller is given by

$$\mathbb{E}\left\{\sum_{t=0}^{\infty}\beta^t\left[-c(q_t)+\mathcal{U}(C_t)-N_t\right]\right\},\tag{2}$$

where the utility cost function $c:\mathbb{R}_+\to\mathbb{R}_+$ is such that $c(0)=0$, $c'(q)>0$, and $c''(q)\geq 0$. Note that DM-buyers and DM-sellers have identical per-period payoff functions in the CM sub-period given by $\mathcal{U}(C)-N$ as both types of agents can consume and produce in this frictionless market.

Information and Trade Since agents in DM have fixed types and production is specialized, agents face a double coincidence problem. Moreover, since buyers and sellers in DM are anonymous, the only incentive compatible form of payment is fiat money. Buyers and sellers have access to two distinct and divisible fiat currencies. Following Kareken and Wallace (1981), and in contrast to mainstream international macroeconomics, we do not impose any restrictions on which of the currencies, nor the compositions thereof, can be used to settle transactions.[14] However, sellers face asymmetric information, as in Li, Rocheteau and Weill (2012), regarding the quality of the currencies when trading in DM. Finally, there is a technology that can detect and destroy counterfeits that circulate in CM.

In the next sections we describe in detail the sub-period trades. The precise information problem that buyers and sellers are facing will be described in more detail below.

3.1 Centralized Market

After trade occurs in DM, agents have access to a frictionless Walrasian market (CM). At the beginning of each CM, DM-buyers have non-negative balances of both currencies which we denote by m_t and m_t^f. Before they make decisions and participate in the CM good, labor and asset markets,

[13] The model has no aggregate random variables, and therefore, it will turn out that the equilibrium distribution of agent types will depend only on the idiosyncratic random-matching probability $\sigma\in(0,1)$, and the equilibrium probabilities concerning the acceptability and genuineness of assets in an exchange, respectively, $\pi\in[0,1]$ and $(\eta,\eta^f)\in[0,1]^2$. The expected utility setup will be made more precise later.

[14] Notable exceptions are Zhang (2014) and Nosal and Rocheteau (2011).

they receive a total lump-sum transfer of each currency denoted by x_t and x_t^f.[15] Then, CM agents can trade goods and rebalance their portfolio of currencies. After trade in the CM closes, but before matching in the DM begins, each buyer has the ability to counterfeit the different fiat currencies. The cost of counterfeiting is common knowledge and is assumed to be a per-period flow utility cost of $\kappa^f > 0$ for the f currency and $\kappa > 0$ for the other fiat money.[16] When trading in DM, sellers are not able to distinguish between genuine and counterfeited currencies. However, following Nosal and Wallace (2007) and Li, Rocheteau and Weill (2012), we assume that a technology exists that can detect and destroy any fraudulent fiat currencies when trading in CM, thus they cannot be exchanged for goods.[17]

CM goods are produced with a linear technology that all agents have access to, thus a medium of exchange in this market is not essential. Agents choose CM labor, end-of-period currency portfolio and consumption of CM goods. Each agent faces a sequential budget constraint given by

$$C_t \leq N_t - \phi_t[m_{t+1} - y_t] - \phi_t e_t[m_{t+1}^f - y_t^f], \tag{3}$$

where C_t denotes consumption of the CM good, and $y_t := m_t + x_t$ and $y_t^f := m_t^f + x_t^f$, respectively, are initial holdings of genuine each currency (including the transfers). The variable e_t represents the current nominal exchange rate which measures the value of one unit of currency f in units of the other currency and m_{t+1} and m_{t+1}^f are the end-of-period currency portfolio of the respective fiat currencies. Finally, ϕ_t ($\phi_t e_t$) denotes the value of a unit of m_t (m_t^f) in terms of the CM good.

Given the sequential nature of markets in this environment, the DM-buyers' currency portfolio and counterfeiting decisions are dynamic. We will defer the discussion of agents' dynamic decision problems until the next section, and only after we have described the random matching and private information bargaining game between potential DM-buyers and DM-sellers. For now, we note that all DM-buyers will exit each CM with the same currency portfolio. Likewise, all DM-sellers will exit with $m_{t+1} = m_{t+1}^f = 0$.

In the next section we describe the one-sided private information bargaining game between a potentially matched buyer and seller in DM. This problem will span from the end of a period-t CM to the end of a period-$(t+1)$ DM. Then we describe the dynamic decision problems of all agents and describe the monetary equilibrium.

3.2 Decentralized Market

Consider the DM sub-period where trade occurs through random bilateral matches. Below we describe the particulars of this frictional environment.

[15]This assumption will do no harm to the results later since agent preferences are quasilinear, so that having transfers made to DM-sellers as well does not matter for the end result. Also note that who gets which seigniorage transfer—i.e., x or x^f—does not matter in this setting given quasilinear preferences and the timing of the CM transfer. This becomes apparent later from equation (7).

[16]Given that CM is closed agents are no longer together so a coalition of buyers to save counterfeit costs is not feasible.

[17]This detecting of fraudulent currency is typically done by banks when clients deposit fiat currency into their accounts. The holder of these counterfeits has them removed (exchanged for nothing) from the economy.

Matching. There are two fixed types of agents in the DM: buyers (b) and sellers (s). The measures of both b– and s–types are equal to 1. At the beginning of each period $t \in \mathbb{N}$, ex-ante anonymous buyers and sellers enter DM where they are randomly and bilaterally matched. With probability $\sigma \in (0,1)$ each buyer is pairwise matched with a seller. Moreover, as agents are anonymous, exchange supported by contracts that promise repayment in the future is not incentive compatible. Therefore, agents trade just with currencies.

Feasible offers. Let $\omega := (q, d, d^f)$ denote the terms of trade that specifies how much a seller must produce in DM (q) in exchange for d^f units of the f currency and/or d units of the other currency. The particulars of the terms of trade ω is an outcome of a bargaining game with private information which we describe below. Denote the set of feasible buyer offers at each aggregate state (ϕ, e) as $\Omega(\phi, e) \ni \omega$.

Given DM preferences and technologies, the corresponding first-best quantity traded is $q^* \in (0, \infty)$ and satisfies $u'(q^*) = c'(q^*)$. For each aggregate state (ϕ, e), there exist maximal finite and positive numbers $\overline{q} := \overline{q}(\phi, e)$, $\overline{m} := \overline{m}(\phi, e)$ and $\overline{m}^f := \overline{m}^f(\phi, e)$ solving $(\overline{m} + e\overline{m}^f)\phi = u(\overline{q}) = c(\overline{q})$, since $u(\cdot)$ and $c(\cdot)$ are monotone and continuous functions on every $[0, \bar{q}(\phi, e)]$. That is, the outcomes $(\overline{q}, \overline{m}, \overline{m}^f)(\phi, e)$ will be finite for every (ϕ, e). Therefore, the set of all feasible offers $\Omega(\phi, e)$ at given (ϕ, e), is a closed and bounded subset of $\mathbb{R}^3_+$, where $\Omega(\phi, e) = [0, \overline{q}(\phi, e)] \times [0, \overline{m}(\phi, e)] \times [0, \overline{m}^f(\phi, e)]$. We summarize this observation in the lemma below.

Lemma 1 *For each given (ϕ, e), the set of feasible buyer offers $\Omega(\phi, e) \subset \mathbb{R}^3_+$ is compact.*

Having specified the set of all possible offers that the buyer can feasibly make in each state of the economy, we now characterize the private information bargaining game.

3.3 Private Information

The DM-buyers' portfolio composition of genuine and fraudulent fiat currencies is private information, so the seller cannot distinguish between them. This private information problem is modeled as a signaling game between pairs of randomly matched buyers (signal sender) and sellers (signal receiver). The game is a one-period extensive form game played out in virtual time between each CM and the following period's DM.

A buyer has private information on his accumulation decision and holdings of the two fiat currencies. A matched seller can observe the terms of trade $\omega := (q, d, d^f)$ offered by the buyer but she is not able to distinguish between genuine and counterfeited currencies.[18] In contrast to standard signalling games, here, signal senders have a choice over their private-information types. These types are defined by the buyer's portfolio choice at the end of each CM. If the buyer decides to counterfeit fiat currencies she will exchange them for DM goods as in the next CM they are going to be detected and destroyed. In what follows next, we first describe and characterize the equilibrium of the game.

[18]Implicit in our environment is that the seller can distinguish between the f currency and the other currency.

3.3.1 Endogenous-type Signalling Game

At the beginning of each DM, a seller s is randomly matched with a buyer b. The seller cannot recognize whether the buyer is offering genuine fiat currencies or not. Next we describe the exact timing of events.

Let CM$(t-1)$ denote the time-$(t-1)$ frictionless Walrasian market and DM(t) represent the time-t decentralized and frictional market. One could also think in terms of a CM(t) and its ensuing DM$(t+1)$, so the timing notation here does not affect the analysis. For every $t \geq 1$, and given prices, (ϕ_t, e_t), the timing of the signalling game is as follows:

1. At the end of CM$(t-1)$ a buyer decides whether or not to costly counterfeit each currency at the one-period fixed costs of $\kappa > 0$ and $\kappa^f > 0$, respectively. This decision is captured by the binary action $\chi, \chi^f \in \{0, 1\}$ where $\chi, \chi^f = 0$ represents "no counterfeiting of currency".

2. The buyer chooses how much CM$(t-1)$ good to produce in exchange for genuine currencies, m and/or m^f.

3. In the subsequent DM(t), a buyer is randomly matched with a seller with probability σ.[19] Upon a successful match, the buyer makes a take-it-or-leave-it (TIOLI) offer (q, d, d^f) to the seller.[20]

4. The seller decides whether to accept the offer or not. If the seller accepts, she produces according to the buyer's TIOLI offer.

The extensive-form game tree of this private information problem is depicted in Figure 1.

As in Li, Rocheteau and Weill (2012), this original extensive-form game has the same payoff-equivalent reduced-form game as the following reverse-ordered extensive-form game. Given prices, (ϕ_t, e_t), we describe the following reverse-ordered game:

1. A DM-buyer signals a TIOLI offer $\omega := (q, d, d^f)$ and commits to ω, before making any (C, N) decisions in CM$(t-1)$.

2. The buyer decides whether or not to counterfeit the fiat currencies, $\chi(\omega), \chi^f(\omega) \in \{0, 1\}$.

3. The buyer decides on portfolio $a(\omega) := (m, m^f)(\omega)$ and (C, N).

4. The buyer enters DM(t) and Nature randomly matches the buyer with a DM-seller with probability σ.

5. The DM-seller chooses whether to reject or accept the offer, $\alpha(\omega) \in \{0, 1\}$.

This reverse-ordered extensive-form game tree is depicted in Figure 2.

This new reverse-ordered game helps refine the set of perfect Bayesian equilibria (PBE) that would arise in the original extensive form game. In and Wright (2011) provide sufficient conditions

[19] For simplicity, double-coincidence-of-wants meetings occur with probability zero.

[20] Implicit in the offer is the buyer signalling that the payment offered consists of genuine assets.

for the existence of a PBE in an original extensive-form game which is an outcome equivalent to the PBE of its simpler reordered game. Such an equilibrium is called a *Reordering-invariant Equilibrium* or RI-equilibrium.[21]

3.3.2 Players and Strategies

To simplify exposition, we let X represent X_t, X_{-1} correspond to X_{t-1}, and X_{+1} stand for X_{t+1}, for any date $t \geq 1$. In the next section we characterize the buyer and seller's strategies.

A DM-buyer in CM$(t-1)$ has individual state, $\mathbf{s}_{-1} := (y_{-1}, y^f_{-1}; \phi_{-1}, e_{-1})$ which is publicly observable in CM$(t-1)$.[22] A DM-seller in CM$(t-1)$ is labelled as $\check{\mathbf{s}}_{-1} := (\check{m}_{-1}, \check{m}^f_{-1}; \phi_{-1}, e_{-1})$. Let $B(\phi, e) := [0, \overline{m}(\phi, e)] \times [0, \overline{m}^f(\phi, e)]$ denote the feasible currency portfolio choice set for a given aggregate state (ϕ, e).

Definition 1 *A pure strategy of a buyer, $\boldsymbol{\sigma}^s$, in the counterfeiting game is a triple $\langle \omega, \aleph(\omega), a(\omega) \rangle$ comprised by the following:*

1. *Offer decision rule, $\mathbf{s}_{-1} \mapsto \omega \equiv \omega(\mathbf{s}_{-1}) \in \Omega(\phi, e)$;*

2. *Binary decision rules on counterfeiting, $\aleph := \left\langle \chi(\omega), \chi^f(\omega) \right\rangle \in \{0, 1\}$, for each currency; and*

3. *Asset accumulation decision, $\omega \mapsto a(\omega) \in B(\phi, e)$, and, $(d, d^f) \leq a(\omega)$.*

A pure strategy of a seller $\boldsymbol{\sigma}^s$ is a binary acceptance rule $(\omega, \check{\mathbf{s}}_{-1}) \mapsto \alpha(\omega, \check{\mathbf{s}}_{-1}) \in \{0, 1\}$.

More generally, we allow players to play behavioral strategies given the buyer's posted offer ω. This is the case as quasilinearity in CM makes the buyer's payoff linear in (d, d^f). This implies that taking a lottery over these payments yields the same utility $u(q)$. Thus, for notational convenience, we drop the lottery over offers when describing a buyer's behavior strategy $\tilde{\boldsymbol{\sigma}}^b$.

Definition 2 *A behavior strategy of a buyer $\tilde{\boldsymbol{\sigma}}^b$ is a triple $\langle \omega, G[a(\omega)|\omega], H(\aleph|\omega) \rangle$, where*

1. *$H(\cdot|\omega) := \langle \eta(\cdot|\omega), \eta^f(\cdot|\omega) \rangle$ specifies marginal probability distributions over the $\{0, 1\}$ spaces of each of the two counterfeiting decisions $\aleph := (\chi, \chi^f)$; and*

2. *$G(\cdot|\omega)$ is a conditional lottery over each set of feasible asset pairs, $B(\phi, e)$.*

[21] See conditions A1-A3 in In and Wright (2011) for more details. Their characterization of equilibria is related to the Cho and Kreps (1987) Intuitive Criterion refinement, in the sense that both approaches are implied by the requirement of strategic stability (see Kohlberg and Mertens, 1986). However, the difference in the class of games considered by In and Wright (2011) to that of standard signalling games using Cho and Kreps, is that the class of games considered by the former admits signal senders who have an additional choice of a private-information action. That is, who chooses the private-information type—i.e. Nature in standard signalling games or a Sender in In and Wright (2011)—matters for the game structure. When a strategic and forward-looking Sender can choose his unobserved type, there will be additional ways he can deviate (but these deviations must be unprofitable in equilibrium). Thus standard PBE may still yield too many equilibria in these games with a signalling of private decisions. Further discussions are available in a separate appendix.

[22] Given exogenous policy outcomes x_{-1} and x^f_{-1}, and through a change of variables, we let $y_{-1} \equiv m_{-1} + x_{-1}$ and $y^f_{-1} \equiv m^f_{-1} + x^f_{-1}$ be the DM-buyer's individual state variables.

A behavior strategy of a seller is $\tilde{\boldsymbol{\sigma}}^s : \pi(\omega)$ which generates a lottery over $\{0,1\} \ni \alpha$.

Finally, we note that buyers in each CM($t-1$) make the same optimal decisions in subsequent periods. This is the case as agents have CM quasilinear preferences so that history does not matter. Likewise, for the sellers' decisions. All agents, conditional on their DM-buyer or DM-seller types, have the same individual state after they leave CM. Therefore, characterizing the equilibrium of the counterfeiting-bargaining game between a matched anonymous buyer and seller pair in DM(t) is tractable. Thus, we just can simply focus on the payoffs of any ex-ante DM-buyer and DM-seller.

3.3.3 Buyers' Payoff

Let $W^b(\cdot)$ denote the value function of a DM-buyer at the beginning of CM(t). Since per-period CM utilities are quasilinear the corresponding CM value function is linear in the buyer's individual state (m, m^f) so that

$$W^b(\mathbf{s}) \equiv W^b(y, y^f; \phi, e) = \phi(y + ey^f) + W^b(0, 0; \phi, e). \tag{4}$$

Let us define $Z(C_{-1}; \mathbf{s}_{-1}) = \mathcal{U}(C_{-1}) - C_{-1} + \phi_{-1}(y_{-1} + e_{-1}y^f_{-1})$ which summarizes the CM($t-1$) flow utility from consuming $(C_{-1}, -N_{-1})$ plus the time-($t-1$) real value of accumulating genuine fiat currencies. Then given prices (ϕ, e) and his belief about the seller's behavior $\hat{\pi}$, the DM-buyer's Bernoulli payoff function, $U^b(\cdot)$, can be written as follows:[23]

$$\begin{aligned} &U^b(C_{-1}, \omega, \eta, \eta^f, G[a(\omega)|\omega], \hat{\pi}|\mathbf{s}_{-1}; \phi, e) = \\ &\qquad \int_{B(\phi,e)} \Big\{ Z(C_{-1}; \mathbf{s}_{-1}) - \phi_{-1}(m + e_{-1}m^f) - \kappa(1-\eta) - \kappa^f(1-\eta^f) \\ &\qquad\qquad + \beta\sigma\hat{\pi} \left[u(q) + W^b(m - \eta d, m^f - \eta^f ed^f; \phi, e) \right] \\ &\qquad\qquad + \beta \left[\sigma(1-\hat{\pi}) + (1-\sigma) \right] W^b(m, m^f; \phi, e) \Big\} dG[a(\omega)|\omega]. \end{aligned} \tag{5}$$

Given the linearity of $W^b(\cdot)$, we can further reduce equation (5) to the following expression

$$\begin{aligned} U^b(C_{-1}, \omega, \eta, \eta^f, G[a(\omega)|\omega], \hat{\pi}|\mathbf{s}_{-1}; \phi, e) &= -\kappa(1-\eta) - \kappa^f(1-\eta^f) \\ + \int_{B(\phi,e)} \Big\{ & Z(C_{-1}; \mathbf{s}_{-1}) - \left(\frac{\phi_{-1}}{\phi} - \beta \right) \phi m - \left(\frac{\phi_{-1} e_{-1}}{\phi e} - \beta \right) \phi e m^f \\ & + \beta\sigma\hat{\pi} \left[u(q) - \phi \left(\eta d + \eta^f ed^f \right) \right] \Big\} dG[a(\omega)|\omega] \end{aligned} \tag{6}$$

which corresponds to the expected total payoff under a given strategy $\tilde{\boldsymbol{\sigma}}^b$ for a DM-buyer in CM($t-1$). Note that the first term of equation (6) is the expected total fixed cost of counterfeiting both currencies. The second term on the right of equation (6) is the utility flow from consuming $(C_{-1}, -N_{-1})$ and the DM(t) continuation value from accumulating currencies in CM($t-1$). The

[23] We have imposed symmetry among all sellers for notational simplicity.

third and fourth term are the expected total cost (equivalently inflation cost) of holding unused currencies between CM($t-1$) and DM(t). The last term is the expected net payoff gain from trades in which the buyer pays for the good q with genuine currencies, with marginal probability measures $H(\omega) =: (\eta, \eta^f)$, and the buyer believes a randomly encountered seller accepts with probability $\hat{\pi}$.

Finally, we still have to take into account the buyer's mixed strategy $G(\cdot|\omega)$. In Supplementary Appendix A we show that in a monetary equilibrium $G(\cdot|\omega)$ is always degenerate, so the buyer's total expected payoff in (6) further simplifies to

$$
\begin{aligned}
U^b[C_{-1}, \omega, &H(\omega), \hat{\pi}|\mathbf{s}_{-1}; \phi, e] = \\
&Z(\mathbf{s}_{-1}) - \left(\frac{\phi_{-1}}{\phi} - \beta\right)\phi m - \left(\frac{\phi_{-1}e_{-1}}{\phi e} - \beta\right)\phi e m^f \\
&- \kappa(1-\eta) - \kappa^f(1-\eta^f) + \beta\sigma\hat{\pi}\left[u(q) - \phi\left(\eta d + \eta^f e d^f\right)\right].
\end{aligned} \tag{7}
$$

Note that initial monetary wealth $y_{-1} := m_{-1} + x_{-1}$ and $y^f := m^f_{-1} + x^f_{-1}$ will not matter for the DM-buyers' CM decisions on the continuation portfolio of assets, (m, m^f), given the linearity of the payoff function in these choices.

