Breaking the Curse of Kareken and Wallace[*]

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Abstract

We study the endogenous choice to accept fiat objects as media of exchange and the implications for nominal exchange rate determination. To this end, we consider an integrated economy with two different monetary authorities and where agents can use any currency to settle transactions. However, currencies can be counterfeited at a fixed cost and the decision to counterfeit is private information. This imposes an equilibrium liquidity constraint on currencies in circulation. We show that the threat of counterfeiting can pin down the nominal exchange rate and break the Kareken-Wallace indeterminacy result. We also show that the mechanism explored in this paper to determine nominal exchange rates, the threat of counterfeiting, is operative in a variety of different trading environments, geographical separations and timing of new money supply transfers. Finally, we show that with appropriate fiscal policies we can enlarge the set of monetary equilibria with determinate nominal exchange rates.

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1 Introduction

During the eighteenth to nineteenth century economic history of the United States, the following facts were well known: Many emerging markets were created and bankers being the principal capitalists, underwrote most business enterprises. As part of the growing financial activity, banks and other trading entities also acted as (local) providers of various media of exchange, even if these notes were not legal tender. One literally observed the emergence and coexistence of media of exchange, in response to limited specie. Indeed, as Mihm (2007) noted:

> By the 1850s, with so many entities commissioning bank notes of their own design, the money supply became a great confluence of more than ten thousand different kinds of paper that continually changed hands, baffled the uninitiated, and fluctuated in value according to the whims of the market. Thousands of different kinds of gold, silver, and copper coins issued by foreign governments and domestic merchants complicated the mix.

In more modern times, we can still observe the coexistence of media of exchange, despite certain media being dominated in rate of return and legal restrictions.

However, such observed phenomena are not readily available from standard theories in international macroeconomics and finance, at least not without some ad-hoc or exogenous assumptions. When agents have unrestricted access to currency markets and are free to use any currency as means of payment, Kareken and Wallace (1981) showed that the rate of return on the two currencies must be identical for both of them to circulate.[1] More surprising is that, in this case, the nominal exchange rate between these currencies is indeterminate. In the last three decades since Kareken and Wallace, no one has been able to generate nominal exchange rate determinacy without imposing *ad hoc* frictions that inhibit trade using one or more of the currencies. We refer to this as 'the curse of Kareken and Wallace'. The frictions include currencies in the utility function, imposing restrictions on the use of currency for certain transactions and assuming differential transaction costs. By assuming unique liquidity properties for each currency, nominal exchange rate determinacy is effectively imposed on the model. A more desirable approach is to have the liquidity properties of currencies determined endogenously implying that the determinacy, or indeterminacy, of the nominal exchange rate is an equilibrium outcome.[2] This is the approach we take in this paper with our main con-

[1] With perfect currency substitution, there is only one single world market clearing condition determining the supplies and demands of all currencies jointly. Thus an indeterminate monetary equilibrium can only be pinned down by an exogenous selection of the nominal exchange rate. This exogenous information is often interpreted as arbitrary speculation.

[2] Consequently, we are trying to adhere to Wallace's dictum (1998). We interpret Wallace's dictum to mean that 1) monetary economists should explain why fiat money is essential, not assume that it is; 2) the value of fiat money should be determined without resorting to ad hoc restrictions; and 3) any "good" model of money

tribution being that we break the curse of Kareken and Wallace without imposing ad hoc restrictions on the liquidity properties of the currencies.

In this paper we study the endogenous choice to accept fiat objects as media of exchange, the fundamentals that drive their acceptance, and their implications for their bilateral nominal exchange rate. To this end, we consider an integrated world economy with two governments where a medium of exchange is essential in the tradition of Rocheteau and Wright (2005) or Lagos and Wright (2005). Agents have no restrictions on what divisible fiat currency can be used to settle transactions nor is there a cost advantage of trading one currency over the other. What renders two fiat currencies imperfect substitutes is the existence of private information regarding their quality. We build on the insights of Li, Rocheteau and Weill (2012) and allow both fiat currencies to be counterfeited at a fixed cost. Since sellers cannot recognize counterfeit currency, in equilibrium they put a limit on how much of each currency they are willing to accept. These limits in turn are endogenous and depend on the relative inflation rates associated with each currency.

Our key results are as follows. When neither limit is binding, the nominal exchange rate is indeterminate. However, if the limit is binding for one or both currencies, then we have nominal exchange rate determinacy. When both limits are binding and the currencies are identical in every respect, i.e., same counterfeiting costs and rate of return, we obtain the surprising result that the nominal exchange rate is the ratio of the two money stocks, which is the standard solution coming out of a two country cash in advance model. Another surprising result is that the first best may not be attainable even if the Friedman rule is implemented for both currencies. The reason is that the counterfeiting limits may still bind such that the first best quantity of goods cannot be purchased. We also show that the mechanism explored in this paper to determine nominal exchange rates, the threat of counterfeiting, is operative in a variety of different environments. This situation allow us to think more generally how counterfeiting in assets denominated in various currencies can help determine nominal exchange rates. We also show that when there is nominal exchange rate indeterminacy there exists fiscal policies that can restore determinacy of the nominal exchange rate. Finally, we also show that our result is robust to an alternative timing and composition of transfers of seigniorage revenue. (This alternative setup results in an explicit international finance model with spatially separated decentralized trades in each country which will be more familiar to the standard literature on international macroeconomics.)

An interesting feature of our results is that there is no counterfeiting in equilibrium. It is the threat of counterfeiting that pins down the nominal exchange rate and because of this

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should have a non-monetary equilibrium as a possibility.

