

BREAKING THE CURSE OF KAREKEN AND WALLACE WITH PRIVATE INFORMATION*

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Abstract

We study the endogenous choice to accept fiat objects as media of exchange and their implications for nominal exchange rate determination. We consider a two-country environment with two currencies which can be used to settle any transactions. However, currencies can be counterfeited at a fixed cost and the decision to counterfeit is private information. This induces equilibrium liquidity constraints on the currencies in circulation. We show that the threat of counterfeiting can pin down the nominal exchange rate even when the currencies are perfect substitutes, thus breaking the Kareken-Wallace indeterminacy result. When the two currencies are not perfect substitutes, an international currency can exist whereby one country has two currencies circulating while the other country uses only one. We also find that with appropriate fiscal policies we can enlarge the set of monetary equilibria with determinate nominal exchange rates. Finally, we show that the threat of counterfeiting can also help determine nominal exchange rates in a variety of different trading environments.

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1 Introduction

When agents have unrestricted access to currency markets and are free to use any currency as means of payment, Kareken and Wallace (1981) showed that the rate of return on the two currencies must be identical for both of them to circulate, which means that these currencies are perfect substitutes.¹ However, in this case the nominal exchange rate between these currencies is indeterminate.

In the three decades since Kareken and Wallace, it has been extremely difficult for models of international monetary economics to generate nominal exchange rate determinacy without imposing *ad hoc* frictions that inhibit trade using one or more of the currencies. Most standard models of international monetary economics build in such restrictions, which include currencies in the utility function, imposing restrictions on the use of currency for certain transactions, assuming differential transaction costs or having differential terms of trade depending on the currency. By assuming distinct liquidity properties for each currency, nominal exchange rate determinacy is effectively imposed on the model by making these currencies imperfect substitutes. We refer to this as ‘the curse of Kareken and Wallace’.

A more desirable approach is to have the liquidity properties of currencies determined endogenously, so that the determinacy, or indeterminacy, of the nominal exchange rate is an equilibrium outcome.² Our main contribution is to break the curse of Kareken and Wallace without imposing exogenous restrictions on the liquidity properties of the currencies so they still remain perfect substitutes. We also show how inflation rates and the severity of the private information problem affect the properties of the nominal exchange rate.

In this paper we study the endogenous choice to accept different fiat objects as media of exchange, the fundamentals that drive their acceptance, and the implications for their bilateral nominal exchange rate. To this end, we consider a stylized model that captures some fundamental frictions that make a medium of exchange essential and that allows various assets to coexist. In particular, agents in this economy have no restrictions on what divisible fiat currency can be used to settle transactions. Potential sellers of goods face a private information problem regarding the quality of what is offered as payment by buyers. More precisely, we allow both fiat currencies to be counterfeited at a fixed cost as in Li, Rocheteau and Weill (2012). An example of such a recognizability problem does arise in the foreign exchange market, where the threat of “fraudulent” currencies has been detected before.³

¹With perfect currency substitution, there is only one single world market clearing condition determining the supplies and demands of all currencies jointly. Thus an indeterminate monetary equilibrium can only be pinned down by an exogenous selection of the nominal exchange rate. This exogenous information is often interpreted as arbitrary speculation.

²Consequently, we are trying to adhere to Wallace’s dictum (1998). We interpret Wallace’s dictum to mean that: 1) monetary economists should explain why fiat money is essential, not assume that it is; 2) the value of fiat money should be determined without resorting to *ad hoc* restrictions; and 3) any “good” model of money should have a non-monetary equilibrium as a possibility.

As a corollary, we argue that: 1) monetary economists should explain why the nominal exchange rate is determinate, not assume that it is; 2) determine its value without resorting to *ad hoc* restrictions on currencies and 3) a “good” model of fiat currencies should have nominal exchange rate indeterminacy as a possible equilibrium outcome.

³In 2005, the Commodity Futures Trading Commission and the Commissioner of Corporations of the State of California sued National Investment Consultants, Inc. and others, in U.S. District Court for the Northern District

Since sellers cannot recognize counterfeited currency, in equilibrium they put a limit on how much of each currency they are willing to accept. These upper bounds in turn are endogenous and depend on the relative inflation rates and the severity of the private information problem. In this environment, possibly imperfect currency substitution occurs as a response to equilibrium liquidity constraints arising from the private information problem. These constraints are such that increased matching efficiency or fixed costs of counterfeiting tighten the buyers' upper bound on payment offers. A critical feature of these liquidity constraints is that the marginal liquidity value of an additional unit of currency beyond the upper bound is zero. These properties stem from hidden actions as we explicitly formalize the illiquidity problem faced by agents. This deeper connection to information frictions allows us to characterize how the equilibrium liquidity constraints depend on policy, and, how pecuniary externalities may arise as a result of these liquidity problems.

We find that there can exist equilibria where both currencies circulate in spite of one being dominated in its rate of return. This arises when the dominated currency faces a binding liquidity constraint. When both liquidity constraints are binding and the currencies are identical in every respect—i.e., same counterfeiting costs and rate of return—we obtain the surprising result that the nominal exchange rate is still determinate and that it is the ratio of the two money stocks, thus breaking the curse of Kareken and Wallace. The latter is observationally equivalent to the standard solution coming out of a two-country cash-in-advance model with ad-hoc local currency restrictions. (Also, this result illustrates that nominal exchange rate determinacy in our model is not an artefact of assuming differences in exogenous features of transactions costs.) Moreover, when the two currencies are not perfect substitutes in equilibrium, an international currency can exist whereby one country has two currencies circulating while the other country uses only one. We also show that when there is nominal exchange rate indeterminacy, there exist fiscal policies that can restore determinacy of the nominal exchange rate. Another surprising result is that the first best may not be attainable even if the Friedman rule is implemented for both currencies. This is the case as the endogenous counterfeiting constraints may still bind such that the first best quantity of goods cannot be traded.

An interesting feature of our results is that there is no counterfeiting in equilibrium. It is the threat of counterfeiting that pins down the nominal exchange rate, and because of this feature, both currencies can circulate even though one of them is dominated in rate of return. This is interesting because empirical evidence suggests that observed counterfeiting of currencies is not a significant problem. This is the case as substantial resources and penalties are applied to those who counterfeit.⁴

of California, for engaging in Forex fraud involving approximately \$2 million in customer funds. In 2006, the Court ordered restitution and fines amounting to \$3.4 million.

⁴Counterfeiting currencies is a punishable criminal offence. Several law enforcement entities like INTERPOL, the United States Secret Service and Europol as well as the European Anti-Fraud Office (OLAF), European Central Bank, the US Federal Reserve Bank, and the Central Bank Counterfeit Deterrence Group provide forensic support, operational assistance and technical databases in order to assist countries in addressing counterfeit currency on a global scale. All these features and efforts substantially reduce the number of circulating counterfeited notes.

Judson and Porter (2010) estimated that the stock of US-dollar counterfeits in the whole world makes up about 0.01 of 1% of the total stock in terms of both note count, and also in terms of dollar value. We take this as a case in point that counterfeiting outcomes are virtually non-existent for the purposes of the theory.

However, our results show that even if counterfeiting outcomes are not important quantitatively, its threat is nevertheless of first-order importance for nominal exchange rate determination. This threat is taken seriously by governments as Central Banks around the world spend resources to prevent counterfeiting by incorporating several security features on fiat currencies. Finally we show that the introduction of credit enlarges the set of equilibria where nominal exchange rate is indeterminate.

2 Related literature

Models in mainstream international monetary economics typically pin down the value of a currency by imposing exogenous assumptions on what objects may be used as media of exchange. For instance, Stockman (1980) and Lucas (1982), among others, assume that in order to buy a good produced by a particular country, only that country's currency can be used. That is, in these environments, the demand for a specific fiat currency is solely driven by the demand for goods produced by that particular country. Devereaux and Shi (2013) study a trading post model under the assumption that there is only bilateral exchange at each trading post. Thus, by assumption, the ability to pay for goods with combinations of currencies is eliminated. Assumptions of this sort are *exogenous* currency constraints. By construction, they yield determinacy in agents' portfolio holdings of any two fiat currencies, and therefore determinacy in their nominal exchange rate.⁵ Other researchers have introduced local currency in the utility function as in Obstfeld and Rogoff (1984), or have assumed differential trading cost advantages through network externalities as in Uribe (1997), or having different terms of trade depending on the currency used when purchasing goods as in Nosal and Rocheteau (2011, Chapter 10), or different costly technologies to recognize currencies as in Zhang (2014). In short, *endogenous* currency choice effectively is assumed away.

In the early search theoretic models of money, agents are able to choose which currencies to accept and use for payment. This literature shows that multiple currencies can circulate even if one is dominated in rate of return and the nominal exchange rate is determinate [see Matsuyama, Kiyotaki and Matsui (1993), Zhou (1997) and Waller and Curtis (2003), Craig and Waller (2004), Camera, Craig and Waller (2004)]. In these models, currency exchange can occur in bilateral matches if agents' portfolios are overly weighted towards one currency or the other. In fact, this leads to a distribution of determinate exchange rates. However, these findings are driven solely by the decentralized nature of exchange, since agents never have access to a centralized market to rebalance their portfolios. Once agents have the ability to rebalance their currency holdings, be it by the large family assumption in Shi (1997) or the periodic centralized market structure in Lagos and Wright (2005), the curse of Kareken and Wallace rears its head. To get around the curse, Head and Shi (2003) consider an environment where the large household can hold a portfolio of currencies but individual buyers are constrained to hold only one currency. So although the household endogenously chooses a portfolio of currencies, bilateral exchange requires using

⁵In another strand of literature coined as the "New Open Economy Macroeconomics", which is partially summarized in Obstfeld and Rogoff (1996) and used extensively for monetary policy prescriptions, similar assumptions are in place.

one currency or the other, but not both simultaneously. In another paper, Liu and Shi (2010) assume that buyers can offer any currency but sellers can only accept one currency. Nosal and Rocheteau (2011), instead, adopt a trading mechanism in decentralized markets whereby a buyer obtains better terms of trade in a country by using the domestic money rather than the foreign one.⁶ The main contribution of our paper relative to this literature is that we include Walrasian markets with centralized exchange while assuming no exogenous restrictions on currency exchange nor differential trading protocols—and yet we can obtain nominal exchange rate determinacy, even when the currencies are perfect substitutes. All we require is a private information problem between buyers and sellers in decentralized exchanges.

The paper closest in spirit to ours is that of Zhang (2014), who considers an open economy search model with multiple competing currencies and governments that require transactions to be made in a local currency. Buyers can always costlessly produce counterfeit currencies while sellers face a recognizability problem, as in Lester, Postlewaite and Wright (2012). The recognizability problem is only in terms of foreign currencies so treating currencies asymmetrically. In order for sellers to detect counterfeits they have to purchase a counterfeit detection technology by incurring a fixed cost each period.⁷ Here, trade in bilateral matches occur under full information—it is common knowledge in a match whether the seller has invested in the detection technology and that sellers do not accept currencies they do not recognize. This allows for strategic complementarities so that multiple equilibria exist.⁸ Because producing a counterfeit when meeting an uninformed seller is a dominant strategy, unrecognizable fiat currency cannot be used as means of payment in a fraction of the matches where sellers do not have the relevant detection technology. Thus, in Zhang (2014) currency-choice outcomes and currency coexistence emerge at the extensive margin as possible equilibrium phenomena.

In contrast to Zhang (2014), we propose an environment that has an explicit private information problem and study its implications for the determination of nominal exchange rates. As a result, we can examine the seller’s decision to accept different fiat currencies at the intensive rather than the extensive margin. Monetary equilibria in our environment have the property that when the liquidity constraint of one currency binds, the marginal value of an additional unit of this currency is zero since the seller will not accept it. At the margin sellers will produce an extra unit of output only for the fiat currency that has a non-binding liquidity constraint. Thus, generally a buyer would offer—as payment for goods—the currency with the best rate of return. If the endogenous liquidity constraint is binding on the higher return currency, the buyer pays for additional units of the good with the lower return currency. This feature is critical in determining the properties of the nominal exchange rate. We can also allow for situations where the two currencies are perfect substitutes—i.e. when the cost of counterfeiting and the inflation rates of the two currencies are the same so that both currencies are treated symmetrically—and yet there is coexistence of the currencies and

⁶By doing so the authors are able to exploit all potential gains from trade as opposed to currency in advance models and determine the nominal exchange rate.

⁷These information costs try to reflect the costly nature of dealing with multiple currencies.

⁸This multiplicity is in the spirit of Rocheteau and Wright (2005) where there exist complementarities between sellers’ decision to enter and buyers’ choice of real balances.

determinacy of their nominal exchange rate.

3 Model

Consider a two-country model with two-sectors. In each country, there is a per-period sequential decentralized-then-centralized market (DM-CM) structure. Trading is anonymous in the DMs as in Lagos and Wright (2005) or Rocheteau and Wright (2005).⁹ In this environment a medium of exchange is essential. Moreover, DM sellers of goods face an asset recognizability problem as in Li, Rocheteau and Weill (2012). That is, sellers in this market face informational asymmetry regarding the quality of the fiat currencies to be exchanged for goods.¹⁰ These currencies are perfectly divisible and can grow at different constant rates. In the second sub-period, all activity occurs in a full information and frictionless international centralized market (CM).

DM production is specialized and agents take on fixed trader types so that they are either buyers (consumers) or sellers (producers) as in Rocheteau and Wright (2005).¹¹ In CM all agents can produce and consume an homogenous perishable good. Agents, in this market can trade the CM good and rebalance their currency portfolio. International trade and asset portfolio rebalancing occurs in the CM, as in a standard international monetary model. We can also interpret this model as one where there is a tradable- and a nontradable-goods sectors as in Gomis-Porqueras, Kam and Lee (2013).

General Description and Notation. Each country is labeled as either Home or Foreign and has a continuum of agents of measure 2. Time is discrete and indexed by $t \in \mathbb{N} := \{0, 1, 2, \dots\}$. Let a variable X (or X^f) denote an object produced in the Home (or Foreign) country, which is held by agents in the Home country. Denote X_\star (or X_\star^f) as an outcome in the Home (or Foreign) country but destined for use by Foreign agents.

At each date $t \in \mathbb{N}$ both countries have synchronized DM(t)-then-CM(t) sequence of submarkets. The sequence of information, markets and actions are as follows: (1) At the start of each date t , the Home-country DM(t) and the Foreign-country DM $_\star^f(t)$ open. Only the buyers and sellers located in each specific country can trade in its DM—i.e. there is no international trade across the two DM submarkets; (2) new monies are created (x_t for Home money and $x_{\star,t}^f$ for Foreign money) and transferred to the respective countries' DM-buyers; (3) matched DM-buyers and DM-sellers in each

⁹The search literature uses the term anonymity to encompass these three frictions: (i) no record-keeping over individual trading histories (“memory”), (ii) no public communication of histories and (iii) insufficient enforcement (or punishment). An environment with any of these frictions imply that credit between buyer and seller is not incentive compatible.

¹⁰We will also be referring to this as a *threat of counterfeiting* problem. Adkisson (2012) reports that the U.S. Commodities Futures Trading Commission was compelled to put out a great deal of warnings about trading in Iraqi Dinars. In some of the Dinar scams, the buyer does not actually get any Dinars, and instead a “certificate of ownership”. This latter situation has the flavor of the *threat* of counterfeiting of assets in our model.

¹¹The justification for this assumption is twofold. First, it allows for a simple description of production specialization as in Alchian (1977). Second, it allows us to abstract away from the additional role of money as a medium of insurance against buyer/seller idiosyncratic shocks. An instance of the latter can be found in the Lagos and Wright (2005) sort of environment.

country resolve their terms of trade, respectively $\omega := (q, d, d^f)$ and $\omega_\star := (q_\star, d_\star, d_\star^f)$, via take-it-or-leave-it (TIOLI) bargaining. Buyers have to signal the quality of their offers to sellers; and (4) the DMs close and agents from each country enter an international frictionless Walrasian CM.

In the CM, labor inputs (N and N_\star^f) are immobile across countries, but forward monetary claims and goods are. That is, in this market all agents from each country have access to the international CM to rebalance their asset portfolios and to trade in goods. Here, agents in each country work, consume and produce a homogeneous general good, where Home and Foreign agents will consume respectively C and C_\star^f . At the end of the CM, DM-buyers of each country make their portfolio and counterfeiting decisions, before they enter their own country's DM. We will expand on the detail of these events later. In what follows, we will describe the model environment from the perspective of the Home country. For each description of payoffs and constraints, there is a symmetric Foreign counterpart.

Preferences. Consider the Home country. (A symmetric description applies to Foreign agents' preferences.) Agents derive utility from DM and CM consumption and disutility from effort. A common discount factor $\beta \in (0, 1)$ applies to utility flows one period ahead. Given the specialization structure in DM, the (discounted) total expected utility of a DM-buyer is given by

$$\mathbb{E} \left\{ \sum_{t=0}^{\infty} \beta^t [u(q_t) + \mathcal{U}(C_t) - N_t] \right\}, \quad (1)$$

where q_t represents DM goods, N_t is the CM labor supply and C_t denotes consumption of perishable CM good.¹² Finally, \mathbb{E} is a linear expectation operator with respect to an equilibrium distribution of idiosyncratic agent types.¹³ The utility function $u : \mathbb{R}_+ \rightarrow \mathbb{R}$ is such that $u(0) = 0$, $u'(q) > 0$ and $u''(q) < 0$, for all $q \in \mathbb{R}_+$. Also, $\mathcal{U} : \mathbb{R}_+ \rightarrow \mathbb{R}$ has the property that $\mathcal{U}(0) = 0$, $\mathcal{U}'(C) > 0$, and $\mathcal{U}''(C) < 0$, for all $C \in \mathbb{R}_+$. We refer the reader to Supplementary Appendix A for the precise formulation of C as an index of tradable intermediate goods.

¹²Given common quasilinear preferences across countries, it will turn out in equilibrium that $C_t = C_{\star,t}^f$ for all dates. We can interpret C as an index of tradable intermediate goods, c and c^f , for the Home country's allocation, or, c_\star^f and c_\star , for the Foreign country. For example, we can assume that this index or aggregator is a function $D(\cdot)$ such that $C_t = D(c, c^f)$, and that the law of one price also holds for these intermediate goods. An example of $D(\cdot)$ is the Armington aggregator function used commonly in international trade models. This was used by Gomis-Porqueras, Kam and Lee, (2013) in a simpler quantitative international business cycle study.

Note that unlike Gomis-Porqueras, Kam and Lee (2013), here we do not need to specify the details of international trade in intermediate goods since this is not necessary for characterizing coexistence of fiat monies and nominal exchange rate determination. That is, these additional details merely introduce additional static relative prices and excess demand conditions for the tradable intermediate goods, and they do not change the co-existence and exchange rate determinacy results. This was also the same rationale of Kareken and Wallace (1981, p.210) for not explicitly modelling the aspect of international trade (in goods) in their model.

¹³The model has no aggregate random variables, and therefore, it will turn out that the equilibrium distribution of agent types will depend only on the idiosyncratic random-matching probability $\sigma \in (0, 1)$, and the equilibrium probabilities concerning the acceptability and genuineness of assets in an exchange, respectively, $\pi \in [0, 1]$ and $(\eta, \eta^f) \in [0, 1]^2$. The expected utility setup will be made more precise later.

The (discounted) total expected utility of a DM-seller is given by

$$\mathbb{E} \left\{ \sum_{t=0}^{\infty} \beta^t [-c(q_t) + \mathcal{U}(C_t) - N_t] \right\}, \quad (2)$$

where the utility cost function $c : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is such that $c(0) = 0$, $c'(q) > 0$, and $c''(q) \geq 0$. Note that DM-buyers and DM-sellers have identical per-period payoff functions in the CM sub-period, which is given by $\mathcal{U}(C) - N$, as both types of agents can consume and produce in this frictionless market.

Information and Trade. Since agents in DM have fixed types and production is specialized, agents face a double coincidence problem. Moreover, since buyers and sellers in DM are anonymous, the only incentive compatible form of payment is fiat money. Buyers and sellers have access to two distinct and divisible fiat currencies. Following Kareken and Wallace (1981), and in contrast to mainstream international macroeconomics, we do not impose any restrictions on which of the currencies, nor the compositions thereof, can be used to settle transactions.¹⁴ However, sellers face asymmetric information, as in Li, Rocheteau and Weill (2012), regarding the quality of the currencies when trading in DM.¹⁵ Finally, there is a technology that can detect and destroy counterfeits that circulate in the frictionless Walrasian CM. The precise information problem that buyers and sellers are facing will be described in more detail below.

3.1 Centralized Market

After trade occurs in DM, agents have access to a frictionless international Walrasian market (CM). At the beginning of each CM, DM-buyers have non-negative balances of both currencies which we denote by m_t and m_t^f . Before they make decisions and participate in the CM good, labor and asset markets, they receive a total lump-sum transfer of each currency denoted by x_t and x_t^f .¹⁶ Then, CM agents can trade goods and rebalance their portfolio of currencies.

After trade in the CM closes, but before matching in the DM begins, each buyer has the ability to counterfeit the different fiat currencies. The cost(s) of counterfeiting is common knowledge: For Home (Foreign) DM-buyers, this is assumed to be some per-period utility fixed cost of $\kappa^f > 0$ ($\kappa_\star^f > 0$) for the Foreign currency, and, $\kappa > 0$ ($\kappa_\star > 0$) for the Home fiat money. Superficially, one might conjecture that because of different counterfeiting costs, we would necessarily obtain the exchange rate determinacy result we are after. Such a conjecture is incorrect. As we show later,

¹⁴Notable exceptions are Zhang (2014) and Nosal and Rocheteau (2011).

¹⁵The observability of the portfolio in the decentralized market is not a critical issue. With fixed costs, one can consider both cases and obtain the same outcome. With proportional costs the buyer has strict incentives to reveal his portfolio. We refer to Li, Rocheteau and Weill (2012) for more on this issue.

¹⁶This assumption will do no harm to the results later since agent preferences are quasilinear, so that having transfers made to DM-sellers as well does not matter for the end result. Also note that who gets which seigniorage transfer—i.e., x or x^f —does not matter in this setting given quasilinear preferences and the timing of the CM transfer. This becomes apparent later from equation (7).

even if the cost parameters are identical there can still be equilibrium determinacy of the exchange rate.