3.3.4 Sellers' Payoff

A DM-seller's payoff function is simpler. Let $W^s(\cdot)$ denote the seller's value function at the start of any CM. The seller also has a linear value function $W^s(\cdot)$ in currency holdings. Let $Z(\breve{C}_{-1}; \breve{\mathbf{s}}_{-1}) = \mathcal{U}(\breve{C}_{-1}) - \breve{C}_{-1} + \phi_{-1}(\breve{m}_{-1} + e_{-1}\breve{m}^f_{-1})$ summarize the CM($t-1$) flow utility from consuming $(\breve{C}_{-1}, -\breve{H}_{-1})$ plus the time-($t-1$) real value of accumulating genuine currencies. Note that the DM-seller will always accumulate zero money holdings, because of inflation and the fact that she knows that she has no use of her money holdings in the ensuing DM.

Let $\left(\hat{\eta}, \hat{\eta}^f\right)$ be the seller's belief about the buyer's behavior with respect to counterfeiting of fiat currencies. Given an offer ω, the seller belief system and the seller's response $\pi(\omega)$, her Bernoulli payoff for the game is given by

$$
\begin{aligned}
U^s(\breve{C}_{-1}, \omega, &\hat{\eta}, \hat{\eta}^f, \pi(\omega)|\breve{\mathbf{s}}_{-1}; \phi, e) = Z(\breve{C}_{-1}; \breve{\mathbf{s}}_{-1}) \\
&+ \beta\sigma\pi(\omega)\left[-c(q) + W^s(\hat{\eta}d, \hat{\eta}^f d^f; \phi, e)\right] \\
&\quad + \beta\left[\sigma\left(1-\pi(\omega)\right) + (1-\sigma)\right]\left[-c(0) + W^s(0,0;\phi, e)\right] \\
&= Z(\breve{C}_{-1}; \breve{\mathbf{s}}_{-1}) + \beta\sigma\pi(\omega)\left[\phi\left(\hat{\eta}d + \hat{\eta}^f e d^f\right) - c(q)\right],
\end{aligned} \tag{8}
$$

where the last equality is a direct consequence of linearity in the seller's CM value function: $W^s(\breve{m}, \breve{m}^f) = \phi(\breve{m} + e\breve{m}^f) + W^s(0,0)$. The last term on the right of the payoff function (8) is the total discounted expected profit arising from the σ-measure of DM(t) exchange, in which the seller accepts an offer ω with probability $\pi(\omega)$ and she anticipates that the buyer pays with genuine assets according to beliefs $(\hat{\eta}, \hat{\eta}^f)$.

3.4 Equilibrium of the Private Information Game

The equilibrium concept for the counterfeiting-bargaining game is Perfect Bayesian in the reordered extensive-form game, as in Li, Rocheteau and Weill (2012). More precisely, we utilize the *RI-equilibrium* refinement proposed by In and Wright (2011). In order to solve the game we proceed by backward induction on the game depicted in Figure 1.

3.4.1 Seller's Problem

Following a (partially) private buyer history $\langle \omega, \aleph(\omega) \rangle$ in which an offer ω is observable and $\aleph$ is not observable, the seller plays a mixed strategy π to maximize her expected pay off which is given by

$$\pi(\omega) \in \left\{ \arg \max_{\pi' \in [0,1]} \pi' \left[\phi \left(\hat{\eta} d + \hat{\eta}^f e d^f \right) - c(q) \right] \right\}. \tag{9}$$

3.4.2 Buyer's Counterfeiting Problem

Given history ω and the buyer's belief about the seller's best response, $\hat{\pi}$, the buyer solves the following cost-minimization problem

$$\begin{aligned} (\eta(\omega), \eta^f(\omega)) \quad = \quad & \arg \max_{\eta, \eta^f \in [0,1]} \Bigg\{ -\kappa(1-\eta) - \kappa^f(1-\eta^f) - \beta\sigma\hat{\pi}\phi \left[\eta d + \eta^f e d^f \right] \\ & - \left(\frac{\phi_{-1}}{\phi} - \beta \right) \phi m - \left(\frac{\phi_{-1} e_{-1}}{\phi e} - \beta \right) \phi e m^f \Bigg\}. \end{aligned} \tag{10}$$

Given that the terms of trade in DM are given by the buyer's TIOLI offer at the beginning of the game, the buyer maximizes her payoff given her conjecture $(\hat{\eta}, \hat{\pi})$ of the continuation play, the buyer commits to an optimal offer $\omega \equiv (q, \hat{d}, \hat{d}^f)$ which is given by

$$\begin{aligned} \omega \quad \in \quad & \Bigg\{ \arg \max_{\omega' \in \Omega(\phi, e)} \Big\{ -\kappa (1-\hat{\eta}) - \kappa^f \left(1 - \hat{\eta}^f\right) + \beta\sigma\hat{\pi} \left[u(q) - \phi \left(\hat{\eta} \hat{d} + \hat{\eta}^f e \hat{d}^f \right) \right] \\ & - \left(\frac{\phi_{-1}}{\phi} - \beta \right) \phi m - \left(\frac{\phi_{-1} e_{-1}}{\phi e} - \beta \right) \phi e m^f \Big\} \Bigg\}. \end{aligned} \tag{11}$$

3.4.3 Equilibrium

Having specified the seller's and buyer's respective problems, we can now characterize the resulting equilibrium in the private-information bargaining game.

Definition 3 *A reordering-invariant (RI-) equilibrium of the original extensive-form game is a perfect Bayesian equilibrium* $\tilde{\boldsymbol{\sigma}} := (\tilde{\boldsymbol{\sigma}}^b, \tilde{\boldsymbol{\sigma}}^s) = \langle \omega, \eta(\omega), \eta^f(\omega), \pi(\omega) \rangle$ *of the reordered game such that (9) and (10) are satisfied.*

The following proposition provides a simple characterization of a RI-equilibrium in the game.

Proposition 1 (RI-equilibrium) *An RI-equilibrium of the counterfeiting-bargaining game is such that*

1. *Each seller accepts with probability* $\hat{\pi} = \pi(\omega) = 1$;

2. *Each buyer does not counterfeit:* $(\hat{\eta}, \hat{\eta^f}) = (\eta(\omega), \eta^f(\omega)) = (1, 1)$; *and*

3. *Each buyer's TIOLI offer* ω *is such that:*

$$
\begin{aligned}
\omega \in \Bigg\{ arg \max_{\omega \in \Omega(\phi, e)} \Bigg[-\left(\frac{\phi_{-1}}{\phi} - \beta\right)\phi m - \left(\frac{\phi_{-1}e_{-1}}{\phi e} - \beta\right)\phi e m^f \\
&+ \beta\sigma\left[u(q) - \phi\left(d + ed^f\right)\right]\Bigg] \qquad s.t. \\
&(\zeta): \quad \phi\left(d + ed^f\right) - c(q) \geq 0, \\
&(\nu): \quad 0 \leq d, \\
&(\mu): \quad d \leq m, \\
&(\nu^f): \quad 0 \leq d^f, \\
&(\mu^f): \quad d^f \leq m^f, \\
&(\ \lambda): \quad \phi d \leq \frac{\kappa}{\phi_{-1}/\phi - \beta(1-\sigma)} \equiv \bar{\kappa}(\phi_{-1}/\phi), \\
&(\lambda^f): \quad \phi e d^f \leq \frac{\kappa^f}{\phi_{-1}e_{-1}/\phi e - \beta(1-\sigma)} \equiv \bar{\kappa}^f(\phi_{-1}e_{-1}/\phi e)\Bigg\}.
\end{aligned}
\tag{12}
$$

and the RI-equilibrium is unique.

As we can see from the RI-equilibrium, ζ represents the Lagrange multiplier associated with the seller's participation constraint, ν (ν^f) is the Lagrange multiplier corresponding to the non-negativity of the payments in the f currency and the other fiat object. Finally, μ^f (μ) represents the feasibility constraint for $m^f(m)$, λ (λ^f) is the Lagrange multiplier corresponding to the liquidity constraint for the f currency (the other fiat object) that arise because of the threat of counterfeiting.

It is important to highlight that the last two constraints are *endogenous* liquidity constraints in that they provide an upper bound on the quantities of *genuine* currencies that the seller will accept. This type of endogenous constraints can also be observed under different trading protocols.[24]

These upper bounds depend positively on the fixed cost of counterfeiting and negatively on the degree of matching efficiency σ. Note that a larger σ implies greater matching efficiency in the DM so that buyers and sellers are more likely to meet and trade. This creates a larger incentive for the buyer to produce counterfeits, thus increasing the information problem. Thus, in equilibrium, in order for sellers to accept buyers' offers, each buyer has a tighter upper-bound on his signal/offer of DM payment. The same logic applies to the effect of the fixed costs of counterfeiting, and, also to the effect of the aggregate returns on holding *genuine* currencies.

[24]Shao (2014) shows that in a modified version of Li, Rocheteau and Weil (2012) with competitive search where sellers set the terms of trade, the exact same liquidity constraint is found as in Li, Rocheteau and Weil (2012). Finally, Berensten, McBride and Rocheteau (2014) propose a way to extend the methodology in Li, Rocheteau and Weil (2012) to the case with proportional bargaining.

A critical feature of these liquidity constraints is that the marginal liquidity value of an additional unit of currency beyond the bound is zero. Thus, if one currency has a higher rate of return (lower inflation rate) but a lower counterfeiting cost, then the buyer will first pay with it up to the bound and use the weaker currency to pay for the remainder of the goods purchased.

3.5 Money supplies and seigniorage transfers

We assume that the supply of the fiat monies, M_t and M_t^f, respectively, grow at a constant rate of γ and γ^f. Lump sum transfers of each currency are transferred uniformly to the DM-buyers as $x_t = M_{t+1} - M_t = (\gamma - 1)\gamma^{t-1} M_0$ and $x_t^f = M_{t+1}^f - M_t^f = (\gamma^f - 1)(\gamma^f)^{t-1} M_0^f$, at the beginning of each CM. The initial stocks M_0 and M_0^f are known.

4 Monetary Equilibrium

We can now embed the equilibrium characterization of the game into the monetary equilibrium of the model. Since preferences are quasilinear, the infinite history of past games between buyers and sellers does not matter for each current period agents' decision problems. This allows us to tractably incorporate the equilibrium characterization of the game previously described, into the overall dynamic general monetary setting. Before we do so, we return to describing the agents' dynamic decision problems.

4.1 Agents' Recursive Problems

DM-buyers' Problem As we previously saw, the beginning-of-CM value function for buyers $W^b(\cdot; \phi, e)$ is linear in the fiat currency portfolio (y, y^f). As a result, the buyer's intertemporal problem, conditional on an equilibrium of the private-information bargaining game, is given by

$$\max_{C_{-1}, q, d, d^f, m, m^f} U^b(C_{-1}, \omega, \eta(\omega), \hat{\pi} | \mathbf{s}_{-1}; \phi, e) \quad \text{s.t.}$$

$$\begin{aligned}
(\eta(\omega), \eta^f(\omega)) = \quad & (1,1), \qquad \hat{\pi} = \pi(\omega) = 1, && \text{(13a)}\\
(\zeta): \quad & \phi\left(d + ed^f\right) - c(q) = 0, && \text{(13b)}\\
(\nu): \quad & 0 \leq d, && \text{(13c)}\\
(\mu): \quad & d \leq m, && \text{(13d)}\\
(\nu^f): \quad & 0 \leq d^f, && \text{(13e)}\\
(\mu^f): \quad & d^f \leq m^f, && \text{(13f)}\\
(\lambda): \quad & \phi d \leq \bar{\kappa}(\phi_{-1}/\phi), && \text{(13g)}\\
(\lambda^f): \quad & \phi e d^f \leq \bar{\kappa}^f(\phi_{-1} e_{-1}/\phi e). && \text{(13h)}
\end{aligned}$$

where the DM-buyer's lifetime expected payoff is given by

$$\begin{aligned}
&U^b(C_{-1},\omega,\eta(\omega),\eta^f(\omega),\hat{\pi}|\mathbf{s}_{-1};\phi,e) \\
&\quad = \mathcal{U}(C_{-1}) - C_{-1} + \phi_{-1}(y_{-1} + e_{-1}y^f_{-1}) - \left(\frac{\phi_{-1}}{\phi} - \beta\right)\phi m \\
&\quad - \left(\frac{\phi_{-1}e_{-1}}{\phi e} - \beta\right)\phi e m^f + \beta\sigma\left[u(q) - \phi\left(d + ed^f\right)\right].
\end{aligned} \tag{14}$$

In contrast to a full information setting, the threat of counterfeits which is private information to buyers, introduces additional endogenous state-dependent liquidity constraints (13g)-(13h) into a buyer's Bellman equation problem. These endogenous liquidity constraints are going to play an important role in determining the coexistence of the two currencies and the determinacy of nominal exchange rates.

The corresponding first order conditions of the rest of the DM-buyers' dynamic decision problem, given the RI-equilibrium, are given by

$$1 = \mathcal{U}'(C), \tag{15}$$

$$0 = \beta\sigma u'(q) - \zeta c'(q), \tag{16}$$

$$\beta\sigma = \zeta + \nu - \mu - \lambda, \tag{17}$$

$$\beta\sigma = \zeta + \nu^f - \mu^f - \lambda^f, \tag{18}$$

$$\mu = \frac{\phi_{-1}}{\phi} - \beta, \tag{19}$$

$$\mu^f = \frac{\phi^f_{-1}}{\phi^f} - \beta. \tag{20}$$

$$\zeta \geq 0, \nu \geq 0, \nu^f \geq 0, \mu \geq 0, \mu^f \geq 0, \lambda \geq 0, \lambda^f \geq 0, \tag{21}$$

for every date $t \geq 1$.

Note that Equation (15) describes the optimal within-period labor versus consumption trade-off in CM, where the marginal disutility of labor is -1 and the real-wage (marginal product of labor) is 1. Equation (16) corresponds to the first order condition for DM output which equates the marginal benefit of consuming and marginal value of the payment to the seller. Since the buyer offers a TIOLI, the payment is equal to the seller's DM production cost. Equations (17) and (18) summarize the optimal choice with respect to the two nominal payments and equate the value of holding a particular fiat currency from one CM to the next versus trading it in DM. Finally, equations (19) and (20) describe the optimal accumulation of each currency which of course depends on their implied rate of return. Equations (17) and (19) (or (18) and (20)) imply a sequence of intertemporal consumption Euler inequalities, where one or both currencies are used as store of value.

DM-sellers' Problem A DM-seller's problem, embedding the game's equilibrium, is simpler as sellers cannot counterfeit. This is given by

$$\max_{C_{-1}} U^s(\breve{C}_{-1}, \omega, \eta(\omega), \eta^f(\omega), \hat{\pi}|\breve{\mathbf{s}}_{-1}; \phi, e) \quad \text{s.t.} \quad (\eta(\omega), \eta^f(\omega)) = (1,1) \text{ and } \hat{\pi} = 1;$$

where each seller's Bernoulli payoff is given by

$$\begin{aligned} &U^s(\breve{C}_{-1}, \omega, \eta(\omega), \eta^f(\omega), \hat{\pi}|\breve{\mathbf{s}}_{-1}; \phi, e) \\ &\qquad = \mathcal{U}(\breve{C}_{-1}) - \breve{C}_{-1} + \phi_{-1}(\breve{m}_{-1} + e_{-1}\breve{m}^f_{-1}) + \beta\sigma \left[\phi \left(d + ed^f\right) - c(q)\right]. \end{aligned} \tag{22}$$

4.2 Steady State Monetary Equilibrium

We will focus on steady-state monetary equilibria where the nominal exchange rate can grow at a constant rate. In fact, if we consider (for now) monetary equilibria where both monies circulate, we have the following intermediate observation:[25]

Proposition 2 *Assume the existence of a monetary equilibrium where both monies circulate. When there is no portfolio restriction on what currencies must serve as a medium of exchange in any country, the equilibrium nominal exchange rate growing in absolute terms at a constant and bounded rate γ_e, i.e.,*

$$\left|\frac{e_{t+1} - e_t}{e_t}\right| = \gamma_e \in [0, +\infty),$$

for all $t \geq 0$, is a (deterministic) monetary equilibrium property.

For the paper we thus focus on monetary equilibria in which the equilibrium exchange rate grows at some constant rate (possibly zero). Also, Proposition 2 will apply in the explicit two-country version of the economy.

We study the implications of the endogenous liquidity constraints for the coexistence of multiple fiat currencies. This also allows us to understand under what conditions there is determinacy of the nominal exchange rate.

Define stationary variables by taking ratios of growing variables as follows $\frac{M}{M_{-1}} = \gamma = \Pi \equiv \frac{\phi_{-1}}{\phi}$; $\frac{M^f}{M^f_{-1}} = \gamma^f = \Pi^f \equiv \frac{\phi^f_{-1}}{\phi^f}$. In steady state, all real quantities are constant implying $\phi M = \phi_{-1}M_{-1}$, and, $e\phi M^f = e_{-1}\phi_{-1}M^f_{-1}$, which yields the steady state home currency (gross) depreciation/appreciation as

$$\frac{e}{e_{-1}} = \frac{\gamma}{\gamma^f} = \frac{\Pi}{\Pi^f}. \tag{23}$$

We now examine monetary equilibrium where money markets clear so that $m = M$ and $m^f = M^f$ and where $d = m$ and $d^f = m^f$ (see Lemma 2 in the Supplementary Appendix), then the steady

[25] We relegate the proof to Supplementary Appendix C.

state of the economy satisfies the following set of equations and weak inequalities:

$$\begin{aligned}
1 &= \mathcal{U}'(C), & (24)\\
0 &= \beta\sigma u'(q) - \zeta c'(q), & (25)\\
\beta\sigma &= \zeta + \nu - \mu - \lambda, & (26)\\
\beta\sigma &= \zeta + \nu^f - \mu^f - \lambda^f, & (27)\\
\mu &= \Pi - \beta \geq 0, & (28)\\
\mu^f &= \Pi^f - \beta \geq 0, & (29)\\
c(q) &= \phi M + e\phi M^f, & (30)\\
\phi M &\leq \frac{\kappa}{\Pi - \beta(1-\sigma)}, & (31)\\
e\phi M^f &\leq \frac{\kappa^f}{\Pi^f - \beta(1-\sigma)}. & (32)
\end{aligned}$$

Some description of this monetary steady state is in order. Equation (24) describes the optimal consumption of the CM good. Equation (25) corresponds to the first order condition for DM output. Equations (26) and (27) summarize the optimal choice with respect to nominal payments. Equations (28) and (29) describe the optimal accumulation of each currency. Finally, equations (30), (31) and (32) correspond to the multipliers $\zeta > 0$, $\lambda \geq 0$ and $\lambda^f \geq 0$. For the rest of the paper we focus on equilibria that satisfy $\zeta > 0$ (i.e. DM-seller's participation constraint (30) binds).

Intuitively, a determinate equilibrium arises if all conditions (24)-(32) hold with strict equality, with the multipliers (μ, μ^f) being strictly positive. However, this may not always hold. In particular, determinacy of the steady state equilibrium, and therefore its nominal exchange rate outcome, depends crucially on cross-country monetary policies (Π, Π^f). It also depends on the economic structure—in particular the counterfeiting costs (κ, κ^f) and matching rates in decentralized and frictional trades σ.

The following Proposition is the main result of the paper which provides sufficient conditions for the two currencies to coexist, and, for the nominal exchange rate to be determinate, even if one of the currencies is dominated in rate of return.