As a corollary, we argue that 1) monetary economists should explain why the nominal exchange rate is determinate, not assume that it is; 2) determine its value without resorting to ad hoc restrictions on currencies and 3) a "good" model of fiat currencies should have nominal exchange rate indeterminacy as a possible equilibrium outcome.

both currencies can circulate even though one of them is dominated in rate of return. This is interesting because empirical evidence suggests that observed counterfeiting of currencies is not a big problem in practice as substantial resources and penalties are applied to those that counterfeit.[3] Our results show that even if counterfeiting is not important quantitatively it is nevertheless of first-order importance for nominal exchange rate determination. We show that the threat of counterfeiting mechanism is robust in determining nominal exchange rates. In particular, when liquidity constraints bind, due to the private information problem, help determine nominal exchange rates: (i) when trade credit is possible, (ii) when currencies can be used as collateral and (ii) when nominal bonds denominated in domestic and foreign currency can be counterfeited.

In what follows, Section 2 reviews the literature and Section 3 describes the model environment. In particular, the key private-information friction giving rise to the endogenous liquidity constraints is described and the equilibrium of an associated signalling game is characterized. The equilibrium characterization of the game is then embedded in a general monetary equilibrium in Section 4. In this section, we also consider the implications of the endogenous liquidity constraints for equilibrium and exchange rate determinacy. In Section 5, we explore the robustness of the proposed mechanism, the threat of counterfeiting, in helping determine nominal exchange rates by considering a variety of trading environments and alternative timing and composition assumptions regarding monetary transfers. We also discuss how cross-country international monetary policies, and, in conjunction with domestic fiscal policy may further rescue the economy from the Kareken and Wallace indeterminacy result. Finally, Section 6 offers some concluding remarks. All proofs are given is a Supplementary Appendix.

2 Related literature

Models in mainstream international monetary economics typically pin down the value of a currency by imposing exogenous assumptions on what objects may be used as media of exchange. For instance, Stockman (1980) and Lucas (1982), among others, assume that in order to buy a good produced by a particular country, only that country's currency can be used. That is, in these environments, the demand for a specific fiat currency is solely driven by the demand for goods produced by that particular country. Devereaux and Shi (2013) study a trading post model under the assumption that there is only bilateral exchange at each trading post. Thus, by assumption, the ability to pay for goods with combinations of currencies is elimi-

[3] Central Banks around the world spend resources to prevent counterfeiting by incorporating several security features on notes. Also, counterfeiting currencies is a punishable criminal offence. Several law enforcement entities like INTERPOL, the United States Secret Service and Europol as well as the European Anti-Fraud Office (OLAF), European Central Bank, the US Federal Reserve Bank, and the Central Bank Counterfeit Deterrence Group provide forensic support, operational assistance, and technical databases in order to assist countries in addressing counterfeit currency on a global scale. All these features and efforts substantially reduce the number of circulating counterfeited notes.

nated. Assumptions of this sort are *exogenous* currency constraints. By construction, they yield determinacy in agents' portfolio holdings of any two fiat currencies, and therefore determinacy in their nominal exchange rate.[4] Other researchers have introduced local currency in the utility function as in Obstfeld and Rogoff (1984) or have assumed differential trading cost advantages through network externalities as in Uribe (1997). In short, *endogenous* currency choice effectively is assumed away in this literature.

In the early search theoretic models of money, agents are able to choose which currencies to accept and use for payment. This literature shows that multiple currencies can circulate even if one is dominated in rate of return and the nominal exchange rate is determinate [see Matsuyama, Kiyotaki and Matsui (1993), Zhou (1997) and Waller and Curtis (2003), Craig and Waller (2004), Camera, Craig and Waller (2004)]. In these models, currency exchange can occur in bilateral matches if agents' portfolios are overly weighted towards one currency or the other. In fact, this leads to a distribution of determinate exchange rates. However, these findings are driven solely by the decentralized nature of exchange since agents never have access to a centralized market for rebalancing their portfolios. Once agents have the ability to rebalance their currency holdings, be it by the large family assumption in Shi (1997) or the periodic centralized market structure in Lagos and Wright (2005), the curse of Kareken and Wallace rears its head. To get around the curse, Head and Shi (2003) consider an environment where the large household can hold a portfolio of currencies but individual buyers are constrained to hold only one currency. So although the household endogenously chooses a portfolio of currencies, bilateral exchange requires using one currency or the other but not both simultaneously. In another paper, Liu and Shi (2010) assume that buyers can offer any currency but sellers can only accept one currency. Thus, the main contribution of our paper relative to this literature is that we have centralized exchange and no exogenous restrictions on currency exchange yet we can obtain nominal exchange rate determinacy.

The paper closest in spirit to ours is that of Zhang (2014), who considers a recognizability problem between currencies *à la* Lester, Postlewaite and Wright (2012) where sellers have to purchase a costly technology to detect counterfeits. In particular, Zhang (2014) allows fiat money to be counterfeited at no cost once the buyer has learnt whether the matched seller has the technology to detect counterfeits. Because producing a counterfeit when the seller is uninformed is a dominant strategy, unrecognizable fiat currency cannot be used as means of payment in a fraction of the matches where sellers do not have the relevant detection technology.[5] To determine the exchange rate Zhang (2014) assumes that local currency is only accepted by local government authorities, making currencies imperfect substitutes by

[4] In another strand of literature coined as the "New Open Economy Macroeconomics", which is partially summarized in Obstfeld and Rogoff (1996) and used extensively for monetary policy prescriptions, similar assumptions are in place.