When trading in DM, sellers are not able to distinguish between genuine and counterfeited currencies. However, following Nosal and Wallace (2007) and Li, Rocheteau and Weill (2012), we assume that a technology exists that can detect and destroy any fraudulent fiat currencies when trading in CM. Thus counterfeited currencies cannot be exchanged for CM goods in the frictionless international market.¹⁷

CM goods are produced with a linear technology that all agents have access to. Since the CM is a complete Walrasian market, a medium of exchange in this market is not essential. Agents choose CM labor, end-of-period currency portfolio and consumption of CM goods. Each agent in the Home country faces a sequential budget constraint given by

$$C_t \leq N_t - \phi_t[m_{t+1} - y_t] - \phi_t e_t[m_{t+1}^f - y_t^f], \quad (3)$$

where C_t denotes consumption of the CM good, and $y_t := m_t$ and $y_t^f := m_t^f$, respectively, are initial holdings of genuine each currency (including the transfers). The variable e_t is the current nominal exchange rate which measures the value of one unit of Foreign currency (f) in units of the Home currency, and, m_{t+1} and m_{t+1}^f are the end-of-period nominal balance of the respective fiat currencies. Finally, ϕ_t ($\phi_t e_t$) denotes the value of a unit of m_t (m_t^f) in terms of the CM good.

Given the sequential nature of markets in this environment, the DM-buyers' currency portfolio and counterfeiting decisions are dynamic. We will defer the discussion of agents' dynamic decision problems until the next section, and only after we have described the random matching and private information bargaining game between potential DM-buyers and DM-sellers. For now, we note that all DM-buyers will exit each CM with the same currency portfolio. Likewise, all DM-sellers will exit with $m_{t+1} = m_{t+1}^f = 0$.

In the next section we describe the one-sided private information bargaining game between a potentially matched buyer and seller in DM. This problem will span from the end of a period- t CM to the end of a period- $(t + 1)$ DM. Then we describe the dynamic decision problems of all agents and describe the monetary equilibrium.

3.2 Decentralized Market

Consider the DM sub-period where trade occurs through random bilateral matches. Below we describe the particulars of this frictional environment.

Matching. There are two fixed types of agents in the DM: buyers (b) and sellers (s). The measures of both b - and s -types are equal to 1. At the beginning of each period $t \in \mathbb{N}$, ex-ante anonymous buyers and sellers enter DM where they are randomly and bilaterally matched. With probability $\sigma \in (0, 1)$ each buyer is pairwise matched with a seller. Moreover, as agents are anonymous,

¹⁷This detecting of fraudulent currency is typically done by banks when clients deposit fiat currency into their accounts. The holder of these counterfeits has them removed (exchanged for nothing) from the economy.

exchange supported by contracts that promise repayment in the future is not incentive compatible. Therefore, agents trade just with fiat monies.

Feasible offers. Let $\omega := (q, d, d^f)$ denote the terms of trade that specifies how much a seller must produce in DM (q) in exchange for d^f units of the f currency and/or d units of the other currency. The particulars of the terms of trade ω is an outcome of a bargaining game with private information which we describe below. Denote the set of feasible buyer offers at each aggregate state (ϕ, e) as $\Omega(\phi, e) \ni \omega$.

Given DM preferences and technologies, the corresponding first-best quantity traded is $q^* \in (0, \infty)$ and satisfies $u'(q^*) = c'(q^*)$. For each aggregate state (ϕ, e) , there exist maximal finite and positive numbers $\bar{q} := \bar{q}(\phi, e)$, $\bar{m} := \bar{m}(\phi, e)$ and $\bar{m}^f := \bar{m}^f(\phi, e)$ solving $(\bar{m} + e\bar{m}^f)\phi = u(\bar{q}) = c(\bar{q})$, since $u(\cdot)$ and $c(\cdot)$ are monotone and continuous functions on every $[0, \bar{q}(\phi, e)]$. That is, the outcomes $(\bar{q}, \bar{m}, \bar{m}^f)(\phi, e)$ will be finite for every (ϕ, e) . Therefore, the set of all feasible offers $\Omega(\phi, e)$ at given (ϕ, e) , is a closed and bounded subset of \mathbb{R}_+^3 , where $\Omega(\phi, e) = [0, \bar{q}(\phi, e)] \times [0, \bar{m}(\phi, e)] \times [0, \bar{m}^f(\phi, e)]$. We summarize this observation in the lemma below.

Lemma 1 *For each given (ϕ, e) , the set of feasible buyer offers $\Omega(\phi, e) \subset \mathbb{R}_+^3$ is compact.*

Having specified the set of all possible offers that the buyer can feasibly make in each state of the economy, we now characterize the private information bargaining game.

3.3 Private Information

The DM-buyers' portfolio composition of genuine and fraudulent fiat currencies is private information, so the seller cannot distinguish between them. This private information problem is modeled as a signaling game between pairs of randomly matched buyers (signal sender) and sellers (signal receiver). The game is a one-period extensive form game played out in virtual time between each CM and the following period's DM.

A buyer has private information on his accumulation decision and holdings of the two fiat currencies. A matched seller can observe the terms of trade $\omega := (q, d, d^f)$ offered by the buyer but she is not able to distinguish between genuine and counterfeited currencies.¹⁸ In contrast to standard signalling games, here, signal senders have a choice over their private-information types. These types are defined by the buyer's portfolio choice at the end of each CM. If the buyer decides to counterfeit fiat currencies she will exchange them for DM goods as in the next CM they are going to be detected and destroyed. In what follows next, we first describe and characterize the equilibrium of the game.

¹⁸Implicit in our environment is that the seller can distinguish between the genuine f currency and the other genuine currency.

3.3.1 Endogenous-type Signalling Game

At the beginning of each DM, a seller s is randomly matched with a buyer b . The seller cannot recognize whether the buyer is offering genuine fiat currencies or not. Next we describe the exact timing of events.

Let $CM(t-1)$ denote the time- $(t-1)$ frictionless Walrasian market and $DM(t)$ represent the time- t decentralized and frictional market. One could also think in terms of a $CM(t)$ and its ensuing $DM(t+1)$, so the timing notation here does not affect the analysis. For every $t \geq 1$, and given prices, (ϕ_t, e_t) , the timing of the signalling game is as follows:

1. At the end of $CM(t-1)$ a buyer decides whether or not to costly counterfeit each currency at the one-period fixed costs of $\kappa > 0$ and $\kappa^f > 0$, respectively. This decision is captured by the binary action $\chi, \chi^f \in \{0, 1\}$ where $\chi = \chi^f = 0$ represents “no counterfeiting of currencies”.
2. The buyer chooses how much $CM(t-1)$ good to produce in exchange for genuine currencies, m and/or m^f .
3. In the subsequent $DM(t)$, a buyer is randomly matched with a seller with probability σ .¹⁹ Upon a successful match, the buyer makes a take-it-or-leave-it (TIOLI) offer (q, d, d^f) to the seller.²⁰
4. The seller decides whether to accept the offer or not. If the seller accepts, she produces according to the buyer’s TIOLI offer.

The extensive-form game tree of this private information problem is depicted in Figure 1.

As in Li, Rocheteau and Weill (2012), this original extensive-form game has the same payoff-equivalent reduced-form game as the following reverse-ordered extensive-form game:

1. A DM-buyer signals a TIOLI offer $\omega := (q, d, d^f)$ and commits to ω , before making any (C, N) decisions in $CM(t-1)$.
2. The buyer decides whether or not to counterfeit the fiat currencies, $\chi(\omega), \chi^f(\omega) \in \{0, 1\}$.
3. The buyer decides on portfolio $a(\omega) := (m, m^f)(\omega)$ and (C, N) .
4. The buyer enters $DM(t)$ and Nature randomly matches the buyer with a DM-seller with probability σ .
5. The DM-seller chooses whether to reject or accept the offer, $\alpha(\omega) \in \{0, 1\}$.

This reverse-ordered extensive-form game tree is depicted in Figure 2.

The reverse-ordered game helps refine the set of perfect Bayesian equilibria (PBE) that would arise in the original extensive form game. In and Wright (2011) provide sufficient conditions for

¹⁹For simplicity, double-coincidence-of-wants meetings occur with probability zero.

²⁰Implicit in the offer is the buyer signalling that the payment offered consists of genuine assets.

the existence of a PBE in an original extensive-form game which is outcome equivalent to the PBE of its simpler reordered game. Such an equilibrium is called a *Reordering-invariant Equilibrium* or RI-equilibrium.²¹

3.3.2 Players and Strategies

To simplify exposition, we let X represent X_t , X_{-1} correspond to X_{t-1} , and X_{+1} stand for X_{t+1} , for any date $t \geq 1$. In the next section we characterize the buyer and seller's strategies.

A DM-buyer in $\text{CM}(t-1)$ has individual state, $\mathbf{s}_{-1} := (y_{-1}, y_{-1}^f; \phi_{-1}, e_{-1})$ which is publicly observable in $\text{CM}(t-1)$. A DM-seller in $\text{CM}(t-1)$ is labelled as $\check{\mathbf{s}}_{-1} := (\check{y}_{-1}, \check{y}_{-1}^f; \phi_{-1}, e_{-1})$. Let $B(\phi, e) := [0, \bar{m}(\phi, e)] \times [0, \bar{m}^f(\phi, e)]$ denote the feasible currency portfolio choice set for a given aggregate state (ϕ, e) .

Definition 1 A pure strategy of a buyer, σ^s , in the counterfeiting game is a triple $\langle \omega, \aleph(\omega), a(\omega) \rangle$ comprised by the following:

1. Offer decision rule, $\mathbf{s}_{-1} \mapsto \omega \equiv \omega(\mathbf{s}_{-1}) \in \Omega(\phi, e)$;
2. Binary decision rules on counterfeiting, $\aleph := \langle \chi(\omega), \chi^f(\omega) \rangle \in \{0, 1\}$, for each currency; and
3. Asset accumulation decision, $\omega \mapsto a(\omega) \in B(\phi, e)$, and, $(d, d^f) \leq a(\omega)$.

A pure strategy of a seller σ^s is a binary acceptance rule $(\omega, \check{\mathbf{s}}_{-1}) \mapsto \alpha(\omega, \check{\mathbf{s}}_{-1}) \in \{0, 1\}$.

More generally, we allow players to play behavioral strategies given the buyer's posted offer ω . This is the case as quasilinearity in CM makes the buyer's payoff linear in (d, d^f) . This implies that taking a lottery over these payments yields the same utility $u(q)$. Thus, for notational convenience, we drop the lottery over offers when describing a buyer's behavior strategy $\tilde{\sigma}^b$.

Definition 2 A behavior strategy of a buyer $\tilde{\sigma}^b$ is a triple $\langle \omega, G[a(\omega)|\omega], H(\aleph|\omega) \rangle$, where

1. $H(\cdot|\omega) := \langle \eta(\cdot|\omega), \eta^f(\cdot|\omega) \rangle$ specifies marginal probability distributions over the $\{0, 1\}$ spaces of each of the two counterfeiting decisions $\aleph := (\chi, \chi^f)$; and
2. $G(\cdot|\omega)$ is a conditional lottery over each set of feasible asset pairs, $B(\phi, e)$.

A behavior strategy of a seller is $\tilde{\sigma}^s : \pi(\omega)$ which generates a lottery over $\{0, 1\} \ni \alpha$.

²¹See conditions A1-A3 in In and Wright (2011) for more details. Their characterization of equilibria is related to the Cho and Kreps (1987) Intuitive Criterion refinement, in the sense that both approaches are implied by the requirement of strategic stability (see Kohlberg and Mertens, 1986). However, the difference in the class of games considered by In and Wright (2011) to that of standard signalling games using Cho and Kreps, is that the class of games considered by the former admits signal senders who have an additional choice of a private-information action. That is, who chooses the private-information type—i.e. Nature in standard signalling games or a Sender in In and Wright (2011)—matters for the game structure. When a strategic and forward-looking Sender can choose his unobserved type, there will be additional ways he can deviate (but these deviations must be unprofitable in equilibrium). Thus standard PBE may still yield too many equilibria in these games with a signalling of private decisions. Further discussions are available in a separate appendix.

Finally, we note that buyers in each $CM(t-1)$ make the same optimal decisions in subsequent periods. This is the case as agents have CM quasilinear preferences so that history does not matter. Likewise, for the sellers' decisions. All agents, conditional on their DM-buyer or DM-seller types, have the same individual state after they leave CM. Therefore, characterizing the equilibrium of the counterfeiting-bargaining game between a matched anonymous buyer and seller pair in $DM(t)$ is tractable. Thus, we just can simply focus on the payoffs of any ex-ante DM-buyer and DM-seller.

3.3.3 Buyers' Payoff

Let $W^b(\cdot)$ denote the value function of a DM-buyer at the beginning of $CM(t)$. Since per-period CM utilities are quasilinear, the corresponding CM value function is linear in the buyer's individual state (m, m^f) so that

$$W^b(\mathbf{s}) \equiv W^b(y, y^f; \phi, e) = \phi(y + ey^f) + W^b(0, 0; \phi, e). \quad (4)$$

Let us define $Z(C_{-1}; \mathbf{s}_{-1}) = \mathcal{U}(C_{-1}) - C_{-1} + \phi_{-1}(y_{-1} + e_{-1}y_{-1}^f)$ which summarizes the $CM(t-1)$ flow utility from consuming $(C_{-1}, -N_{-1})$ plus the time- $(t-1)$ real value of accumulating genuine fiat currencies. Then given prices (ϕ, e) and his belief about the seller's behavior $\hat{\pi}$, the DM-buyer's Bernoulli payoff function, $U^b(\cdot)$, can be written as follows:²²

$$\begin{aligned} U^b(C_{-1}, \omega, \eta, \eta^f, G[a(\omega)|\omega], \hat{\pi}|\mathbf{s}_{-1}; \phi, e) = \\ \int_{B(\phi, e)} \left\{ Z(C_{-1}; \mathbf{s}_{-1}) - \phi_{-1}(m + e_{-1}m^f) - \kappa(1 - \eta) - \kappa^f(1 - \eta^f) \right. \\ \left. + \beta\sigma\hat{\pi} \left[u(q) + W^b(m - \eta d, m^f - \eta^f e d^f; \phi, e) \right] \right. \\ \left. + \beta[\sigma(1 - \hat{\pi}) + (1 - \sigma)] W^b(m, m^f; \phi, e) \right\} dG[a(\omega)|\omega]. \end{aligned} \quad (5)$$

Given the linearity of $W^b(\cdot)$, we can further reduce equation (5) to the following expression

$$\begin{aligned} U^b(C_{-1}, \omega, \eta, \eta^f, G[a(\omega)|\omega], \hat{\pi}|\mathbf{s}_{-1}; \phi, e) = -\kappa(1 - \eta) - \kappa^f(1 - \eta^f) \\ + \int_{B(\phi, e)} \left\{ Z(C_{-1}; \mathbf{s}_{-1}) - \left(\frac{\phi_{-1}}{\phi} - \beta \right) \phi m - \left(\frac{\phi_{-1}e_{-1}}{\phi e} - \beta \right) \phi e m^f \right. \\ \left. + \beta\sigma\hat{\pi} \left[u(q) - \phi(\eta d + \eta^f e d^f) \right] \right\} dG[a(\omega)|\omega] \end{aligned} \quad (6)$$

which corresponds to the expected total payoff under a given strategy $\tilde{\sigma}^b$ for a DM-buyer in $CM(t-1)$. Note that the first term of equation (6) is the expected total fixed cost of counterfeiting both currencies. The second term on the right of equation (6) is the utility flow from consuming $(C_{-1}, -N_{-1})$ and the $DM(t)$ continuation value from accumulating currencies in $CM(t-1)$. The third and fourth term are the expected total cost (equivalently inflation cost) of holding unused

²²We have imposed symmetry among all sellers for notational simplicity.

currencies between $CM(t-1)$ and $DM(t)$. The last term is the expected net payoff gain from trades in which the buyer pays for the good q with genuine currencies, with marginal probability measures $H(\omega) := (\eta, \eta^f)$, and the buyer believes a randomly encountered seller accepts with probability $\hat{\pi}$.

Finally, we still have to take into account the buyer's mixed strategy $G(\cdot|\omega)$. In Supplementary Appendix B we show that in a monetary equilibrium $G(\cdot|\omega)$ is always degenerate, so the buyer's total expected payoff in (6) further simplifies to

$$\begin{aligned} U^b[C_{-1}, \omega, H(\omega), \hat{\pi}|\mathbf{s}_{-1}; \phi, e] = \\ Z(\mathbf{s}_{-1}) - \left(\frac{\phi_{-1}}{\phi} - \beta \right) \phi m - \left(\frac{\phi_{-1}e_{-1}}{\phi e} - \beta \right) \phi e m^f \\ - \kappa(1 - \eta) - \kappa^f(1 - \eta^f) + \beta \sigma \hat{\pi} \left[u(q) - \phi \left(\eta d + \eta^f e d^f \right) \right]. \end{aligned} \quad (7)$$

Note that initial monetary wealth $y_{-1} := m_{-1}$ and $y^f := m_{-1}^f$ will not matter for the DM-buyers' CM decisions on the continuation portfolio of assets, (m, m^f) , given the linearity of the payoff function in these choices. The money demands carried by Home DM-buyers into a subsequent Home DM faces the feasibility constraints: $d \leq m + x$ and $d^f \leq m^f$. In other words, how much can be offered as payments in $DM(t)$ exchange are now bounded above by what Home DM-buyers have accumulated at the end of each $CM(t-1)$, respectively m and m^f , including the nominal transfer from the Home government x .

Symmetrically, the Foreign country's DM_\star^f -buyer's total expected lifetime payoff can be derived as

$$\begin{aligned} U^b[C_{\star-1}^f, \omega_\star^f, H(\omega_\star^f), \hat{\pi}_\star^f|\mathbf{s}_{-1}; \phi, e] = \\ Z_\star^f(\mathbf{s}_{-1}) - \left(\frac{\phi_{-1}}{\phi} - \beta \right) \phi m_\star - \left(\frac{\phi_{-1}e_{-1}}{\phi e} - \beta \right) \phi e m_\star^f \\ - \kappa_\star(1 - \eta_\star) - \kappa_\star^f(1 - \eta_\star^f) + \beta \sigma \hat{\pi}_\star^f \left[u(q_\star^f) - \phi \left(\eta_\star d_\star + \eta_\star^f e d_\star^f \right) \right], \end{aligned} \quad (7_\star)$$

with payment feasibility constraints $d_\star^f \leq m_\star^f + x_\star^f$ and $d_\star \leq m_\star$. Note that since x and x_\star^f are exogenous to DM-buyers in each country, they do not affect their marginal decisions, however, they will matter for the global adding-up conditions for monies supplied and demanded in equilibrium.

3.3.4 Sellers' Payoff

A DM-seller's payoff function is simpler. Let $W^s(\cdot)$ denote the seller's value function at the start of any CM. The seller also has a linear value function $W^s(\cdot)$ in currency holdings. Let $Z(\check{C}_{-1}; \check{\mathbf{s}}_{-1}) = \mathcal{U}(\check{C}_{-1}) - \check{C}_{-1} + \phi_{-1}(\check{m}_{-1} + e_{-1}\check{m}_{-1}^f)$ summarize the $CM(t-1)$ flow utility from consuming $(\check{C}_{-1}, -\check{H}_{-1})$ plus the time- $(t-1)$ real value of accumulating genuine currencies. Note that the DM-seller will always accumulate zero money holdings, because of the opportunity cost of holding currencies and the fact that she knows that she has no use of her money holdings in the ensuing DM.

Let $(\hat{\eta}, \hat{\eta}^f)$ be the seller's belief about the buyer's behavior with respect to counterfeiting of fiat

currencies. Given an offer ω , the seller belief system and the seller's response $\pi(\omega)$, her Bernoulli payoff for the game is given by

$$\begin{aligned}
U^s(\check{C}_{-1}, \omega, \hat{\eta}, \hat{\eta}^f, \pi(\omega) | \check{s}_{-1}; \phi, e) &= Z(\check{C}_{-1}; \check{s}_{-1}) \\
&+ \beta \sigma \pi(\omega) \left[-c(q) + W^s(\hat{\eta}d, \hat{\eta}^f d^f; \phi, e) \right] \\
&+ \beta [\sigma (1 - \pi(\omega)) + (1 - \sigma)] [-c(0) + W^s(0, 0; \phi, e)] \\
&= Z(\check{C}_{-1}; \check{s}_{-1}) + \beta \sigma \pi(\omega) \left[\phi \left(\hat{\eta}d + \hat{\eta}^f e d^f \right) - c(q) \right],
\end{aligned} \tag{8}$$

where the last equality is a direct consequence of linearity in the seller's CM value function: $W^s(\check{m}, \check{m}^f) = \phi(\check{m} + e\check{m}^f) + W^s(0, 0)$. The last term on the right of the payoff function (8) is the total discounted expected profit arising from the σ -measure of DM(t) exchange, in which the seller accepts an offer ω with probability $\pi(\omega)$ and she anticipates that the buyer pays with genuine assets according to beliefs $(\hat{\eta}, \hat{\eta}^f)$.

3.4 Equilibrium of the Private Information Game

The equilibrium concept for the counterfeiting-bargaining game is Perfect Bayesian in the reordered extensive-form game, as in Li, Rocheteau and Weill (2012). More precisely, we utilize the *RI-equilibrium* refinement proposed by In and Wright (2011). In order to solve the game we proceed by backward induction on the game depicted in Figure 2. Again, we focus on the events in the Home country as a similar conditions can be derived for the Foreign country.