Proposition 3 (Equilibria and Coexistence) *Depending on the relative inflation rates of the two fiat currencies, there are three cases charaterizing a steady-state monetary equilibrium:*

1. *When currency f dominates in rate of return* $(\Pi > \Pi^f)$ *and*

   (a) *when neither liquidity constraints bind* $(\lambda = \lambda^f = 0)$*, or when only the liquidity constraint on the dominated fiat currency binds* $(\lambda > 0, \lambda^f = 0)$*, then a monetary equilibrium exists with the unique outcome where only the low inflation currency circulates; or,*

   (b) *the liquidity constraint on currency f binds* $(\lambda^f > \lambda = 0)$*, then there exists a monetary equilibrium with a unique outcome where the currencies coexist and the nominal exchange* *rate is determinate*

$$e = \frac{M}{M^f} \frac{\kappa^f}{c(q)\left[\Pi^f - \beta(1-\sigma)\right] - \kappa^f};$$

where q solves

$$\frac{\Pi - \beta}{\sigma\beta} = \frac{u'(q) - c'(q)}{c'(q)}.$$

(c) *both liquidity constraints bind ($\lambda^f > 0, \lambda > 0$), then there exists a unique monetary equilibrium where the currencies coexist and the nominal exchange rate is determinate*

$$e = \frac{\kappa^f M}{\kappa M^f} \frac{\Pi - \beta(1-\sigma)}{\Pi^f - \beta(1-\sigma)}.$$

2. *When currency f is dominated in rate of return ($\Pi^f > \Pi$), the coexistence results are the symmetric opposite to those of Case 1.*

3. *When the currencies have the same rate of return ($\Pi^f = \Pi$) and*

   (a) *neither liquidity constraints bind ($\lambda = \lambda^f = 0$), then the two fiat monies coexist but the individual's currency portfolio composition and the nominal exchange rate are indeterminate;*

   (b) *both liquidity constraints bind ($\lambda = \lambda^f > 0$), then the currencies coexist. The individual's currency portfolio composition is unique, and thus there is a unique nominal exchange rate*

$$e = \frac{\kappa^f M}{\kappa M^f}.$$

The key point of this Proposition is that, although there is no counterfeiting in equilibrium, the threat of counterfeiting is all that is required to generate the coexistence and determinacy of the nominal exchange rate as long as one or both of the liquidity constraints, (31) and/or (32), bind. This is true even if the currencies are equivalent in all respects; i.e., the inflation rates and the counterfeiting costs are the same. Thus we have broken the curse of Kareken and Wallace.

What is actually shocking about Case 3b is that when the two currencies are identical in every respect, i.e., $\Pi = \Pi^f$ and $\kappa = \kappa^f$, and both can be used simultaneously as payment, then the nominal exchange rate is simply the ratio of the two money stocks. This is the standard exchange rate solution that comes from a symmetric, two country, domestic cash-in-advance (CIA) model as in Stockman (1980) and Lucas (1982). While one may be tempted to say that we have provided a 'micro-foundation' for the two country CIA model, this would be incorrect for two reasons. First, our result only holds for a limited set of parameter values. Second, in the standard CIA model only one of the currencies is used per transaction, whereas here both currencies are used in the same transaction. Thus, equivalent exchange rate solutions should not be confused with equivalent results elsewhere. Nevertheless, it is interesting to note that it can be derived under certain conditions.

Proposition 3 also contemplates the possibility of just one currency circulating. It is important to highlight that this Proposition describes equilibria that have the property that when the liquidity constraint binds, the marginal value of an additional unit of this currency is zero since the seller will not accept it. At the margin DM-sellers will produce an extra unit of output only for the fiat currency that has a non-binding liquidity constraint. Under these circumstances, the buyer first offers the currency with the best rate of return. Once the endogenous liquidity constraint is binding, the buyer pays for additional units of the DM good with the lower return currency. This intensive margin is key in generating the results in Proposition 3.

In addition to breaking the curse of Kareken and Wallace, we are also able to establish some results regarding efficiency.

Proposition 4 (First Best) *When both inflation rates are at the Friedman rule,* $\Pi = \Pi^f = \beta$, *the first best quantity* q^* ***may not*** *be attainable, Case (3b), and the nominal exchange rate may not be determinate, Case (3a).*

To demonstrate this result, it suffices to provide a counter-example to the claim that the Friedman rule is always optimal. Indeed we show that when there is the threat of counterfeiting, the Friedman Rule may no longer be able to achieve the first best as each DM-seller is not willing to produce more output than what can be afforded by a DM-buyer faced with binding endogenous liquidity constraints. Note that in environments without private information nor bargaining inefficiencies, resulting from Nash Bargaining, the Friedman rule is able to correct for the intertemporal distortion and achieve first best allocations as it does not distort the saving decisions of buyers.[26] However, in this environment having the highest rate of return on fiat currency is not enough as sellers have an upper bound of how much currency they are willing to accept. This liquidity constraint is a direct consequence of the counterfeiting problem they face. As a result, it is possible that sellers do not accept currencies consistent with first best production even at the Friedman rule.

4.3 Sensitivity of equilibrium coexistence

We briefly discuss how the fixed costs (κ and κ^f) consistent with coexistence equilibria vary with inflation (we consider Π^f), market/matching friction (σ), DM-buyers' intertemporal elasticity of substitution (parameterized as $1/\theta := u'(q)/u''(q) \cdot q$) and the convexity of sellers' production cost (parametrized as α). We focus only on Case 1 where $\Pi > \Pi^f > \beta$. The different equilibria to be considered are: (Case 1a) where $\lambda = \lambda^f = 0$, or, where $\lambda > 0, \lambda^f = 0$, (Case 1b) $\lambda = 0, \lambda^f > 0$, and (Case 1c) $\lambda > 0, \lambda^f > 0$.

It was shown in Proposition 3 that in Case 1a, only currency f is used as a medium of exchange. However, in Case 1b, we know that there is a unique $\hat{q}$ that solves $\beta\sigma[u'(q)/c'(q) - 1] = \Pi - \beta$. It is straightforward to arrive at the following set of inequalities that define feasible counterfeiting costs (κ, κ^f) consistent with equilibrium coexistence of the two monies in Case 1b. These are given by:

$$\kappa^f \cdot \frac{[\Pi - \beta(1-\sigma)]}{\Pi^f - \beta(1-\sigma)} \leq [\Pi - \beta(1-\sigma)]c(\hat{q}) < \kappa^f \cdot \frac{\Pi - \beta(1-\sigma)}{\Pi^f - \beta(1-\sigma)} + \kappa. \tag{33}$$

[26] For more on the distortions induced by Nash bargaining and how monetary and fiscal policy can restore the first best we refer to Gomis-Porqueras et al (2010).

The first weak inequality is obtained by combining the DM-seller's binding participation constraint, the binding liquidity constraint on currency f and the requirement that currency holdings are non-negative. The second strict inequality is a direct consequence of the slack liquidity constraint on holding the dominated currency.

In Case 1c, the coexistence equilibrium (given $\Pi > \Pi^f$) has the following restrictions on (κ, κ^f):

$$\kappa^f \cdot \frac{\Pi - \beta(1-\sigma)}{\Pi^f - \beta(1-\sigma)} + \kappa \leq [\Pi - \beta(1-\sigma)]c(\hat{q}), \qquad \kappa > 0, \qquad \kappa^f > 0. \tag{34}$$

These cases and their corresponding inequalities are derived in Supplementary Appendix F.

If we define $k^f(\hat{q}, \Pi^f, \beta, \sigma) := c(\hat{q})[\Pi^f - \beta(1-\sigma)]$ and $k(\hat{q}, \Pi, \beta, \sigma) := c(\hat{q})[\Pi - \beta(1-\sigma)]$, in the space of (κ, κ^f) pairs, then we can deduce the threshold levels of the pair (κ, κ^f) that are required to sustain coexistence equilibria of Case 1b and Case 1c.

To illustrate the equilibria consistent with different values of the parameter space, let us consider $u(q) := [(q + \underline{q})^{1-\theta} - \underline{q}^{1-\theta}/(1-\theta)$, where $\underline{q} > 0$ and $\theta > 0$, and $c(q) = q^\alpha$, where $\alpha \geq 1$ is depicted in Figure 3 as the baseline setting.[27]

Consider raising Π^f, as shown in Figure 4, while holding all else equal. This has a tendency to reduce the opportunity cost of holding the dominated fiat currency. That is, DM-buyers now have relatively more incentive to counterfeit this currency. At the same time, a higher Π^f also means that the liquidity constraint on currency f is even tighter. These two imply that sellers will be less accepting of either currencies offered as genuine payments. Thus the coexistence equilibria of Case 1c can only be sustained with even higher costs of counterfeiting.

Consider next Figure 5, where all else the same, we increase the DM matching probability σ. A higher σ lowers the inefficiency wedge $[\Pi - \beta(1-\sigma)]/\beta\sigma$, between the monetary equilibrium outcome $\hat{q}$ in Case 1b (or $\tilde{q}$ in Case 1c) and the first-best q^*. This means that both currencies will circulate more in either Case 1b or Case 1c, to support a higher allocation of $\hat{q}$ (and $\tilde{q}$). That is buyers are more likely to use their portfolio of currencies to spend in DM meetings and thus have more incentive to counterfeit both currencies, when we perturb their environment in terms of raising σ. Thus to ensure equilibrium acceptability and coexistence of both currencies, one needs to have relatively higher thresholds for (κ, κ^f).

Figure 6 illustrates the comparative static with respect to buyers' appetite for intertemporal consumption smoothing, θ. The larger θ is, the less tolerant they are of substitution in q across periods. In other words, DM-buyers would like to smooth out their consumption as much as possible over time, and the only means of intertemporal smoothing is money. Thus, there is a higher incentive to counterfeit both monies, the larger is θ. However, to ensure that there are offers of genuine monies in equilibrium where both monies coexist, then the cost of counterfeiting sustaining the equilibria must be higher.

Figure 7 depicts the comparative static with respect to the convexity of production cost of DM-sellers. The faster production cost rises at the margin for producers, the more willing they are to

[27] `Python` code producing these comparisons are available from the authors' public repository at `https://github.com/phantomachine/_gkwcurse.git`.

accept offers of payment in both currencies, so that lower counterfeiting costs are required to sustain truthful or genuine offers of both monies as payment for DM goods in the coexistence equilibria.

Put another way, consider an economy that has low enough inflation (Π) on one currency relative to anothers' (Π^f), low market frictions, agents with large intertemporal insurance motives, or has sufficiently non-accelerating marginal costs of production. These comparative static analyses suggest that in such an economy, one must have an institution of high enough regulatory or technological costs to creating counterfeit media of exchange, so that multiple means of payments can co-exist, and so that there is no currency flight to quality.

5 Extensions

In this section we explore the robustness of the threat of counterfeiting in determining nominal exchange rates. Using the previous benchmark model we investigate how fiscal policies can help determine nominal exchange rate when monetary policy alone and the fundamentals of the economy are not able to do so. We then explore whether our proposed mechanism can help determine nominal exchange rates in a variety of environments. In particular, we consider the following modifications to the closed economy environment: (i) one where credit is possible and (ii) one where there are two countries.

5.1 Fiscal Policies

In this section we study how fiscal policies can restore determinacy of the nominal exchange rates in instances when they are not, as outlined in Proposition 3. Finding such policies is crucial for policy analysis as an environment with indeterminacy requires the selection of a specific allocation and prices consistent with an equilibrium. Establishing an appropriate selection rule is extremely difficult.

Let us now consider our benchmark environment and fiscal authorities that can impose a tax on CM production. For simplicity, we assume that the tax revenues fund wasteful government expenditures. In this new environment the buyer's sequential budget constraint for each CM is given by

$$C_{-1} \leq (1-\tau)N_{-1} - \phi(m - m_{-1}) - \phi e(m^f - m^f_{-1}),$$

where τ is the income tax rate. The same taxation assumption applies to the DM-seller's and their corresponding CM budget constraints. This labor tax does not give any of the currencies a distinct advantage over the other. In short, we are not specifying how taxes are paid. To be able to determine the nominal exchange rate we only need that income taxes are ad valorem.

It is straightforward to show that the resulting liquidity constraints for a stationary monetary equilibrium for both currencies are given by

$$\frac{\phi M}{1-\tau} \leq \frac{\kappa}{\Pi - \beta(1-\sigma)};$$

$$\frac{e\phi M^f}{1-\tau} \leq \frac{\kappa^f}{\Pi^f - \beta(1-\sigma)}.$$

As we can see from these new endogenous liquidity constraints, taxation affects the value of fiat currency which in turn affects the incentives to counterfeit. Thus for a given monetary allocation, we can always find a tax rate $\bar{\tau}$ such that one of the liquidity constraints bind. Thus fiscal policies in coordination with monetary policy can increase the set of equilibria where the nominal exchange rate outcome is determinate. This parallels the analysis of fiscal-monetary policy interdependence for determinacy of equilibria of Leeper (1991) in frictional economies of another kind.

This type of fiscal intervention can also be employed to increase the set of equilibria where the nominal exchange rate is determinate in the different environments that are examined in the next subsections.

5.2 Trade Credit

We consider the benchmark model and allow agents to have access to credit in certain states of the world. This allows us to examine how this new option to settle transactions affects the determinacy of the nominal exchange rate.

As in Aruoba, Waller and Wright (2011), we consider the possibility that in some matches credit is possible as record-keeping and enforcement services are available.[28] In particular, we assume that conditional on buyers being matched with a seller, the exogenous probability that a buyer or seller would engage in an exchange where trade credit is possible is change $(1-\rho) \in [0,1]$. That is, the event that a buyer can buy a good in the DM using credit occurs with probability $\sigma(1-\rho)$. Since credit is assumed to be enforceable in such an event, a buyer is willing to take (and a seller is willing to give) out a loan, which we denote by l, in exchange for the DM good, say q_c. This loan is required to be repaid in full in the following CM. The particular terms of trade when credit is available are determined by a buyer take it or leave it offer.

The possibility that in some matches trade credit can be used reduces the *anonymity* frictions in DM which can be interpreted as an improvement in the record-keeping and enforcement powers. In this new environment the payoff for the buyer that is offered a nominal loan l in terms of the domestic currency is given by

$$Z(C_{-1}, \mathbf{s}_{-1}) - \left(\frac{\phi_{-1}}{\phi} - \beta\right)\phi m - \left(\frac{\phi_{-1}e_{-1}}{\phi e} - \beta\right)\phi e m^f - \kappa(1-\eta) - \kappa^f(1-\eta^f)$$
$$+ \beta\sigma\left(\rho\hat{\pi}\left[u(q) - \phi\left(\eta d + \eta^f e d^f\right)\right] + (1-\rho)\left[u(q_c) - \phi l\right]\right),$$

while for the seller the payoff is given by

$$Z(\breve{C}_{-1}; \breve{\mathbf{s}}_{-1}) + \beta\sigma\left(\rho\pi(\omega)\left[\phi\left(\hat{\eta} d + \hat{\eta}^f e d^f\right) - c(q)\right] + (1-\rho)\left[\phi l - c(q_c)\right]\right),$$

[28] Recall that agents in our environment, in the absence of monitoring, enforcement and record-keeping, are said to be anonymous.

where q_c is the quantity traded when credit is possible, $Z(C_{-1}, \mathbf{s}_{-1})$ and $Z(\check{C}_{-1}; \check{\mathbf{s}}_{-1})$ are defined as in the benchmark model.

Since the loan l is repaid in CM the possibility of paying with counterfeit currencies is not feasible as the government can easily detect and destroy them. The seller then faces no private information problems when using credit as a means of payment. Thus allowing this additional payment option does not really change the private information problem of counterfeiting currencies agents face in DM whenever in a credit match all goods are paid with a private claim. As a result, as in the previous section, the PBE re-ordering equilibrium of the counterfeiting game is such that each seller accepts with probability one and each buyer does not counterfeit.[29]

It can be shown that the resulting monetary steady state equilibria in this new environment is given by

$$
\begin{aligned}
1 &= \mathcal{U}'(C); &(35)\\
u'(q_c) &= c'(q_c), &(36)\\
\phi l &= c(q_c), &(37)\\
\beta\sigma\gamma u'(q) &= \zeta c'(q), &(38)\\
\beta\sigma\gamma &= \zeta - \mu - \lambda, &(39)\\
\beta\sigma\gamma &= \zeta - \mu^f - \lambda^f, &(40)\\
\mu &= \Pi - \beta \geq 0, &(41)\\
\mu^f &= \Pi^f - \beta \geq 0; &(42)\\
c(q) &= \phi M + e\phi M^f, &(43)\\
\phi M &\leq \frac{\kappa}{\Pi - \beta(1-\sigma\rho)}, &(44)\\
e\phi M^f &\leq \frac{\kappa^f}{\Pi^f - \beta(1-\sigma\rho)}. &(45)
\end{aligned}
$$

As we can see relative to the benchmark economy without credit, given by equations (24)-(32), we now have two extra equilibrium conditions that specify the quantity and corresponding payment of the DM specialized good when credit is feasible. These are given by equations (36) and (37), respectively. This new credit environment also changes the matching probability for cash trades from σ to $\rho\sigma$. Thus the insights of Proposition 3 regarding the coexistence of fiat currencies and determinacy of nominal exchange rates still hold when credit is available. In particular, in order to obtain determinacy of nominal exchange rates, liquidity constraints arising from the signalling-bargaining game need to bind.[30]

Finally, we note that if agents have more access to credit (i.e. ρ is lower), then buyers and sellers are less likely to meet and trade with fiat currencies. This creates a smaller incentive for

[29]Note that even if buyers faced some limited commitment which would induce an endogenous borrowing constraints, the private information problem would not change as the settlement of the loan would occur in CM where all counterfeits when traded can be detectable.

[30]The particular values of the equilibrium observables are of course different from those of the benchmark model.

the buyer to produce counterfeits, thus decreases the information problem and relaxes the liquidity constraints. This implies that as credit markets develop the possibility of having nominal exchange rate indeterminacy increases. This prediction is consistent with Keynes (1964) and Friedman (1956) who have suggested that interactions between the conduct of monetary policy and the financial system create considerable scope for endogenous volatility and indeterminacy.

5.3 An explicit two-country setup

In the previous setup and its variations, we have seigniorage revenue transfers x_t occurring at the beginning of each CM(t), and the fact that preferences are quasilinear, we end up with an equilibrium property that it does not matter who receives the transfers (i.e., Home or Foreign agents), nor which transfers (i.e., Home or Foreign currency). This fact allowed us to abstract from modelling explicitly multiple country-specific decentralized markets. That is, we could focus on a single integrated world economy.

We now consider a more explicit two-country variation which is closer to a standard two-country general equilibrium international macro-finance model. Some additional notation is required for this setting. Looking ahead, we obtain a more general version of Proposition 3 on monetary coexistence and nominal exchange rate determinacy.

Let a variable X (or X^f) denote an object produced in the Home (or Foreign) country, which is held by agents in the Home country. Denote $X_\star$ (or $X_\star^f$) as an outcome produced in the Home (or Foreign) country but destined for use by Foreign agents.

Now we are ready to consider the details of this model. There are two countries, each with the DM(t)-then-CM(t) sequence of submarkets in each date $t \in \mathbb{N}$ as before. The sequence of information, markets, and actions are as follows: (1) At the start of each date t, the Home-country DM(t) and the Foreign-country $\text{DM}_\star^f(t)$ open. Only the buyers and sellers located in each specific country can trade in its DM—i.e. there is no international trade across the two DM submarkets;[31] (2) new monies created—x_t for Home money and $x_{\star,t}^f$ for Foreign money—are transferred to the respective countries' DM-buyers; (3) matched DM-buyers and DM-sellers in each country resolve their terms of trade, respectively $\omega := (q, d, d^f)$ and $\omega_\star := (q_\star, d_\star, d_\star^f)$, via TIOLI bargaining in which buyers have to signal the quality of their offers, as in the previous model; and (4) the DMs close and agents from each country enter their respective country-specific CM. In each domestic CM agents work, consume and produce a Home and Foreign country-specific general good, respectively C and $C_\star^f$. In the CM, labor inputs (N and $N_\star^f$) are immobile across countries, but forward contracts, monetary claims and goods are.[32] At the end of the CM, DM-buyers of each country make their

[31]Each country's DM has an equivalent interpretation of a non-traded goods sector (see e.g., Gomis-Porqueras, Kam and Lee, 2013).

[32]Equivalently, there is just one international CM in which goods and assets are internationally mobile, but labor and hence production is country specific. Note that unlike Gomis-Porqueras, Kam and Lee (2013), here we do not need to specify the details of international trade in goods since this detail is not necessary for characterizing coexistence of fiat monies and nominal exchange rate determination. This is also the same rationale used in Kareken and Wallace (1981, p.210). We can easily model C and $C_\star^f$ as composites of many intermediate goods that are internationally tradable. We refer readers to Gomis-Porqueras, Kam and Lee (2013) for more details. However, this additional

portfolio and counterfeiting decisions as in the previous model, before they enter their own country's DM.