[5] The technology adoption decision of sellers allows for strategic complementarities so that multiple equilibria exist.

assumption.[6]

3 Model

We consider an environment where agents can trade in an integrated world economy with two governments that can issue two independent fiat currencies. In this environment a medium of exchange is essential and agents face private information in some markets. We assume a per-period sequential decentralized-then-centralized market structure and anonymous trading in decentralized markets as in Lagos and Wright (2005) so that a medium of exchange is essential.

General Description The integrated world economy has a continuum of agents of measure 2. Time is discrete and indexed by $t \in \mathbb{N} := \{0, 1, 2, ...\}$. Each period is divided into two sub-periods with different trading protocols and informational frictions. In the first subperiod, anonymous agents meet pairwise and at random in a decentralized market (DM).[7] Sellers in this market face informational asymmetry regarding the quality of the fiat currencies to be exchanged for goods.[8] These currencies are perfectly divisible and are the only assets in the economy which can grow potentially at different constant rates. In the second sub-period, all activity occurs in a full information and frictionless centralized market (CM).[9]

DM production is specialized and agents take on fixed trader types so that they are either buyers (consumers) or sellers (producers).[10] In CM all agents can produce and consume an homogenous perishable good. Agents, in this market can trade the CM good and rebalance their portfolio.

Preferences Agents derive utility from DM and CM consumption and some disutility from effort. A common discount factor $\beta \in (0, 1)$ applies to utility flows one period ahead. Given the specialization structure in DM where buyers want to consume but cannot produce and

[6]Zhang (2014) shows that the nominal exchange rate could still be determined even both currencies are imperfectly recognizable (though buyers can costlessly counterfeit them, so that in equilibrium sellers only accept recognizable currencies), each with potentially different (or even the same) fixed costs of recognizing.

[7]The search literature uses the term anonymity to encompass these three frictions: (i) no record-keeping over individual trading histories ("memory"), (ii) no public communication of histories and (iii) insufficient enforcement (or punishment). An environment with any of these frictions imply that credit between buyer and seller is not incentive compatible.

[8]The integrated world economy has locations (countries) that are identical within each submarket and all agents can freely move to any locations. In other words, we can interpret this integrated world economy as a closed economy.

[9]The two different governments could use their respective different currencies to finance some of their expenditures of the CM homogenous good.

[10]The justification for this assumption is twofold. First, it allows for a simple description of production specialization as in Alchian (1977). Second, it allows us to abstract away from the additional role of money as a medium of insurance against buyer/seller idiosyncratic shocks. An instance of the latter can be found in the Lagos and Wright (2005) sort of environment.

sellers can produce but do not want to consume. The (discounted) total expected utility of a DM-buyer is given by

$$\mathbb{E}\left\{\sum_{t=0}^{\infty}\beta^{t}\left[u(q_t)+\mathcal{U}(C_t)-N_t\right]\right\}, \tag{1}$$

where q_t represents DM goods, N_t is the CM labor supply and C_t denotes consumption of perishable CM good. Finally, $\mathbb{E}$ is a linear expectations operator with respect to an equilibrium distribution of idiosyncratic agent types.[11] The utility function $u : \mathbb{R}_+ \to \mathbb{R}$ is such that $u(0) = 0$, $u'(q) > 0$ and $u''(q) < 0$, for all $q \in \mathbb{R}_+$. Also, $\mathcal{U} : \mathbb{R}_+ \to \mathbb{R}$ has the property that $\mathcal{U}(0) = 0$, $\mathcal{U}'(C) > 0$, and $\mathcal{U}''(C) < 0$, for all $C \in \mathbb{R}_+$.

The (discounted) total expected utility of a DM-seller is given by

$$\mathbb{E}\left\{\sum_{t=0}^{\infty}\beta^{t}\left[-c(q_t)+\mathcal{U}(C_t)-N_t\right]\right\}, \tag{2}$$

where the utility cost function $c : \mathbb{R}_+ \to \mathbb{R}_+$ is such that $c(0) = 0$, $c'(q) > 0$, and $c''(q) \geq 0$. Note that DM-buyers and DM-sellers have identical per-period payoff functions in the CM subperiod given by $\mathcal{U}(C) - N$ as both types of agents can consume and produce in this frictionless market.

Information and Trade Since agents in DM have fixed types and production is specialized, agents face a double coincidence problem. Moreover, since buyers and sellers in DM are anonymous, the only incentive compatible form of payment is fiat money. Buyers and sellers have access to two different and divisible fiat currencies: domestic and foreign. Following Kareken and Wallace (1981), and in contrast to mainstream international macroeconomics, we do not impose any restrictions on which of the currencies, nor the compositions thereof, can be used to settle transactions. However, sellers face asymmetric information, as in Li, Rocheteau and Weill (2012), regarding the quality of the currencies when trading in DM. In the next sections we describe in detail the sub-period trades. The precise information problem that buyers and sellers are facing will be described in more detail below.