3.4.1 Home DM-Seller's Problem

Following a (partially) private buyer history $\langle \omega, \aleph(\omega) \rangle$ in which an offer ω is observable and \aleph is not observable, the seller plays a mixed strategy π to maximize her expected pay off which is given by

$$\pi(\omega) \in \left\{ \arg \max_{\pi' \in [0,1]} \pi' \left[\phi \left(\hat{\eta}d + \hat{\eta}^f e d^f \right) - c(q) \right] \right\}. \tag{9}$$

3.4.2 Home DM-Buyer's Counterfeiting Problem

Given history ω and the buyer's belief about the seller's best response, $\hat{\pi}$, the buyer solves the following cost-minimization problem

$$\begin{aligned}
(\eta(\omega), \eta^f(\omega)) &= \arg \max_{\eta, \eta^f \in [0,1]} \left\{ -\kappa(1 - \eta) - \kappa^f(1 - \eta^f) - \beta \sigma \hat{\pi} \phi \left[\eta d + \eta^f e d^f \right] \right. \\
&\quad \left. - \left(\frac{\phi - 1}{\phi} - \beta \right) \phi m - \left(\frac{\phi - 1 e - 1}{\phi e} - \beta \right) \phi e m^f \right\}.
\end{aligned} \tag{10}$$

Given that the terms of trade in DM are given by the buyer's TIOLI offer at the beginning of the game, the buyer maximizes her payoff given her conjecture $(\hat{\eta}, \hat{\pi})$ of the continuation play, the

buyer commits to an optimal offer $\omega \equiv (q, \hat{d}, \hat{d}^f)$ which is given by

$$\omega \in \left\{ \arg \max_{\omega' \in \Omega(\phi, e)} \left\{ -\kappa(1 - \hat{\eta}) - \kappa^f(1 - \hat{\eta}^f) + \beta\sigma\hat{\pi} \left[u(q) - \phi(\hat{\eta}\hat{d} + \hat{\eta}^f e \hat{d}^f) \right] - \left(\frac{\phi_{-1}}{\phi} - \beta \right) \phi m - \left(\frac{\phi_{-1}e_{-1}}{\phi e} - \beta \right) \phi e m^f \right\} \right\}. \quad (11)$$

3.4.3 Equilibrium

Having specified the seller's and buyer's respective problems, we can now characterize the resulting equilibrium in the private-information bargaining game in each country. We show this characterization for the Home country. (Again, a symmetric description can be written out for the Foreign country.)

Definition 3 *A reordering-invariant (RI-) equilibrium of the original extensive-form game in the Home country is a perfect Bayesian equilibrium $\tilde{\sigma} := (\tilde{\sigma}^b, \tilde{\sigma}^s) = \langle \omega, \eta(\omega), \eta^f(\omega), \pi(\omega) \rangle$ of the re-ordered game such that (9) and (10) are satisfied.*

The following proposition provides a simple characterization of a RI-equilibrium in the game.

Proposition 1 (RI-equilibrium) *An RI-equilibrium of the counterfeiting-bargaining game is such that*

1. *Each seller accepts with probability $\hat{\pi} = \pi(\omega) = 1$;*
2. *Each buyer does not counterfeit: $(\hat{\eta}, \hat{\eta}^f) = (\eta(\omega), \eta^f(\omega)) = (1, 1)$; and*
3. *Each buyer's TIOLI offer ω is such that:*

$$\begin{aligned} \omega \in \left\{ \arg \max_{\omega \in \Omega(\phi, e)} \left[- \left(\frac{\phi_{-1}}{\phi} - \beta \right) \phi m - \left(\frac{\phi_{-1}e_{-1}}{\phi e} - \beta \right) \phi e m^f \right. \right. \\ \left. \left. + \beta\sigma \left[u(q) - \phi(d + e d^f) \right] \right] \right\} \quad s.t. \\ (\zeta): \quad \phi(d + e d^f) - c(q) \geq 0, \\ (\nu): \quad 0 \leq d, \\ (\mu): \quad d \leq m + x, \\ (\nu^f): \quad 0 \leq d^f, \\ (\mu^f): \quad d^f \leq m^f, \\ (\lambda): \quad \phi d \leq \frac{\kappa}{\phi_{-1}/\phi - \beta(1 - \sigma)} \equiv \bar{\kappa}(\phi_{-1}/\phi), \\ (\lambda^f): \quad \phi e d^f \leq \frac{\kappa^f}{\phi_{-1}e_{-1}/\phi e - \beta(1 - \sigma)} \equiv \bar{\kappa}^f(\phi_{-1}e_{-1}/\phi e) \left. \right\}. \end{aligned} \quad (12)$$

and the RI-equilibrium is unique.

The proof of this is given in Appendix C. Here we will give an intuitive sketch of this unique RI-equilibrium result, and in particular, we discuss why the beliefs in equilibrium are uniquely coordinated on the no-counterfeiting outcome. Consider the subgame, where a buyer has committed to an offer ω . The buyers' expected payoff for the continuation game is summarized by (10). Given a seller's belief (and probability of acceptance of the offer ω), $\hat{\pi}$, the buyer has to compare his expected cost of counterfeiting (the first two terms on the RHS in (10)), and his expected gain (the remainder terms of (10)). Given $\hat{\pi}$, the response of the buyer in choosing not to counterfeit with probability one, $(\eta, \eta^f) = (1, 1)$ is the best response, since if the buyer were to ever counterfeit with some probability, his expected profit can always be increased at the margin by reducing the probability on counterfeiting events. Given any lottery over pure counterfeiting actions by the buyer $(\hat{\eta}, \hat{\eta}^f)$, we can see immediately from the seller's problem in (9), that the seller's best response is to always accept with probability one, if there is positive probability that the buyer never counterfeits. In a Perfect Bayesian Equilibrium, we have that there is a positive probability of no counterfeiting, the seller accepts with probability one, and by backward induction, the buyer's maximal payoff is attained by pushing the lottery over counterfeiting all the way to zero weights on all counterfeiting events.

As we can see from the RI-equilibrium, ζ represents the Lagrange multiplier associated with the seller's participation constraint, ν (ν^f) is the Lagrange multiplier corresponding to the non-negativity of the payments in the f currency and the other fiat object. Finally, μ^f (μ) represents the feasibility constraint for $m^f(m)$, λ (λ^f) is the Lagrange multiplier corresponding to the liquidity constraint for the f currency (the other fiat object) that arise because of the threat of counterfeiting.

It is important to highlight that the last two constraints are *endogenous* liquidity constraints in that they provide an upper bound on the quantities of *genuine* currencies that the seller will accept in exchange for the specialised good. These endogenous constraints are robust to different trading protocols under similar environments.²³

The upper bounds resulting from the private information problem depend positively on the fixed cost of counterfeiting and negatively on the degree of matching efficiency σ . Note that greater matching efficiency in the DM implies that buyers and sellers are more likely to meet and trade. This creates a larger incentive for the buyer to produce counterfeits, thus increasing the information problem. As a result, in equilibrium, in order for sellers to accept buyers' offers, each buyer has a tighter upper-bound on his signal/offer of DM payment. The same logic applies to the effect of the fixed costs of counterfeiting, and, also to the effect of the aggregate returns on holding *genuine* currencies.

A critical feature of these liquidity constraints is that the marginal liquidity value of an additional unit of currency beyond the upper bound is zero. Thus, if one currency has a higher rate of return (lower inflation rate) but a lower counterfeiting cost, then the buyer will first pay with it up to the

²³Shao (2014) shows that in a modified version of Li, Rocheteau and Weil (2012) with competitive search where sellers set the terms of trade, a similar equilibrium liquidity constraint arises. Also, Berentsen, McBride and Rocheteau (2014) propose a way to extend the methodology in Li, Rocheteau and Weil (2012) to the case with proportional bargaining.

bound and use the weaker currency to pay for the remainder of the goods purchased.

3.5 Money supplies and seigniorage transfers

We assume that the supply of the fiat monies, M_t and M_t^f , respectively, grow at a constant rate of γ and γ^f . Lump sum (seigniorage revenue) transfers of Home and Foreign monies, respectively $x_t = M_{t+1} - M_t = (\gamma - 1)\gamma^{t-1}M_0$ and $x_t^f = M_{t+1}^f - M_t^f = (\gamma^f - 1)(\gamma^f)^{t-1}M_0^f$, are made to the respective country's DM-buyers at the beginning of each DM (or end of CM).²⁴ The initial stocks M_0 and M_0^f are known.

4 Monetary Equilibrium

We can now embed the equilibrium characterization of the games in both Home and Foreign into the monetary equilibrium of the two-country model. Since preferences are quasilinear, the infinite history of past games between buyers and sellers does not matter for each current period agents' decision problems. This allows us to tractably incorporate the equilibrium characterization of the game previously described, into the overall dynamic general monetary setting. Before we do so, we return to describing the Home agents' dynamic decision problems.

4.1 Home Agents' Recursive Problems

DM-buyers' Problem As we previously saw, the beginning-of-CM value function for Home DM-buyers, $W^b(\cdot; \phi, e)$, is linear in the fiat currency portfolio (y, y^f) . As a result, the buyer's intertemporal problem, conditional on an equilibrium of the private-information bargaining game, is given by

$$\max_{C_{-1}, q, d, d^f, m, m^f} U^b(C_{-1}, \omega, \eta(\omega), \hat{\pi}|\mathbf{s}_{-1}; \phi, e) \quad \text{s.t.}$$

²⁴Observe that no transfers are made to DM-sellers as they would have no use for it in the immediate DM. These DM-sellers would just bring the additional money transfer with zero inflation cost into the CM, since this occurs within the same period. In turn, they can afford to work less in the CM without altering their consumption allocation since they have quasilinear preferences. Thus it is without loss to our result that we assume that transfers are made to DM-buyers only. Also, if we were to assume that seigniorage revenues were transferred to agents at the start of the CM, the effect of money supply growth would just wash out in the CM in terms of having any real effects since with quasilinear preferences, all agents would just work less in the CM.

$$(\eta(\omega), \eta^f(\omega)) = (1, 1), \quad \hat{\pi} = \pi(\omega) = 1, \quad (13a)$$

$$(\zeta) : \quad \phi(d + ed^f) - c(q) = 0, \quad (13b)$$

$$(\nu) : \quad 0 \leq d, \quad (13c)$$

$$(\mu) : \quad d \leq m + x, \quad (13d)$$

$$(\nu^f) : \quad 0 \leq d^f, \quad (13e)$$

$$(\mu^f) : \quad d^f \leq m^f, \quad (13f)$$

$$(\lambda) : \quad \phi d \leq \bar{\kappa}(\phi_{-1}/\phi), \quad (13g)$$

$$(\lambda^f) : \quad \phi ed^f \leq \bar{\kappa}^f(\phi_{-1}e_{-1}/\phi e). \quad (13h)$$

where the DM-buyer's lifetime expected payoff is given by

$$\begin{aligned} U^b(C_{-1}, \omega, \eta(\omega), \eta^f(\omega), \hat{\pi}|\mathbf{s}_{-1}; \phi, e) \\ = \mathcal{U}(C_{-1}) - C_{-1} + \phi_{-1}(y_{-1} + e_{-1}y_{-1}^f) - \left(\frac{\phi_{-1}}{\phi} - \beta\right)\phi m \\ - \left(\frac{\phi_{-1}e_{-1}}{\phi e} - \beta\right)\phi em^f + \beta\sigma \left[u(q) - \phi(d + ed^f)\right]. \end{aligned} \quad (14)$$

In contrast to a full information setting, the threat of counterfeits which is private information to buyers, introduces additional endogenous state-dependent liquidity constraints (13g)-(13h) into a buyer's Bellman equation problem. These endogenous liquidity constraints are going to play an important role in determining the coexistence of the two currencies and the determinacy of nominal exchange rates.

The corresponding first order conditions of the rest of the Home DM-buyers' dynamic decision problem, given the RI-equilibrium, are given by

$$1 = \mathcal{U}'(C), \quad (15)$$

$$0 = \beta\sigma u'(q) - \zeta c'(q), \quad (16)$$

$$\beta\sigma = \zeta + \nu - \mu - \lambda, \quad (17)$$

$$\beta\sigma = \zeta + \nu^f - \mu^f - \lambda^f, \quad (18)$$

$$\mu = \frac{\phi_{-1}}{\phi} - \beta, \quad (19)$$

$$\mu^f = \frac{\phi_{-1}^f}{\phi^f} - \beta. \quad (20)$$

$$\zeta \geq 0, \nu \geq 0, \nu^f \geq 0, \mu \geq 0, \mu^f \geq 0, \lambda \geq 0, \lambda^f \geq 0, \quad (21)$$

for every date $t \geq 1$.

Note that Equation (15) describes the optimal within-period labor versus consumption trade-

off in CM, where the marginal disutility of labor is -1 and the real-wage (marginal product of labor) is 1. Equation (16) corresponds to the first order condition for DM output which equates the marginal benefit of consuming and the marginal value of the payment to the seller. Since the buyer makes a TIOLI offers, the payment is equal to the seller's DM production cost. Equations (17) and (18) summarize the optimal choice with respect to the two nominal payments and equate the value of holding a particular fiat currency from one CM to the next versus trading it in DM. Finally, equations (19) and (20) describe the optimal accumulation of each currency which of course depends on their implied rate of return. Equations (17) and (19) (or (18) and (20)) imply a sequence of intertemporal consumption Euler inequalities, where one or both currencies are used as store of value.

Home DM-sellers' Problem A DM-seller's problem, embedding the game's equilibrium, is simpler as sellers cannot counterfeit. The Home DM-sellers lifetime payoff function is given by

$$\max_{C_{-1}} U^s(\check{C}_{-1}, \omega, \eta(\omega), \eta^f(\omega), \hat{\pi}|\check{s}_{-1}; \phi, e) \quad \text{s.t.} \quad (\eta(\omega), \eta^f(\omega)) = (1, 1) \quad \text{and} \quad \hat{\pi} = 1;$$

where each seller's Bernoulli payoff is given by

$$\begin{aligned} U^s(\check{C}_{-1}, \omega, \eta(\omega), \eta^f(\omega), \hat{\pi}|\check{s}_{-1}; \phi, e) \\ = \mathcal{U}(\check{C}_{-1}) - \check{C}_{-1} + \phi_{-1}(\check{m}_{-1} + e_{-1}\check{m}_{-1}^f) + \beta\sigma \left[\phi \left(d + ed^f \right) - c(q) \right]. \end{aligned} \quad (22)$$

A similar function can be written down for their Foreign counterpart.

4.2 Steady State Monetary Equilibrium

We now focus on steady-state monetary equilibria where the nominal exchange rate can grow at a constant rate. In fact, if we consider monetary equilibria where both monies circulate, we have the following intermediate observation.²⁵

Proposition 2 *Assume the existence of a monetary equilibrium where both monies circulate. When there is no portfolio restriction on what currencies must serve as a medium of exchange in any country, the equilibrium nominal exchange rate growing in absolute terms at a constant and bounded rate γ_e —i.e., $|(e_{t+1} - e_t)/e_t| = \gamma_e \in [0, +\infty)$, for all $t \geq 0$ —is a (deterministic) monetary equilibrium property.*

For the rest of the paper we thus focus on monetary equilibria in which the equilibrium nominal exchange rate grows at some constant rate (possibly zero). We can now study the implications of the endogenous liquidity constraints for the coexistence of multiple fiat currencies. This also allows us to understand under what conditions there is determinacy of the nominal exchange rate.

²⁵We relegate the proof to Supplementary Appendix D.

Define stationary variables by taking ratios of growing variables as follows $\frac{M}{M_{-1}} = \gamma = \Pi \equiv \frac{\phi_{-1}}{\phi}$; $\frac{M^f}{M_{-1}^f} = \gamma^f = \Pi^f \equiv \frac{\phi_{-1}^f}{\phi^f}$. In steady state, all real quantities are constant implying $\phi M = \phi_{-1} M_{-1}$, and, $e \phi M^f = e_{-1} \phi_{-1} M_{-1}^f$, which yields the steady state home currency (gross) depreciation/appreciation as

$$\frac{e}{e_{-1}} = \frac{\gamma}{\gamma^f} = \frac{\Pi}{\Pi^f}. \quad (23)$$

After some algebra, the necessary conditions for such an equilibrium is given by the Home DM-buyers' Euler conditions and the DM-sellers' participation constraint, respectively,

$$\beta \sigma \left[\frac{u'(q)}{c'(q)} - 1 \right] = \lambda - \nu + (\Pi - \beta) = \lambda^f - \nu^f + (\Pi^f - \beta), \quad (24a)$$

and,

$$c(q) = \phi d + \phi^f d^f, \quad (24b)$$

where λ (λ^f) is the Home liquidity constraint on Home (Foreign) real money balance, and ν (ν^f) is the non-negativity constraint on d (d^f). The resulting Karush-Kuhn-Tucker (KKT) conditions are:

$$\begin{aligned} \lambda \cdot [\bar{\kappa}(\Pi) - \phi d] &= 0, & \lambda &\geq 0, & \phi d &\leq \bar{\kappa}(\Pi), \\ \lambda^f \cdot [\bar{\kappa}^f(\Pi^f) - \phi^f d^f] &= 0, & \lambda^f &\geq 0, & \phi^f d^f &\leq \bar{\kappa}^f(\Pi^f), \\ -\nu \cdot d &= 0, & d &\geq 0, & \nu &\geq 0, \\ -\nu^f \cdot d^f &= 0, & d^f &\geq 0, & \nu^f &\geq 0, \end{aligned} \quad (24c)$$

where $\bar{\kappa}(\Pi) := \kappa/[\Pi - \beta(1 - \sigma)]$ and $\bar{\kappa}^f(\Pi^f) := \kappa^f/[\Pi^f - \beta(1 - \sigma)]$.

Similarly, we have the Foreign block as

$$\beta \sigma \left[\frac{u'(q_\star^f)}{c'(q_\star^f)} - 1 \right] = \lambda_\star - \nu_\star + (\Pi - \beta) = \lambda_\star^f - \nu_\star^f + (\Pi^f - \beta), \quad (25a)$$

and,

$$c(q_\star^f) = \phi^f d_\star^f + \phi d_\star, \quad (25b)$$

where λ_\star (or λ_\star^f) is the Foreign liquidity constraint on Home (Foreign) real money balance, and ν_\star (or ν_\star^f) is the non-negativity constraint on d_\star (or d_\star^f). The KKT conditions are

$$\begin{aligned} \lambda_\star \cdot [\bar{\kappa}_\star(\Pi) - \phi d_\star] &= 0, & \lambda_\star &\geq 0, & \phi d_\star &\leq \bar{\kappa}_\star(\Pi), \\ \lambda_\star^f \cdot [\bar{\kappa}_\star^f(\Pi^f) - \phi^f d_\star^f] &= 0, & \lambda_\star^f &\geq 0, & \phi^f d_\star^f &\leq \bar{\kappa}_\star^f(\Pi^f), \\ -\nu_\star \cdot d_\star &= 0, & d_\star &\geq 0, & \nu_\star &\geq 0, \\ -\nu_\star^f \cdot d_\star^f &= 0, & d_\star^f &\geq 0, & \nu_\star^f &\geq 0, \end{aligned} \quad (25c)$$

where $\bar{\kappa}_\star(\Pi) := \kappa_\star/[\Pi - \beta(1 - \sigma)]$ and $\bar{\kappa}_\star^f(\Pi^f) := \kappa_\star^f/[\Pi^f - \beta(1 - \sigma)]$, with $\kappa_\star > 0$ and $\kappa_\star^f > 0$. In the two-country setting, the following adding-up conditions for Home and Foreign monies must hold between demand for real balances across countries and their supplies. Respectively, these are

$$\phi d_\star = \phi(M - d), \quad (26a)$$

and,

$$\phi d_\star^f = \phi^f(M^f - d^f). \quad (26b)$$

The following is our main result on coexistence and nominal exchange rate determinacy. In the interest of space, we omit the symmetric case where the foreign fiat money is dominated in rate of return (i.e. $\Pi < \Pi^f$).

Proposition 3 (Monetary Coexistence and Exchange Rates) *Given monetary policies γ and γ^f :*

Case 1 When Home fiat money is dominated in rate of return ($\Pi > \Pi^f$), there exists a monetary equilibrium with a determinate nominal exchange rate with

- the two currencies coexisting in only one country when some liquidity constraints bind; or*
- both currencies circulating in the Home and Foreign countries when all liquidity constraints, or at least those on the higher-return currency, bind in both countries.*

Case 2 When Home fiat money dominates in rate of return ($\Pi^f > \Pi$), the coexistence results are the symmetric opposite to those of Case 1.

Case 3 When Home fiat money has the same rate of return as the Foreign currency ($\Pi = \Pi^f$), there exist a monetary equilibrium with a determinate nominal exchange rate. In this equilibrium, both currencies circulate in the Home and Foreign countries when all liquidity constraints bind in both countries.

The details of these possible equilibrium configurations are summarized in detail in Table 1.

There are more than one configuration in which both currencies circulate in both countries, in spite of the fact that one currency is dominated in rate of return—see Cases 1(b)(7), (11) and (16) in Table 1 (of Proposition 3). In these scenarios, it is always the case that the higher return currency is liquidity constrained from the perspective of DM-buyers in both countries' DMs.

Finally, consider case 1(b)(16) from Table 1, or Case 3(a)(1), when the two currencies are identical in terms of their returns, i.e., $\Pi = \Pi^f$. Further, suppose that $\kappa = \kappa^f$ so both currencies are perfect substitutes. This scenario allows us to show that the coexistence result is not an artefact of assuming different counterfeiting costs as parametrized by differential κ and κ^f . In other words,

Table 1: Equilibrium cases in two-country model (Proposition 3).