Compared to the benchmark economy earlier, we have just made three new assumptions here. First, we assumed that transfers of fiat currency happen at the beginning of each DM, after CM portfolio decisions are made and after the international CM closes. This is crucial for breaking the presumption that we can think of the problem in terms of a closed economy with two independent currency suppliers, as we did earlier. What this assumption does is to allow for wealth effects (from the transfers) to matter for the DM-buyers' feasible offer set in the subsequent DMs. In particular, it matters in terms of which country-specific agents receive which seigniorage transfers. We made the following second assumption: Only Home DM-buyers receive a uniform lump sum transfer of the home-currency seigniorage revenue $x := M_t - M_{t-1}$. Likewise, only Foreign $\mathrm{DM}^f_\star$-buyers receive the uniform transfer $x^f_\star := M^f_{\star,t} - M^f_{\star,t-1}$. This is innocuous for the results we will obtain later.[33] Third, we have doubled up on the DMs and made them sectors in which there is no international trade. This third assumption will imply that the nominal exchange rate (if it exists and is determinate) in some equilibria, may depend on both DM outcomes through the equilibrium relative prices ϕ and ϕ^f.

From the model described earlier, now the initial state of a DM-buyer in the Home country at the start of each CM(t) is $y_t \equiv m_t$ and $y^f_t \equiv m^f_t$, denoting the buyer's residual money balances upon exiting a domestic DM, since transfers are no longer made at the beginning of CM. Nevertheless, the version of the DM-buyer's Bernoulli payoff from (7) is now:

$$
\begin{aligned}
U^b[C_{-1}, \omega, &H(\omega), \hat{\pi} | \mathbf{s}_{-1}; \phi, e] = \\
&Z(\mathbf{s}_{-1}) - \left(\frac{\phi_{-1}}{\phi} - \beta \right) \phi m - \left(\frac{\phi_{-1} e_{-1}}{\phi e} - \beta \right) \phi e m^f \\
&- \kappa(1-\eta) - \kappa^f (1 - \eta^f) + \beta \sigma \hat{\pi} \left[u(q) - \phi \left(\eta d + \eta^f e d^f \right) \right].
\end{aligned}
\tag{7'}
$$

The RI-equilibrium characterization will appear similar to that in Proposition 1, with the exception that the feasibility constraints on the monies will now say $d \leq m + x$ and $d^f \leq m^f$. That is, what are to be offered as payments in DM(t) exchange are now bounded above by what Home DM-buyers have accumulated at the end of each CM($t-1$), respectively m and m^f, including the nominal transfer from the Home government x.

Symmetrically, there is another characterization of an RI-equilibrium for the Foreign country,

detail will not change the main result—on coexistence of fiat monies and (in)determinacy of the nominal exchange rate—that follows.

[33] If all Home agents receive the transfer uniformly then each DM-buyer gets $x/2$ and each DM-seller gets $x/2$, since each population is of measure one. DM-sellers have no use of the transfer $x/2$ and will only spend it in the ensuing CM, and decide to work less there. From the perspective of the agents, this will only alter the DM-buyers' initial wealth by $x/2$, which is taken as parametric by the agents, and thus will not affect their optimal decision margins. The same argument applies to the Foreign country.

and the Foreign country's $\text{DM}^f_\star$-buyer's total discounted payoff is

$$
\begin{aligned}
U^b[C^f_{\star-1}, \omega^f_\star, &H(\omega^f_\star), \hat{\pi}^f_\star | \mathbf{s}_{-1}; \phi, e] = \\
&Z^f_\star(\mathbf{s}_{-1}) - \left(\frac{\phi_{-1}}{\phi} - \beta\right)\phi m_\star - \left(\frac{\phi_{-1}e_{-1}}{\phi e} - \beta\right)\phi e m^f_\star \\
&- \kappa_\star(1-\eta_\star) - \kappa^f_\star(1-\eta^f_\star) + \beta\sigma\hat{\pi}^f_\star\left[u(q^f_\star) - \phi\left(\eta_\star d_\star + \eta^f_\star e d^f_\star\right)\right],
\end{aligned}
\tag{$7_\star$}
$$

with payment feasibility constraints $d^f_\star \leq m^f_\star + x^f_\star$ and $d_\star \leq m_\star$. Note that since x and $x^f_\star$ are exogenous to DM-buyers in each country, they do not affect their marginal decisions, however, they will matter for the global adding-up conditions for monies supplied and demanded in equilibrium.

A Home DM-seller's problem, incorporating the RI-equilibrium, is still the same as (22) embedding the RI-equilibrium with $\pi(\omega) = 1$. A Foreign DM-seller's problem is symmetric and we omit its description here.

Given the familiarity with the simpler integrated world economy setting earlier, we can now jump ahead to describing a steady-state monetary equilibrium for this explicit two-country variation. After some algebra, the necessary conditions for such an equilibrium is given by the Home DM-buyers' Euler conditions and the DM-sellers' participation constraint, respectively,

$$
\beta\sigma\left[\frac{u'(q)}{c'(q)} - 1\right] = \lambda - \nu + (\Pi - \beta) = \lambda^f - \nu^f + (\Pi^f - \beta), \tag{46a}
$$

and,

$$
c(q) = \phi d + \phi^f d^f, \tag{46b}
$$

where λ (λ^f) is the Home liquidity constraint on Home (Foreign) real money balance, and ν (ν^f) is the nonnegativity constraint on d (d^f). The Karush-Kuhn-Tucker (KKT) conditions are:

$$
\begin{aligned}
\lambda \cdot [\bar{\kappa}(\Pi) - \phi d] = 0, \qquad & \lambda \geq 0, \qquad \phi d \leq \bar{\kappa}(\Pi), \\
\lambda^f \cdot \left[\bar{\kappa}^f(\Pi^f) - \phi^f d^f\right] = 0, \qquad & \lambda^f \geq 0, \qquad \phi^f d^f \leq \bar{\kappa}^f(\Pi^f), \\
-\nu \cdot d = 0, \qquad & d \geq 0, \qquad \nu \geq 0, \\
-\nu^f \cdot d^f = 0, \qquad & d^f \geq 0, \qquad \nu^f \geq 0.
\end{aligned}
\tag{46c}
$$

where $\bar{\kappa}(\Pi) := \kappa/[\Pi - \beta(1-\sigma)]$ and $\bar{\kappa}^f(\Pi^f) := \kappa^f/[\Pi^f - \beta(1-\sigma)]$.

Similarly, we have the Foreign block as

$$
\beta\sigma\left[\frac{u'(q^f_\star)}{c'(q^f_\star)} - 1\right] = \lambda_\star - \nu_\star + (\Pi - \beta) = \lambda^f_\star - \nu^f_\star + (\Pi^f - \beta), \tag{47a}
$$

and,

$$
c(q^f_\star) = \phi^f d^f_\star + \phi d_\star, \tag{47b}
$$

where $\lambda_\star$ (or $\lambda^f_\star$) is the Foreign liquidity constraint on Home (Foreign) real money balance, and $\nu_\star$

(or $\nu_\star^f$) is the non-negativity constraint on $d_\star$ (or $d_\star^f$). The KKT conditions are:

$$\begin{aligned}
\lambda_\star \cdot [\bar{\kappa}_\star(\Pi) - \phi d_\star] = 0, \qquad & \lambda \geq 0, \qquad \phi d_\star \leq \bar{\kappa}_\star(\Pi), \\
\lambda_\star^f \cdot \left[\bar{\kappa}_\star^f(\Pi^f) - \phi^f d_\star^f\right] = 0, \qquad & \lambda_\star^f \geq 0, \qquad \phi^f d_\star^f \leq \bar{\kappa}_\star^f(\Pi^f), \\
-\nu_\star \cdot d_\star = 0, \qquad & d_\star \geq 0, \qquad \nu_\star \geq 0, \\
-\nu_\star^f \cdot d_\star^f = 0, \qquad & d_\star^f \geq 0, \qquad \nu_\star^f \geq 0.
\end{aligned} \tag{47c}$$

where $\bar{\kappa}_\star(\Pi) := \kappa_\star/[\Pi - \beta(1-\sigma)]$ and $\bar{\kappa}_\star^f(\Pi^f) := \kappa_\star^f/[\Pi^f - \beta(1-\sigma)]$, with $\kappa_\star > 0$ and $\kappa_\star^f > 0$. In the explicit two-country setting, the following adding-up conditions for Home and Foreign monies, must hold between demand for real balances across countries and their supplies. Respectively, these are

$$\phi d_\star = \phi(M - d), \tag{48a}$$

and,

$$\phi d_\star^f = \phi^f(M^f - d^f). \tag{48b}$$

The following states a generalization of our main result on coexistence, nominal exchange rate determinacy (Proposition 3). (In the interest of space, we omit the symmetric case where the foreign fiat money is dominated in rate of return.)

Proposition 5 (Two-Country Monetary Coexistence and Exchange Rate) *Assume the two-country world economy with independent monetary policies Π and Π^f, where injections/withdrawals of lump sum monetary take place after CM trades and before the country-specific DMs opens. Then:*

- *When Home fiat money is dominated in rate of return ($\Pi > \Pi^f$), there exist a monetary equilibrium with a determinate nominal exchange rate with*
  - *the two currencies coexisting in only one country when some liquidity constraints bind, or*
  - *both currencies circulating in the Home and Foreign countries when all liquidity constraints, or at least those on the higher-return currency, bind in both countries.*
- *When Home fiat money dominates in rate of return ($\Pi^f > \Pi$), the coexistence results are the symmetric opposite to those of Case 1.*
- *When Home fiat money has the same rate of return as the Foreign currency ($\Pi = \Pi^f$), there exist a monetary equilibrium with a determinate nominal exchange rate. In this equilibrium, both currencies circulate in the Home and Foreign countries when all liquidity constraints bind in both countries.*

These possible equilibrium configurations are summarized in detail in Table 1.

As in Zhang (2014), in this economy there exist equilibria where one country may have two currencies circulating while the other country uses only one. In our setting, this situation can only be obtained when the two currencies are not perfect substitutes. Also, there are more than one configuration in which both currencies circulate in both countries, in spite of the fact that one currency is dominated in rate of return—see Cases 1(b)(7), (11) and (16) in Table 1. In these cases, it is always the case that the higher return currency is liquidity constrained from the perspective of DM-buyers in both countries' DMs. This again, is a generalization of what was observed earlier with the simple integrated world economy setting.

Finally, consider cases 1(b)(16) or 3(a)(1) from Table 1 for this generalized result, and the special setting when the two currencies are identical in every respect, i.e., $\Pi = \Pi^f$ and $\kappa = \kappa^f$. In this case, both currencies are perfect substitutes as payment instruments, and the (determinate) nominal exchange rate is simply the ratio of the two money stocks. Again, one may compare this with the nominal exchange rate arising out of standard CIA models. However, what we get here is—as discussed earlier—a special parametric characterization of a particular equilibrium regime in which there is coexistence and determinacy of the exchange rate. More importantly, the CIA-like equilibrium outcome is an artefact of a special-case equilibrium, not a direct consequence of an ad hoc restriction on payment instruments (as is the case in CIA models).

6 Conclusion

In this paper we present a search theoretic model of two fiat currencies to study the properties of nominal exchange rates when agents face private information. Agents have no restrictions on what divisible fiat currency can be used to settle transactions. Buyers may counterfeit both fiat currencies at some fixed cost while sellers sub-period distinguish between counterfeit and genuine fiat currencies. Thus counterfeiting is private information to buyers. This informational problem gives rise to endogenous liquidity constraints that specify a seller's upper bound on how much fiat currency is willing to accept.

An interesting feature of our results is that there is no counterfeiting in equilibrium. It is the threat of counterfeiting that pins down the nominal exchange rate. A critical feature of these liquidity constraints is that the marginal liquidity value of an additional unit of currency beyond the endogenous binding liquidity constraint is zero. This property is key in determining the properties of nominal exchange rates.

When endogenous liquidity constraints on both currencies are binding and the currencies are identical in every respect, we obtain the surprising result that the nominal exchange rate is the ratio of the two money stocks. When the foreign currency has a higher rate of return but a lower counterfeiting cost, then the buyer will first pay with the foreign currency up to the bound and use domestic currency to pay for the remainder of the goods purchased. Because of this, both currencies can circulate even though one currency is dominated in the rate of return. We also show that when there is nominal exchange rate indeterminacy, there exist fiscal policies that can restore

Table 1: Equilibrium cases in two-country model (Proposition 5).

Equilibrium Cases[1]		Liquidity constraints[2] $(\lambda, \lambda^f, \lambda_\star^f, \lambda_\star)$	Circulation[3] $(m, m^f, m_\star^f, m_\star)$	Exchange rate[4] e
1(a): $\Pi > \Pi^f$	(1):	$(0,0,0,0)$	$(0,+,+,0)$	n/a
	(2):	$(+,0,0,0)$	$(0,+,+,0)$	n/a
	(3):	$(0,0,0,+)$	$(0,+,+,0)$	n/a
	(4):	$(+,0,0,+)$	$(0,+,+,0)$	n/a
1(b): $\Pi > \Pi^f$	(5):	$(+,+,0,0)$	$(+,+,+,0)$	$\frac{\bar{\kappa}^f + c(q_\star^f)}{\bar{\kappa}} \frac{M}{M^f}$
	(6):	$(0,+,0,0)$	$(+,+,+,0)$	$\frac{-\bar{\kappa}^f + c(q_\star^f)}{c(q) - \bar{\kappa}^f} \frac{M}{M^f}$
	(7):	$(0,+,+,0)$	$(+,+,+,+)$	$\frac{\bar{\kappa}^f + \bar{\kappa}_\star^f}{c(q) + c(q_\star^f) - [\bar{\kappa}^f + \bar{\kappa}_\star^f]} \frac{M}{M^f}$
	(8):	$(0,+,0,+)$	$(+,+,+,0)$	$\frac{\bar{\kappa}^f + c(q_\star^f)}{c(q) - \bar{\kappa}^f} \frac{M}{M^f}$
	(9):	$(+,0,+,0)$	$(0,+,+,+)$	$\frac{\bar{\kappa}^f - c(q)}{c(q_\star^f) - \bar{\kappa}^f} \frac{M}{M^f}$
	(10):	$(+,+,0,+)$	$(+,+,+,0)$	$\frac{\bar{\kappa}^f + c(q_\star^f)}{\bar{\kappa}} \frac{M}{M^f}$
	(11):	$(+,+,+,0)$	$(+,+,+,+)$	$\frac{\bar{\kappa}^f + \bar{\kappa}_\star^f}{c(q_\star^f) - \bar{\kappa}_\star^f + \bar{\kappa}^f} \frac{M}{M^f}$
	(12):	$(0,+,+,+)$	$(+,+,+,+)$	$\frac{\bar{\kappa}^f + \bar{\kappa}_\star^f}{c(q_\star^f) + \bar{\kappa}_\star^f - \bar{\kappa}^f} \frac{M}{M^f}$
	(13):	$(+,0,+,+)$	$(0,+,+,+)$	$\frac{\bar{\kappa}_\star^f + c(q)}{\bar{\kappa}_\star^f} \frac{M}{M^f}$
	(14):	$(0,0,+,+)$	$(0,+,+,+)$	$\frac{\bar{\kappa}_\star^f + c(q)}{\bar{\kappa}_\star^f} \frac{M}{M^f}$
	(15):	$(0,0,+,0)$	$(+,+,+,0)$	$\frac{\bar{\kappa}_\star^f + c(q)}{c(q_\star^f) + \bar{\kappa}_\star^f} \frac{M}{M^f}$
	(16):	$(+,+,+,+)$	$(+,+,+,+)$	$\frac{\bar{\kappa}_\star^f + \bar{\kappa}^f}{\bar{\kappa}_\star + \bar{\kappa}} \frac{M}{M^f}$
3(a): $\Pi = \Pi^f$	(1):	$(+,+,+,+)$	$(+,+,+,+)$	$\frac{\bar{\kappa}_\star^f + \bar{\kappa}^f}{\bar{\kappa}_\star + \bar{\kappa}} \frac{M}{M^f}$
3(c): $\Pi = \Pi^f$	(14):	$(0,0,0,0)$	$(+,+,+,+)$	$\in [0, +\infty)$
	(15):	$(+,+,0,0)$	$(+,+,+,+)$	$\in [0, +\infty)$
	(16):	$(0,0,+,+)$	$(+,+,+,+)$	$\in [0, +\infty)$

Notes:

1. Equilibria under Case 2 (not shown) are symmetrical opposites to Case 1. In the proof to Proposition 5 in Appendix G, Case 3(b), not shown above, comprise of configurations that cannot exist in an equilibrium.
2. This refers to the Lagrange multipliers on respective endogenous liquidity constraints on both countries, for both countries. For example, $(\lambda, \lambda^f, \lambda_\star^f, \lambda_\star) \equiv (0,+,+,0)$ refer to only the liquidity constraints on holding/paying with the foreign currency binding, for both Home DM-buyers and Foreign DM$_\star$-buyers.
3. Circulation refers to steady-state equilibrium circulation of currecies in each country. For example, $(m, m^f, m_\star^f, m_\star) \equiv (0,+,+,0)$ refers to only the Foreign currency circulating in both countries.
4. The equilibrium nominal exchange rate in each possible steady-state equilibrium is either not applicable (n/a), determinate and finitely positive-valued, or indeterminate in the set $[0, +\infty)$. We use abbreviations $\bar{\kappa} := \bar{\kappa}(\Pi)$, $\bar{\kappa}^f := \bar{\kappa}^f(\Pi^f)$, $\bar{\kappa}_\star^f := \bar{\kappa}_\star^f(\Pi^f)$, and, $\bar{\kappa}_\star := \bar{\kappa}_\star(\Pi)$.

determinacy of the nominal exchange rate.

In this environment, the first best allocation may not be attainable even if the Friedman rule is implemented for both currencies. We also show that the private information problem considered in this paper can help determine nominal exchange rates when trade credit is possible. When agents have more access to credit, there is a smaller incentive for the buyer to produce counterfeits, increasing the possibility of inducing more indeterminate exchange rate equilibria. Finally, when an explicit two country model is considered, we find that there exist equilibria where one country may have two currencies circulating while the other country uses only one. This situation is only observed when the two currencies are not perfect substitutes.

The private information problem explored in this paper allows us to break the Kareken and Wallace indeterminacy result and provides a rationalization of why currencies with dominated rates of return remain in circulation (apart from obvious explanations in terms of legal restrictions) as media of exchange.

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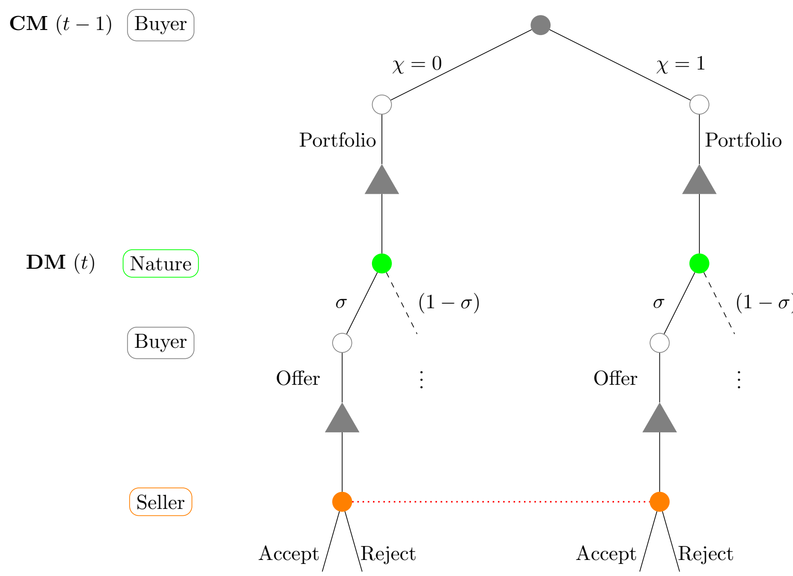
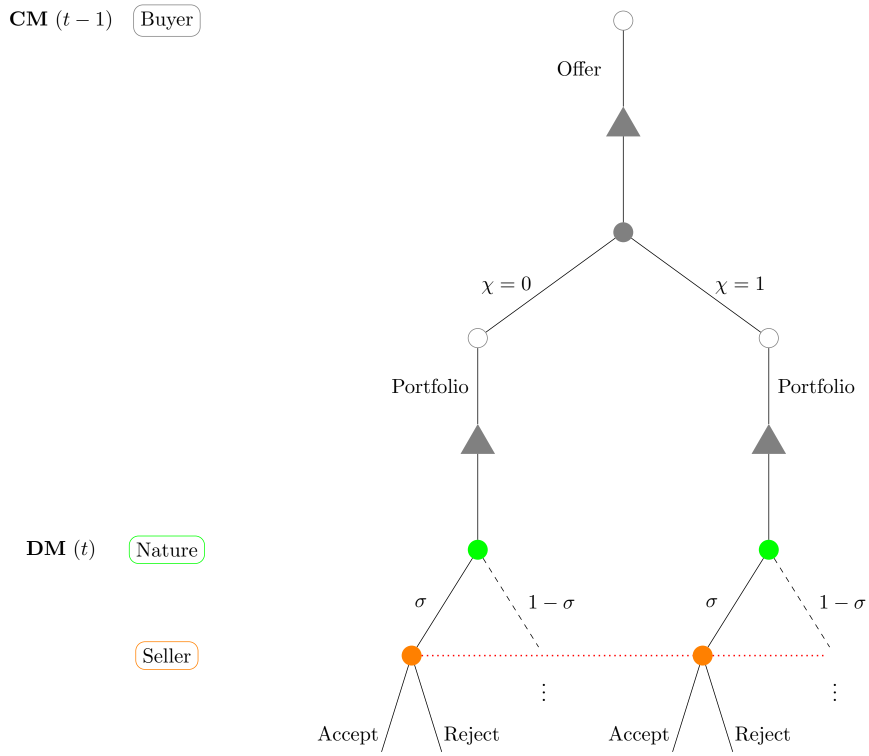
Figure 1: Original extensive-form game.