3.1 Centralized Market

After trade occurs in DM, agents have access to a frictionless Walrasian world integrated market (CM). At the beginning of each CM, DM-buyers have (possibly) positive balances of domestic and foreign fiat monies, respectively, m_t and m_t^f. Before they make decisions and participate in the CM good, labor and international asset markets, they receive a total

[11] The model has no aggregate random variables, and therefore, it will turn out that the equilibrium distribution of agent types will depend only on the idiosyncratic random-matching probability $\sigma \in (0, 1)$, and the equilibrium probabilities concerning the acceptability and genuineness of assets in an exchange, respectively, $\pi \in [0, 1]$ and $(\eta, \eta^f) \in [0, 1]^2$. The expected utility setup will be made more precise later.

lump-sum transfer of seigniorage revenues in domestic and foreign currencies, respectively, x_t and x_t^f.[12] Then, CM agents can trade goods and rebalance their portfolio of domestic and foreign currencies and decide whether to counterfeit or not. In particular, before trading in the next DM the each buyer has to decide whether to counterfeit these fiat currencies or not in the current CM. Following Nosal and Wallace (2007) and Li, Rocheteau and Weill (2012), we assume that no fraudulent fiat currencies can be traded for goods in CM as these are detectable by the two governments who can confiscate and destroy them.[13] CM goods are produced with a linear technology that all agents have access to, thus a medium of exchange in this market is not essential.

Each DM agent faces a sequential budget constraint given by

$$C_t \leq N_t - \phi_t[m_{t+1} - y_t] - \phi_t e_t[m_{t+1}^f - y_t^f], \tag{3}$$

where C_t denotes consumption of the CM good, and $y_t := m_t + x_t$ and $y_t^f := m_t^f + x_t^f$, respectively, are initial holdings of genuine domestic and foreign money (including the transfers). The variable e_t represents the current nominal exchange rate which measures how much of the domestic currency exchanges for one unit of the foreign currency and m_{t+1} and m_{t+1}^f are the end-of-period currency portfolio.[14] Finally, ϕ_t ($\phi_t e_t$) denotes the value of domestic (foreign) currency in units of the CM good C_t.

DM-buyers and DM-sellers choose CM labor, end-of-period currency portfolio and consumption of CM goods. In this frictionless market DM-buyers have the possibility to costly counterfeit both fiat currencies. The cost of counterfeiting is common knowledge and is assumed to be a per-period fixed cost $\kappa > 0$ ($\kappa^f > 0$) for domestic (foreign) currency. However, when trading in DM, sellers are not able to distinguish between genuine and counterfeited currencies.

Given the sequential nature of markets in this environment, the DM-buyers' currency portfolio and counterfeiting decisions are dynamic. We will defer the discussion of agents' dynamic decision problem until the next section, and only after we have described the random matching and private information bargaining game between potential DM-buyers and DM-sellers. For now, we note that all DM-buyers will exit each CM with the same currency portfolio. Likewise, all DM-sellers will exit with $m_{t+1} = m_{t+1}^f = 0$.

In the next section we describe the one-sided private information bargaining game between

[12] This assumption will do no harm to the results later since agent preferences are quasilinear, so that having transfers made to DM-sellers as well does not matter for the end result. Also note that who gets which seigniorage transfer—i.e., x or x^f—does not matter in this setting given quasilinear preferences and the timing of the CM transfer. This becomes apparent later from equation (7).

[13] This detecting of fraudulent currency is typically done by banks when clients deposit fiat currency into their accounts. The holder of these counterfeits has them removed (exchanged for nothing) from the economy.

[14] Note that given we have an integrated economy which fiat currency is labelled domestic and foreign is completely arbitrary.

a potentially matched buyer and seller in DM. This problem will span from the end of a period-t CM to the end of a period-$(t+1)$ DM. Then we describe the dynamic decision problems of all agents and describe the monetary equilibrium.

3.2 Decentralized Market

Consider the DM subperiod where trade occurs through random bilateral matches. Below we describe the particulars of this frictional environment.

Matching. There are two fixed types of agents in the DM: buyers (b) and sellers (s). The measures of both b– and s–types are equal to 1. At the beginning of each period $t \in \mathbb{N}$, ex-ante anonymous buyers and sellers enter DM where they are randomly and bilaterally matched. With probability $\sigma \in (0,1)$ each buyer of one country is pairwise matched with a seller of the same country. Moreover, as agents are anonymous, exchange supported by contracts that promise repayment in the future is not incentive compatible. Therefore, agents trade with domestic (m) and foreign fiat (m^f) currency.

Feasible offers. Let $\omega := (q, d, d^f)$ denote the terms of trade that specifies how much a seller must produce in DM (q) in exchange for domestic (d) and/or foreign (d^f) fiat currencies. The particulars of the terms of trade ω is an outcome of a bargaining game with private information which we describe below. Denote the set of feasible buyer offers at each aggregate state (ϕ, e) as $\Omega(\phi, e) \ni \omega$.

Given DM preferences and technologies, the corresponding first-best quantity traded is $q^* \in (0, \infty)$ and satisfies $u'(q^*) = c'(q^*)$. For each aggregate state (ϕ, e), there exist maximal finite and positive numbers $\overline{q} := \overline{q}(\phi, e)$, $\overline{m} := \overline{m}(\phi, e)$ and $\overline{m}^f := \overline{m}^f(\phi, e)$ solving $(\overline{m} + e\overline{m}^f)\phi = u(\overline{q}) = c(\overline{q})$, since $u(\cdot)$ and $c(\cdot)$ are monotone and continuous functions on every $[0, \bar{q}(\phi, e)]$. That is, the outcomes $(\overline{q}, \overline{m}, \overline{m}^f)(\phi, e)$ will be finite for every (ϕ, e). Therefore, the set of all feasible offers $\Omega(\phi, e)$ at given (ϕ, e), is a closed and bounded subset of $\mathbb{R}^3_+$, where $\Omega(\phi, e) = [0, \overline{q}(\phi, e)] \times [0, \overline{m}(\phi, e)] \times [0, \overline{m}^f(\phi, e)]$. We summarize this observation in the lemma below.