Equilibrium Cases ¹	Liquidity constraints ² ($\lambda, \lambda^f, \lambda_\star^f, \lambda_\star$)	Circulation ³ ($m, m^f, m_\star^f, m_\star$)	Exchange rate ⁴ e
1(a): $\Pi > \Pi^f$	(1): (0, 0, 0, 0)	(0, +, +, 0)	n/a
	(2): (+, 0, 0, 0)	(0, +, +, 0)	n/a
	(3): (0, 0, 0, +)	(0, +, +, 0)	n/a
	(4): (+, 0, 0, +)	(0, +, +, 0)	n/a
1(b): $\Pi > \Pi^f$	(5): (+, +, 0, 0)	(+, +, +, 0)	$\frac{\bar{\kappa}^f + c(q_\star^f)}{\bar{\kappa}} \frac{M}{M^f}$
	(6): (0, +, 0, 0)	(+, +, +, 0)	$\frac{-\bar{\kappa}^f + c(q_\star^f)}{c(q) - \bar{\kappa}^f} \frac{M}{M^f}$
	(7): (0, +, +, 0)	(+, +, +, +)	$\frac{\bar{\kappa}^f + \bar{\kappa}_\star^f}{c(q) + c(q_\star^f) - [\bar{\kappa}^f + \bar{\kappa}_\star^f]} \frac{M}{M^f}$
	(8): (0, +, 0, +)	(+, +, +, 0)	$\frac{\bar{\kappa}^f + c(q_\star^f)}{c(q) - \bar{\kappa}^f} \frac{M}{M^f}$
	(9): (+, 0, +, 0)	(0, +, +, +)	$\frac{\bar{\kappa}^f - c(q)}{c(q_\star^f) - \bar{\kappa}^f} \frac{M}{M^f}$
	(10): (+, +, 0, +)	(+, +, +, 0)	$\frac{\bar{\kappa}^f + c(q_\star^f)}{\bar{\kappa}} \frac{M}{M^f}$
	(11): (+, +, +, 0)	(+, +, +, +)	$\frac{\bar{\kappa}^f + \bar{\kappa}_\star^f}{c(q_\star^f) - \bar{\kappa}_\star^f + \bar{\kappa}^f} \frac{M}{M^f}$
	(12): (0, +, +, +)	(+, +, +, +)	$\frac{\bar{\kappa}^f + \bar{\kappa}_\star^f}{c(q_\star^f) + \bar{\kappa}_\star^f - \bar{\kappa}^f} \frac{M}{M^f}$
	(13): (+, 0, +, +)	(0, +, +, +)	$\frac{\bar{\kappa}_\star^f + c(q)}{\bar{\kappa}_\star^f} \frac{M}{M^f}$
	(14): (0, 0, +, +)	(0, +, +, +)	$\frac{\bar{\kappa}_\star^f + c(q)}{\bar{\kappa}_\star^f} \frac{M}{M^f}$
	(15): (0, 0, +, 0)	(+, +, +, 0)	$\frac{\bar{\kappa}_\star^f + c(q)}{c(q_\star^f) + \bar{\kappa}_\star^f} \frac{M}{M^f}$
	(16): (+, +, +, +)	(+, +, +, +)	$\frac{\bar{\kappa}_\star^f + \bar{\kappa}^f}{\bar{\kappa}_\star + \bar{\kappa}} \frac{M}{M^f}$
3(a): $\Pi = \Pi^f$	(1): (+, +, +, +)	(+, +, +, +)	$\frac{\bar{\kappa}_\star^f + \bar{\kappa}^f}{\bar{\kappa}_\star + \bar{\kappa}} \frac{M}{M^f}$
3(c): $\Pi = \Pi^f$	(14): (0, 0, 0, 0)	(+, +, +, +)	$\in [0, +\infty)$
	(15): (+, +, 0, 0)	(+, +, +, +)	$\in [0, +\infty)$
	(16): (0, 0, +, +)	(+, +, +, +)	$\in [0, +\infty)$

Notes:

1. Equilibria under Case 2 (not shown) are symmetrical opposites to Case 1. In the proof to Proposition 3 in Appendix E, Case 3(b), not shown above, comprise of configurations that cannot exist in an equilibrium.
2. This refers to the Lagrange multipliers on respective endogenous liquidity constraints on both countries, for both countries. For example, $(\lambda, \lambda^f, \lambda_\star^f, \lambda_\star) \equiv (0, +, +, 0)$ refer to only the liquidity constraints on holding/paying with the foreign currency binding, for both Home DM-buyers and Foreign DM_★-buyers.
3. Circulation refers to steady-state equilibrium circulation of currencies in each country. For example, $(m, m^f, m_\star^f, m_\star) \equiv (0, +, +, 0)$ refers to only the Foreign currency circulating in both countries.
4. The equilibrium nominal exchange rate in each possible steady-state equilibrium is either not applicable (n/a), determinate and finitely positive-valued, or indeterminate in the set $[0, +\infty)$. We use abbreviations $\bar{\kappa} := \bar{\kappa}(\Pi)$, $\bar{\kappa}^f := \bar{\kappa}^f(\Pi^f)$, $\bar{\kappa}_\star^f := \bar{\kappa}_\star^f(\Pi^f)$, and, $\bar{\kappa}_\star := \bar{\kappa}_\star(\Pi)$.

our approach here should not be compared with an older literature that assumes differential and reduced-form transactions costs to using different currencies.

As we can see, when both currencies are perfect substitutes as payment instruments, and yet the nominal exchange rate is determinate and equal to the ratio of the two money stocks. This is observationally equivalent to an outcome from a standard exchange rate solution in a symmetric, two-country cash-in-advance (CIA) model as in Stockman (1980) and Lucas (1982). While one may be tempted to say that we have provided a ‘micro-foundation’ for the two-country CIA model, this would be incorrect for two reasons. First, our result only holds for a limited set of parameter values. Second, in the standard CIA model only one of the currencies is used per transaction (by assumption), whereas here both currencies are used in the same transaction. Thus, equivalent exchange rate solutions should not be confused with equivalent results elsewhere. Nevertheless, it is interesting to note that it can be derived under certain conditions.

The key point of this Proposition is that, although there is no counterfeiting in equilibrium, the threat of counterfeiting is all that is required to generate the coexistence and determinacy of the nominal exchange rate as long as some of the liquidity constraints bind. This is true even if the currencies are equivalent in all respects; i.e., the inflation rates and the counterfeiting costs are the same. Thus we have broken the curse of Kareken and Wallace.

Proposition 3 also contemplates the possibility of just one currency circulating. It is important to highlight that this Proposition describes equilibria that have the property that when the liquidity constraint binds, the marginal value of an additional unit of this currency is zero since the seller will not accept it. At the margin DM-sellers will produce an extra unit of output only for the fiat currency that has a non-binding liquidity constraint. Under these circumstances, the buyer first offers the currency with the best rate of return. Once the endogenous liquidity constraint is binding, the buyer pays for additional units of the DM good with the lower return currency. This intensive margin is key in generating the results in Proposition 3.

Moreover, from Proposition 3 (or Table 1), we also have equilibria in which a single international currency can emerge alongside a local currency. That is, one currency has global circulation, whereas the other currency only has local circulation.²⁶ In our setting, this situation can only be obtained when the two currencies are, in equilibrium, not perfect substitutes in particular ways that share one common feature: The internationally circulating currency is the higher return money but it is liquidity constrained in at least one country. The latter necessitates the use of the inferior-return currency as a “top-up payment” in equilibrium exchange.

In addition to breaking the curse of Kareken and Wallace, we also establish some results regarding efficiency.

Proposition 4 (First Best) *When both inflation rates are at the Friedman rule, $\Pi = \Pi^f = \beta$, the first best quantity q^* **may not** be attainable, and the nominal exchange rate may not be determinate.*

²⁶Derereux and Shi (2013) and Zhang (2014) also obtain this type of equilibria by exploiting a network externalities and strategic complementarities, respectively.

To demonstrate this result, it suffices to provide a counter-example to the claim that the Friedman rule is always optimal. Indeed we show that when there is the threat of counterfeiting, the Friedman Rule may no longer be able to achieve the first best as each DM-seller is not willing to produce more output than what can be afforded by a DM-buyer faced with binding endogenous liquidity constraints. Note that in environments without private information nor bargaining inefficiencies, resulting from Nash Bargaining, the Friedman rule is able to correct for the intertemporal distortion and achieve first best allocations as it does not distort the saving decisions of buyers.²⁷ However, in this environment having the highest rate of return on fiat currency is not enough as sellers have an upper bound of how much currency they are willing to accept. This liquidity constraint is a direct consequence of the counterfeiting problem they face. As a result, it is possible that sellers do not accept currencies consistent with first best production even at the Friedman rule.

5 Fixed Costs, Inflation and Existence

In this section we explore the impact of the various parameters of the model on the determination of nominal exchange rates. Given the large number of parameters associated with the two-country model, here we consider a special case of the two-country model that retains the essence of our main result on coexistence of monies and exchange rate determinacy. Suppose we consider an integrated world economy where agents can also move between the DMs in each country. We now have an international CM with no geographical restriction on the trade of labor, as is the case for goods. In the DM, agents trade anonymously, and the DM-buyers still need to signal to DM-sellers that their monies—in either or both currencies—are genuine. There are still two independent monopoly suppliers of the two monies.²⁸ Also, without loss of insight, we assume that $\kappa = \kappa_\star$ and $\kappa^f = \kappa_\star^f$, and seigniorage transfers of both currencies are distributed uniformly. As a result, in the resulting equilibrium we do not have to distinguish between quantities allocated to Home or Foreign agents. With this special setting, we can illustrate how the fixed costs (κ and κ^f) consistent with coexistence equilibria depend on (relative) inflation (Π^f), market/matching friction (σ), DM-buyers' intertemporal elasticity of substitution (parameterized as $1/\theta := u'(q)/u''(q) \cdot q$) and the convexity of sellers' production cost (parametrized as α).

The following proposition retains the insight of Proposition 3:

Proposition 5 (Equilibria and Coexistence) *Depending on the relative inflation rates of the two fiat currencies, there are three cases characterizing a steady-state monetary equilibrium:*

1. *When currency f dominates in rate of return ($\Pi > \Pi^f$) and*
 - (a) *when neither liquidity constraints bind ($\lambda = \lambda^f = 0$), or when only the liquidity constraint on the dominated fiat currency binds ($\lambda > 0, \lambda^f = 0$), then a monetary equilibrium exists with the unique outcome where only the low inflation currency circulates; or,*

²⁷For more on the distortions induced by Nash bargaining and how monetary and fiscal policy can restore the first best we refer to Gomis-Porqueras et al (2010).

²⁸This specialization does not detract from the explicit two-country model earlier.

- (b) the liquidity constraint on currency f binds ($\lambda^f > \lambda = 0$), then there exists a monetary equilibrium with a unique outcome where the currencies coexist and the nominal exchange rate is determinate

$$e = \frac{M}{M^f} \frac{\kappa^f}{c(q) [\Pi^f - \beta(1 - \sigma)] - \kappa^f};$$

where q solves

$$\frac{\Pi - \beta}{\sigma\beta} = \frac{u'(q) - c'(q)}{c'(q)}.$$

- (c) both liquidity constraints bind ($\lambda^f > 0, \lambda > 0$), then there exists a unique monetary equilibrium where the currencies coexist and the nominal exchange rate is determinate

$$e = \frac{\kappa^f M}{\kappa M^f} \frac{\Pi - \beta(1 - \sigma)}{\Pi^f - \beta(1 - \sigma)}.$$

2. When currency f is dominated in rate of return ($\Pi^f > \Pi$), the coexistence results are the symmetric opposite to those of Case 1.
3. When the currencies have the same rate of return ($\Pi^f = \Pi$) and

- (a) neither liquidity constraints bind ($\lambda = \lambda^f = 0$), then the two fiat monies coexist but the individual's currency portfolio composition and the nominal exchange rate are indeterminate;
- (b) both liquidity constraints bind ($\lambda = \lambda^f > 0$), then the currencies coexist. The individual's currency portfolio composition is unique, and thus there is a unique nominal exchange rate

$$e = \frac{\kappa^f M}{\kappa M^f}.$$

In Case 1a, only currency f is used as a medium of exchange. However, in Case 1b, we know that there is a unique \hat{q} that solves $\beta\sigma[u'(q)/c'(q) - 1] = \Pi - \beta$. It is straightforward to arrive at the following set of inequalities that define feasible counterfeiting costs (κ, κ^f) consistent with equilibrium coexistence of the two monies in Case 1b. These are given by:

$$\kappa^f \cdot \frac{[\Pi - \beta(1 - \sigma)]}{\Pi^f - \beta(1 - \sigma)} \leq [\Pi - \beta(1 - \sigma)]c(\hat{q}) < \kappa^f \cdot \frac{\Pi - \beta(1 - \sigma)}{\Pi^f - \beta(1 - \sigma)} + \kappa. \quad (27)$$

The first weak inequality is obtained by combining the DM-seller's binding participation constraint, the binding liquidity constraint on currency f and the requirement that currency holdings are non-negative. The second strict inequality is a direct consequence of the slack liquidity constraint on holding the dominated currency.

In Case 1c, the coexistence equilibrium (given $\Pi > \Pi^f$) has the following restrictions on (κ, κ^f) :

$$\kappa^f \cdot \frac{\Pi - \beta(1 - \sigma)}{\Pi^f - \beta(1 - \sigma)} + \kappa \leq [\Pi - \beta(1 - \sigma)]c(\hat{q}), \quad \kappa > 0, \quad \kappa^f > 0. \quad (28)$$

These cases and their corresponding inequalities are derived in Supplementary Appendix H.

If we define $k^f(\hat{q}, \Pi^f, \beta, \sigma) := c(\hat{q})[\Pi^f - \beta(1 - \sigma)]$ and $k(\hat{q}, \Pi, \beta, \sigma) := c(\hat{q})[\Pi - \beta(1 - \sigma)]$, in the space of (κ, κ^f) pairs, then we can deduce the threshold levels of the pair (κ, κ^f) that are required to sustain coexistence equilibria of Case 1b and Case 1c.

To illustrate the equilibria consistent with different values of the parameter space, let us consider $u(q) := [(q + \underline{q})^{1-\theta} - \underline{q}^{1-\theta}]/(1 - \theta)$, where $\underline{q} > 0$ and $\theta > 0$, and $c(q) = q^\alpha$, where $\alpha \geq 1$ is depicted in Figure 3 as the baseline setting.²⁹

Consider raising Π^f , as shown in Figure 4, while holding all else equal. This has a tendency to reduce the opportunity cost of holding the dominated fiat currency. That is, DM-buyers now have relatively more incentive to counterfeit this currency. At the same time, a higher Π^f also means that the liquidity constraint on currency f is even tighter. These two imply that sellers will be less accepting of either currencies offered as genuine payments. Thus the coexistence equilibria of Case 1c can only be sustained with even higher costs of counterfeiting.

Consider next Figure 5, where all else the same, we increase the DM matching probability σ . A higher σ lowers the inefficiency wedge $[\Pi - \beta(1 - \sigma)]/\beta\sigma$, between the monetary equilibrium outcome \hat{q} in Case 1b (or \tilde{q} in Case 1c) and the first-best q^* . This means that both currencies will circulate more in either Case 1b or Case 1c, to support a higher allocation of \hat{q} (and \tilde{q}). That is buyers are more likely to use their portfolio of currencies to spend in DM meetings and thus have more incentive to counterfeit both currencies, when we perturb their environment in terms of raising σ . Thus to ensure equilibrium acceptability and coexistence of both currencies, one needs to have relatively higher thresholds for (κ, κ^f) .

Figure 6 illustrates the comparative static with respect to buyers' appetite for intertemporal consumption smoothing, θ . The larger θ is, the less tolerant they are of substitution in q across periods. In other words, DM-buyers would like to smooth out their consumption as much as possible over time, and the only means of intertemporal smoothing is money. Thus, there is a higher incentive to counterfeit both monies, the larger is θ . However, to ensure that there are offers of genuine monies in equilibrium where both monies coexist, then the cost of counterfeiting sustaining the equilibria must be higher.

Figure 7 depicts the comparative static with respect to the convexity of production cost of DM-sellers. The faster production cost rises at the margin for producers, the more willing they are to accept offers of payment in both currencies, so that lower counterfeiting costs are required to sustain truthful or genuine offers of both monies as payment for DM goods in the coexistence equilibria.

The key takeaway is as follows: Consider an economy that has low enough inflation (Π) on one currency relative to another's (Π^f), low market frictions (σ), agents with large intertemporal insurance motives (θ), or has sufficiently non-accelerating marginal costs of production (α). These comparative static analyses suggest that in such an economy, one must have an institution of high enough regulatory or technological costs to creating counterfeit media of exchange, so that multiple means of payments can co-exist, and so that there is no currency flight to quality. Indeed, in many

²⁹Python code producing these comparisons are available from the authors' public repository at https://github.com/phantomachine/_gkwcourse.git.

advanced countries with low inflation and highly centralized markets, we see that counterfeiting is virtually non-existent and there can still be coexistence of currencies used in exchange (see Judson and Porter, 2010).

6 Extensions

In this section we explore the robustness of the threat of counterfeiting in determining nominal exchange rates. Using the benchmark model we investigate how fiscal policies can help determine nominal exchange rate when monetary policy alone and the fundamentals of the economy are not able to do so. We then analyze whether our proposed mechanism can help determine nominal exchange rates when credit is possible.

6.1 Fiscal Policies

In this section we study how active fiscal policies can restore determinacy of the nominal exchange rates. Finding such policies is crucial for policy analysis as an environment with indeterminacy requires the selection of a specific allocation and prices consistent with equilibrium. Establishing an appropriate selection rule is extremely difficult.

Let us now consider our benchmark environment and fiscal authorities that can impose a tax on CM production. For simplicity, we assume that the tax revenues fund wasteful government expenditures. In this new environment the buyer's sequential budget constraint for the Home country is given by

$$C_{-1} \leq (1 - \tau_{-1})N_{-1} - \phi(m - m_{-1}) - \phi e(m^f - m_{-1}^f),$$

where τ is the income tax rate faced by Home buyers and sellers. Symmetrically, we have similar budget constraints for the Foreign buyers and sellers.

It is worth emphasizing that the introduction of an income tax in CM does not change the nature of the private information problem that sellers face in DM. Thus the previous results outlined in Proposition 3 regarding existence of the game of private information problem still hold.

It is worth noting that in contrast to Zhang (2014), the income tax imposed by the Home (and Foreign) country does not give any of the currencies a distinct advantage over the other. In short, we are not specifying how taxes are paid.³⁰ As is typical in environments where agents trade in frictionless and centralized markets, how transactions are settled need not be specified. All that is required is for the agent to exhaust her budget constraint to finance CM consumption, rebalance the portfolio and pay her income taxes.

Under this new fiscal regime, it is straightforward to show that the resulting equilibrium liquidity

³⁰To be able to determine the nominal exchange rate we only need that income taxes are ad valorem.

constraints (for the Home country) are given by

$$\frac{\phi d}{1 - \tau_{-1}} \leq \frac{\kappa}{\phi_{-1}/\phi - \beta(1 - \sigma)};$$

$$\frac{\phi e d^f}{1 - \tau_{-1}} \leq \frac{\kappa^f}{\phi_{-1} e_{-1}/\phi e - \beta(1 - \sigma)}.$$

Similar constraints are obtained for Foreign buyers.

As we can see from these modified endogenous liquidity constraints, the income tax rate affects the real value of the payment made in Home and Foreign currency. Fiscal measures can actively affect the incentives to counterfeit fiat currencies. Thus for a given monetary steady state allocation, Home and Foreign monetary policies, γ and γ^f , and Foreign fiscal policy, τ_* , we can always find a Home income tax rate $\bar{\tau}$ such that one of the liquidity constraints bind. Thus a coordinated Home fiscal policy can increase the set of equilibria where the nominal exchange rate is determinate.

For expositional purposes, let us consider a stationary monetary equilibria where all fiat currencies are perfect substitutes, so that $\gamma = \gamma^f$ and $\kappa = \kappa^f = \kappa_* = \kappa_*^f = \hat{\kappa}$, and the endogenous liquidity constraints do not bind when there are no income taxes.³¹ In such scenario we then have that

$$\phi e d^f = \phi d < \frac{\hat{\kappa}}{\Pi - \beta(1 - \sigma)}.$$

The Home fiscal authority can then impose an income tax rate ($\bar{\tau}$) such that the new endogenous liquidity constraint binds so that it can yield determinate nominal exchange rates. For this economy, the critical income tax rate is equal to

$$\bar{\tau} = 1 - (\Pi - \beta(1 - \sigma))$$

the relative price is $\phi = 2\hat{\kappa}/M$ the nominal exchange rate is $e = M/M^f$. A similar line of reasoning can be applied to the rest of the cases in Proposition 3 where liquidity constraints do not bind and the exchange rate is indeterminate. Note that the use of fiscal policy to increase the set of monetary equilibria that are determinate in this paper parallels the analysis of fiscal-monetary policy interdependence studied by Leeper (1991).

6.2 Trade Credit

We now consider an environment where agents have access to credit in certain states of the world. This allows us to examine how this new option to settle transactions affects the determinacy of the nominal exchange rate.

As in Aruoba, Waller and Wright (2011), we consider the possibility that in some matches credit is possible as record-keeping and enforcement services are available.³² In particular, we assume that

³¹Note that in this case both countries are perfectly symmetric and identical.

³²Recall that agents in our environment, in the absence of monitoring, enforcement and record-keeping, are said to be anonymous.

conditional on buyers being matched with a seller, the exogenous probability that a buyer or seller would engage in an exchange where trade credit is possible is $(1 - \rho) \in [0, 1]$. That is, the event that a buyer can buy a good in the DM using credit occurs with probability $\sigma(1 - \rho)$. Similarly for the Foreign country.

Since credit is assumed to be enforceable in such an event, a buyer is willing to take (and a seller is willing to give) out a loan, which we denote by l , in exchange for the DM good, say q_c . This loan is required to be repaid in full in the following CM. As in the two-country model presented previously, the particular terms of trade when credit is available are determined by a buyer take it or leave it offer.

The possibility that in some matches trade credit can be used reduces the *anonymity* frictions in DM. This situation can be interpreted as an improvement in the record-keeping and enforcement powers. In this new environment, the payoff for the buyer that is offered a real loan l is given by

$$Z(C_{-1}, \mathbf{s}_{-1}) - \left(\frac{\phi_{-1}}{\phi} - \beta \right) \phi m - \left(\frac{\phi_{-1}e_{-1}}{\phi e} - \beta \right) \phi e m^f - \kappa(1 - \eta) - \kappa^f(1 - \eta^f) + \beta \sigma \left(\rho \hat{\pi} \left[u(q) - \phi \left(\eta d + \eta^f e d^f \right) \right] + (1 - \rho) [u(q_c) - l] \right),$$

while for the seller the payoff is given by

$$Z(\check{C}_{-1}; \check{\mathbf{s}}_{-1}) + \beta \sigma \left(\rho \pi(\omega) \left[\phi \left(\hat{\eta} d + \hat{\eta}^f e d^f \right) - c(q) \right] + (1 - \rho) [l - c(q_c)] \right),$$

where q_c is the quantity traded when credit is possible, $Z(C_{-1}, \mathbf{s}_{-1})$ and $Z(\check{C}_{-1}; \check{\mathbf{s}}_{-1})$ are defined as in Section 3.