Note: ● Buyer's discrete decision node; ○ Buyer's continuation to next decision node; ▲ Buyer's continuous decision node; ● Nature's discrete decision node; ● Seller's discrete decision node; ··· Information set.

Figure 2: Reverse-order extensive-form game.



Note: ● Buyer's discrete decision node; ○ Buyer's continuation to next decision node; ▲ Buyer's continuous decision node; ● Nature's discrete decision node; ● Seller's discrete decision node; ··· Information set.

Figure 3: Equilibrium coexistence or non-coexistence of monies when $\Pi > \Pi^f$ and counterfeiting costs—baseline setting.

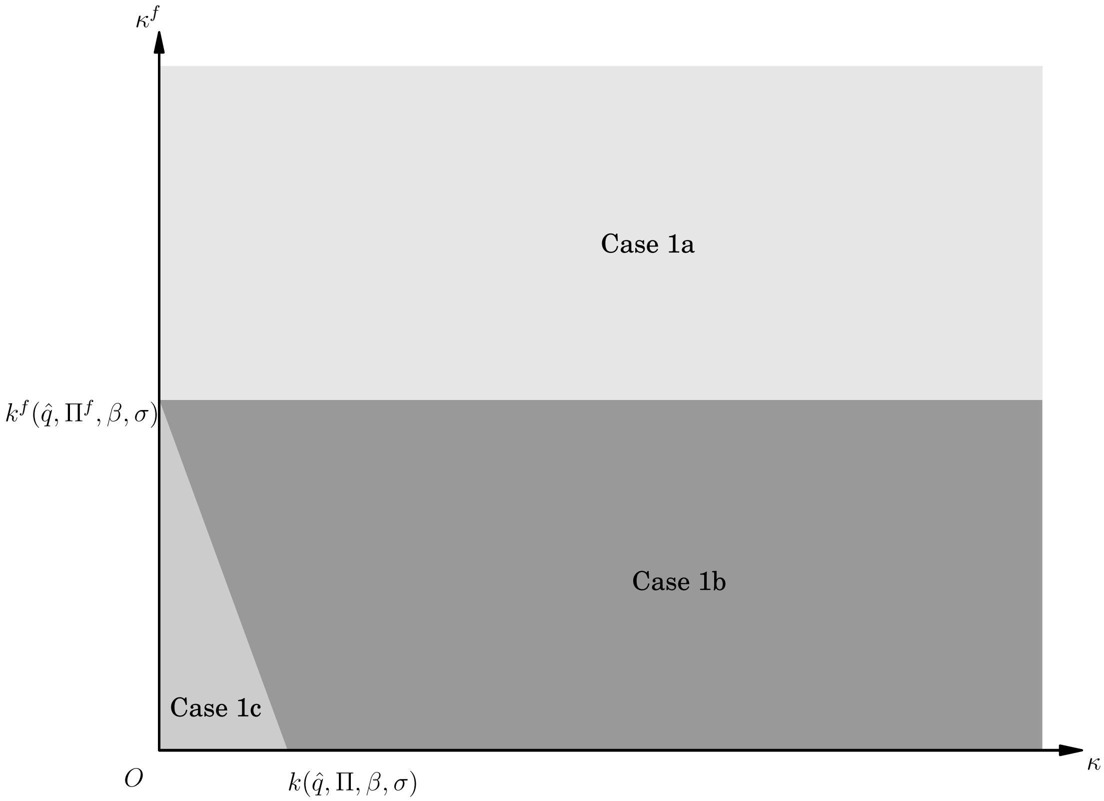


Figure 4: Equilibrium coexistence or non-coexistence of monies when $\Pi > \Pi^f$ and counterfeiting costs. Comparative static with higher Π^f.

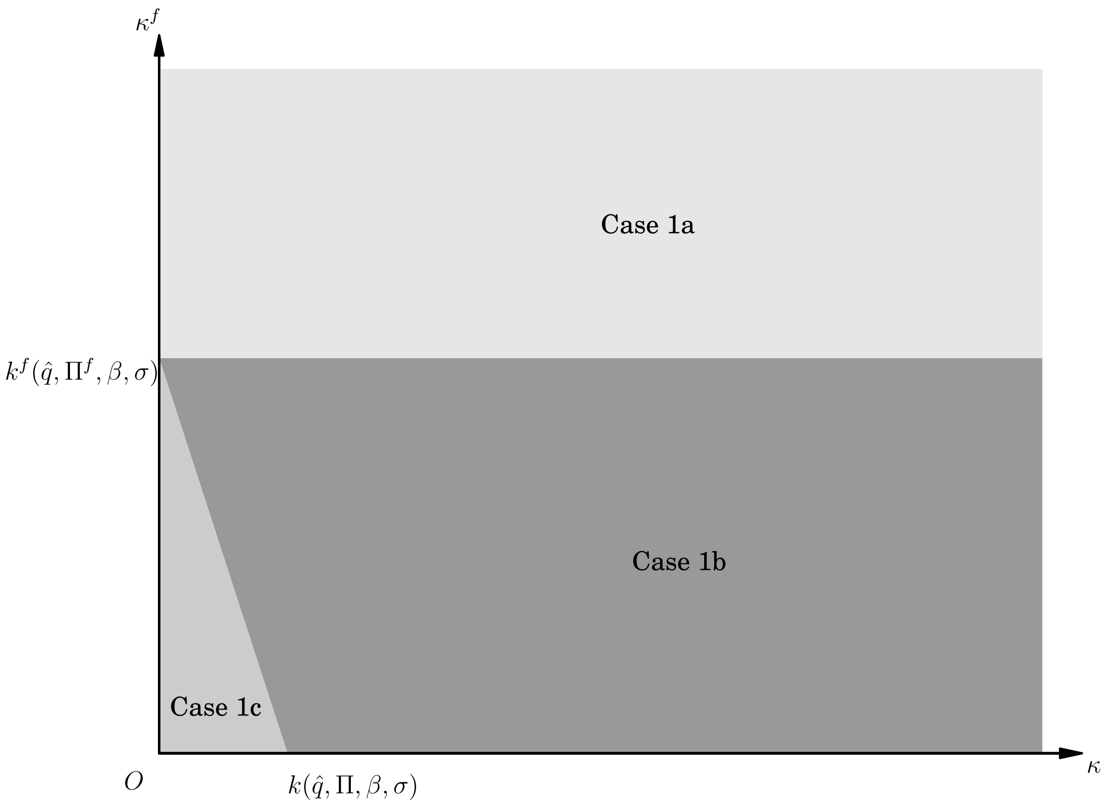


Figure 5: Equilibrium coexistence or non-coexistence of monies when $\Pi > \Pi^f$ and counterfeiting costs. Comparative static with higher matching probability σ.

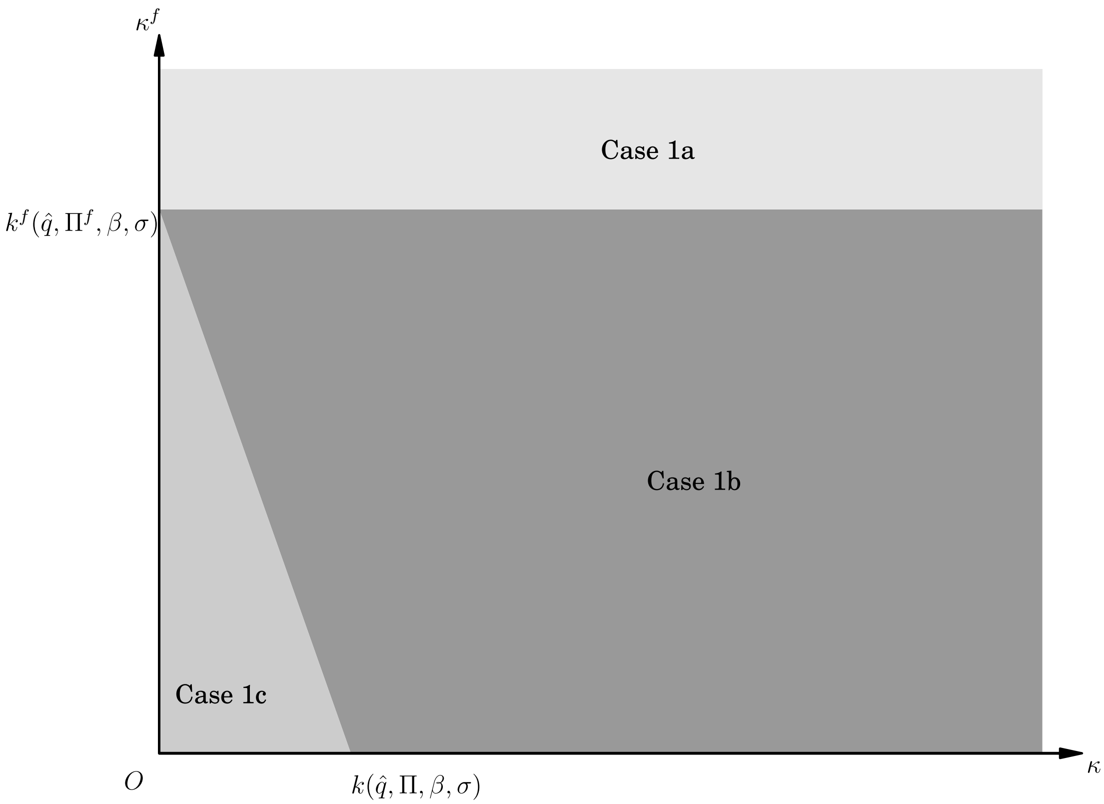


Figure 6: Equilibrium coexistence or non-coexistence of monies when $\Pi > \Pi^f$ and counterfeiting costs. Comparative static with higher risk aversion θ.

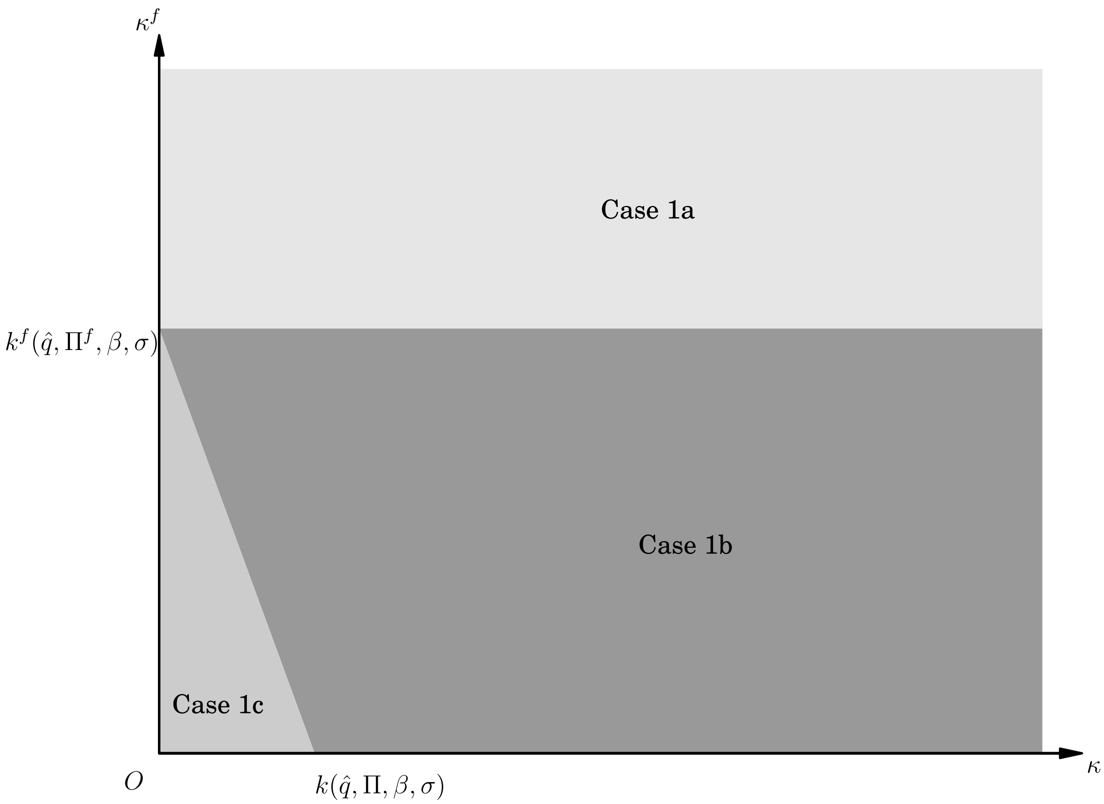
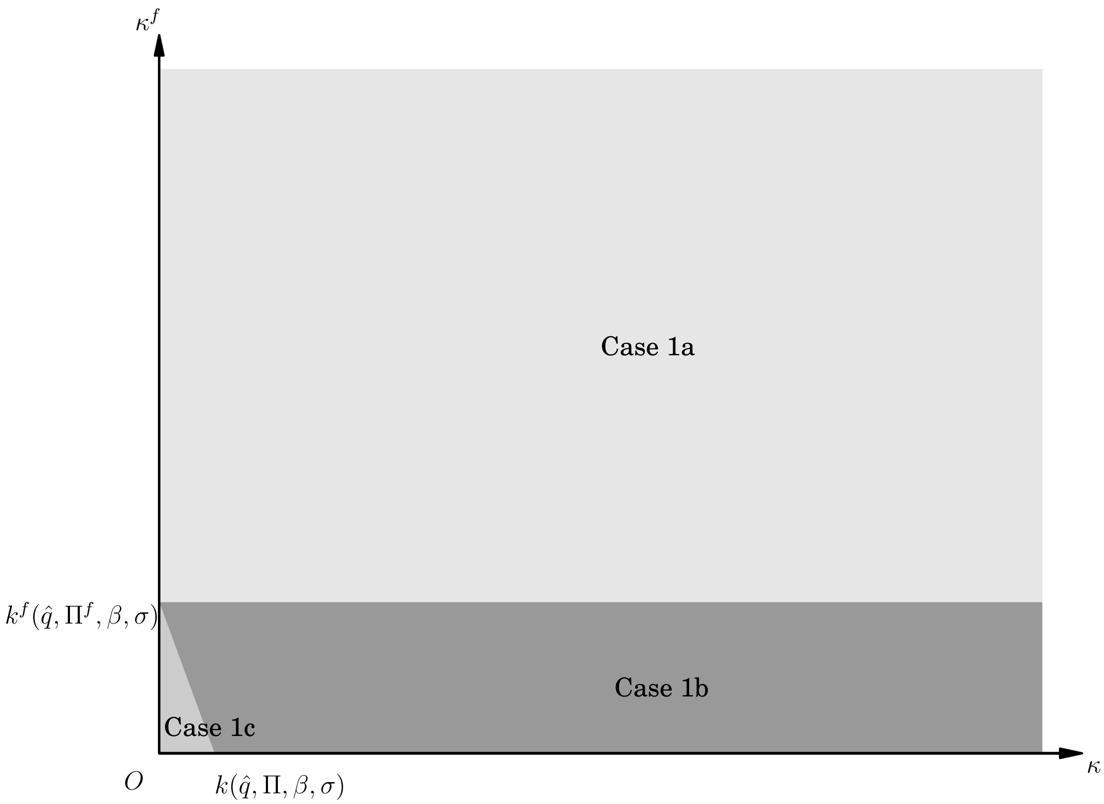


Figure 7: Equilibrium coexistence or non-coexistence of monies when $\Pi > \Pi^f$ and counterfeiting costs. Comparative static with more convex cost of DM production α.



— Supplementary (Online) Appendix —

A Degeneracy of asset portfolios

Given that we are interested in monetary equilibria from now on we restrict attention to economies where $\phi_{t-1}/\phi_t \geq \beta$; and $\phi^f_{t-1}/\phi^f_t \equiv e_{t-1}\phi_{t-1}/e_t\phi_t \geq \beta$. The following lemma allows us to simplify the Bernoulli payoff function given by equation (6).

Lemma 2 *Under any optimal measurable strategy $\tilde{\boldsymbol{\sigma}}^b$, genuine portfolio choices are always such that:*

$$m \begin{cases} = \chi d, & \text{if } \phi_{-1}/\phi > \beta \\ \geq \chi d, & \text{if } \phi_{-1}/\phi = \beta \end{cases}; \text{ and, } m^f \begin{cases} = \chi^f d^f, & \text{if } \phi_{-1}e_{-1}/\phi e > \beta \\ \geq \chi^f d^f, & \text{if } \phi_{-1}e_{-1}/\phi e = \beta \end{cases}.$$

Moreover, whenever $\phi_{-1}/\phi = \beta$ (or $\phi_{-1}e_{-1}/\phi e = \beta$), demanding $m > \chi d$ (or $m^f > \chi^f d^f$) is U^b-payoff equivalent for the buyer to demanding $m = \chi d$ (or $m^f = \chi^f d^f$).

Proof. First consider the cases where the returns of either (or both) assets are strictly dominated by β. Then, holding either (or both) assets beyond what is necessary for payments in the DM (i.e. d and d^f) is intertemporally costly since the price levels ϕ^{-1}, and, $(\phi^f)^{-1}$ are respectively growing at the rates $\gamma - 1$ and $\gamma^f - 1$. Thus holding only $m = (1-\chi)d$ or (and) $m^f = (1-\chi^f)d^f$ is optimal for the DM-buyer under any optimal strategy $\tilde{\boldsymbol{\sigma}}^b$.

Second, consider the cases where the returns of either (or both) assets are equal to β. Then any portfolio demand comprising $m \geq (1-\chi)d$ or (and) $m^f \geq (1-\chi^f)d^f$ is optimal. However, since W is linear, the only terms involving m and m^f in the buyer's payoff function U^b in (6) are the expected costs of holding unused genuine assets, given by the linear functions

$$\left(\frac{\phi_{-1}}{\phi} - \beta\right)\phi m - \left(\frac{\phi_{-1}e_{-1}}{\phi e} - \beta\right)\phi e m^f.$$

Observe that in the cases where the asset returns are equal to β, the value of these costs are zero. Therefore, the second statement in the Lemma is true. ■

This result stems from two observations: (i) if the returns on the two fiat currencies are strictly below the discount factor β, then, holding these assets are intertemporally costly; and (ii) if their returns are equal to β, the linearity of $W(\cdot)$ ensures that any excess asset demands beyond what is necessary for trade in the DM is inconsequential to the payoff $U^b(\cdot)$.

Lemma 2 also implies that each $G(\cdot|\omega)$ consistent with $\tilde{\boldsymbol{\sigma}}^b$ is degenerate, as far as characterizing the Bernoulli payoff function U^b is concerned. That is, given the realization of $\aleph(\omega) := (\chi(\omega), \chi^f(\omega))$, we have the following

$$G[a(\omega)|\omega] = \delta_{\{(1-\chi)d,(1-\chi^f)d^f\}}, \qquad \forall \aleph \in \{0,1\}^2, \tag{49}$$

where δ_E denotes the Dirac delta function defined to be everywhere zero-valued except on singleton events E, on which the function has value 1. In short, we can characterize the buyer's mixed strategy $G(\cdot|\omega)$ (over portfolio accumulation) in the subgame following the buyer's finite history of play, $\langle \omega, \aleph(\omega) \rangle$, prior to comprehensively describing equilibrium in the game.