Lemma 1 *For each given (ϕ, e), the set of feasible buyer offers $\Omega(\phi, e) \subset \mathbb{R}^3_+$ is compact.*

Having specified the set of all possible offers that the buyer can feasibly make in each state of the economy, we now characterize the private information bargaining game.

3.3 Private Information

The DM-buyers' portfolio composition of genuine and fraudulent fiat currencies is private information as the seller can not distinguish them. This private information problem is modeled as a signaling game between pairs of randomly matched buyers (signal sender) and sellers (signal receiver). The game is a one-period extensive form game played out in virtual time between each CM and the following period's DM.

A buyer has private information on his accumulation decision and holdings of the two fiat currencies. A matched seller can observe the terms of trade $\omega := (q, d, d^f)$ offered by the buyer but she is not able to distinguish between genuine and counterfeited currencies. In contrast to standard signalling games, here, signal senders have a choice over their private-information types. These types are defined by the buyer's portfolio choice at the end of each CM. If the buyer decides to counterfeit fiat currencies she will exchange them for DM goods as in the next CM they are going to be detectable by government authorities and destroyed. In what follows next, we first describe and characterize the equilibrium of the game.

3.3.1 Endogenous-type Signalling Game

At the beginning of each DM, a seller s is randomly matched with a buyer b. The seller cannot recognize whether the buyer is offering genuine fiat currencies or not. Next we describe the exact timing of events.

Let CM($t-1$) denote the time-($t-1$) frictionless Walrasian inetgerated market and DM(t) represent the time-t domestic decentralized and frictional market. One could also think in terms of a CM(t) and its ensuing DM($t+1$), so the timing notation here does not affect the analysis. For every $t \geq 1$, and given prices, (ϕ_t, e_t), the timing of the signalling game is as follows:

1. In CM($t-1$) a buyer decides whether or not to costly counterfeit domestic or foreign fiat currency at a one-period fixed cost $\kappa > 0$ and $\kappa^f > 0$, respectively. This decision is captured by the binary action $\chi, \chi^f \in \{0, 1\}$ where $\chi, \chi^f = 0$ represents "no counterfeiting of currency".

2. The buyer chooses how much CM($t-1$) good to produce in exchange for genuine currencies, m and/or m^f.

3. In the subsequent DM(t), a buyer is randomly matched with a seller with probability σ.[15] Upon a successful match, the buyer makes a take-it-or-leave-it (TIOLI) offer (q, d, d^f) to the seller.[16]

4. The seller decides whether to accept the offer or not. If the seller accepts, she produces according to the buyer's TIOLI offer.

The extensive-form game tree of this private information problem is depicted in Figure 1.

[15]For simplicity, double-coincidence-of-wants meetings occur with probability zero.

[16]Implicit in the offer is the buyer signalling that the payment offered consists of genuine assets.

As in Li, Rocheteau and Weill (2012), this original extensive-form game has the same payoff-equivalent reduced-form game as the following reverse-ordered extensive-form game. Given prices, (ϕ_t, e_t), we describe the following reverse-ordered game:

1. A DM-buyer signals a TIOLI offer $\omega := (q, d, d^f)$ and commits to ω, before making any (C, N) decisions in CM$(t-1)$.

2. The buyer decides whether or not to counterfeit the fiat currencies, $\chi(\omega), \chi^f(\omega) \in \{0, 1\}$.

3. The buyer decides on portfolio $a(\omega) := (m, m^f)(\omega)$ and (C, N).

4. The buyer enters DM(t) and Nature randomly matches the buyer with a DM-seller with probability σ.

5. The DM-seller chooses whether to reject or accept the offer, $\alpha(\omega) \in \{0, 1\}$.

This reverse-ordered extensive-form game tree is depicted in Figure 2.

This new reverse-ordered game helps refine the set of perfect Bayesian equilibria (PBE) that would arise in the original extensive form game. In and Wright (2011) provide sufficient conditions for the existence of a PBE in an original extensive-form game which is an outcome equivalent to the PBE of its simpler reordered game. Such an equilibrium is called a *Reordering-invariant Equilibrium* or RI-equilibrium.[17]

3.3.2 Players and Strategies

To simplify exposition, we let X represent X_t, X_{-1} correspond to X_{t-1}, and X_{+1} stand for X_{t+1}, for any date $t \geq 1$. In the next section we characterize the buyer and seller's strategies.

A DM-buyer in CM$(t-1)$ has individual state, $\mathbf{s}_{-1} := (y_{-1}, y^f_{-1}; \phi_{-1}, e_{-1})$ which is publicly observable in CM$(t-1)$.[18] A DM-seller in CM$(t-1)$ is labelled as $\check{\mathbf{s}}_{-1} := (\check{m}_{-1}, \check{m}^f_{-1}; \phi_{-1}, e_{-1})$. Let $B(\phi, e) := [0, \overline{m}(\phi, e)] \times [0, \overline{m}^f(\phi, e)]$ denote the feasible currency portfolio choice set for a given aggregate state (ϕ, e).