Since the loan l is repaid in CM the possibility of paying with counterfeit currencies is not feasible as they can easily be detected and destroyed. The seller then faces no private information problems when using credit. Thus allowing this additional payment instrument does not really change the private information problem of counterfeiting currencies agents face in DM. As a result, as in the previous section, the PBE re-ordering equilibrium of the counterfeiting game is such that each seller accepts with probability one and each buyer does not counterfeit.³³

It is easy to show that the resulting monetary steady state equilibria in this new environment is similar to what we derived before. In particular these new liquidity constraints are being influenced by the credit trading opportunities which are given by

$$\phi d \leq \frac{\kappa}{\Pi - \beta(1 - \sigma\rho)}, \quad (29)$$

$$e\phi d^f \leq \frac{\kappa^f}{\Pi^f - \beta(1 - \sigma\rho)}. \quad (30)$$

This new credit environment changes the matching probability for cash trades from σ to $\rho\sigma$. Thus

³³Note that even if buyers faced some limited commitment which would induce an endogenous borrowing constraints, the private information problem would not change as the settlement of the loan would occur in CM where all counterfeits when traded can be detectable.

the insights of Proposition 3 regarding the coexistence of fiat currencies and determinacy of nominal exchange rates still hold when credit is available. In particular, in order to obtain determinacy of nominal exchange rates, liquidity constraints arising from the signalling-bargaining game need to bind.

Finally, it is worth emphasizing that if agents have more access to credit (i.e. ρ is lower), then buyers and sellers are less likely to meet and trade with fiat currencies. This creates a smaller incentive for the buyer to produce counterfeits, thus decreases the information problem. This in turn relaxes the liquidity constraints faced by buyers. This property then implies that as credit markets develop the possibility of having nominal exchange rate indeterminacy increases. This prediction is consistent with the views of Keynes (1964) and Friedman (1956) who have suggested that interactions between the conduct of monetary policy and the financial system create considerable scope for endogenous volatility and indeterminacy.

7 Conclusion

In this paper we present a search theoretic model of two fiat currencies to study the properties of nominal exchange rates when agents face private information. Agents have no restrictions on what divisible fiat currency can be used to settle transactions. Buyers may counterfeit both fiat currencies at some fixed cost while sellers can not distinguish between counterfeit and genuine fiat currencies. Thus counterfeiting is private information to buyers. This informational problem gives rise to endogenous liquidity constraints that specify a seller's upper bound on how much fiat currency is willing to accept.

The private information problem faced by sellers yield some endogenous liquidity constraints. These liquidity constraints have the property that the marginal liquidity value of an additional unit of currency beyond the endogenous binding liquidity constraint is zero. The binding nature of these liquidity constraints are key in determining nominal exchange rates.

When endogenous liquidity constraints on both currencies are binding and the currencies are identical in every respect, we obtain the surprising result that the nominal exchange rate is the ratio of the two money stocks thus breaking the curse of Kareken and Wallace. When the foreign currency has a higher rate of return but a lower counterfeiting cost, then the buyer will first pay with the foreign currency up to the bound and use domestic currency to pay for the remainder of the goods purchased. Because of this, both currencies can circulate even though one currency is dominated in its rate of return. Also, because of this, we can have equilibrium cases in which only one currency emerges as an international currency, which the other circulates only locally. We also show that when there is nominal exchange rate indeterminacy, there exist fiscal policies that can restore determinacy of the nominal exchange rate.

In this environment, we also find that the first best allocation may not be attainable even if the Friedman rule is implemented for both currencies. We also show that the private information problem considered in this paper can help determine nominal exchange rates when trade credit

is possible. When agents have more access to credit, there is a smaller incentive for the buyer to produce counterfeits, increasing the possibility of inducing more indeterminate exchange rate equilibria.

The private information problem explored in this paper allows us to break the Kareken and Wallace indeterminacy result and provides a rationalization of why currencies with dominated rates of return remain in circulation (apart from obvious explanations in terms of legal restrictions) as media of exchange.

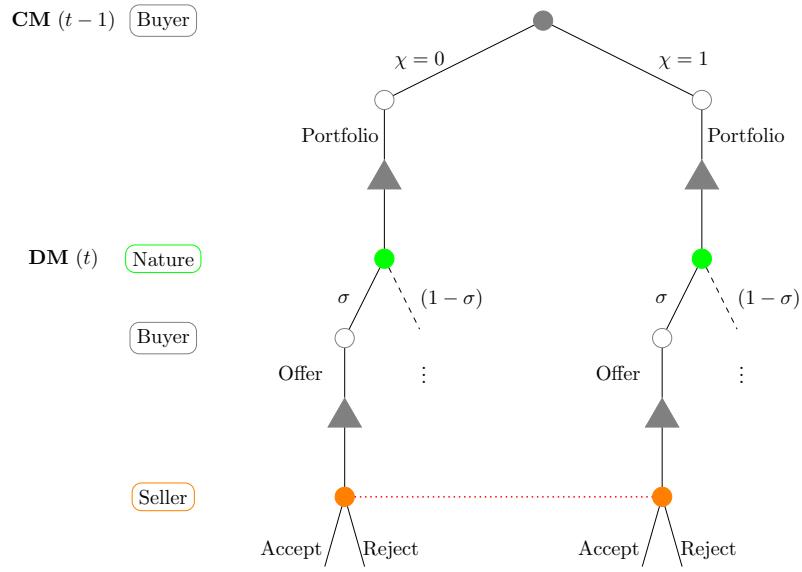
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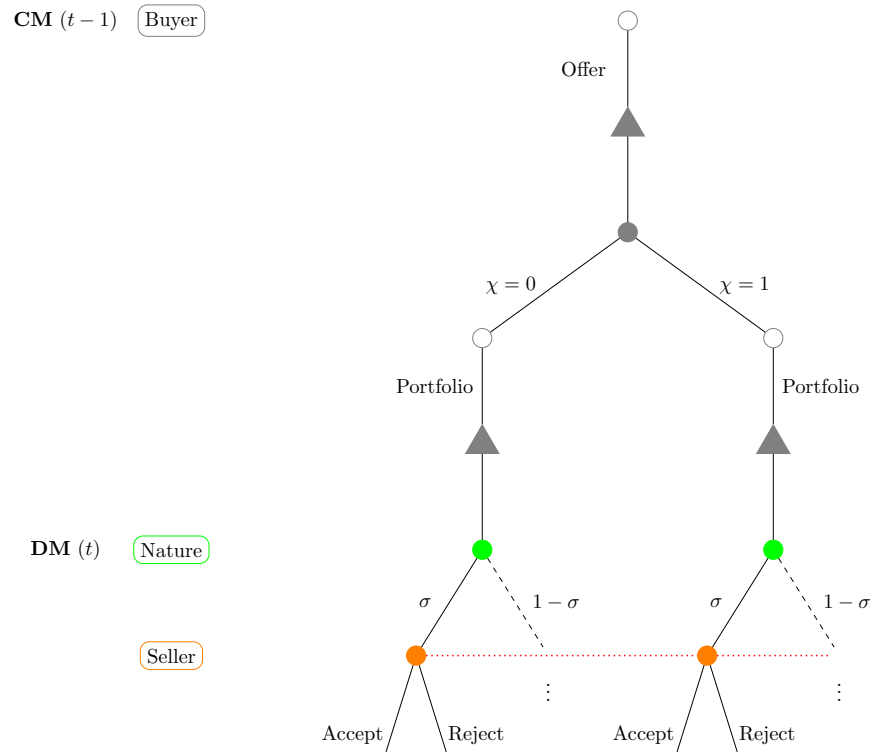
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Figure 1: Original extensive-form game.



Note: \bullet Buyer's discrete decision node; \circ Buyer's continuation to next decision node; \blacktriangle Buyer's continuous decision node; \bullet Nature's discrete decision node; \bullet Seller's discrete decision node; \cdots Information set.

Figure 2: Reverse-order extensive-form game.



Note: • Buyer's discrete decision node; ○ Buyer's continuation to next decision node; ▲ Buyer's continuous decision node; ● Nature's discrete decision node; ● Seller's discrete decision node; ⋯ Information set.

Figure 3: Equilibrium coexistence or non-coexistence of monies when $\Pi > \Pi^f$ and counterfeiting costs—baseline setting.

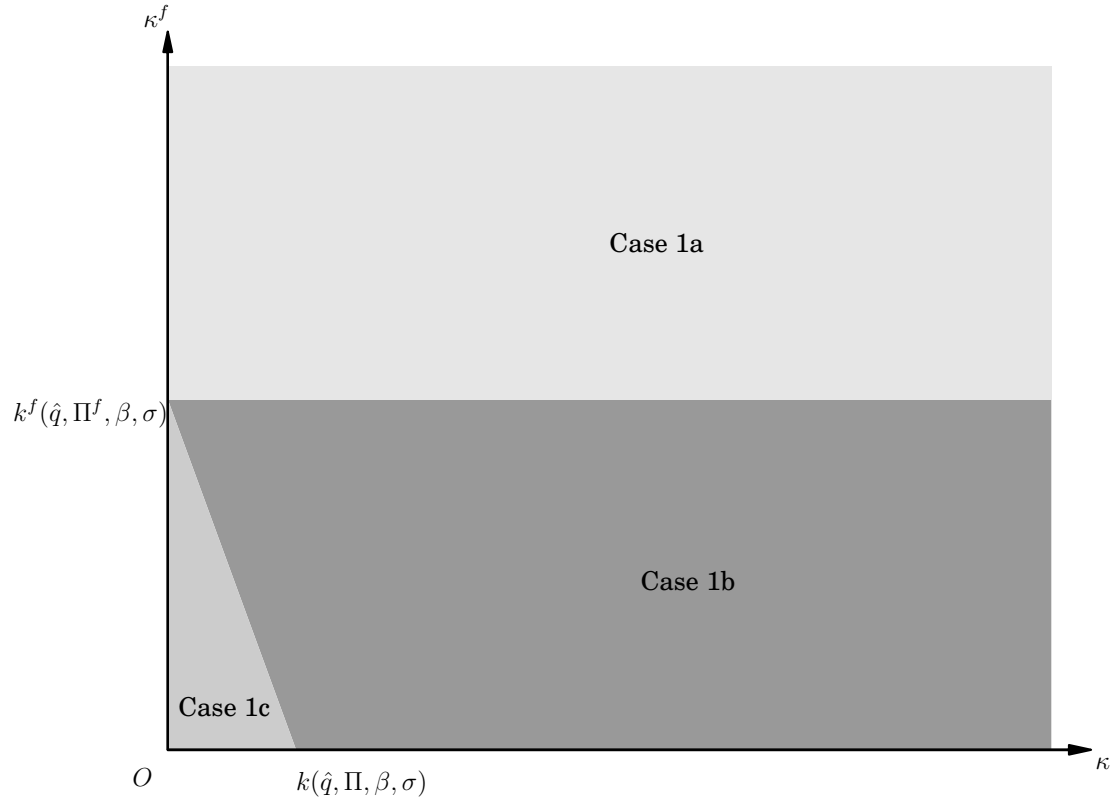


Figure 4: Equilibrium coexistence or non-coexistence of monies when $\Pi > \Pi^f$ and counterfeiting costs. Comparative static with higher Π^f .

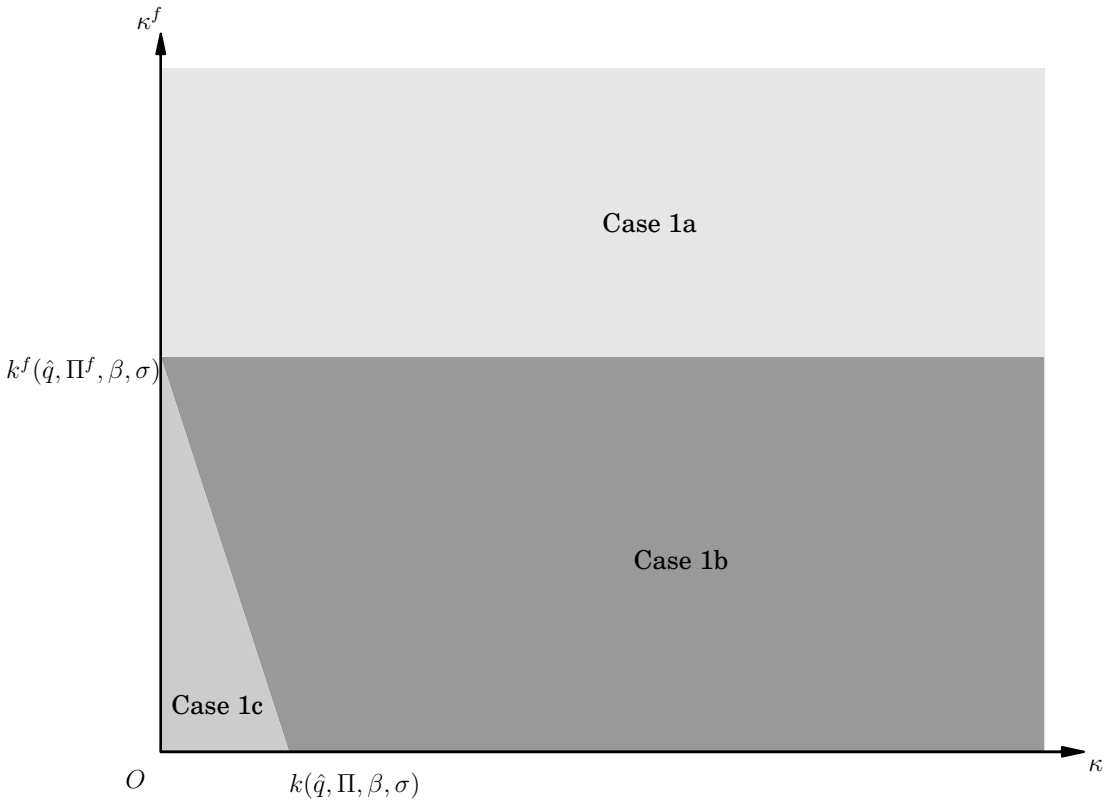


Figure 5: Equilibrium coexistence or non-coexistence of monies when $\Pi > \Pi^f$ and counterfeiting costs. Comparative static with higher matching probability σ .

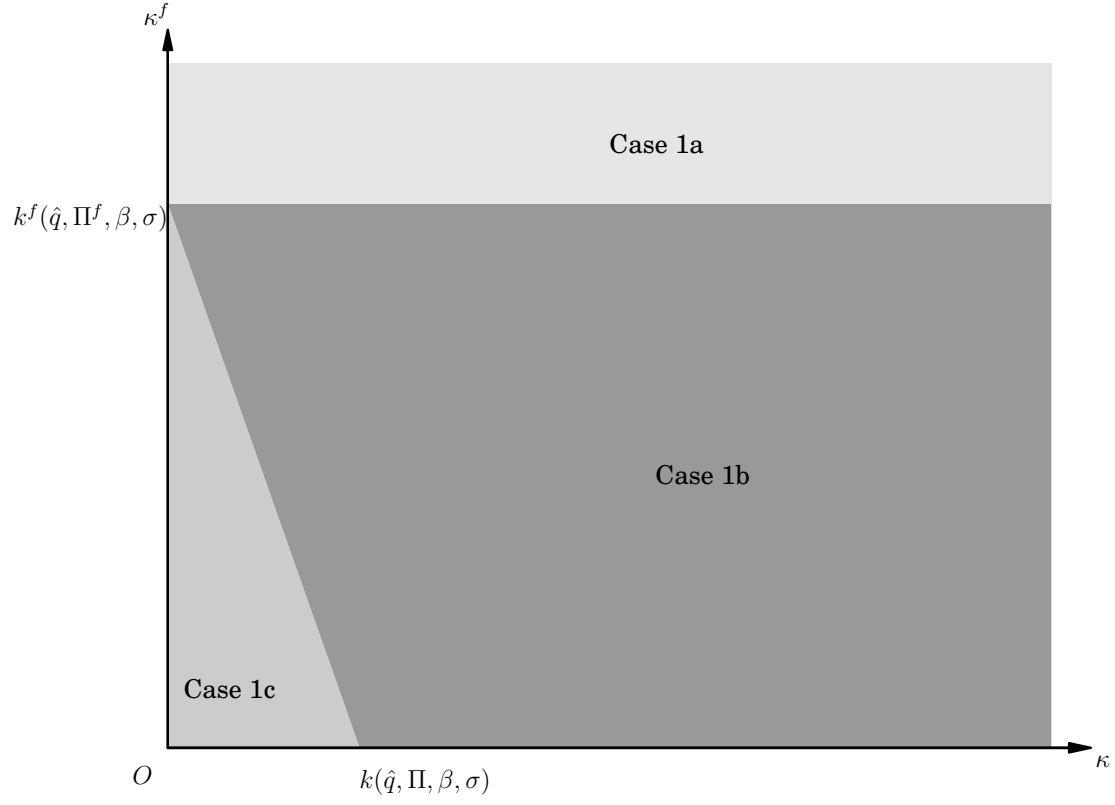


Figure 6: Equilibrium coexistence or non-coexistence of monies when $\Pi > \Pi^f$ and counterfeiting costs. Comparative static with higher risk aversion θ .

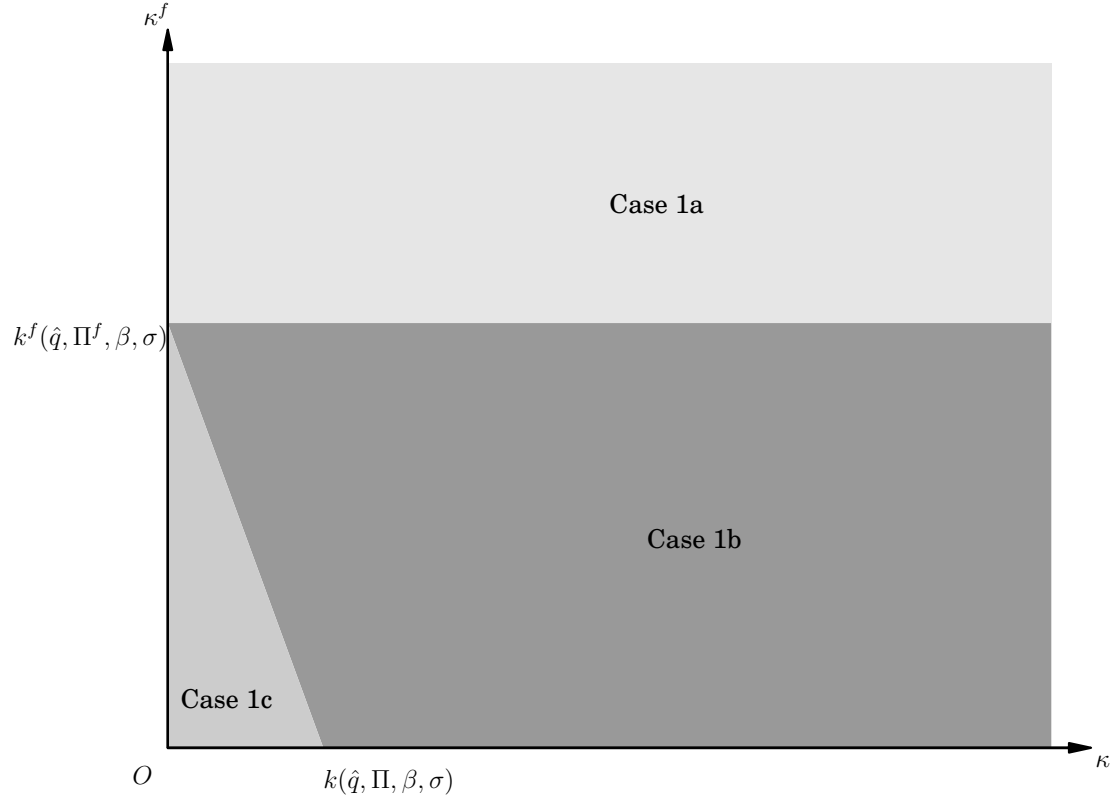
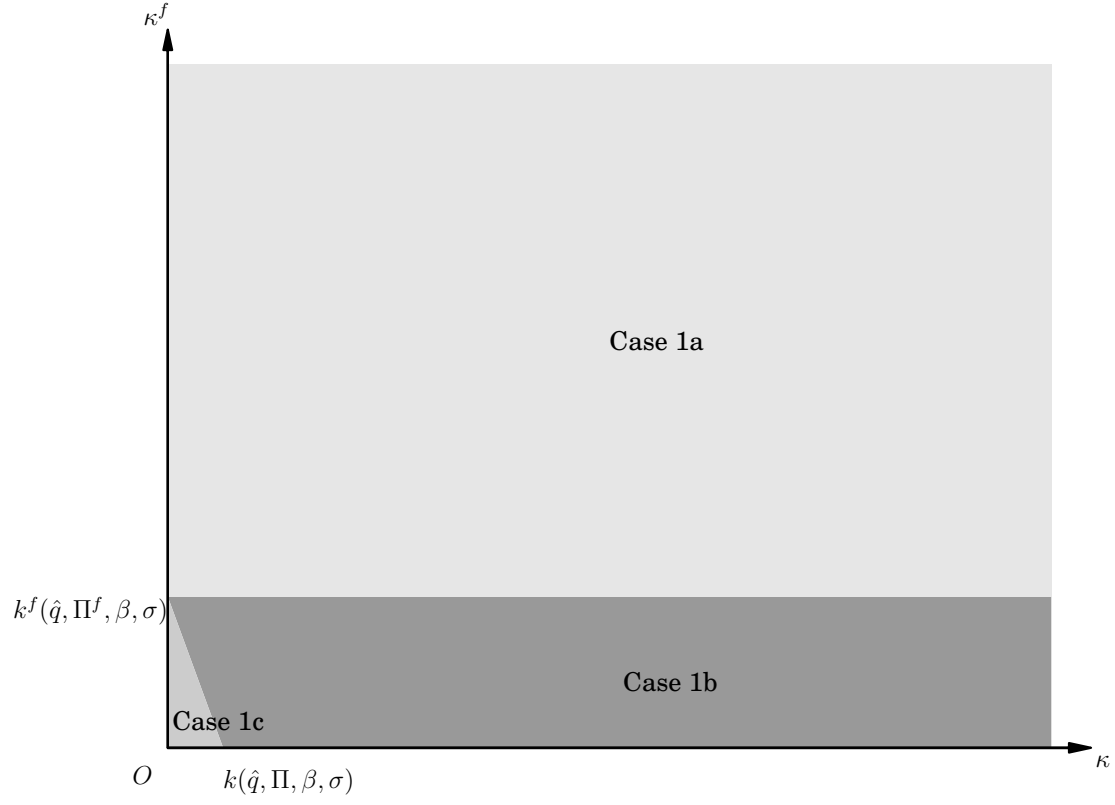


Figure 7: Equilibrium coexistence or non-coexistence of monies when $\Pi > \Pi^f$ and counterfeiting costs. Comparative static with more convex cost of DM production α .