B Proof of Proposition 1 (RI-equilibrium)

Denote the maximum value of the program in (12), when $\hat{\pi} = \pi(\omega) = 1$ and $H = (\hat{\eta}, \hat{\eta^f}) = (\eta(\omega), \eta^f(\omega)) = (1,1)$, as $(U^b)^*$. The aim is to show that an equilibrium $\tilde{\boldsymbol{\sigma}}$ yields the same value as $(U^b)^*$, and it satisfies the characterization in Proposition 1 (Case 1); and that any other candidate strategy $\tilde{\boldsymbol{\sigma}}' := \langle \omega', H', \pi' \rangle$ such that $\hat{\pi}' = \pi'(\omega) \neq 1$ and/or $\hat{H}' \neq (1,1)$ will induce a buyer's valuation that is strictly less than $(U^b)^*$, and therefore cannot constitute an equilibrium (Cases 2-5).

Consider the subgame following offer ω. Let $\rho(\chi, \chi^f)$ denote the joint probability measure on events $\{(\chi, \chi^f)\}$, where the pure actions over counterfeiting are $(\chi, \chi^f) \in \{0,1\}^2$. Denote $P := 2^{\{0,1\}^2}$ as the power set of $\{0,1\}^2$. By the definition of probability measures, it must be that $\sum_{\{z\} \in P} \rho(z) = 1$.

The seller's problem in (9) is equivalent to:

$$\pi(\omega) \in \left\{ \arg \max_{\pi' \in [0,1]} \pi' \left[\phi \left([1 - \hat{\rho}(1,0) - \hat{\rho}(1,1)]d \right. \right. \right. \\ \left. \left. \left. + [1 - \hat{\rho}(0,1) - \hat{\rho}(1,1)]ed^f \right) - c(q) \right] \right\}. \tag{50}$$

This is a linear programming problem in π, given the seller's rational belief system $\hat{\rho}$ and buyer's offer ω. Thus the seller's best response satisfies:

$$\left(\phi \left([1 - \hat{\rho}(1,0) - \hat{\rho}(1,1)]d + [1 - \hat{\rho}(0,1) - \hat{\rho}(1,1)]ed^f \right) - c(q) \begin{cases} > 0 \\ < 0 \\ = 0 \end{cases} \right) \\ \Rightarrow \left(\pi(\omega) \begin{cases} = 1 \\ = 0 \\ \in [0,1] \end{cases} \right). \tag{51}$$

Let $U^b_{\{z\}} \equiv U^b[\omega, \{z\}, \hat{\pi} | \mathbf{s}_{-1}, \phi, e]$ denote the buyer's expected payoff from *realizing* pure actions (χ^h, χ^f), given offer ω and rational belief system $\hat{\pi} \in [0,1]$, where $\{z\} \in P$. We have the following possible payoffs following each event $\{z\}$:

$$U^b_{\{(0,0)\}} = -\left(\frac{\phi_{-1}}{\phi} - \beta\right)\phi d - \left(\frac{\phi_{-1}e_{-1}}{\phi e} - \beta\right)\phi e d^f + \beta\sigma\hat{\pi}\left[u(q) - \phi\left(d + ed^f\right)\right]; \tag{52}$$

$$U^b_{\{(0,1)\}} = -\kappa^f - \left(\frac{\phi_{-1}}{\phi} - \beta\right)\phi d + \beta\sigma\hat{\pi}\left[u(q) - \phi d\right]; \tag{53}$$

$$U^b_{\{(1,0)\}} = -\kappa - \left(\frac{\phi_{-1}e_{-1}}{\phi e} - \beta\right)\phi e d^f + \beta\sigma\hat{\pi}\left[u(q) - \phi e d^f\right]; \tag{54}$$

$$U^b_{\{(1,1)\}} = -\kappa^f - \kappa + \beta\sigma\hat{\pi}u(q). \tag{55}$$

Observe that

$$U^b_{\{(0,1)\}} + U^b_{\{(1,0)\}} = U^b_{\{(0,0)\}} + U^b_{\{(1,1)\}}. \tag{56}$$

There are five cases to consider.

Case 1. Suppose there is a set of candidate equilibria such that $\rho(0,0) = 1$ and $\rho(z) = 0$, for all $\{z\} \in P$ and $z \neq (0,0)$. Then, we have $U^b_{\{(0,0)\}} > \max\{U^b_{\{(1,0)\}}, U^b_{\{(0,1)\}}, U^b_{\{(1,1)\}}\}$. Since $U^b_{\{(0,0)\}} > U^b_{\{(1,0)\}}$ and $U^b_{\{(0,0)\}} > U^b_{\{(0,1)\}}$, then, from (52)-(55) we can derive that

$$\phi d < \frac{\kappa}{\frac{\phi_{-1}}{\phi} - \beta(1 - \sigma\hat{\pi})}, \tag{57}$$

and,

$$\phi e d^f < \frac{\kappa^f}{\frac{\phi_{-1}e_{-1}}{\phi e} - \beta(1 - \sigma\hat{\pi})}. \tag{58}$$

The interpretation from (57) and (58) is that the liquidity constraints on either monies are slack. Therefore the buyer's expected payoff in this case can be evaluated from (52). If $\hat{\pi} < 1$, then from the seller's decision rule (51) we can deduce $\omega \equiv (q, d, d^f)$ must be such that the seller's participation/incentive constraint binds:

$$c(q) = \phi(d + ed^f). \tag{59}$$

Since (59) holds, all we need to do is verify the buyer's payoff. Since, the buyer's liquidity constraints (57) and (58) do not bind at $\hat{\pi} < 1$, a small increment in either payment offered, d or d^f, relaxes (59) and this raises $\hat{\pi}$, and thus the buyer's payoff (52). The maximal payoff to the buyer, keeping the seller in participation, is when $\pi(\omega) = \hat{\pi} = 1$, and the offer $\overline{\omega}$ is such that

$$\overline{U}^b \equiv U^b_{\{(0,0)\}}[\overline{\omega}|\pi(\omega) = \hat{\pi} = 1] = \sup_{\omega} \Bigg\{ U^b_{\{(0,0)\}}[\omega|\pi(\omega) = \hat{\pi} = 1] : \\ \phi d \leq \frac{\kappa}{\frac{\phi_{-1}}{\phi} - \beta(1-\sigma)}, \\ \phi e d^f \leq \frac{\kappa^f}{\frac{\phi_{-1}e_{-1}}{\phi e} - \beta(1-\sigma)}, \\ c(q) \leq \phi(d + ed^f) \Bigg\}.$$

Then it is easily verified that this maximal value coincides with the maximum value of the program given in (12) in Proposition 1, i.e. $\overline{U}^b = (U^b)^*$, since the payoff function is continuous, and the constraints also define a nonempty, compact subset of the feasible set $\Omega(\phi, e) \ni \overline{\omega}$. Since the seller has no incentive to deviate from $\pi(\overline{\omega}) = 1$, then a behavior strategy $\tilde{\boldsymbol{\sigma}} = \langle \overline{\omega}, (1,1), 1 \rangle$ inducing the TIOLI payoff $\overline{U}^b$ is a PBE.

Case 2. Note that in any equilibrium, a seller will never accept an offer if $\rho(1,1) = 1$, and, a buyer will never counterfeit both assets with probability 1—counterfeiting for sure costs $\kappa + \kappa^f$ and the buyer gains nothing. Therefore, $\rho(1,1) < 1$ is a necessary condition for an equilibrium in the subgame following ω. Likewise, all unions of disjoint events with this event of counterfeiting all assets—i.e. $\{(\chi, \chi^f)\} \in \{(0,1)\} \cup \{(1,1)\}$ or $\{(\chi, \chi^f)\} \in \{(1,0)\} \cup \{(1,1)\}$—such that $\rho(0,1) + \rho(1,1) = 1$ or $\rho(1,0) + \rho(1,1) = 1$, respectively, cannot be on any equilibrium path.

Case 3. Suppose instead we have equilibria in which $\rho(0,0) + \rho(1,0) = 1$, $\rho(1,0) \neq 0$, and $\rho(1,1) + \rho(0,1) = 0$, so $U^b_{\{(1,0)\}} = U^b_{\{(0,0)\}} > \max\{U^b_{\{(0,1)\}}, U^b_{\{(1,1)\}}\}$.

Given this case, and from (56), we have $U^b_{\{(0,1)\}} = U^b_{\{(1,1)\}}$. From $U^b_{\{(1,0)\}} = U^b_{\{(0,0)\}}$, and (52) and (54), respectively, we have:

$$\hat{\pi} = \frac{\kappa - (\phi_{-1}/\phi - \beta)\phi d}{\beta\sigma\phi d}, \tag{60}$$

and,

$$\phi e d^f < \frac{\kappa^f}{\frac{\phi_{-1}e_{-1}}{\phi e} - \beta(1 - \sigma\hat{\pi})}. \tag{61}$$

If $\hat{\pi} < 1$, then from the seller's decision rule (51) we can deduce $\omega \equiv (q, d, d^f)$ must be such that the seller's participation/incentive constraint binds:

$$\begin{aligned} c(q) &= \phi[(1 - \rho(1,0) - \rho(1,1))d + (1 - \rho(0,1) - \rho(1,1))ed^*] \\ &= \phi[(1 - \rho(1,0))d + ed^*]. \end{aligned} \tag{62}$$

The buyer's payoff can be evaluated from (54). If $\hat{\pi} < 1$, then reducing d infinitesimally will increase $\hat{\pi}$ in (60), and this increase the buyer's payoff in (54). The buyer would like to attain $\hat{\pi} = 1$ since the seller's participation constraint will still be respected:

$$c(q) \leq \phi[(1-\rho(1,0))d + ed^f]. \tag{63}$$

Let the maximum of the buyer's TIOLI value (54) such that the constraints (60), (61) and (63) are respected, in this case be $(U^b)^\dagger$. However, since $\rho(1,0) \neq 0$, it is easily verified that $(U^b)^\dagger < U^b_{\{(0,0)\}}[\overline{\omega}|\pi(\omega) = \hat{\pi} = 1; \rho(1,0) = 0] = \sup_{\omega,\rho(1,0)}\{U^b_{\{(1,0)\}}|(60),(61),(63)\} = (U^b)^*$, in which the last equality is attained when $\rho(1,0) = 0$. This contradicts the claim that $\rho(0,0)+\rho(1,0) = 1$ and $\rho(1,0) \neq 0$ is a component of a PBE.

Case 4. Suppose there are equilibria consisting of $\rho(0,0) + \rho(0,1) = 1$ with $\rho(0,1) \neq 0$, and $\rho(1,0) = \rho(1,1) = 0$. The buyer's payoff is such that $U^b_{\{(0,1)\}} = U^b_{\{(0,0)\}} > \max\{U^b_{\{(1,0)\}}, U^b_{\{(1,1)\}}\}$. Given this assumption, we have from (56) that $U^b_{\{(1,0)\}} = U^b_{\{(1,1)\}}$. From (52) and (53), we can derive

$$\hat{\pi} = \frac{\kappa^f - (\phi_{-1}e_{-1}/\phi e - \beta)\phi ed^f}{\beta\sigma\phi ed^f}. \tag{64}$$

From the case that $U^b_{\{(0,0)\}} > U^b_{\{(1,0)\}}$ and (52)-(54), we have:

$$\phi d < \frac{\kappa}{\frac{\phi_{-1}}{\phi} - \beta(1-\sigma\hat{\pi})}. \tag{65}$$

The buyer's payoff can be evaluated from (53). If $\hat{\pi} < 1$, from (51), we can deduce that the seller's participation constraint is binding. If $\hat{\pi} < 1$, then reducing d^f infinitesimally will increase $\hat{\pi}$ in (64), and this increase the buyer's payoff in (53). The buyer would like to attain $\hat{\pi} = 1$ since the seller's participation constraint will still be respected at that point:

$$c(q) \leq \phi[d + (1-\rho(0,1))ed^f]. \tag{66}$$

Let the maximum of the buyer's TIOLI value (53) such that the constraints (64), (65) and (66) are respected, in this case be $(U^b)^{\dagger\dagger}$. However, since $\rho(1,0) \neq 0$, it is easily verified that $(U^b)^{\dagger\dagger} < U^b_{\{(0,0)\}}[\overline{\omega}|\pi(\omega) = \hat{\pi} = 1; \rho(0,1) = 0] = \sup_{\omega}\{U^b_{\{(0,1)\}}|(64),(65),(66)\} = (U^b)^*$, in which the last equality is attained when $\rho(0,1) = 0$. This contradicts the claim that $\rho(0,0) + \rho(0,1) = 1$ and $\rho(0,1) \neq 0$ is a component of a PBE.

Case 5. Suppose a candidate equilibrium is such that $\sum_{\{z\}\in P}\rho(z) = 1$, $\rho(z) \neq 0$ for all $\{z\} \in P$, and that $U^b_{\{(0,1)\}} = U^b_{\{(0,0)\}} = U^b_{\{(1,0)\}} = U^b_{\{(1,1)\}}$. Then from (53) and (54), we can derive

$$\hat{\pi} = \frac{\kappa^f - (\phi_{-1}e_{-1}/\phi e - \beta)\phi ed^f}{\beta\sigma\phi ed^f} = \frac{\kappa - (\phi_{-1}/\phi - \beta)\phi d}{\beta\sigma\phi d}. \tag{67}$$

If the payment offered (d, d^f) are such that $\hat{\pi} < 1$, then from the seller's decision rule (51) we can deduce $\omega \equiv (q, d, d^f)$ must be such that the seller's participation/incentive constraint binds:

$$c(q) = \phi[(1-\rho(1,0)-\rho(1,1))d + (1-\rho(0,1)-\rho(1,1))ed^f]. \tag{68}$$

However, the buyer can increase his expected payoff in (55) by reducing both (d, d^*), thus raising $\hat{\pi}$ in (67) while still ensuring that the seller participates, until $\hat{\pi} = 1$, where

$$c(q) \leq \phi[(1 - \rho(1,0) - \rho(1,1))d + (1 - \rho(0,1) - \rho(1,1))ed^f]. \tag{69}$$

Let the maximum of the buyer's TIOLI value (55) such that the constraints (67) and (69) are respected, in this case be $(U^b)^\ddagger$. However, since $\rho(1,0), \rho(0,1), \rho(1,1) \neq 0$, it is easily verified that $(U^b)^\ddagger < U^b_{\{(1,1)\}}[\overline{\omega}|\pi(\omega) = \hat{\pi} = 1; \rho(0,0) = 1] = \sup_\omega\{U^b_{\{(1,1)\}}|(67), (66)\} = (U^b)^*$, in which the last equality is attained when $\rho(0,0) = 1$. This contradicts the claim that $\sum_{\{z\}\in P} \rho(z) = 1$, $\rho(z) \neq 0$ for all $\{z\} \in P$, is a component of a PBE.

Summary. From Cases 1 to 5, we have shown that the only mixed-strategy Nash equilibrium in the subgame following an offer ω must be one such that $\langle \rho(0,0), \pi \rangle = \langle 1, 1 \rangle$, and that the offer ω satisfies the program in (12) in Proposition 1.

Finally, since $u(.)$ and $-c(.)$ are strictly quasiconcave functions and the inequality constraints in program (12) define a convex feasible set, the program (12) has a unique solution.

C Proof of Proposition 2 (Constant exchange rate growth)

A monetary equilibrium implies that the general goods market, asset (money) markets and labor markets must clear in every country. This also implies that each DM-buyer's sequential (and therefore intertemporal) budget constraint must hold. We will evaluate these budget constraints at a monetary equilibrium. For the treatment below, we assume that seigniorage revenue x_t is transferred to the DM-buyers uniformly at the beginning of each CM.

First, we re-write the date-t budget constraint of a DM-buyer for end-of-period change in domestic real money holding:

$$\phi_t(m_{t+1} - m_t) = N_t - C_t + \phi_t x_t - \phi_t^f \left(m_{t+1}^f - m_t^f \right), \qquad \forall t \geq 0, \tag{70}$$

where m_t (or m_t^f) is the initial stock of domestic (or foreign) currency held by the DM-buyer. Also, ϕ_t (or $\phi_t^f = e_t\phi_t$), given an equilibrium nominal exchange rate e_t, is the value of a unit of domestic (or foreign) currency in units of the domestic (or foreign) CM consumption good C_t (or C_t^f). In equilibrium N_t the amount of labor supplied is also the labor income to the DM-buyer.

Summing these sequential budget constraints up to some finite date $T > 0$, multiplying both sides by a constant $\beta^T \in (0,1)$, and taking the limit of $T \nearrow +\infty$, we have:

$$\begin{aligned}\lim_{T\nearrow+\infty} \beta^T \phi_{T+1} m_{T+1} = \lim_{T\nearrow+\infty} \beta^T \left(\frac{\phi_T}{\phi_0}\right) \Bigg\{ \phi_0(m_0 + e_0 m_0^f) + \sum_{t=0}^{T} \left(\frac{\phi_0}{\phi_t}\right) [\phi_t x_t + N_t - C_t] \\ + \phi_0 \sum_{t=0}^{T} \left(\frac{e_{t+1} - e_t}{e_t}\right) e_t m_{t+1}^f \Bigg\},\end{aligned} \tag{71}$$

where $(e_{t+1} - e_t)/e_t \equiv (\phi^f_{t+1}/\phi_{t+1} - \phi^f_t/\phi_t)/(\phi^f_t/\phi_t)$ is the one-period growth rate in the nominal exchange rate.

The first term on the LHS of (71) equals zero by the transversality condition on assets. That is, in the infinite-horizon limit, the discounted value of accumulated domestic real money balances must be zero in a monetary equilibrium. The first term on the RHS also goes to zero, since in any monetary equilibrium the T-period return on domestic money ϕ_T/ϕ_0 must be finite-valued, and, the given initial real balances on both monies $\phi_0(m_0 + e_0 m^f_0)$ are also finite. The second term on the RHS must equal zero since the infinite series of real wealth (including real seigniorage transfers) net of CM consumption expenditure must be finite in any well-defined equilibrium. Since $\beta \in (0, 1)$, then this term must be zero in the infinite-horizon limiting economy. Therefore, that leaves one with the final term on the RHS which must then equal zero to satisfy the conditions for any monetary general equilibrium. We thus arrive at the following observation:

Lemma 3 *Assuming monetary equilibria with coexistence of both monies, then in any infinite-horizon monetary equilibrium, the discounted total changes in the real value of foreign money holdings of any DM-buyer must be zero,*

$$\lim_{T \nearrow +\infty} \left\{ \beta^T \phi_T \sum_{t=0}^{T} \left(\frac{e_{t+1} - e_t}{e_t} \right) e_t m^f_{t+1} \right\} = 0.$$

This fact will allow us to deduce an admissible property of the *nominal exchange rate* path consistent with the existence of a monetary equilibrium (with coexistence of both monies). In particular we have the following result summarized in Proposition 2.

Proof of Proposition 2. Lemma 3 and the fact that $\beta^T \in (0, 1)$ for any $T = 0, 1, 2, ...,$ implies that for a monetary equilibrium to exist, it is necessary to have

$$\lim_{T \nearrow +\infty} \left| \sum_{t=0}^{T} \left(\frac{e_{t+1}}{e_t} - 1 \right) \left(\frac{\phi_T}{\phi_t} \right) \phi^f_t m_{t+1} \right| < +\infty.$$

Let the one-period growth rate in the nominal exchange rate and the end of date-t CM holding of foreign currency real balance (in units of date-T CM good), respectively, be denoted as $a_t := (e_{t+1}/e_t - 1)$, and, $b_t := (\phi_T/\phi_t)\, \phi^f_t m^f_{t+1}$.