Definition 2 *A pure strategy of a buyer, $\boldsymbol{\sigma}^s$, in the counterfeiting game is a triple $\langle \omega, \aleph(\omega), a(\omega) \rangle$ comprised by the following:*

[17] See conditions A1-A3 in In and Wright (2011) for more details.Their characterization of equilibria is related to the Cho and Kreps (1987) Intuitive Criterion refinement, in the sense that both approaches are implied by the requirement of strategic stability (see Kohlberg and Mertens, 1986). However, the difference in the class of games considered by In and Wright (2011) to that of standard signalling games using Cho and Kreps, is that the class of games considered by the former admits signal senders who have an additional choice of a private-information action. That is, who chooses the private-information type—i.e. Nature in standard signalling games or a Sender in In and Wright (2011)—matters for the game structure. When a strategic and forward-looking Sender can choose his unobserved type, there will be additional ways he can deviate (but these deviations must be unprofitable in equilibrium). Thus standard PBE may still yield too many equilibria in these games with a signalling of private decisions. Further discussions are available in a separate appendix.

[18] Given exogenous policy outcomes x_{-1} and x^f_{-1}, and through a change of variables, we let $y_{-1} \equiv m_{-1} + x_{-1}$ and $y^f_{-1} \equiv m^f_{-1} + x^f_{-1}$ be the DM-buyer's individual state variables.

1. *Offer decision rule,* $\mathbf{s}_{-1} \mapsto \omega \equiv \omega(\mathbf{s}_{-1}) \in \Omega(\phi, e)$*;*

2. *Binary decision rules on counterfeiting,* $\aleph := \left\langle \chi(\omega), \chi^f(\omega) \right\rangle \in \{0,1\}$*, for each currency; and*

3. *Asset accumulation decision,* $\omega \mapsto a(\omega) \in B(\phi, e)$*, and,* $(d, d^f) \leq a(\omega)$*.*

A pure strategy of a seller $\boldsymbol{\sigma}^s$ *is a binary acceptance rule* $(\omega, \check{\mathbf{s}}_{-1}) \mapsto \alpha(\omega, \check{\mathbf{s}}_{-1}) \in \{0,1\}$*.*

More generally, we allow players to play behavioral strategies given the buyer's posted offer ω. This is the case as quasilinearity in CM makes the buyer's payoff linear in (d, d^f). This implies that taking a lottery over these payments yields the same utility $u(q)$. Thus, for notational convenience, we drop the lottery over offers when describing a buyer's behavior strategy $\tilde{\boldsymbol{\sigma}}^b$.

Definition 3 *A behavior strategy of a buyer* $\tilde{\boldsymbol{\sigma}}^b$ *is a triple* $\langle \omega, G[a(\omega)|\omega], H(\aleph|\omega) \rangle$*, where*

1. $H(\cdot|\omega) := \langle \eta(\cdot|\omega), \eta^f(\cdot|\omega) \rangle$ *specifies marginal probability distributions over the* $\{0,1\}$ *spaces of each of the two counterfeiting decisions* $\aleph := (\chi, \chi^f)$*; and*

2. $G(\cdot|\omega)$ *is a conditional lottery over each set of feasible asset pairs,* $B(\phi, e)$*.*

A behavior strategy of a seller is $\tilde{\boldsymbol{\sigma}}^s : \pi(\omega)$ *which generates a lottery over* $\{0,1\} \ni \alpha$*.*

Finally, we note that buyers in each CM$(t-1)$ make the same optimal decisions in subsequent periods. This is the case as agents have CM quasilinear preferences so that history does not matter. Likewise, for the sellers' decisions. All agents, conditional on their DM-buyer or DM-seller types, have the same individual state after they leave CM. Therefore, characterizing the equilibrium of the counterfeiting-bargaining game between a matched anonymous buyer and seller pair in DM(t) is tractable. Thus, we just can simply focus on the payoffs of any ex-ante DM-buyer and DM-seller.

3.3.3 Buyers' Payoff

Let $W^b(\cdot)$ denote the value function of a DM-buyer at the beginning of CM(t). Since per-period CM utilities are quasilinear the corresponding CM value function is linear in the buyer's individual state (m, m^f) so that

$$W^b(\mathbf{s}) \equiv W^b(y, y^f; \phi, e) = \phi(y + ey^f) + W^b(0, 0; \phi, e). \tag{4}$$

Let us define $Z(C_{-1}; \mathbf{s}_{-1}) = \mathcal{U}(C_{-1}) - C_{-1} + \phi_{-1}(y_{-1} + e_{-1}y^f_{-1})$ which summarizes the CM$(t-1)$ flow utility from consuming $(C_{-1}, -N_{-1})$ plus the time-$(t-1)$ real value of accumulating genuine fiat currencies. Then given prices (ϕ, e) and his belief about the seller's behavior $\hat{\pi}$, the DM-buyer's Bernoulli payoff function, $U^b(\cdot)$, can be written as follows:[19]

$$\begin{aligned} U^b(C_{-1}, \omega, \eta, \eta^f, G[a(\omega)|\omega], \hat{\pi}|\mathbf{s}_{-1}; \phi, e) = \\ \int_{B(\phi,e)} \Big\{ Z(C_{-1}; \mathbf{s}_{-1}) - \phi_{-1}(m + e_{-1}m^f) - \kappa(1-\eta) - \kappa^f(1-\eta^f) \\ + \beta\sigma\hat{\pi}\left[u(q) + W^b(m - \eta d, m^f - \eta^f e d^f; \phi, e)\right] \\ + \beta\left[\sigma(1-\hat{\pi}) + (1-\sigma)\right] W^b(m, m^f; \phi, e) \Big\} dG[a(\omega)|\omega]. \end{aligned} \tag{5}$$