— Supplementary (Online) Appendix —

A International goods trade in CM

Following international trade models we assume that a CM good is a Dixit-Stiglitz CES aggregator of Home and Foreign inputs which is given by

$$C = D(c, c^f),$$

where $D : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ is a consumption aggregator with the following properties: (i) $D_x(x, y) > 0$, $D_y(x, y) > 0$; (ii) $D_{xx}(x, y) < 0$, $D_{yy}(x, y) < 0$, and $D_{xy}(x, y) > 0$; (iii) $\lim_{x \rightarrow 0} D_x(x, y) = \lim_{y \rightarrow 0} D_x(x, y) = +\infty$; and (iv) $\lim_{x \rightarrow +\infty} D_x(x, y) = \lim_{y \rightarrow +\infty} D_x(x, y) = 0$, for any $(x, y) \in \mathbb{R}_+^2$.

Intermediate goods in CM are produced using a technology that is linear in labour input. The total labor supplied in CM is the sum of labor effort of DM-buyers and DM-sellers, $H + \check{H}$. This total effort is then transformed into total units of domestic inputs which are demanded domestically (c) and from abroad (c_\star). A symmetric setting applies to the Foreign agents.

DM-buyers and DM-sellers choose CM labor, end-of-period currency portfolio and a bundle of domestic and foreign produced intermediate goods. These determine their composite consumption C_t through their respective sequential budget constraints. Note that for in every date and state, given the dynamic allocation of C_t , the optimal decision problem on traded intermediate goods is static. That is, given C_t agents solve a dual expenditure minimization problem given by

$$\min_{c_t^h, c_t^f} \left\{ P_t^h c_t^h + e_t P_t^f c_t^f + \phi_t^{-1} \left[C_t - D(c_t^h, c_t^f) \right] \right\}; \quad (31)$$

where P_t^h (P_t^f) correspond to the price of the domestic (foreign) input in the Home economy. Note that here we do not specify exactly how the Home and Foreign goods are paid. Any real currency balance or good can be used, which is in sharp contrast to the traditional international monetary models that impose a currency restrictions when buying inputs. Finally we note that this minimization problem, because it is static, can be characterized separately from the agents' dynamic decision problem as in standard international models. A similar problem is faced by Foreign agents. (That is why, in the main text, we had abstracted from characterizing the international trade side of the two-country model.)

The equilibrium prices for the Home and Foreign inputs are determined by the respective market clearing conditions that impose that the total demand equals its corresponding supply, internationally. It is important to highlight that the international goods trade details of the environment do not change the mechanism that allows us to deliver determinate nominal exchange rates which solely relies on the private information problem that sellers face.

B Degeneracy of asset portfolios

Given that we are interested in monetary equilibria from now on we restrict attention to economies where $\phi_{t-1}/\phi_t \geq \beta$; and $\phi_{t-1}^f/\phi_t^f \equiv e_{t-1}\phi_{t-1}/e_t\phi_t \geq \beta$. The following lemma allows us to simplify the Bernoulli payoff function given by equation (6).

Lemma 2 *Under any optimal measurable strategy $\tilde{\sigma}^b$, genuine portfolio choices are always such that:*

$$m \begin{cases} = \chi d, & \text{if } \phi_{-1}/\phi > \beta \\ \geq \chi d, & \text{if } \phi_{-1}/\phi = \beta \end{cases}; \text{ and, } m^f \begin{cases} = \chi^f d^f, & \text{if } \phi_{-1}e_{-1}/\phi e > \beta \\ \geq \chi^f d^f, & \text{if } \phi_{-1}e_{-1}/\phi e = \beta \end{cases}.$$

Moreover, whenever $\phi_{-1}/\phi = \beta$ (or $\phi_{-1}e_{-1}/\phi e = \beta$), demanding $m > \chi d$ (or $m^f > \chi^f d^f$) is U^b -payoff equivalent for the buyer to demanding $m = \chi d$ (or $m^f = \chi^f d^f$).

Proof. First consider the cases where the returns of either (or both) assets are strictly dominated by β . Then, holding either (or both) assets beyond what is necessary for payments in the DM (i.e. d and d^f) is intertemporally costly since the price levels ϕ^{-1} , and, $(\phi^f)^{-1}$ are respectively growing at the rates $\gamma - 1$ and $\gamma^f - 1$. Thus holding only $m = (1 - \chi)d$ or (and) $m^f = (1 - \chi^f)d^f$ is optimal for the DM-buyer under any optimal strategy $\tilde{\sigma}^b$.

Second, consider the cases where the returns of either (or both) assets are equal to β . Then any portfolio demand comprising $m \geq (1 - \chi)d$ or (and) $m^f \geq (1 - \chi^f)d^f$ is optimal. However, since W is linear, the only terms involving m and m^f in the buyer's payoff function U^b in (6) are the expected costs of holding unused genuine assets, given by the linear functions

$$\left(\frac{\phi_{-1}}{\phi} - \beta\right) \phi m - \left(\frac{\phi_{-1}e_{-1}}{\phi e} - \beta\right) \phi e m^f.$$

Observe that in the cases where the asset returns are equal to β , the value of these costs are zero. Therefore, the second statement in the Lemma is true. ■

This result stems from two observations: (i) if the returns on the two fiat currencies are strictly below the discount factor β , then, holding these assets are intertemporally costly; and (ii) if their returns are equal to β , the linearity of $W(\cdot)$ ensures that any excess asset demands beyond what is necessary for trade in the DM is inconsequential to the payoff $U^b(\cdot)$.

Lemma 2 also implies that each $G(\cdot|\omega)$ consistent with $\tilde{\sigma}^b$ is degenerate, as far as characterizing the Bernoulli payoff function U^b is concerned. That is, given the realization of $\aleph(\omega) := (\chi(\omega), \chi^f(\omega))$, we have the following

$$G[a(\omega)|\omega] = \delta_{\{(1-\chi)d, (1-\chi^f)d^f\}}, \quad \forall \aleph \in \{0, 1\}^2, \quad (32)$$

where δ_E denotes the Dirac delta function defined to be everywhere zero-valued except on singleton events E , on which the function has value 1. In short, we can characterize the buyer's mixed strategy $G(\cdot|\omega)$ (over portfolio accumulation) in the subgame following the buyer's finite history of play, $\langle \omega, \aleph(\omega) \rangle$, prior to comprehensively describing equilibrium in the game.

C Proof of Proposition 1 (RI-equilibrium)

Denote the maximum value of the program in (12), when $\hat{\pi} = \pi(\omega) = 1$ and $H = (\hat{\eta}, \hat{\eta}^f) = (\eta(\omega), \eta^f(\omega)) = (1, 1)$, as $(U^b)^*$. The aim is to show that an equilibrium $\tilde{\sigma}$ yields the same value as $(U^b)^*$, and it satisfies the characterization in Proposition 1 (Case 1); and that any other candidate strategy $\tilde{\sigma}' := \langle \omega', H', \pi' \rangle$ such that $\hat{\pi}' = \pi'(\omega) \neq 1$ and/or $\hat{H}' \neq (1, 1)$ will induce a buyer's valuation that is strictly less than $(U^b)^*$, and therefore cannot constitute an equilibrium (Cases 2-5).

Consider the subgame following offer ω . Let $\rho(\chi, \chi^f)$ denote the joint probability measure on events $\{(\chi, \chi^f)\}$, where the pure actions over counterfeiting are $(\chi, \chi^f) \in \{0, 1\}^2$. Denote $P := 2^{\{0, 1\}^2}$ as the power set of $\{0, 1\}^2$. By the definition of probability measures, it must be that $\sum_{\{z\} \in P} \rho(z) = 1$.

The seller's problem in (9) is equivalent to:

$$\pi(\omega) \in \left\{ \arg \max_{\pi' \in [0, 1]} \pi' \left[\phi \left([1 - \hat{\rho}(1, 0) - \hat{\rho}(1, 1)]d + [1 - \hat{\rho}(0, 1) - \hat{\rho}(1, 1)]ed^f \right) - c(q) \right] \right\}. \quad (33)$$

This is a linear programming problem in π , given the seller's rational belief system $\hat{\rho}$ and buyer's offer ω . Thus the seller's best response satisfies:

$$\begin{aligned} & \left(\phi \left([1 - \hat{\rho}(1, 0) - \hat{\rho}(1, 1)]d + [1 - \hat{\rho}(0, 1) - \hat{\rho}(1, 1)]ed^f \right) - c(q) \right) \begin{cases} > 0 \\ < 0 \\ = 0 \end{cases} \\ \Rightarrow & \left(\pi(\omega) \begin{cases} = 1 \\ = 0 \\ \in [0, 1] \end{cases} \right). \end{aligned} \quad (34)$$

Let $U_{\{z\}}^b \equiv U^b[\omega, \{z\}, \hat{\pi}|s_{-1}, \phi, e]$ denote the buyer's expected payoff from *realizing* pure actions (χ^h, χ^f) , given offer ω and rational belief system $\hat{\pi} \in [0, 1]$, where $\{z\} \in P$. We have the following

possible payoffs following each event $\{z\}$:

$$U_{\{(0,0)\}}^b = -\left(\frac{\phi-1}{\phi} - \beta\right)\phi d - \left(\frac{\phi-1e-1}{\phi e} - \beta\right)\phi ed^f + \beta\sigma\hat{\pi}\left[u(q) - \phi(d + ed^f)\right]; \quad (35)$$

$$U_{\{(0,1)\}}^b = -\kappa^f - \left(\frac{\phi-1}{\phi} - \beta\right)\phi d + \beta\sigma\hat{\pi}[u(q) - \phi d]; \quad (36)$$

$$U_{\{(1,0)\}}^b = -\kappa - \left(\frac{\phi-1e-1}{\phi e} - \beta\right)\phi ed^f + \beta\sigma\hat{\pi}[u(q) - \phi ed^f]; \quad (37)$$

$$U_{\{(1,1)\}}^b = -\kappa^f - \kappa + \beta\sigma\hat{\pi}u(q). \quad (38)$$

Observe that

$$U_{\{(0,1)\}}^b + U_{\{(1,0)\}}^b = U_{\{(0,0)\}}^b + U_{\{(1,1)\}}^b. \quad (39)$$

There are five cases to consider.

Case 1. Suppose there is a set of candidate equilibria such that $\rho(0,0) = 1$ and $\rho(z) = 0$, for all $\{z\} \in P$ and $z \neq (0,0)$. Then, we have $U_{\{(0,0)\}}^b > \max\{U_{\{(1,0)\}}^b, U_{\{(0,1)\}}^b, U_{\{(1,1)\}}^b\}$. Since $U_{\{(0,0)\}}^b > U_{\{(1,0)\}}^b$ and $U_{\{(0,0)\}}^b > U_{\{(0,1)\}}^b$, then, from (35)-(38) we can derive that

$$\phi d < \frac{\kappa}{\frac{\phi-1}{\phi} - \beta(1 - \sigma\hat{\pi})}, \quad (40)$$

and,

$$\phi ed^f < \frac{\kappa^f}{\frac{\phi-1e-1}{\phi e} - \beta(1 - \sigma\hat{\pi})}. \quad (41)$$

The interpretation from (40) and (41) is that the liquidity constraints on either monies are slack. Therefore the buyer's expected payoff in this case can be evaluated from (35). If $\hat{\pi} < 1$, then from the seller's decision rule (34) we can deduce $\omega \equiv (q, d, d^f)$ must be such that the seller's participation/incentive constraint binds:

$$c(q) = \phi(d + ed^f). \quad (42)$$

Since (42) holds, all we need to do is verify the buyer's payoff. Since, the buyer's liquidity constraints (40) and (41) do not bind at $\hat{\pi} < 1$, a small increment in either payment offered, d or d^f , relaxes (42) and this raises $\hat{\pi}$, and thus the buyer's payoff (35). The maximal payoff to the buyer, keeping

the seller in participation, is when $\pi(\omega) = \hat{\pi} = 1$, and the offer $\bar{\omega}$ is such that

$$\begin{aligned} \bar{U}^b \equiv U_{\{(0,0)\}}^b[\bar{\omega}|\pi(\omega) = \hat{\pi} = 1] = \sup_{\omega} \Big\{ & U_{\{(0,0)\}}^b[\omega|\pi(\omega) = \hat{\pi} = 1] : \\ & \phi d \leq \frac{\kappa}{\frac{\phi-1}{\phi} - \beta(1-\sigma)}, \\ & \phi e d^f \leq \frac{\kappa^f}{\frac{\phi-1e-1}{\phi e} - \beta(1-\sigma)}, \\ & c(q) \leq \phi(d + e d^f) \Big\}. \end{aligned}$$

Then it is easily verified that this maximal value coincides with the maximum value of the program given in (12) in Proposition 1, i.e. $\bar{U}^b = (U^b)^*$, since the payoff function is continuous, and the constraints also define a nonempty, compact subset of the feasible set $\Omega(\phi, e) \ni \bar{\omega}$. Since the seller has no incentive to deviate from $\pi(\bar{\omega}) = 1$, then a behavior strategy $\tilde{\sigma} = \langle \bar{\omega}, (1, 1), 1 \rangle$ inducing the TIOLI payoff \bar{U}^b is a PBE.

Case 2. Note that in any equilibrium, a seller will never accept an offer if $\rho(1, 1) = 1$, and, a buyer will never counterfeit both assets with probability 1—counterfeiting for sure costs $\kappa + \kappa^f$ and the buyer gains nothing. Therefore, $\rho(1, 1) < 1$ is a necessary condition for an equilibrium in the subgame following ω . Likewise, all unions of disjoint events with this event of counterfeiting all assets—i.e. $\{(\chi, \chi^f)\} \in \{(0, 1)\} \cup \{(1, 1)\}$ or $\{(\chi, \chi^f)\} \in \{(1, 0)\} \cup \{(1, 1)\}$ —such that $\rho(0, 1) + \rho(1, 1) = 1$ or $\rho(1, 0) + \rho(1, 1) = 1$, respectively, cannot be on any equilibrium path.

Case 3. Suppose instead we have equilibria in which $\rho(0, 0) + \rho(1, 0) = 1$, $\rho(1, 0) \neq 0$, and $\rho(1, 1) + \rho(0, 1) = 0$, so $U_{\{(1,0)\}}^b = U_{\{(0,0)\}}^b > \max\{U_{\{(0,1)\}}^b, U_{\{(1,1)\}}^b\}$.

Given this case, and from (39), we have $U_{\{(0,1)\}}^b = U_{\{(1,1)\}}^b$. From $U_{\{(1,0)\}}^b = U_{\{(0,0)\}}^b$, and (35) and (37), respectively, we have:

$$\hat{\pi} = \frac{\kappa - (\phi_{-1}/\phi - \beta)\phi d}{\beta\sigma\phi d}, \quad (43)$$

and,

$$\phi e d^f < \frac{\kappa^f}{\frac{\phi_{-1}e-1}{\phi e} - \beta(1 - \sigma\hat{\pi})}. \quad (44)$$

If $\hat{\pi} < 1$, then from the seller's decision rule (34) we can deduce $\omega \equiv (q, d, d^f)$ must be such that the seller's participation/incentive constraint binds:

$$\begin{aligned} c(q) &= \phi[(1 - \rho(1, 0) - \rho(1, 1))d + (1 - \rho(0, 1) - \rho(1, 1))e d^*] \\ &= \phi[(1 - \rho(1, 0))d + e d^*]. \end{aligned} \quad (45)$$

The buyer's payoff can be evaluated from (37). If $\hat{\pi} < 1$, then reducing d infinitesimally will increase $\hat{\pi}$ in (43), and this increase the buyer's payoff in (37). The buyer would like to attain $\hat{\pi} = 1$ since

the seller's participation constraint will still be respected:

$$c(q) \leq \phi[(1 - \rho(1, 0))d + ed^f]. \quad (46)$$

Let the maximum of the buyer's TIOLI value (37) such that the constraints (43), (44) and (46) are respected, in this case be $(U^b)^\dagger$. However, since $\rho(1, 0) \neq 0$, it is easily verified that $(U^b)^\dagger < U_{\{(0,0)\}}^b[\bar{\omega}|\pi(\omega) = \hat{\pi} = 1; \rho(1, 0) = 0] = \sup_{\omega, \rho(1,0)} \{U_{\{(1,0)\}}^b | (43), (44), (46)\} = (U^b)^*$, in which the last equality is attained when $\rho(1, 0) = 0$. This contradicts the claim that $\rho(0, 0) + \rho(1, 0) = 1$ and $\rho(1, 0) \neq 0$ is a component of a PBE.

Case 4. Suppose there are equilibria consisting of $\rho(0, 0) + \rho(0, 1) = 1$ with $\rho(0, 1) \neq 0$, and $\rho(1, 0) = \rho(1, 1) = 0$. The buyer's payoff is such that $U_{\{(0,1)\}}^b = U_{\{(0,0)\}}^b > \max\{U_{\{(1,0)\}}^b, U_{\{(1,1)\}}^b\}$. Given this assumption, we have from (39) that $U_{\{(1,0)\}}^b = U_{\{(1,1)\}}^b$. From (35) and (36), we can derive

$$\hat{\pi} = \frac{\kappa^f - (\phi_{-1}e_{-1}/\phi e - \beta)\phi ed^f}{\beta\sigma\phi ed^f}. \quad (47)$$

From the case that $U_{\{(0,0)\}}^b > U_{\{(1,0)\}}^b$ and (35)-(37), we have:

$$\phi d < \frac{\kappa}{\frac{\phi_{-1}}{\phi} - \beta(1 - \sigma\hat{\pi})}. \quad (48)$$

The buyer's payoff can be evaluated from (36). If $\hat{\pi} < 1$, from (34), we can deduce that the seller's participation constraint is binding. If $\hat{\pi} < 1$, then reducing d^f infinitesimally will increase $\hat{\pi}$ in (47), and this increase the buyer's payoff in (36). The buyer would like to attain $\hat{\pi} = 1$ since the seller's participation constraint will still be respected at that point:

$$c(q) \leq \phi[d + (1 - \rho(0, 1))ed^f]. \quad (49)$$

Let the maximum of the buyer's TIOLI value (36) such that the constraints (47), (48) and (49) are respected, in this case be $(U^b)^{\dagger\dagger}$. However, since $\rho(1, 0) \neq 0$, it is easily verified that $(U^b)^{\dagger\dagger} < U_{\{(0,0)\}}^b[\bar{\omega}|\pi(\omega) = \hat{\pi} = 1; \rho(0, 1) = 0] = \sup_{\omega} \{U_{\{(0,1)\}}^b | (47), (48), (49)\} = (U^b)^*$, in which the last equality is attained when $\rho(0, 1) = 0$. This contradicts the claim that $\rho(0, 0) + \rho(0, 1) = 1$ and $\rho(0, 1) \neq 0$ is a component of a PBE.

Case 5. Suppose a candidate equilibrium is such that $\sum_{\{z\} \in P} \rho(z) = 1$, $\rho(z) \neq 0$ for all $\{z\} \in P$, and that $U_{\{(0,1)\}}^b = U_{\{(0,0)\}}^b = U_{\{(1,0)\}}^b = U_{\{(1,1)\}}^b$. Then from (36) and (37), we can derive

$$\hat{\pi} = \frac{\kappa^f - (\phi_{-1}e_{-1}/\phi e - \beta)\phi ed^f}{\beta\sigma\phi ed^f} = \frac{\kappa - (\phi_{-1}/\phi - \beta)\phi d}{\beta\sigma\phi d}. \quad (50)$$

If the payment offered (d, d^f) are such that $\hat{\pi} < 1$, then from the seller's decision rule (34) we can deduce $\omega \equiv (q, d, d^f)$ must be such that the seller's participation/incentive constraint binds:

$$c(q) = \phi[(1 - \rho(1, 0) - \rho(1, 1))d + (1 - \rho(0, 1) - \rho(1, 1))ed^f]. \quad (51)$$

However, the buyer can increase his expected payoff in (38) by reducing both (d, d^*) , thus raising $\hat{\pi}$ in (50) while still ensuring that the seller participates, until $\hat{\pi} = 1$, where

$$c(q) \leq \phi[(1 - \rho(1, 0) - \rho(1, 1))d + (1 - \rho(0, 1) - \rho(1, 1))ed^f]. \quad (52)$$

Let the maximum of the buyer's TIOLI value (38) such that the constraints (50) and (52) are respected, in this case be $(U^b)^\dagger$. However, since $\rho(1, 0), \rho(0, 1), \rho(1, 1) \neq 0$, it is easily verified that $(U^b)^\dagger < U_{\{(1,1)\}}^b[\bar{\omega}|\pi(\omega) = \hat{\pi} = 1; \rho(0, 0) = 1] = \sup_{\omega} \{U_{\{(1,1)\}}^b | (\omega), (49)\} = (U^b)^*$, in which the last equality is attained when $\rho(0, 0) = 1$. This contradicts the claim that $\sum_{\{z\} \in P} \rho(z) = 1$, $\rho(z) \neq 0$ for all $\{z\} \in P$, is a component of a PBE.

Summary. From Cases 1 to 5, we have shown that the only mixed-strategy Nash equilibrium in the subgame following an offer ω must be one such that $\langle \rho(0, 0), \pi \rangle = \langle 1, 1 \rangle$, and that the offer ω satisfies the program in (12) in Proposition 1.

Finally, since $u(\cdot)$ and $-c(\cdot)$ are strictly quasiconcave functions and the inequality constraints in program (12) define a convex feasible set, the program (12) has a unique solution.

D Proof of Proposition 2 (Constant exchange rate growth)

A monetary equilibrium implies that the general goods market, asset (money) markets and labor markets must clear in every country. This also implies that each DM-buyer's sequential (and therefore intertemporal) budget constraint must hold. We will evaluate these budget constraints at a monetary equilibrium. For the treatment below, we assume that seigniorage revenue x_t is transferred to the DM-buyers uniformly at the beginning of each CM.