Now let the partial sum $\sum_{s=0}^{t} b_t =: B_t$. By Abel's Lemma (summation by parts), the date-T series can be transformed as follows:

$$\begin{aligned} S_T \equiv \sum_{t=0}^{T} a_t b_t &= a_0 b_0 - a_0 B_0 + a_T B_T + \sum_{t=0}^{T-1} B_t (a_t - a_{t+1}) \\ &= a_T B_T - \sum_{t=0}^{T-1} B_t (a_{t+1} - a_t). \end{aligned}$$

For any integer $t \geq 0$, any integer $k \geq 0$, and some finite real number a, we can write

$$S_{t+k} - S_t = (a_{t+k} - a)B_{t+k} + (a_t - a)B_t + a(B_{t+k} - B_t) + \sum_{s=0}^{t+k-1} B_s(a_s - a_{s+1}).$$

Suppose the partial sums B_t do not converge: $|B_t| \to +\infty$. Then $\lim_{t \nearrow +\infty} |S_{t+k} - S_t| \neq 0$, and, by the Cauchy convergence criterion, this implies that

$$\lim_{T \nearrow +\infty} |S_T| \equiv \lim_{T \nearrow +\infty} \left| \sum_{t=0}^{T} \left(\frac{e_{t+1}}{e_t} - 1 \right) \left(\frac{\phi_T}{\phi_t} \right) \phi_t^f m_{t+1} \right| = +\infty.$$

But this violates Lemma 3 in a monetary equilibrium. Thus, the B_t partial sums must be convergent in any monetary equilibrium.

Therefore, we may assume some upper bound $0 \leq B < +\infty$ for the sequences of absolute partial sums, $\{|B_t|\}_{t \geq 0}$. Note that

$$\begin{aligned} |S_{t+k} - S_t| &= \left| (a_{t+k} - a)B_{t+k} + (a_t - a)B_t + a(B_{t+k} - B_t) + \sum_{s=0}^{t+k-1} B_s(a_s - a_{s+1}) \right| \\ &\leq |a_{t+k} - a||B_{t+k}| + |a_t - a||B_t| + \left| \sum_{s=0}^{t+k-1} B_s(a_s - a_{s+1}) \right| \\ &\leq |a_{t+k} - a|B + |a_t - a|B + |a_{t+k} - a_t|B. \end{aligned}$$

Thus, by applying the Cauchy convergence criterion again, the nominal exchange rate's absolute growth rate converging to a constant $\gamma_e \equiv a \in [0, +\infty)$ in the infinite horizon economy, i.e. $(a_t \to a) \Leftrightarrow |a_{t+k} - a_t| \to 0$, then $|S_{t+k} - S_t| \to 0 \Leftrightarrow \lim_{T \nearrow +\infty} S_T < +\infty$, satisfies the requirement of Lemma 3. That is, it satisfies CM market clearing and agents' intertemporal budget constraints in any monetary equilibrium. ∎

Remark 1 *In the rest of the paper, we can thus focus on monetary equilibria in which the equilibrium exchange rate grows at some constant rate. Also, Proposition 2 will apply in the explicit two-country version of the world economy below. All that is required is to set $x_t = 0$ for every date t in the sequential CM budget constraints, since x_t will later appear in the DM feasibility constraint, since transfers will occur after each CM closes.*

D Proof of Proposition 3 (Equilibria and Coexistence)

Recall from (12), we denoted

$$\frac{\kappa}{\phi_{-1}/\phi - \beta(1-\sigma)} =: \bar{\kappa}(\phi_{-1}/\phi); \qquad \text{and,} \qquad \frac{\kappa^f}{\phi_{-1}e_{-1}/\phi e - \beta(1-\sigma)} =: \bar{\kappa}^f(\phi_{-1}e_{-1}/\phi e).$$

In steady state we have these, respectively, as $\bar{\kappa}(\Pi)$ and $\bar{\kappa}^f(\Pi^f)$.

After some elementary algebra, the necessary first-order conditions for a (steady-state) monetary equilibrium together with their Karush-Kuhn-Tucker (KKT) conditions are as follows: The set of Euler operators evaluated at steady state are

$$\beta\sigma\left[\frac{u'(q)}{c'(q)}-1\right]=\lambda-\nu+(\Pi-\beta)=\lambda^f-\nu^f+(\Pi^f-\beta). \tag{72a}$$

The DM-sellers' participation constraint is binding ($\zeta>0$), so that

$$c(q)=\phi d+\phi^f d^f, \tag{72b}$$

and the KKT conditions are:

$$\begin{aligned}
\lambda\cdot[\bar{\kappa}(\Pi)-\phi d]&=0, \qquad \lambda\geq 0, \qquad \phi d\leq\bar{\kappa}(\Pi),\\
\lambda^f\cdot\left[\bar{\kappa}^f(\Pi^f)-\phi^f d^f\right]&=0, \qquad \lambda^f\geq 0, \qquad \phi^f d^f\leq\bar{\kappa}^f(\Pi^f),\\
-\nu\cdot d&=0, \qquad d\geq 0, \qquad \nu\geq 0,\\
-\nu^f\cdot d^f&=0, \qquad d^f\geq 0, \qquad \nu^f\geq 0.
\end{aligned} \tag{72c}$$

We must consider different cases depending on when the endogenous liquidity constraint associated with the local and foreign currency bind or not, given domestic (foreign) inflation rate, money supply, counterfeiting costs and the matching probability. We focus on equilibria that satisfy $\zeta>0$, and at least one of ν and ν^f is zero, so that buyers and sellers trade in DM, and $\mu,\mu^f>0$.

Case 1(a) $\Pi-\Pi^f>0$ **and**

(i) $\lambda=0,\lambda^f=0$: This case is trivial to check. When both liquidity constraints are not binding, and the foreign currency dominates in rate of return, buyers demand only the foreign fiat money, so that only M^f is in circulation.

(ii) $\lambda>0,\lambda^f=0$. Since $\lambda>0$ and $\lambda^f=0$ and $\Pi^f<\Pi$, all agents optimally demand zero domestic fiat money, so that $d=0$ or by complementary slackness $\nu>0$, and, $d^f=M^f$ with $\nu^f=0$.

In both subcases, we have the following characterization of a unique equilibrium outcome. First, from the equilibrium Euler condition (72a), q solves

$$\beta\sigma\left[\frac{u'(q)}{c'(q)}-1\right]-(\Pi^f-\beta)=0.$$

Second, since u' (c') is a continuous and monotone decreasing (non-decreasing) function on a compact set $[0,q^*]$, and since it is optimal to consume $q>0$ and given that $\Pi^f-\beta>0$, then there exists a unique solution $q\in(0,q^*)$ where q^* satisfies the first-best solution $u'(q)=c'(q)$. Therefore, there is a unique price level, $1/\phi^f$, determined from the DM-sellers' participation constraint (72b): $\phi^f=c(q)/M^f$.

Case 1(b): $\lambda = 0, \lambda^f > 0$ **and** $\Pi - \Pi^f > 0$**.** From the result in Lemma 2, we have in any monetary equilibrium, a buyer at the end of every CM will make offers of payments $\left(d, d^f\right)$ up to the respective limits of their portfolio components $\left(m, m^f\right)$—i.e. $d = m \geq 0$ and $d^f = m^f \geq 0$—which implies that the multipliers on payment upper-bounds are strictly positive:

$$\mu = \Pi - \beta > 0,$$

$$\mu^f = \Pi^f - \beta > 0.$$

In this case, the Euler conditions (72a) reduce to

$$\beta\sigma \left[\frac{u'(q)}{c'(q)} - 1\right] - (\Pi - \beta) = 0.$$

which solve for a unique q, following similar arguments in Case 1(a), except that now we also have $\Pi - \beta > 0$.

Next, we show that there is coexistence of the home currency with the foreign currency, in spite of the former being dominated in its return, $\Pi > \Pi^f$. By construction the highest sustainable allocation of q is a $q^* > 0$ satisfying the first best trade-off: $u'(q^*) = c'(q^*)$. Comparing the first-best condition with the monetary equilibrium condition for q above, we can easily deduce that $q < q^*$ since $\Pi - \beta > 0$. Since the liquidity constraint on the foreign currency payment is binding, then, from the seller's participation constraint we can re-write as:

$$\phi m = c(q) - \bar{\kappa}^f(\Pi^f) \geq 0.$$

Suppose to the contrary that the demand for home currency were zero, $m = 0$. Then we have $q = c^{-1}\left(\bar{\kappa}^f(\Pi^f)\right) < q^*$. Since each buyer can increase his lifetime payoff by accumulating more domestic money ($\lambda = 0$) and offering it to the seller in the DM to consume more q; and the seller would willingly accept it by producing more q while ensuring that her participation constraint is still binding, then we have in this equilibrium positive demand for home real currency, $\phi m = c(q) - \bar{\kappa}^f > 0$. In equilibrium in the integrated economy, $m = M$ and $m^f = M^f$.

From the DM-sellers' participation constraint (72b), there is a unique value for domestic fiat money:

$$\phi = \frac{c(q) - \bar{\kappa}^f(\Pi^f)}{M}.$$

Since $\lambda^f > 0$ or that the foreign money liquidity constraint binds (72c), then there is a unique value for the foreign fiat money:

$$\phi^f = \frac{\bar{\kappa}^f(\Pi^f)}{M^f}.$$

Therefore, there is a unique monetary equilibrium outcome in this case, with the nominal exchange determined as

$$e := \frac{\phi^f}{\phi} = \frac{M}{M^f} \cdot \frac{\bar{\kappa}^f(\Pi^f)}{c(q) - \bar{\kappa}^f(\Pi^f)}.$$

Case 1(c): $\lambda > 0, \lambda^f > 0$ **and** $\Pi - \Pi^f > 0$ It is easy to show that the resulting system of equations has a unique solution for $\{\mu, \mu^f, \phi, e, q, \lambda, \zeta\}$. For a given domestic and foreign inflation rates, money supplies, counterfeited costs and matching probability, the relevant block of the steady state equilibrium conditions is given as follows. First, as in the previous cases,

$$\mu = \Pi - \beta > 0$$

$$\mu^f = \Pi^f - \beta > 0.$$

Second, since the DM-buyer is liquidity constrained in both currencies, then in real terms, he would demand and offer payments up to the limits of both constraints: $\phi M = \bar{\kappa}(\Pi) > 0$ and $\phi e M^f = \bar{\kappa}^f(\Pi^f) > 0$ as measured in units of the home CM good. From the home currency liquidity constraint, we can solve for

$$\phi = \frac{\bar{\kappa}(\Pi)}{M};$$

and then using this in the foreign currency liquidity constraint, we can derive a unique equilibrium nominal exchange rate

$$e = \frac{M}{M^f} \frac{\bar{\kappa}^f(\Pi^f)}{\bar{\kappa}(\Pi)}.$$

Finally, the other relevant equilibrium conditions:

$$c(q) = \bar{\kappa}^f(\Pi^f) + \bar{\kappa}(\Pi);$$

$$\sigma\beta \frac{u'(q) - c'(q)}{c'(q)} = \mu + \lambda;$$

$$\zeta = \beta\sigma + \Pi - \beta + \lambda;$$

$$\lambda^f = \Pi - \Pi^* + \lambda,$$

pin down a unique q, λ, ζ and λ^f, respectively. Therefore, in this case, there is a determinate monetary equilibrium, with a unique nominal exchange rate, and coexistence of both currencies.

Case 2: $\Pi - \Pi^f < 0$ This is the symmetric opposite to the analyses in Case 1. Therefore there can exist a unique steady state e and coexistence of the two currencies, in spite of $\Pi < \Pi^f$.

Case 3(a): $\lambda = \lambda^f = 0$ **and** $\Pi - \Pi^f = 0$**.** This case corresponds to the indeterminacy result in Kareken and Wallace. Since both liquidity constraints are not binding, and both currencies yield equal rates of return, then buyers are indifferent as to which currency to hold and sellers' participation constraint binds for any composition of payments offered.

Case 3(b): $\lambda \neq \lambda^f > 0$ **and** $\Pi - \Pi^f = 0$**.** In this case, when both liquidity constraints bind, the analysis is similar to Case 1(c) above. Therefore we have coexistence of the two currencies and determinacy of the equilibrium nominal exchange rate.

Case 3(c): $\Pi - \Pi^f = 0$ **and** ($\lambda > 0$ **and** $\lambda^f = 0$) **or** ($\lambda = 0$ **and** $\lambda^f > 0$). We can rule out these two cases as equilibria. Since one liquidity constraint binds in either subcases, then $\nu = \nu^f = 0$, but this implies that either $\lambda + \Pi - \beta > \lambda^f + \Pi^f - \beta$ in subcase (i), or, $\lambda + \Pi - \beta < \lambda^f + \Pi^f - \beta$, which respectively, is an impossibility since $\Pi = \Pi^f$ so that this violates the Euler conditions (72a). Therefore these two cases cannot be an equilibrium.

E Proof of Proposition 4 (Non First Best)

It suffices to construct a counterexample. Consider Case 3 of Proposition Proposition 3. The relevant block characterizing steady-state monetary equilibrium is

$$\mu = \Pi - \beta,$$

$$\mu^f = \Pi^f - \beta,$$

$$c(q) = \phi M + e\phi M^f,$$

$$\phi M \leq \frac{\bar{\kappa}}{\beta\sigma},$$

$$e\phi M^f \leq \frac{\bar{\kappa}^f}{\beta\sigma},$$

$$\sigma\beta\frac{u'(q) - c'(q)}{c'(q)} = \mu + \lambda,$$

$$\zeta = \beta\sigma + \Pi - \beta + \lambda,$$

$$\lambda^f = \Pi - \Pi^* + \lambda.$$

When $\Pi = \beta$ and $\Pi^f = \beta$ it implies that $\mu = 0$ so that

$$\sigma\beta\frac{u'(q) - c'(q)}{c'(q)} = \lambda.$$

Notice that the DM first best q^*, which satisfies $u'(q^*) = c'(q^*)$, can only occur if $\lambda = 0$. However, in order for the first best to be a monetary equilibrium, the participation constraint for the seller has to be satisfied and the nominal exchange rate has to be positive. These two conditions are respectively given by

$$c(q^*) \leq \frac{\bar{\kappa}}{\beta\sigma} + \frac{\bar{\kappa}^f}{\beta\sigma},$$

$$c(q^*) \geq \frac{\bar{\kappa}}{\beta\sigma}.$$

Thus, even when both the domestic and foreign inflation rates converge to the Friedman rule, the DM first best may not be attainable and the nominal exchange rate may not be determinate.

F Coexistence, Return Dominance and Counterfeiting Costs

Here we derive the inequalities shown in (33) and (34) in Section 4.3, under the assumption $\Pi > \Pi^f$. Recall Case 1a and its subcases imply no coexistence with only the foreign (high return) money in equilibrium circulation. Case 1a is complementary to the following two.

Case 1b: $\Pi > \Pi^f > \beta$ **and** $\lambda^f > \lambda = 0$**.** From the Euler conditions at steady state, the equilibrium allocation $\hat{q}$ satisfies

$$u'(q) = \frac{\Pi - \beta(1-\sigma)}{\beta\sigma} c'(q).$$

Since the inefficiency wedge, $[\Pi - \beta(1-\sigma)]/\beta\sigma$, is strictly positive due to $\Pi > \beta(1-\sigma)$, and since the derivative function c' is continuous and nondecreasing, and u' is continuous and monotonically decreasing, then the monetary equilibrium solution $\hat{q} < q^*$ where q^* is the efficient solution to $u'(q) = c'(q)$.

Since the foreign money liquidity constraint binds, then the seller's participation constraint in this equilibrium is

$$c(\hat{q}) = \phi M + \frac{\kappa^f}{\Pi^f - \beta(1-\sigma)}.$$

This condition also used the equilibrium requirement in the integrated world economy of $d = m = M$.

Since real balance ϕM have to be non-negative, and since the domestic liquidity constraint is slack, then we have the two respective inequalities

$$0 \leq \phi M = c(\hat{q}) - \frac{\kappa^f}{\Pi^f - \beta(1-\sigma)} < \frac{\kappa}{\Pi - \beta(1-\sigma)}.$$

Tidying up yields the expression in (33).

Case 1c: $\Pi > \Pi^f > \beta$**,** $\lambda^f > 0$**, and** $\lambda > 0$**.** Since both liquidity constraints are binding in this case, then the seller's participation constraint implies a unique solution $\tilde{q}$ such that

$$c(\tilde{q}) = \frac{\kappa}{\Pi - \beta(1-\sigma)} + \frac{\kappa^f}{\Pi^f - \beta(1-\sigma)}.$$

Also, since both liquidity constraints are binding, and the seller's participation constraint holds with equality, the equilibrium outcome is $\tilde{q} \leq \hat{q}$, where $\hat{q}$ was defined in the solution to Case 1b's equilibrium. Therefore,

$$c(\tilde{q}) = \frac{\kappa}{\Pi - \beta(1-\sigma)} + \frac{\kappa^f}{\Pi^f - \beta(1-\sigma)} \leq c(\hat{q}),$$

given that $\kappa > 0$ and $\kappa^f > 0$. Rearranging, we have the inequalities restricting (κ, κ^f) in (34).

G Proof of Proposition 5 (Two-country Variation)

We will only prove the two cases of $\Pi > \Pi^f$ (Case 1) and $\Pi = \Pi^f$ (Case 3). Given each case, there are $\binom{4}{4} + \binom{4}{3} + \binom{4}{2} + \binom{4}{1} + \binom{4}{0} = 16$ subcases (i.e., combinations of the total of four liquidity constraints) to consider. However, depending on the magnitude of Π relative to Π^f, we can easily show that some of the subcases cannot exist as an equilibrium.

Case 1(a): $\Pi > \Pi^f$ **with the following subcases.**

1. $\lambda = \lambda^f = \lambda_\star^f = \lambda_\star = 0$;
2. $\lambda > 0$ and $\lambda^f = \lambda_\star^f = \lambda_\star = 0$;
3. $\lambda = \lambda^f = 0, \lambda_\star^f = 0, \lambda_\star > 0$; or
4. $\lambda > 0$, $\lambda^f = \lambda_\star^f = 0$ and $\lambda_\star > 0$.

The characterization is similar to Case 1(a) of Proposition 3. Since $\Pi > \Pi^f$ and the liquidity constraint on holding Foreign money in both countries are slack, then all agents in all countries will only demand the Foreign fiat money. Thus, there is a unique monetary equilibrium with only the low inflation (Foreign) fiat money in circulation in both countries.

Case 1(b): $\Pi > \Pi^f$ **with all the following subcases.** All these exist as a unique monetary equilibrium exhibiting coexistence of both fiat monies.

5. $\lambda > 0, \lambda^f > 0, \lambda_\star^f = \lambda_\star = 0$:

   Since both liquidity constraints for Home DM-buyers bind, then from the Home DM-seller's participation constraint (46b) and the first two KKT conditions in (46c), we have a unique solution to $q = c^{-1}[\bar{\kappa}(\Pi) + \bar{\kappa}(\Pi^f)]$. Also, from the global money market clearing condition (48a) on Home fiat currency we have that $\phi d_\star = \phi M - \bar{\kappa}(\Pi)$. However, since the Foreign fiat money dominates in rate of return, and Foreign DM-buyers are not liquidity constrained ($\lambda_\star^f = 0$), then $d_\star = 0$, and we can deduce that $\phi = \bar{\kappa}(\Pi)/M$.

   Also, since Foreign DM-buyers demand all of their national fiat money, $d_\star^f > 0$ (or $\nu_\star^f = 0$ by complementary slackness), then there is a unique $q_\star^f \in (0, q^*)$ solving the Euler condition (47a): $\beta\sigma[u'(q)/c'(q) - 1] = \Pi^f - \beta$. From the Foreign DM-seller's participation constraint, we have $\phi^f d^f = \phi^f M^f - c(q_\star^f)$. Since $\lambda^f > 0$, we have $\phi^f d^f = \bar{\kappa}^f(\Pi^f) = \phi^f M^f - c(q_\star^f)$. Rearranging, we have $\phi^f = [\bar{\kappa}^f(\Pi^f) + c(q_\star^f)]/M^f$.

   Therefore there is a unique nominal exchange rate

$$e := \frac{\phi^f}{\phi} = \left[\frac{\bar{\kappa}^f(\Pi^f) + c(q_\star^f)}{\bar{\kappa}(\Pi)}\right] \cdot \frac{M}{M^f}.$$

   There is only coexistence of both monies in the Home country's DM. Only the Foreign currency circulates in the Foreign DM.