Given the linearity of $W^b(\cdot)$, we can further reduce equation (5) to the following expression

$$\begin{aligned} U^b(C_{-1}, \omega, \eta, \eta^f, G[a(\omega)|\omega], \hat{\pi}|\mathbf{s}_{-1}; \phi, e) = -\kappa(1-\eta) - \kappa^f(1-\eta^f) \\ + \int_{B(\phi,e)} \Big\{ Z(C_{-1}; \mathbf{s}_{-1}) - \left(\frac{\phi_{-1}}{\phi} - \beta\right)\phi m - \left(\frac{\phi_{-1}e_{-1}}{\phi e} - \beta\right)\phi e m^f \\ + \beta\sigma\hat{\pi}\left[u(q) - \phi\left(\eta d + \eta^f e d^f\right)\right] \Big\} dG[a(\omega)|\omega] \end{aligned} \tag{6}$$

which corresponds to the expected total payoff under a given strategy $\tilde{\boldsymbol{\sigma}}^b$ for a DM-buyer in CM$(t-1)$. Note that the first term of equation (6) is the expected total fixed cost of counterfeiting both currencies. The second term on the right of equation (6) is the utility flow from consuming $(C_{-1}, -N_{-1})$ and the DM(t) continuation value from accumulating currencies in CM$(t-1)$. The third and fourth term are the expected total cost (equivalently inflation cost) of holding unused currencies between CM$(t-1)$ and DM(t). The last term is the expected net payoff gain from trades in which the buyer pays for the good q with genuine currencies, with marginal probability measures $H(\omega) =: (\eta, \eta^f)$, and the buyer believes a randomly encountered seller accepts with probability $\hat{\pi}$.

Finally, we still have to take into account the buyer's mixed strategy $G(\cdot|\omega)$. In the Appendix we show that in a monetary equilibrium $G(\cdot|\omega)$ is always degenerate, so the buyer's total expected payoff in (6) further simplifies to

$$\begin{aligned} U^b[C_{-1}, \omega, H(\omega), \hat{\pi}|\mathbf{s}_{-1}; \phi, e] = \\ Z(\mathbf{s}_{-1}) - \left(\frac{\phi_{-1}}{\phi} - \beta\right)\phi m - \left(\frac{\phi_{-1}e_{-1}}{\phi e} - \beta\right)\phi e m^f \\ - \kappa(1-\eta) - \kappa^f(1-\eta^f) + \beta\sigma\hat{\pi}\left[u(q) - \phi\left(\eta d + \eta^f e d^f\right)\right]. \end{aligned} \tag{7}$$

Note that initial monetary wealth $y_{-1} := m_{-1} + x_{-1}$ and $y^f := m^f_{-1} + x^f_{-1}$ will not matter for the DM-buyers' CM decisions on the continuation portfolio of assets, (m, m^f), given the linearity of the payoff function in these choices. This confirms our earlier assertion that given

[19] We have imposed symmetry among all sellers for notational simplicity.

quasilinear preferences and the timing of seigniorage revenue transfers, it does not matter that we do not make explicit geographical distinctions in the DM market (i.e. as in standard international models with explicit Home and Foreign agents).[20]

3.3.4 Sellers' Payoff

A DM-seller's payoff function is simpler. Let $W^s(\cdot)$ denote the seller's value function at the start of any CM. The seller also has a linear value function $W^s(\cdot)$ in currency holdings. Let $Z(\breve{C}_{-1};\breve{\mathbf{s}}_{-1}) = \mathcal{U}(\breve{C}_{-1}) - \breve{C}_{-1} + \phi_{-1}(\breve{m}_{-1} + e_{-1}\breve{m}^f_{-1})$ summarize the CM$(t-1)$ flow utility from consuming $(\breve{C}_{-1}, -\breve{H}_{-1})$ plus the time-$(t-1)$ real value of accumulating genuine currencies. Note that the DM-seller will always accumulate zero money holdings, because of inflation and the fact that she knows that she has no use of money in the ensuing DM.

Let $\left(\hat{\eta}, \hat{\eta}^f\right)$ be the seller's belief about the buyer's behavior with respect to counterfeiting of fiat currencies. Given an offer ω, the seller belief system and the seller's response $\pi(\omega)$, her Bernoulli payoff for the game is given by

$$\begin{aligned} U^s(\breve{C}_{-1}, \omega, \hat{\eta}, \hat{\eta}^f, \pi(\omega)|\breve{\mathbf{s}}_{-1}; \phi, e) &= Z(\breve{C}_{-1};\breve{\mathbf{s}}_{-1}) \\ &+ \beta\sigma\pi(\omega)\left[-c(q) + W^s(\hat{\eta}d, \hat{\eta}^f d^f; \phi, e)\right] \\ &\quad + \beta\left[\sigma\left(1-\pi(\omega)\right) + (1-\sigma)\right]\left[-c(0) + W^s(0,0;\phi,e)\right] \\ &= Z(\breve{C}_{-1};\breve{\mathbf{s}}_{-1}) + \beta\sigma\pi(\omega)\left[\phi\left(\hat{\eta}d + \hat{\eta}^f e d^f\right) - c(q)\right], \end{aligned} \tag{8}$$

where the last equality is a direct consequence of linearity in the seller's CM value function: $W^s(\breve{m}, \breve{m}^f) = \phi(\breve{m} + e\breve{m}^f) + W^s(0,0)$. The last term on the right of the payoff function (8) is the total discounted expected profit arising from the σ-measure of DM(t) exchange, in which the seller accepts an offer ω with probability $\pi(\omega)$ and she anticipates that the buyer pays with genuine assets according to beliefs $(\hat{\eta}, \hat{\eta}^f)$.