First, we re-write the date- t budget constraint of a DM-buyer for end-of-period change in domestic real money holding:

$$\phi_t(m_{t+1} - m_t) = N_t - C_t + \phi_t x_t - \phi_t^f (m_{t+1}^f - m_t^f), \quad \forall t \geq 0, \quad (53)$$

where m_t (or m_t^f) is the initial stock of domestic (or foreign) currency held by the DM-buyer. Also, ϕ_t (or $\phi_t^f = e_t \phi_t$), given an equilibrium nominal exchange rate e_t , is the value of a unit of domestic (or foreign) currency in units of the domestic (or foreign) CM consumption good C_t (or C_t^f). In equilibrium N_t the amount of labor supplied is also the labor income to the DM-buyer.

Summing these sequential budget constraints up to some finite date $T > 0$, multiplying both sides by a constant $\beta^T \in (0, 1)$, and taking the limit of $T \nearrow +\infty$, we have:

$$\begin{aligned} \lim_{T \nearrow +\infty} \beta^T \phi_{T+1} m_{T+1} &= \lim_{T \nearrow +\infty} \beta^T \left(\frac{\phi_T}{\phi_0} \right) \left\{ \phi_0(m_0 + e_0 m_0^f) + \sum_{t=0}^T \left(\frac{\phi_0}{\phi_t} \right) [\phi_t x_t + N_t - C_t] \right. \\ &\quad \left. + \phi_0 \sum_{t=0}^T \left(\frac{e_{t+1} - e_t}{e_t} \right) e_t m_{t+1}^f \right\}, \end{aligned} \quad (54)$$

where $(e_{t+1} - e_t)/e_t \equiv (\phi_{t+1}^f/\phi_{t+1} - \phi_t^f/\phi_t)/(\phi_t^f/\phi_t)$ is the one-period growth rate in the nominal exchange rate.

The first term on the LHS of (54) equals zero by the transversality condition on assets. That is, in the infinite-horizon limit, the discounted value of accumulated domestic real money balances must be zero in a monetary equilibrium. The first term on the RHS also goes to zero, since in any monetary equilibrium the T -period return on domestic money ϕ_T/ϕ_0 must be finite-valued, and, the given initial real balances on both monies $\phi_0(m_0 + e_0 m_0^f)$ are also finite. The second term on the RHS must equal zero since the infinite series of real wealth (including real seigniorage transfers) net of CM consumption expenditure must be finite in any well-defined equilibrium. Since $\beta \in (0, 1)$, then this term must be zero in the infinite-horizon limiting economy. Therefore, that leaves one with the final term on the RHS which must then equal zero to satisfy the conditions for any monetary general equilibrium. We thus arrive at the following observation:

Lemma 3 *Assuming monetary equilibria with coexistence of both monies, then in any infinite-horizon monetary equilibrium, the discounted total changes in the real value of foreign money holdings of any DM-buyer must be zero,*

$$\lim_{T \nearrow +\infty} \left\{ \beta^T \phi_T \sum_{t=0}^T \left(\frac{e_{t+1} - e_t}{e_t} \right) e_t m_{t+1}^f \right\} = 0.$$

This fact will allow us to deduce an admissible property of the *nominal exchange rate* path consistent with the existence of a monetary equilibrium (with coexistence of both monies). In particular we have the following result summarized in Proposition 2.

Proof of Proposition 2. Lemma 3 and the fact that $\beta^T \in (0, 1)$ for any $T = 0, 1, 2, \dots$, implies that for a monetary equilibrium to exist, it is necessary to have

$$\lim_{T \nearrow +\infty} \left| \sum_{t=0}^T \left(\frac{e_{t+1}}{e_t} - 1 \right) \left(\frac{\phi_T}{\phi_t} \right) \phi_t^f m_{t+1}^f \right| < +\infty.$$

Let the one-period growth rate in the nominal exchange rate and the end of date- t CM holding of foreign currency real balance (in units of date- T CM good), respectively, be denoted as $a_t := (e_{t+1}/e_t - 1)$, and, $b_t := (\phi_T/\phi_t) \phi_t^f m_{t+1}^f$.

Now let the partial sum $\sum_{s=0}^t b_t =: B_t$. By Abel's Lemma (summation by parts), the date- T series can be transformed as follows:

$$\begin{aligned} S_T &\equiv \sum_{t=0}^T a_t b_t = a_0 b_0 - a_0 B_0 + a_T B_T + \sum_{t=0}^{T-1} B_t (a_t - a_{t+1}) \\ &= a_T B_T - \sum_{t=0}^{T-1} B_t (a_{t+1} - a_t). \end{aligned}$$

For any integer $t \geq 0$, any integer $k \geq 0$, and some finite real number a , we can write

$$S_{t+k} - S_t = (a_{t+k} - a)B_{t+k} + (a_t - a)B_t + a(B_{t+k} - B_t) + \sum_{s=0}^{t+k-1} B_s(a_s - a_{s+1}).$$

Suppose the partial sums B_t do not converge: $|B_t| \rightarrow +\infty$. Then $\lim_{t \nearrow +\infty} |S_{t+k} - S_t| \neq 0$, and, by the Cauchy convergence criterion, this implies that

$$\lim_{T \nearrow +\infty} |S_T| \equiv \lim_{T \nearrow +\infty} \left| \sum_{t=0}^T \left(\frac{e_{t+1}}{e_t} - 1 \right) \left(\frac{\phi_T}{\phi_t} \right) \phi_t^f m_{t+1} \right| = +\infty.$$

But this violates Lemma 3 in a monetary equilibrium. Thus, the B_t partial sums must be convergent in any monetary equilibrium.

Therefore, we may assume some upper bound $0 \leq B < +\infty$ for the sequences of absolute partial sums, $\{|B_t|\}_{t \geq 0}$. Note that

$$\begin{aligned} |S_{t+k} - S_t| &= \left| (a_{t+k} - a)B_{t+k} + (a_t - a)B_t + a(B_{t+k} - B_t) + \sum_{s=0}^{t+k-1} B_s(a_s - a_{s+1}) \right| \\ &\leq |a_{t+k} - a||B_{t+k}| + |a_t - a||B_t| + \left| \sum_{s=0}^{t+k-1} B_s(a_s - a_{s+1}) \right| \\ &\leq |a_{t+k} - a|B + |a_t - a|B + |a_{t+k} - a_t|B. \end{aligned}$$

Thus, by applying the Cauchy convergence criterion again, the nominal exchange rate's absolute growth rate converging to a constant $\gamma_e \equiv a \in [0, +\infty)$ in the infinite horizon economy, i.e. $(a_t \rightarrow a) \Leftrightarrow |a_{t+k} - a_t| \rightarrow 0$, then $|S_{t+k} - S_t| \rightarrow 0 \Leftrightarrow \lim_{T \nearrow +\infty} S_T < +\infty$, satisfies the requirement of Lemma 3. That is, it satisfies CM market clearing and agents' intertemporal budget constraints in any monetary equilibrium. ■

Remark 1 *In the rest of the paper, we can thus focus on monetary equilibria in which the equilibrium exchange rate grows at some constant rate. Also, Proposition 2 will apply in the explicit two-country version of the world economy below. All that is required is to set $x_t = 0$ for every date t in the sequential CM budget constraints, since x_t will later appear in the DM feasibility constraint, since transfers will occur after each CM closes.*

E Proof of Proposition 3 (Equilibria and Coexistence)

We will only prove the two cases of $\Pi > \Pi^f$ (Case 1) and $\Pi = \Pi^f$ (Case 3). Given each case, there are $\binom{4}{4} + \binom{4}{3} + \binom{4}{2} + \binom{4}{1} + \binom{4}{0} = 16$ subcases (i.e., combinations of the total of four liquidity constraints) to consider. However, depending on the magnitude of Π relative to Π^f , we can easily show that some of the subcases cannot exist as an equilibrium.

Case 1(a): $\Pi > \Pi^f$ with the following subcases.

1. $\lambda = \lambda^f = \lambda_\star^f = \lambda_\star = 0$;
2. $\lambda > 0$ and $\lambda^f = \lambda_\star^f = \lambda_\star = 0$;
3. $\lambda = \lambda^f = 0, \lambda_\star^f = 0, \lambda_\star > 0$; or
4. $\lambda > 0, \lambda^f = \lambda_\star^f = 0$ and $\lambda_\star > 0$.

The characterization is similar to Case 1(a) of Proposition 5. Since $\Pi > \Pi^f$ and the liquidity constraint on holding Foreign money in both countries are slack, then all agents in all countries will only demand the Foreign fiat money. Thus, there is a unique monetary equilibrium with only the low inflation (Foreign) fiat money in circulation in both countries.

Case 1(b): $\Pi > \Pi^f$ with all the following subcases. All these exist as a unique monetary equilibrium exhibiting coexistence of both fiat monies.

5. $\lambda > 0, \lambda^f > 0, \lambda_\star^f = \lambda_\star = 0$:

Since both liquidity constraints for Home DM-buyers bind, then from the Home DM-seller's participation constraint (24b) and the first two KKT conditions in (24c), we have a unique solution to $q = c^{-1}[\bar{\kappa}(\Pi) + \bar{\kappa}(\Pi^f)]$. Also, from the global money market clearing condition (26a) on Home fiat currency we have that $\phi d_\star = \phi M - \bar{\kappa}(\Pi)$. However, since the Foreign fiat money dominates in rate of return, and Foreign DM-buyers are not liquidity constrained ($\lambda_\star^f = 0$), then $d_\star = 0$, and we can deduce that $\phi = \bar{\kappa}(\Pi)/M$.

Also, since Foreign DM-buyers demand all of their national fiat money, $d_\star^f > 0$ (or $\nu_\star^f = 0$ by complementary slackness), then there is a unique $q_\star^f \in (0, q^*)$ solving the Euler condition (25a): $\beta\sigma[u'(q)/c'(q) - 1] = \Pi^f - \beta$. From the Foreign DM-seller's participation constraint, we have $\phi^f d^f = \phi^f M^f - c(q_\star^f)$. Since $\lambda^f > 0$, we have $\phi^f d^f = \bar{\kappa}^f(\Pi^f) = \phi^f M^f - c(q_\star^f)$. Rearranging, we have $\phi^f = [\bar{\kappa}^f(\Pi^f) + c(q_\star^f)]/M^f$.

Therefore there is a unique nominal exchange rate

$$e := \frac{\phi^f}{\phi} = \left[\frac{\bar{\kappa}^f(\Pi^f) + c(q_\star^f)}{\bar{\kappa}(\Pi)} \right] \cdot \frac{M}{M^f}.$$

There is only coexistence of both monies in the Home country's DM. Only the Foreign currency circulates in the Foreign DM.

6. $\lambda = 0, \lambda^f > 0, \lambda_\star^f = \lambda_\star = 0$:

From the Home DM-buyers' Euler conditions (24a), we have that $\nu = \Pi - (\Pi^f + \lambda^f) = 0$ since the DM-buyer is constrained in holding the Foreign fiat money. Thus, q solves $\beta\sigma[u'(q)/c'(q) - 1] = \Pi - \beta$ uniquely. From the Home DM-seller's participation constraint (24b) and the Home DM-buyers' binding liquidity constraint on Foreign real money balance, we have $c(q) -$

$\bar{\kappa}^f(\Pi^f) = \phi d$. Substituting this into the global clearing condition on Home fiat money (26a), and since no Foreign DM-buyers will optimally demand Home fiat money ($d_\star = 0$), we have $\phi = [c(q) - \bar{\kappa}^f(\Pi^f)]/M$. Also since Foreign DM-buyers demand only Foreign fiat money, then q_\star^f uniquely solves the Foreign Euler condition (25a): $\beta\sigma[u'(q)/c'(q) - 1] = \Pi^f - \beta$. Using this fact in the Foreign DM-sellers' participation constraint (25b), combined with the global adding-up condition on Foreign fiat money (26b) and the fact that $\phi^f d^f = \bar{\kappa}^f(\Pi^f)$, we have a determination of the inverse Foreign price level as $\phi^f = [c(q_\star^f) - \bar{\kappa}^f(\Pi^f)]/M^f$. Therefore there is a unique equilibrium nominal exchange rate

$$e := \frac{\phi^f}{\phi} = \left[\frac{c(q_\star^f) - \bar{\kappa}^f(\Pi^f)}{c(q) - \bar{\kappa}^f(\Pi^f)} \right] \cdot \frac{M}{M^f}.$$

There is only coexistence of both monies in the Home country's DM. The Home currency does not circulate in the Foreign DM.

7. $\lambda = 0, \lambda^f > 0, \lambda_\star^f > 0, \lambda_\star = 0$:

Since the Home DM-buyers' liquidity constraint on Foreign money is binding ($\lambda^f > 0$), and that on Home money is not ($\lambda = 0$), from the Home DM-buyers' Euler conditions (24a), we have that $\nu = \Pi - (\Pi^f + \lambda^f) = 0$. Thus, q solves $\beta\sigma[u'(q)/c'(q) - 1] = \Pi - \beta$ uniquely. From the Home DM-seller's participation constraint (24b) and the Home DM-buyers' binding liquidity constraint on Foreign real money balance, we have $c(q) - \bar{\kappa}^f(\Pi^f) = \phi d$. Substituting this into the global clearing condition on Home fiat money (26a), then Foreign DM-buyers' real demand for Home fiat money is $\phi d_\star = \phi M - [c(q) - \bar{\kappa}^f(\Pi^f)]$. Since the Foreign agents' demand for Foreign fiat money faces a binding constraint ($\lambda_\star^f > 0$), then from the Foreign Euler conditions (25a) under $\nu_\star = \lambda_\star = 0$, we have q_\star^f solving $\beta\sigma[u'(q_\star^f)/c'(q_\star^f) - 1] = \Pi - \beta$ uniquely. From the Foreign DM-sellers' participation constraint, together with $c(q) - \bar{\kappa}^f(\Pi^f) = \phi d$, we can derive the inverse Home price level as $\phi = [c(q_\star^f) - \bar{\kappa}_\star^f(\Pi^f) + c(q) - \bar{\kappa}^f(\Pi^f)]/M$. From the global Foreign money adding-up condition, we have $\bar{\kappa}_\star^f(\Pi^f) + \bar{\kappa}^f(\Pi^f) = \phi^f M^f$, which implies that $\phi^f = [\bar{\kappa}_\star^f(\Pi^f) + \bar{\kappa}^f(\Pi^f)]/M^f$. So then there is a unique nominal exchange rate

$$e := \frac{\phi^f}{\phi} = \left[\frac{\bar{\kappa}_\star^f(\Pi^f) + \bar{\kappa}^f(\Pi^f)}{c(q_\star^f) + c(q) - [\bar{\kappa}_\star^f(\Pi^f) + \bar{\kappa}^f(\Pi^f)]} \right] \cdot \frac{M}{M^f},$$

and the coexistence of both monies in both countries' DM.

8. $\lambda = 0, \lambda^f > 0, \lambda_\star^f = 0, \lambda_\star > 0$:

Since Foreign DM-buyers are not liquidity constrained in their demand for the high return money ($\lambda_\star^f = 0$), so that $\nu_\star^f = 0$ and their optimal portfolio consists of this asset (i.e., $d_\star^f > 0$ and $d_\star = 0$), then from their Euler conditions (25a), there is a unique q_\star^f satisfying $\beta\sigma[u'(q_\star^f)/c'(q_\star^f) - 1] = \Pi^f - \beta$. Since $d_\star = 0$, from the Foreign DM-sellers' participation constraint (25b), we have $\phi^f d_\star^f = c(q_\star^f)$. Using this result, together with $\phi^f d^f = \bar{\kappa}^f(\Pi^f)$ in

the global clearing condition on Foreign fiat money (26b), we have $\phi^f = [c(q_\star^f) + \bar{\kappa}^f(\Pi^f)]/M^f$. Also, from the global clearing condition on Home fiat money (26a) together with the fact that $d_\star = 0$, we have $\phi = [c(q) - \bar{\kappa}^f(\Pi^f)]/M$. Therefore, there is coexistence (only in the Home country), and only the Foreign currency circulates in the Foreign DM, and the nominal exchange rate is

$$e := \frac{\phi^f}{\phi} = \left[\frac{c(q_\star^f) + \bar{\kappa}^f(\Pi^f)}{c(q) - \bar{\kappa}^f(\Pi^f)} \right] \cdot \frac{M}{M^f}.$$

9. $\lambda > 0, \lambda^f = 0, \lambda_\star^f > 0, \lambda_\star = 0$:

Since DM-buyers in Home are not liquidity constrained on the higher return money, they will only demand the Foreign fiat money, and $d = 0$. From their Euler conditions, (24a), q solves $\beta\sigma[u'(q)/c'(q) - 1] = \Pi^f - \beta$, and from the Home DM-sellers' participation constraint (24b), we have Home DM-buyers' demand for real Foreign fiat money balance as $\phi^f d^f = c(q)$. Since the Foreign DM-buyers are liquidity constrained on the Foreign money, $\phi^f d_\star^f = \bar{\kappa}_\star^f(\Pi^f)$, then the global clearing condition on the Foreign fiat money (26b) implies a unique value for the Foreign fiat money $\phi^f = [\bar{\kappa}_\star^f(\Pi^f) - c(q)]/M^f$. Also, since $\lambda_\star^f > 0$ and $\lambda_\star = 0$, then q_\star^f uniquely satisfies $\beta\sigma[u'(q_\star^f)/c'(q_\star^f) - 1] = \Pi - \beta$ in the Foreign DM-buyers' Euler equation in (25a). From the global clearing condition on the Home fiat money (26a), we thus have $\phi = [c(q_\star^f) - \bar{\kappa}_\star^f(\Pi^f)]/M$. In this case, there is a unique equilibrium nominal exchange rate

$$e := \frac{\phi^f}{\phi} = \left[\frac{\bar{\kappa}_\star^f(\Pi^f) - c(q)}{c(q_\star^f) - \bar{\kappa}_\star^f(\Pi^f)} \right] \cdot \frac{M}{M^f},$$

and there is coexistence of both monies in the Foreign DM, while only the Foreign money circulates in the Home DM.

10. $\lambda > 0, \lambda^f > 0, \lambda_\star^f = 0, \lambda_\star > 0$:

Since Home DM-buyers are liquidity constrained on both monies, then from the Home DM-sellers' participation constraint (24b), we can uniquely deduce q such that $c(q) = \bar{\kappa}(\Pi) + \bar{\kappa}^f(\Pi^f)$. Since Foreign DM-buyers will only demand Foreign money so that $d_\star = 0$, then from the global clearing condition on the Home fiat money (26a), we thus have $\phi = \bar{\kappa}(\Pi)/M$. In the Foreign DM-seller's participation constraint (25b), we have $\phi^f d_\star^f = c(q_\star^f)$, where q_\star^f uniquely solves $\beta\sigma[u'(q_\star^f)/c'(q_\star^f) - 1] = \Pi^f - \beta$ in their Euler condition (25a). Using this—and the fact that Home DM-agents are also liquidity constrained on holding the Foreign fiat money—the global clearing condition on the Foreign fiat money (26b), implies $\phi^f = [c(q_\star^f) + \bar{\kappa}^f(\Pi)]/M^f$. Therefore, there is a unique nominal exchange rate

$$e := \frac{\phi^f}{\phi} = \left[\frac{c(q_\star^f) + \bar{\kappa}^f(\Pi^f)}{\bar{\kappa}(\Pi)} \right] \cdot \frac{M}{M^f},$$

and both monies circulate in the Home DM, but only the Foreign money circulates in the

Foreign DM.

11. $\lambda > 0, \lambda^f > 0, \lambda_\star^f > 0, \lambda_\star = 0$:

Here, Home DM-buyers are liquidity constrained on both monies, so that $\phi d = \bar{\kappa}(\Pi)$ and $\phi^f d^f = \bar{\kappa}^f(\Pi^f)$. For Foreign DM-buyer, since $\lambda_\star = 0$ and $\lambda_\star^f > 0$, then q_\star^f can be uniquely determined as $\beta\sigma[u'(q_\star^f)/c'(q_\star^f) - 1] = \Pi - \beta$ in their Euler condition (25a). Then from the Foreign DM-sellers' participation constraint (25b), we have $\phi d_\star = c(q_\star^f) - \bar{\kappa}_\star^f(\Pi^f)$, since DM-buyers are liquidity constrained on Foreign money, i.e. $\lambda_\star^f > 0$ or that $\phi^f d_\star^f = \bar{\kappa}_\star^f(\Pi^f)$. Using this in the global clearing condition on the Home fiat money (26a), we have $\phi = [c(q_\star^f) - \bar{\kappa}_\star^f(\Pi^f) + \bar{\kappa}(\Pi)]/M$. Since the Foreign fiat money has a liquidity constraint binding in each country, then from the the global clearing condition on the Foreign fiat money (26b), we have $\phi^f = [\bar{\kappa}_\star^f(\Pi^f) + \bar{\kappa}^f(\Pi^f)]/M^f$. Thus, there is a unique nominal exchange rate

$$e := \frac{\phi^f}{\phi} = \left[\frac{\bar{\kappa}_\star^f(\Pi^f) + \bar{\kappa}^f(\Pi^f)}{c(q_\star^f) - \bar{\kappa}_\star^f(\Pi^f) + \bar{\kappa}(\Pi)} \right] \cdot \frac{M}{M^f},$$

and both monies circulate in both Home and Foreign DMs.

12. $\lambda = 0, \lambda^f > 0, \lambda_\star^f > 0, \lambda_\star > 0$:

The proof here is similar to that of the previous Case 1(b)(11). In this case, there is a unique nominal exchange rate

$$e := \frac{\phi^f}{\phi} = \left[\frac{\bar{\kappa}_\star^f(\Pi^f) + \bar{\kappa}^f(\Pi^f)}{c(q) + \bar{\kappa}_\star^f(\Pi^f) - \bar{\kappa}(\Pi)} \right] \cdot \frac{M}{M^f},$$

and both monies circulate in both Home and Foreign DMs.