6. $\lambda = 0, \lambda^f > 0, \lambda_\star^f = \lambda_\star = 0$:

   From the Home DM-buyers' Euler conditions (46a), we have that $\nu = \Pi - (\Pi^f + \lambda^f) = 0$ since the DM-buyer is constrained in holding the Foreign fiat money. Thus, q solves $\beta\sigma[u'(q)/c'(q) - 1] = \Pi - \beta$ uniquely. From the Home DM-seller's participation constraint (46b) and the Home DM-buyers' binding liquidity constraint on Foreign real money balance, we have $c(q) - \bar{\kappa}^f(\Pi^f) = \phi d$. Substituting this into the global clearing condition on Home fiat money (48a), and since no Foreign DM-buyers will optimally demand Home fiat money ($d_\star = 0$), we have $\phi = [c(q) - \bar{\kappa}^f(\Pi^f)]/M]$. Also since Foreign DM-buyers demand only Foreign fiat money, then $q_\star^f$ uniquely solves the Foreign Euler condition (47a): $\beta\sigma[u'(q)/c'(q) - 1] = \Pi^f - \beta$. Using this fact in the Foreign DM-sellers' participation constraint (47b), combined with the global adding-up condition on Foreign fiat money (48b) and the fact that $\phi^f d^f = \bar{\kappa}^f(\Pi^f)$, we have a determination of the inverse Foreign price level as $\phi^f = [c(q_\star^f) - \bar{\kappa}^f(\Pi^f)]/M^f$. Therefore there is a unique equilibrium nominal exchange rate

   $$e := \frac{\phi^f}{\phi} = \left[\frac{c(q_\star^f) - \bar{\kappa}^f(\Pi^f)}{c(q) - \bar{\kappa}^f(\Pi^f)}\right] \cdot \frac{M}{M^f}.$$

   There is only coexistence of both monies in the Home country's DM. The Home currency does not circulate in the Foreign DM.

7. $\lambda = 0, \lambda^f > 0, \lambda_\star^f > 0, \lambda_\star = 0$:

   Since the Home DM-buyers' liquidity constraint on Foreign money is binding ($\lambda^f > 0$), and that on Home money is not ($\lambda = 0$), from the Home DM-buyers' Euler conditions (46a), we have that $\nu = \Pi - (\Pi^f + \lambda^f) = 0$. Thus, q solves $\beta\sigma[u'(q)/c'(q) - 1] = \Pi - \beta$ uniquely. From the Home DM-seller's participation constraint (46b) and the Home DM-buyers' binding liquidity constraint on Foreign real money balance, we have $c(q) - \bar{\kappa}^f(\Pi^f) = \phi d$. Substituting this into the global clearing condition on Home fiat money (48a), then Foreign DM-buyers' real demand for Home fiat money is $\phi d_\star = \phi M - [c(q) - \bar{\kappa}^f(\Pi^f)]$. Since the Foreign agents' demand for Foreign fiat money faces a binding constraint ($\lambda_\star^f > 0$), then from the Foreign Euler conditions (47a) under $\nu_\star = \lambda_\star = 0$, we have $q_\star^f$ solving $\beta\sigma[u'(q_\star^f)/c'(q_\star^f) - 1] = \Pi - \beta$ uniquely. From the Foreign DM-sellers' participation constraint, together with $c(q) - \bar{\kappa}^f(\Pi^f) = \phi d$, we can derive the inverse Home price level as $\phi = [c(q_\star^f - \bar{\kappa}_\star^f(\Pi^f) + c(q) - \bar{\kappa}^f(\Pi^f)]/M$. From the global Foreign money adding-up condition, we have $\bar{\kappa}_\star^f(\Pi^f) + \bar{\kappa}^f(\Pi^f) = \phi^f M^f$, which implies that $\phi^f = [\bar{\kappa}_\star^f(\Pi^f) + \bar{\kappa}^f(\Pi^f)]/M^f$. So then there is a unique nominal exchange rate

   $$e := \frac{\phi^f}{\phi} = \left[\frac{\bar{\kappa}_\star^f(\Pi^f) + \bar{\kappa}^f(\Pi^f)}{c(q_\star^f) + c(q) - [\bar{\kappa}_\star^f(\Pi^f) + \bar{\kappa}^f(\Pi^f)]}\right] \cdot \frac{M}{M^f},$$

   and the coexistence of both monies in both countries' DM.

8. $\lambda = 0, \lambda^f > 0, \lambda_\star^f = 0, \lambda_\star > 0$:

Since Foreign DM-buyers are not liquidity constrained in their demand for the high return money ($\lambda_\star^f = 0$), so that $\nu_\star^f = 0$ and their optimal portfolio consists of this asset (i.e., $d_\star^f > 0$ and $d_\star = 0$), then from their Euler conditions (47a), there is a unique $q_\star^f$ satisfying $\beta\sigma[u'(q_\star^f)/c'(q_\star^f) - 1] = \Pi^f - \beta$. Since $d_\star = 0$, from the Foreign DM-sellers' participation constraint (47b), we have $\phi^f d_\star^f = c(q_\star^f)$. Using this result, together with $\phi^f d^f = \bar{\kappa}^f(\Pi^f)$ in the global clearing condition on Foreign fiat money (48b), we have $\phi^f = [c(q_\star^f) + \bar{\kappa}^f(\Pi^f)]/M^f$. Also, from the global clearing condition on Home fiat money (48a) together with the fact that $d_\star = 0$, we have $\phi = [c(q) - \bar{\kappa}^f(\Pi^f)]/M$. Therefore, there is coexistence (only in the Home country), and only the Foreign currency circulates in the Foreign DM, and the nominal exchange rate is

$$e := \frac{\phi^f}{\phi} = \left[\frac{c(q_\star^f) + \bar{\kappa}^f(\Pi^f)}{c(q) - \bar{\kappa}^f(\Pi^f)}\right] \cdot \frac{M}{M^f}.$$

9. $\lambda > 0, \lambda^f = 0, \lambda_\star^f > 0, \lambda_\star = 0$:

   Since DM-buyers in Home are not liquidity constrained on the higher return money, they will only demand the Foreign fiat money, and $d = 0$. From their Euler conditions, (46a), q solves $\beta\sigma[u'(q)/c'(q) - 1] = \Pi^f - \beta$, and from the Home DM-sellers' participation constraint (46b), we have Home DM-buyers' demand for real Foreign fiat money balance as $\phi^f d^f = c(q)$. Since the Foreign DM-buyers are liquidity constrained on the Foreign money, $\phi^f d_\star^f = \bar{\kappa}_\star^f(\Pi^f)$, then the global clearing condition on the Foreign fiat money (48b) implies a unique value for the Foreign fiat money $\phi^f = [\bar{\kappa}_\star^f(\Pi^f) - c(q)]/M^f$. Also, since $\lambda_\star^f > 0$ and $\lambda_\star = 0$, then $q_\star^f$ uniquely satisfies $\beta\sigma[u'(q_\star^f)/c'(q_\star^f) - 1] = \Pi - \beta$ in the Foreign DM-buyers' Euler equation in (47a). From the global clearing condition on the Home fiat money (48a), we thus have $\phi = [c(q_\star^f) - \bar{\kappa}_\star^f(\Pi^f)]/M$. In this case, there is a unique equilibrium nominal exchange rate

   $$e := \frac{\phi^f}{\phi} = \left[\frac{\bar{\kappa}^f(\Pi^f) - c(q)}{c(q_\star^f) - \bar{\kappa}^f(\Pi^f)}\right] \cdot \frac{M}{M^f},$$

   and there is coexistence of both monies in the Foreign DM, while only the Foreign money circulates in the Home DM.

10. $\lambda > 0, \lambda^f > 0, \lambda_\star^f = 0, \lambda_\star > 0$:

    Since Home DM-buyers are liquidity constrained on both monies, then from the Home DM-sellers' participation constraint (46b), we can uniquely deduce q such that $c(q) = \bar{\kappa}(\Pi) + \bar{\kappa}^f(\Pi^f)$. Since Foreign DM-buyers will only demand Foreign money so that $d_\star = 0$, then from the global clearing condition on the Home fiat money (48a), we thus have $\phi = \bar{\kappa}(\Pi)/M$. In the Foreign DM-seller's participation constraint (47b), we have $\phi^f d_\star^f = c(q_*^f)$, where $q_\star^f$ uniquely solves $\beta\sigma[u'(q_\star^f)/c'(q_\star^f) - 1] = \Pi^f - \beta$ in their Euler condition (47a). Using this—and the fact that Home DM-agents are also liquidity constrained on holding the Foreign fiat money—the global clearing condition on the Foreign fiat money (48b), implies $\phi^f = [c(q_*^f) + \bar{\kappa}^f(\Pi)]/M^f$.

Therefore, there is a unique nominal exchange rate

$$e := \frac{\phi^f}{\phi} = \left[\frac{c(q_\star^f) + \bar{\kappa}^f(\Pi^f)}{\bar{\kappa}(\Pi)}\right] \cdot \frac{M}{M^f},$$

and both monies circulate in the Home DM, but only the Foreign money circulates in the Foreign DM.

11. $\lambda > 0, \lambda^f > 0, \lambda_\star^f > 0, \lambda_\star = 0$:

    Here, Home DM-buyers are liquidity constrained on both monies, so that $\phi d = \bar{\kappa}(\Pi)$ and $\phi^f d^f = \bar{\kappa}^f(\Pi^f)$. For Foreign DM-buyer, since $\lambda_\star = 0$ and $\lambda_\star^f > 0$, then $q_\star^f$ can be uniquely determined as $\beta\sigma[u'(q_\star^f)/c'(q_\star^f) - 1] = \Pi - \beta$ in their Euler condition (47a). Then from the Foreign DM-sellers' participation constraint (47b), we have $\phi d_\star = c(q_*^f) - \bar{\kappa}_\star^f(\Pi^f)$, since DM-buyers are liquidity constrained on Foreign money, i.e. $\lambda_\star^f > 0$ or that $\phi^f d_\star^f = \bar{\kappa}_\star^f(\Pi^f)$. Using this in the global clearing condition on the Home fiat money (48a), we have $\phi = [c(q_*^f) - \bar{\kappa}_\star^f(\Pi^f) + \bar{\kappa}(\Pi)]/M$. Since the Foreign fiat money has a liquidity constraint binding in each country, then from the the global clearing condition on the Foreign fiat money (48b), we have $\phi^f = [\bar{\kappa}_\star^f(\Pi^f) + \bar{\kappa}^f(\Pi^f)]/M^f$. Thus, there is a unique nominal exchange rate

    $$e := \frac{\phi^f}{\phi} = \left[\frac{\bar{\kappa}_\star^f(\Pi^f) + \bar{\kappa}^f(\Pi^f)}{c(q_*^f) - \bar{\kappa}_\star^f(\Pi^f) + \bar{\kappa}(\Pi)}\right] \cdot \frac{M}{M^f},$$

    and both monies circulate in both Home and Foreign DMs.

12. $\lambda = 0, \lambda^f > 0, \lambda_\star^f > 0, \lambda_\star > 0$:

    The proof here is similar to that of the previous Case 1(b)(11). In this case, there is a unique nominal exchange rate

    $$e := \frac{\phi^f}{\phi} = \left[\frac{\bar{\kappa}_\star^f(\Pi^f) + \bar{\kappa}^f(\Pi^f)}{c(q) + \bar{\kappa}_\star^f(\Pi^f) - \bar{\kappa}(\Pi)}\right] \cdot \frac{M}{M^f},$$

    and both monies circulate in both Home and Foreign DMs.

13. $\lambda > 0, \lambda^f = 0, \lambda_\star^f > 0, \lambda_\star > 0$:

    The proof here is opposite to that of the previous Case 1(b)(10). We can easily show that there is a unique nominal exchange rate

    $$e := \frac{\phi^f}{\phi} = \left[\frac{c(q) + \bar{\kappa}_\star^f(\Pi^f)}{\bar{\kappa}_\star(\Pi)}\right] \cdot \frac{M}{M^f},$$

    and both monies circulate in the Foreign DM, but only the Foreign money circulates in the Home DM.

14. $\lambda = \lambda^f = 0, \lambda_\star^f > 0, \lambda_\star > 0$:

The proof here is opposite to that of the previous Case 1(b)(5). Therefore there is a unique nominal exchange rate

$$e := \frac{\phi^f}{\phi} = \left[\frac{\bar{\kappa}_\star^f(\Pi^f) + c(q)}{\bar{\kappa}_\star(\Pi)}\right] \cdot \frac{M}{M^f}.$$

There is only coexistence of both monies in the Foreign country's DM. Only the Foreign currency circulates in the Home DM.

15. $\lambda = \lambda^f = 0, \lambda_\star^f > 0, \lambda_\star = 0$:

    Since Home DM-buyers are not liquidity constrained in their demand for the high return money ($\lambda^f = 0$), so that $\nu^f = 0$ and their optimal portfolio consists of this asset (i.e., $d^f > 0$ and $d = 0$), then from their Euler conditions (46a), there is a unique q satisfying $\beta\sigma[u'(q)/c'(q) - 1] = \Pi^f - \beta$. Since $d = 0$, from the Home DM-sellers' participation constraint (46b), we have $\phi^f d^f = c(q)$. Using this result, together with $\phi_\star^f d_\star^f = \bar{\kappa}_\star^f(\Pi^f)$ in the global clearing condition on Foreign fiat money (48b), we have $\phi^f = [c() + \bar{\kappa}_\star^f(\Pi^f)]/M^f$. Since Foreign DM-buyers are constrained on holding Foreign money ($\lambda_\star^f > 0, \lambda_\star = 0$), then $q_\star^f$ can be uniquely determined as $\beta\sigma[u'(q_\star^f)/c'(q_\star^f) - 1] = \Pi - \beta$ in their Euler condition (47a). Also, from the global clearing condition on Home fiat money (48a) together with the fact that $d = 0$, we have $\phi = [c(q_\star^f) - \bar{\kappa}_\star^f(\Pi^f)]/M$. Therefore, there is coexistence (only in the Home country), and only the Foreign currency circulates in the Foreign DM, and the nominal exchange rate is

    $$e := \frac{\phi^f}{\phi} = \left[\frac{c(q) + \bar{\kappa}_\star^f(\Pi^f)}{c(q_\star^f) - \bar{\kappa}_\star^f(\Pi^f)}\right] \cdot \frac{M}{M^f}.$$

16. $\lambda > 0, \lambda^f > 0, \lambda_\star^f > 0, \lambda_\star > 0$:

    This case is obvious. When all liquidity constraints are binding in all countries, we have from the two adding up conditions on global monies, (48a) and (48b), a determination of relative prices as $\phi^f = [\bar{\kappa}^f(\Pi^f) + \bar{\kappa}_\star^f(\Pi^f)]/M^f$ and $\phi = [\bar{\kappa}(\Pi) + \bar{\kappa}_\star(\Pi)]/M$, respectively. The nominal exchange rate is

    $$e := \frac{\phi^f}{\phi} = \left[\frac{\bar{\kappa}^f(\Pi^f) + \bar{\kappa}_\star^f(\Pi^f)}{\bar{\kappa}(\Pi) + \bar{\kappa}_\star(\Pi)}\right] \cdot \frac{M}{M^f},$$

    and both monies circulate in both DMs.

Case 3(a): $\Pi = \Pi^f$ with all the following subcases.

1. $\lambda > 0, \lambda^f > 0, \lambda_\star^f > 0, \lambda_\star > 0$:

    This case is obvious. When all liquidity constraints are binding in all countries, we have from the two adding up conditions on global monies, (48a) and (48b), a determination of relative prices as $\phi^f = [\bar{\kappa}^f(\Pi^f) + \bar{\kappa}_\star^f(\Pi^f)]/M^f$ and $\phi = [\bar{\kappa}(\Pi) + \bar{\kappa}_\star(\Pi)]/M$, respectively. The nominal exchange rate is

    $$e := \frac{\phi^f}{\phi} = \left[\frac{\bar{\kappa}^f(\Pi^f) + \bar{\kappa}_\star^f(\Pi^f)}{\bar{\kappa}(\Pi) + \bar{\kappa}_\star(\Pi)}\right] \cdot \frac{M}{M^f},$$

and both monies circulate in both DMs.

Case 3(b): $\Pi = \Pi^f$ with all the following subcases. Any of the following configurations, in which at least one country's liquidity constraint is active (inactive) while all of its other liquidity constraints are inactive (active), i.e.,

2. $\lambda > 0$, $\lambda^f = \lambda_\star^f = 0$ and $\lambda_\star > 0$;
3. $\lambda = 0, \lambda^f > 0, \lambda_\star^f = \lambda_\star = 0$;
4. $\lambda = 0, \lambda^f > 0, \lambda_\star^f > 0, \lambda_\star = 0$;
5. $\lambda = 0, \lambda^f > 0, \lambda_\star^f = 0, \lambda_\star > 0$;
6. $\lambda > 0$ and $\lambda^f = \lambda_\star^f = \lambda_\star = 0$;
7. $\lambda = \lambda^f = 0, \lambda_\star^f = 0, \lambda_\star > 0$;
8. $\lambda > 0, \lambda^f = 0, \lambda_\star^f > 0, \lambda_\star = 0$;
9. $\lambda > 0, \lambda^f > 0, \lambda_\star^f = 0, \lambda_\star > 0$;
10. $\lambda > 0, \lambda^f > 0, \lambda_\star^f > 0, \lambda_\star = 0$;
11. $\lambda = 0, \lambda^f > 0, \lambda_\star^f > 0, \lambda_\star > 0$;
12. $\lambda > 0, \lambda^f = 0, \lambda_\star^f > 0, \lambda_\star > 0$; or
13. $\lambda = \lambda^f = 0, \lambda_\star^f > 0, \lambda_\star = 0$,

cannot exist as equilibrium.

Consider the equilibrium Euler conditions (46a) and (47a), which would reduce to the respective conditions $\lambda - \nu = \lambda^f - \nu^f$ and $\lambda_\star^f - \nu_\star^f = \lambda_\star - \nu_\star$. For example, suppose Case 3(b)(1) were an equilibrium. Then, since agents are indifferent between either currency, then we may assume $\nu = \nu^f = \nu_\star^f = \nu_\star = 0$. If so, we have

$$\begin{aligned}\beta\sigma[u'(q)/c'(q) - 1] &= \lambda - \nu + \Pi - \beta \\ &> \lambda^f - \nu^f + \Pi^f - \beta = \beta\sigma[u'(q)/c'(q) - 1],\end{aligned}$$

which is a contradiction. A similar argument applies to all the other configurations.

Case 3(c): $\Pi = \Pi^f$ with the following subcases.

14. $\lambda = \lambda^f = \lambda_\star^f = \lambda_\star = 0$;
15. $\lambda > 0, \lambda^f > 0, \lambda_\star^f = \lambda_\star = 0$ or

16. $\lambda = \lambda^f = 0, \lambda_\star^f > 0, \lambda_\star > 0,$

   In all of these cases, there is a monetary equilibrium where both monies can coexist in both countries' DMs, but the nominal exchange rate is indeterminate.

In case 3(c)(14), since no liquidity constraints bind, and since all monies have the same rate of return, then $q = q_\star^f$ solves the same Euler conditions (46a) and (47a), so that $\beta\sigma[u'(q)/c'(q) - 1] = \Pi - \beta = \Pi^f - \beta$. Given $q = q_\star^f$, then the equilibrium conditions reduce to the global clearing conditions on Foreign and Home fiat money, respectively (48b) and (48a), and the seller's constraints in both contries, (46b) and (47b), determining $(d, d^f, d_\star^f, d_\star)$ for any arbitrary ratio $e := \phi^f/\phi \in [0, \infty)$.

In case 3(c)(15), since only the Home DM-buyers' liquidity constraints bind, we can show that any positive and finite (ϕ, ϕ^*), such that there is a continuum of $e := \phi^f/\phi \in [0, \infty)$ that satisfies one equilibrium condition:

$$c(q_\star^f) = \phi M + \phi^f M^f - [\bar{\kappa}(\Pi) + \bar{\kappa}^f(\Pi^f)].$$

In case 3(c)(16), since only the Foreign DM-buyers' liquidity constraints bind, we can show that any positive and finite (ϕ, ϕ^*), such that there is a continuum of $e := \phi^f/\phi \in [0, \infty)$ that satisfies one equilibrium condition:

$$c(q) = \phi M + \phi^f M^f - [\bar{\kappa}_\star^f(\Pi^f) + \bar{\kappa}_\star(\Pi)].$$