13. $\lambda > 0, \lambda^f = 0, \lambda_\star^f > 0, \lambda_\star > 0$:

The proof here is opposite to that of the previous Case 1(b)(10). We can easily show that there is a unique nominal exchange rate

$$e := \frac{\phi^f}{\phi} = \left[\frac{c(q) + \bar{\kappa}_\star^f(\Pi^f)}{\bar{\kappa}_\star(\Pi)} \right] \cdot \frac{M}{M^f},$$

and both monies circulate in the Foreign DM, but only the Foreign money circulates in the Home DM.

14. $\lambda = \lambda^f = 0, \lambda_\star^f > 0, \lambda_\star > 0$:

The proof here is opposite to that of the previous Case 1(b)(5). Therefore there is a unique nominal exchange rate

$$e := \frac{\phi^f}{\phi} = \left[\frac{\bar{\kappa}_\star^f(\Pi^f) + c(q)}{\bar{\kappa}_\star(\Pi)} \right] \cdot \frac{M}{M^f}.$$

There is only coexistence of both monies in the Foreign country's DM. Only the Foreign currency circulates in the Home DM.

15. $\lambda = \lambda^f = 0, \lambda_\star^f > 0, \lambda_\star = 0$:

Since Home DM-buyers are not liquidity constrained in their demand for the high return money ($\lambda^f = 0$), so that $\nu^f = 0$ and their optimal portfolio consists of this asset (i.e., $d^f > 0$ and $d = 0$), then from their Euler conditions (24a), there is a unique q satisfying $\beta\sigma[u'(q)/c'(q) - 1] = \Pi^f - \beta$. Since $d = 0$, from the Home DM-sellers' participation constraint (24b), we have $\phi^f d^f = c(q)$. Using this result, together with $\phi_\star^f d_\star^f = \bar{\kappa}_\star^f(\Pi^f)$ in the global clearing condition on Foreign fiat money (26b), we have $\phi^f = [c(q) + \bar{\kappa}_\star^f(\Pi^f)]/M^f$. Since Foreign DM-buyers are constrained on holding Foreign money ($\lambda_\star^f > 0, \lambda_\star = 0$), then q_\star^f can be uniquely determined as $\beta\sigma[u'(q_\star^f)/c'(q_\star^f) - 1] = \Pi - \beta$ in their Euler condition (25a). Also, from the global clearing condition on Home fiat money (26a) together with the fact that $d = 0$, we have $\phi = [c(q_\star^f) - \bar{\kappa}_\star^f(\Pi^f)]/M$. Therefore, there is coexistence (only in the Home country), and only the Foreign currency circulates in the Foreign DM, and the nominal exchange rate is

$$e := \frac{\phi^f}{\phi} = \left[\frac{c(q) + \bar{\kappa}_\star^f(\Pi^f)}{c(q_\star^f) - \bar{\kappa}_\star^f(\Pi^f)} \right] \cdot \frac{M}{M^f}.$$

16. $\lambda > 0, \lambda^f > 0, \lambda_\star^f > 0, \lambda_\star > 0$:

This case is obvious. When all liquidity constraints are binding in all countries, we have from the two adding up conditions on global monies, (26a) and (26b), a determination of relative prices as $\phi^f = [\bar{\kappa}^f(\Pi^f) + \bar{\kappa}_\star^f(\Pi^f)]/M^f$ and $\phi = [\bar{\kappa}(\Pi) + \bar{\kappa}_\star(\Pi)]/M$, respectively. The nominal exchange rate is

$$e := \frac{\phi^f}{\phi} = \left[\frac{\bar{\kappa}^f(\Pi^f) + \bar{\kappa}_\star^f(\Pi^f)}{\bar{\kappa}(\Pi) + \bar{\kappa}_\star(\Pi)} \right] \cdot \frac{M}{M^f},$$

and both monies circulate in both DMs.

Case 3(a): $\Pi = \Pi^f$ with all the following subcases.

1. $\lambda > 0, \lambda^f > 0, \lambda_\star^f > 0, \lambda_\star > 0$:

This case is obvious. When all liquidity constraints are binding in all countries, we have from the two adding up conditions on global monies, (26a) and (26b), a determination of relative prices as $\phi^f = [\bar{\kappa}^f(\Pi^f) + \bar{\kappa}_\star^f(\Pi^f)]/M^f$ and $\phi = [\bar{\kappa}(\Pi) + \bar{\kappa}_\star(\Pi)]/M$, respectively. The nominal exchange rate is

$$e := \frac{\phi^f}{\phi} = \left[\frac{\bar{\kappa}^f(\Pi^f) + \bar{\kappa}_\star^f(\Pi^f)}{\bar{\kappa}(\Pi) + \bar{\kappa}_\star(\Pi)} \right] \cdot \frac{M}{M^f},$$

and both monies circulate in both DMs.

Case 3(b): $\Pi = \Pi^f$ with all the following subcases. Any of the following configurations, in which at least one country's liquidity constraint is active (inactive) while all of its other liquidity constraints are inactive (active), i.e.,

2. $\lambda > 0, \lambda^f = \lambda_\star^f = 0$ and $\lambda_\star > 0$;
3. $\lambda = 0, \lambda^f > 0, \lambda_\star^f = \lambda_\star = 0$;
4. $\lambda = 0, \lambda^f > 0, \lambda_\star^f > 0, \lambda_\star = 0$;
5. $\lambda = 0, \lambda^f > 0, \lambda_\star^f = 0, \lambda_\star > 0$;
6. $\lambda > 0$ and $\lambda^f = \lambda_\star^f = \lambda_\star = 0$;
7. $\lambda = \lambda^f = 0, \lambda_\star^f = 0, \lambda_\star > 0$;
8. $\lambda > 0, \lambda^f = 0, \lambda_\star^f > 0, \lambda_\star = 0$;
9. $\lambda > 0, \lambda^f > 0, \lambda_\star^f = 0, \lambda_\star > 0$;
10. $\lambda > 0, \lambda^f > 0, \lambda_\star^f > 0, \lambda_\star = 0$;
11. $\lambda = 0, \lambda^f > 0, \lambda_\star^f > 0, \lambda_\star > 0$;
12. $\lambda > 0, \lambda^f = 0, \lambda_\star^f > 0, \lambda_\star > 0$; or
13. $\lambda = \lambda^f = 0, \lambda_\star^f > 0, \lambda_\star = 0$,

cannot exist as equilibrium.

Consider the equilibrium Euler conditions (24a) and (25a), which would reduce to the respective conditions $\lambda - \nu = \lambda^f - \nu^f$ and $\lambda_\star^f - \nu_\star^f = \lambda_\star - \nu_\star$. For example, suppose Case 3(b)(1) were an equilibrium. Then, since agents are indifferent between either currency, then we may assume $\nu = \nu^f = \nu_\star^f = \nu_\star = 0$. If so, we have

$$\begin{aligned} \beta\sigma[u'(q)/c'(q) - 1] &= \lambda - \nu + \Pi - \beta \\ &> \lambda^f - \nu^f + \Pi^f - \beta = \beta\sigma[u'(q)/c'(q) - 1], \end{aligned}$$

which is a contradiction. A similar argument applies to all the other configurations.

Case 3(c): $\Pi = \Pi^f$ with the following subcases.

14. $\lambda = \lambda^f = \lambda_\star^f = \lambda_\star = 0$;
15. $\lambda > 0, \lambda^f > 0, \lambda_\star^f = \lambda_\star = 0$ or
16. $\lambda = \lambda^f = 0, \lambda_\star^f > 0, \lambda_\star > 0$,

In all of these cases, there is a monetary equilibrium where both monies can coexist in both countries' DMs, but the nominal exchange rate is indeterminate.

In case 3(c)(14), since no liquidity constraints bind, and since all monies have the same rate of return, then $q = q_\star^f$ solves the same Euler conditions (24a) and (25a), so that $\beta\sigma[u'(q)/c'(q) - 1] = \Pi - \beta = \Pi^f - \beta$. Given $q = q_\star^f$, then the equilibrium conditions reduce to the global clearing conditions on Foreign and Home fiat money, respectively (26b) and (26a), and the seller's constraints in both contries, (24b) and (25b), determining $(d, d^f, d_\star^f, d_\star)$ for any arbitrary ratio $e := \phi^f/\phi \in [0, \infty)$.

In case 3(c)(15), since only the Home DM-buyers' liquidity constraints bind, we can show that any positive and finite (ϕ, ϕ^*) , such that there is a continuum of $e := \phi^f/\phi \in [0, \infty)$ that satisfies one equilibrium condition:

$$c(q_\star^f) = \phi M + \phi^f M^f - [\bar{\kappa}(\Pi) + \bar{\kappa}^f(\Pi^f)].$$

In case 3(c)(16), since only the Foreign DM-buyers' liquidity constraints bind, we can show that any positive and finite (ϕ, ϕ^*) , such that there is a continuum of $e := \phi^f/\phi \in [0, \infty)$ that satisfies one equilibrium condition:

$$c(q) = \phi M + \phi^f M^f - [\bar{\kappa}_\star^f(\Pi^f) + \bar{\kappa}_\star(\Pi)].$$

F Proof of Proposition 4 (Non First Best)

It suffices to construct a counterexample. Consider Case 3(a)(1) of Proposition 3. In this case, all liquidity constraints are binding, or $(\lambda, \lambda^f, \lambda_\star^f, \lambda_\star)$ are all strictly positive, and $\Pi = \Pi^f$. Now assume that the Friedman rule holds: $\Pi = \Pi^f = \beta$. From the steady-state monetary equilibrium characterizations of money demand, (24a) for Home, and, (25a) for Foreign, we can easily deduce that $u(q) - c'(q) > 0$ and $u(q_\star) - c'(q_\star) > 0$. In this case the nominal exchange rate is determinate but the Friedman rule is not optimal.

Now considers Cases 3(c)(15) and 3(c)(16) of Proposition 3. In the former case, by inspecting (24a), we have $u(q) - c'(q) > 0$, so that q is below what is attainable under a first best \tilde{q} which satisfies $u'(\tilde{q}) = c'(\tilde{q})$. In the latter case, from (25a), we have that $u(q_\star) - c'(q_\star) > 0$. In both cases, the nominal exchange rate is indeterminate. Thus, even when both the domestic and foreign inflation rates are set at the Friedman rule, the DM first best may not be attainable and the nominal exchange rate may not be determinate.

G Proof of Proposition 5 (Integrated World Economy)

Recall from (12), we denoted

$$\frac{\kappa}{\phi_{-1}/\phi - \beta(1 - \sigma)} =: \bar{\kappa}(\phi_{-1}/\phi); \quad \text{and,} \quad \frac{\kappa^f}{\phi_{-1}e_{-1}/\phi e - \beta(1 - \sigma)} =: \bar{\kappa}^f(\phi_{-1}e_{-1}/\phi e).$$

In steady state we have these, respectively, as $\bar{\kappa}(\Pi)$ and $\bar{\kappa}^f(\Pi^f)$.

After some elementary algebra, the necessary first-order conditions for a (steady-state) monetary equilibrium together with their Karush-Kuhn-Tucker (KKT) conditions are as follows: The set of Euler operators evaluated at steady state are

$$\beta\sigma \left[\frac{u'(q)}{c'(q)} - 1 \right] = \lambda - \nu + (\Pi - \beta) = \lambda^f - \nu^f + (\Pi^f - \beta). \quad (55a)$$

The DM-sellers' participation constraint is binding ($\zeta > 0$), so that

$$c(q) = \phi d + \phi^f d^f, \quad (55b)$$

and the KKT conditions are:

$$\begin{aligned} \lambda \cdot [\bar{\kappa}(\Pi) - \phi d] &= 0, & \lambda &\geq 0, & \phi d &\leq \bar{\kappa}(\Pi), \\ \lambda^f \cdot [\bar{\kappa}^f(\Pi^f) - \phi^f d^f] &= 0, & \lambda^f &\geq 0, & \phi^f d^f &\leq \bar{\kappa}^f(\Pi^f), \\ -\nu \cdot d &= 0, & d &\geq 0, & \nu &\geq 0, \\ -\nu^f \cdot d^f &= 0, & d^f &\geq 0, & \nu^f &\geq 0. \end{aligned} \quad (55c)$$

We must consider different cases depending on when the endogenous liquidity constraint associated with the local and foreign currency bind or not, given domestic (foreign) inflation rate, money supply, counterfeiting costs and the matching probability. We focus on equilibria that satisfy $\zeta > 0$, and at least one of ν and ν^f is zero, so that buyers and sellers trade in DM, and $\mu, \mu^f > 0$.

Case 1(a) $\Pi - \Pi^f > 0$ and

- (i) $\lambda = 0, \lambda^f = 0$: This case is trivial to check. When both liquidity constraints are not binding, and the foreign currency dominates in rate of return, buyers demand only the foreign fiat money, so that only M^f is in circulation.
- (ii) $\lambda > 0, \lambda^f = 0$. Since $\lambda > 0$ and $\lambda^f = 0$ and $\Pi^f < \Pi$, all agents optimally demand zero domestic fiat money, so that $d = 0$ or by complementary slackness $\nu > 0$, and, $d^f = M^f$ with $\nu^f = 0$.

In both subcases, we have the following characterization of a unique equilibrium outcome. First, from the equilibrium Euler condition (55a), q solves

$$\beta\sigma \left[\frac{u'(q)}{c'(q)} - 1 \right] - (\Pi^f - \beta) = 0.$$

Second, since $u' (c')$ is a continuous and monotone decreasing (non-decreasing) function on a compact set $[0, q^*]$, and since it is optimal to consume $q > 0$ and given that $\Pi^f - \beta > 0$, then there exists a unique solution $q \in (0, q^*)$ where q^* satisfies the first-best solution $u'(q) = c'(q)$. Therefore, there is a unique price level, $1/\phi^f$, determined from the DM-sellers' participation constraint (55b): $\phi^f = c(q)/M^f$.

Case 1(b): $\lambda = 0, \lambda^f > 0$ and $\Pi - \Pi^f > 0$. From the result in Lemma 2, we have in any monetary equilibrium, a buyer at the end of every CM will make offers of payments (d, d^f) up to the respective limits of their portfolio components (m, m^f) —i.e. $d = m \geq 0$ and $d^f = m^f \geq 0$ —which implies that the multipliers on payment upper-bounds are strictly positive:

$$\mu = \Pi - \beta > 0,$$

$$\mu^f = \Pi^f - \beta > 0.$$

In this case, the Euler conditions (55a) reduce to

$$\beta \sigma \left[\frac{u'(q)}{c'(q)} - 1 \right] - (\Pi - \beta) = 0.$$

which solve for a unique q , following similar arguments in Case 1(a), except that now we also have $\Pi - \beta > 0$.

Next, we show that there is coexistence of the home currency with the foreign currency, in spite of the former being dominated in its return, $\Pi > \Pi^f$. By construction the highest sustainable allocation of q is a $q^* > 0$ satisfying the first best trade-off: $u'(q^*) = c'(q^*)$. Comparing the first-best condition with the monetary equilibrium condition for q above, we can easily deduce that $q < q^*$ since $\Pi - \beta > 0$. Since the liquidity constraint on the foreign currency payment is binding, then, from the seller's participation constraint we can re-write as:

$$\phi m = c(q) - \bar{\kappa}^f(\Pi^f) \geq 0.$$

Suppose to the contrary that the demand for home currency were zero, $m = 0$. Then we have $q = c^{-1}(\bar{\kappa}^f(\Pi^f)) < q^*$. Since each buyer can increase his lifetime payoff by accumulating more domestic money ($\lambda = 0$) and offering it to the seller in the DM to consume more q ; and the seller would willingly accept it by producing more q while ensuring that her participation constraint is still binding, then we have in this equilibrium positive demand for home real currency, $\phi m = c(q) - \bar{\kappa}^f > 0$. In equilibrium in the integrated economy, $m = M$ and $m^f = M^f$.

From the DM-sellers' participation constraint (55b), there is a unique value for domestic fiat money:

$$\phi = \frac{c(q) - \bar{\kappa}^f(\Pi^f)}{M}.$$

Since $\lambda^f > 0$ or that the foreign money liquidity constraint binds (55c), then there is a unique value for the foreign fiat money:

$$\phi^f = \frac{\bar{\kappa}^f(\Pi^f)}{M^f}.$$

Therefore, there is a unique monetary equilibrium outcome in this case, with the nominal exchange determined as

$$e := \frac{\phi^f}{\phi} = \frac{M}{M^f} \cdot \frac{\bar{\kappa}^f(\Pi^f)}{c(q) - \bar{\kappa}^f(\Pi^f)}.$$

Case 1(c): $\lambda > 0, \lambda^f > 0$ and $\Pi - \Pi^f > 0$ It is easy to show that the resulting system of equations has a unique solution for $\{\mu, \mu^f, \phi, e, q, \lambda, \zeta\}$. For a given domestic and foreign inflation rates, money supplies, counterfeited costs and matching probability, the relevant block of the steady state equilibrium conditions is given as follows. First, as in the previous cases,

$$\mu = \Pi - \beta > 0$$

$$\mu^f = \Pi^f - \beta > 0.$$

Second, since the DM-buyer is liquidity constrained in both currencies, then in real terms, he would demand and offer payments up to the limits of both constraints: $\phi M = \bar{\kappa}(\Pi) > 0$ and $\phi e M^f = \bar{\kappa}^f(\Pi^f) > 0$ as measured in units of the home CM good. From the home currency liquidity constraint, we can solve for

$$\phi = \frac{\bar{\kappa}(\Pi)}{M};$$

and then using this in the foreign currency liquidity constraint, we can derive a unique equilibrium nominal exchange rate

$$e = \frac{M}{M^f} \frac{\bar{\kappa}^f(\Pi^f)}{\bar{\kappa}(\Pi)}.$$

Finally, the other relevant equilibrium conditions:

$$c(q) = \bar{\kappa}^f(\Pi^f) + \bar{\kappa}(\Pi);$$

$$\sigma \beta \frac{u'(q) - c'(q)}{c'(q)} = \mu + \lambda;$$

$$\zeta = \beta \sigma + \Pi - \beta + \lambda;$$

$$\lambda^f = \Pi - \Pi^* + \lambda,$$

pin down a unique q, λ, ζ and λ^f , respectively. Therefore, in this case, there is a determinate monetary equilibrium, with a unique nominal exchange rate, and coexistence of both currencies.

Case 2: $\Pi - \Pi^f < 0$ This is the symmetric opposite to the analyses in Case 1. Therefore there can exist a unique steady state e and coexistence of the two currencies, in spite of $\Pi < \Pi^f$.

Case 3(a): $\lambda = \lambda^f = 0$ and $\Pi - \Pi^f = 0$. This case corresponds to the indeterminacy result in Kareken and Wallace. Since both liquidity constraints are not binding, and both currencies yield equal rates of return, then buyers are indifferent as to which currency to hold and sellers' participation constraint binds for any composition of payments offered.

Case 3(b): $\lambda \neq \lambda^f > 0$ and $\Pi - \Pi^f = 0$. In this case, when both liquidity constraints bind, the analysis is similar to Case 1(c) above. Therefore we have coexistence of the two currencies and determinacy of the equilibrium nominal exchange rate.

Case 3(c): $\Pi - \Pi^f = 0$ **and** $(\lambda > 0 \text{ and } \lambda^f = 0) \text{ or } (\lambda = 0 \text{ and } \lambda^f > 0)$. We can rule out these two cases as equilibria. Since one liquidity constraint binds in either subcases, then $\nu = \nu^f = 0$, but this implies that either $\lambda + \Pi - \beta > \lambda^f + \Pi^f - \beta$ in subcase (i), or, $\lambda + \Pi - \beta < \lambda^f + \Pi^f - \beta$, which respectively, is an impossibility since $\Pi = \Pi^f$ so that this violates the Euler conditions (55a). Therefore these two cases cannot be an equilibrium.

H Coexistence, Return Dominance and Counterfeiting Costs

Here we derive the inequalities shown in (27) and (28) in Section 5, under the assumption $\Pi > \Pi^f$. Recall Case 1a and its subcases imply no coexistence with only the foreign (high return) money in equilibrium circulation. Case 1a is complementary to the following two.

Case 1b: $\Pi > \Pi^f > \beta$ **and** $\lambda^f > \lambda = 0$. From the Euler conditions at steady state, the equilibrium allocation \hat{q} satisfies

$$u'(q) = \frac{\Pi - \beta(1 - \sigma)}{\beta\sigma} c'(q).$$

Since the inefficiency wedge, $[\Pi - \beta(1 - \sigma)]/\beta\sigma$, is strictly positive due to $\Pi > \beta(1 - \sigma)$, and since the derivative function c' is continuous and nondecreasing, and u' is continuous and monotonically decreasing, then the monetary equilibrium solution $\hat{q} < q^*$ where q^* is the efficient solution to $u'(q) = c'(q)$.

Since the foreign money liquidity constraint binds, then the seller's participation constraint in this equilibrium is

$$c(\hat{q}) = \phi M + \frac{\kappa^f}{\Pi^f - \beta(1 - \sigma)}.$$

This condition also used the equilibrium requirement in the integrated world economy of $d = m = M$.

Since real balance ϕM have to be non-negative, and since the domestic liquidity constraint is slack, then we have the two respective inequalities

$$0 \leq \phi M = c(\hat{q}) - \frac{\kappa^f}{\Pi^f - \beta(1 - \sigma)} < \frac{\kappa}{\Pi - \beta(1 - \sigma)}.$$

Tidying up yields the expression in (27).

Case 1c: $\Pi > \Pi^f > \beta$, $\lambda^f > 0$, **and** $\lambda > 0$. Since both liquidity constraints are binding in this case, then the seller's participation constraint implies a unique solution \tilde{q} such that

$$c(\tilde{q}) = \frac{\kappa}{\Pi - \beta(1 - \sigma)} + \frac{\kappa^f}{\Pi^f - \beta(1 - \sigma)}.$$

Also, since both liquidity constraints are binding, and the seller's participation constraint holds with equality, the equilibrium outcome is $\tilde{q} \leq \hat{q}$, where \hat{q} was defined in the solution to Case 1b's equilibrium. Therefore,

$$c(\tilde{q}) = \frac{\kappa}{\Pi - \beta(1 - \sigma)} + \frac{\kappa^f}{\Pi^f - \beta(1 - \sigma)} \leq c(\hat{q}),$$

given that $\kappa > 0$ and $\kappa^f > 0$. Rearranging, we have the inequalities restricting (κ, κ^f) in (28).