3.4 Equilibrium of the Private Information Game

The equilibrium concept for the counterfeiting-bargaining game is Perfect Bayesian in the reordered extensive-form game, as in Li, Rocheteau and Weill (2012). More precisely, we utilize the *RI-equilibrium* refinement proposed by In and Wright (2011). In order to solve the game we proceed by backward induction on the game depicted in Figure 1.

[20] In Section 5.5, we consider an alternative setting where the transfers are made after each CM closes, so that the distinction between Home and Foreign DM markets are crucial to the model setup. We show that despite this more complicated alternative environment, the main result we get out of this simpler integrated economy setting still survives (subject to additional notation).

3.4.1 Seller's Problem

Following a (partially) private buyer history $\langle \omega, \aleph(\omega) \rangle$ in which an offer ω is observable and $\aleph$ is not observable, the seller plays a mixed strategy π to maximize her expected pay off which is given by

$$\pi(\omega) \in \left\{ \arg \max_{\pi' \in [0,1]} \pi' \left[\phi \left(\hat{\eta} d + \hat{\eta}^f e d^f \right) - c(q) \right] \right\}. \tag{9}$$

3.4.2 Buyer's Counterfeiting Problem

Given history ω and the buyer's belief about the seller's best response, $\hat{\pi}$, the buyer solves the following cost-minimization problem

$$\begin{aligned} (\eta(\omega), \eta^f(\omega)) \quad = \quad & \arg \max_{\eta, \eta^f \in [0,1]} \Bigg\{ - \kappa(1-\eta) - \kappa^f(1-\eta^f) - \beta\sigma\hat{\pi}\phi \left[\eta d + \eta^f e d^f \right] \\ & - \left(\frac{\phi_{-1}}{\phi} - \beta \right) \phi m - \left(\frac{\phi_{-1} e_{-1}}{\phi e} - \beta \right) \phi e m^f \Bigg\}. \end{aligned} \tag{10}$$

Given that the terms of trade in DM are given by the buyer's TIOLI offer at the beginning of the game, the buyer maximizes her payoff given her conjecture $(\hat{\eta}, \hat{\pi})$ of the continuation play, the buyer commits to an optimal offer $\omega \equiv (q, \hat{d}, \hat{d}^f)$ which is given by

$$\begin{aligned} \omega \quad \in \quad & \Bigg\{ \arg \max_{\omega' \in \Omega(\phi, e)} \Big\{ -\kappa\left(1-\hat{\eta}\right) - \kappa^f \left(1-\hat{\eta}^f\right) + \beta\sigma\hat{\pi} \left[u(q) - \phi \left(\hat{\eta}\hat{d} + \hat{\eta}^f e \hat{d}^f \right) \right] \\ & - \left(\frac{\phi_{-1}}{\phi} - \beta \right) \phi m - \left(\frac{\phi_{-1} e_{-1}}{\phi e} - \beta \right) \phi e m^f \Big\} \Bigg\}. \end{aligned} \tag{11}$$

3.4.3 Equilibrium

Having specified the seller's and buyer's respective problems, we can now characterize the resulting equilibrium in the private-information bargaining game.

Definition 4 *A reordering-invariant (RI-) equilibrium of the original extensive-form game is a perfect Bayesian equilibrium* $\tilde{\boldsymbol{\sigma}} := (\tilde{\boldsymbol{\sigma}}^b, \tilde{\boldsymbol{\sigma}}^s) = \langle \omega, \eta(\omega), \eta^f(\omega), \pi(\omega) \rangle$ *of the reordered game such that (9) and (10) are satisfied.*

The following proposition provides a simple characterization of a RI-equilibrium in the game.

Proposition 5 (RI-equilibrium) *An RI-equilibrium of the counterfeiting-bargaining game is such that*

1. *Each seller accepts with probability* $\hat{\pi} = \pi(\omega) = 1$*;*
2. *Each buyer does not counterfeit:* $(\hat{\eta}, \hat{\eta^f}) = (\eta(\omega), \eta^f(\omega)) = (1, 1)$*; and*

3. Each buyer's TIOLI offer ω is such that:

$$
\begin{aligned}
\omega \in \Bigg\{ arg \max_{\omega \in \Omega(\phi,e)} \Bigg[& -\left(\frac{\phi_{-1}}{\phi} - \beta\right)\phi m - \left(\frac{\phi_{-1}e_{-1}}{\phi e} - \beta\right)\phi e m^f \\
& + \beta\sigma \left[u(q) - \phi\left(d + ed^f\right)\right]\Bigg] \qquad s.t. \\
& (\zeta): \quad \phi\left(d + ed^f\right) - c(q) \geq 0, \\
& (\nu): \quad 0 \leq d, \\
& (\mu): \quad d \leq m, \\
& (\nu^f): \quad 0 \leq d^f, \\
& (\mu^f): \quad d^f \leq m^f, \\
& (\ \lambda): \quad \phi d \leq \frac{\kappa}{\phi_{-1}/\phi - \beta(1-\sigma)} \equiv \bar{\kappa}(\phi_{-1}/\phi), \\
& (\lambda^f): \quad \phi e d^f \leq \frac{\kappa^f}{\phi_{-1}e_{-1}/\phi e - \beta(1-\sigma)} \equiv \bar{\kappa}^f(\phi_{-1}e_{-1}/\phi e)\Bigg\}.
\end{aligned}
\tag{12}
$$

and the RI-equilibrium is unique.

As we can see from the RI-equilibrium, ζ represents the Lagrange multiplier associated with the seller's participation constraint, ν (ν^f) is the Lagrange multiplier corresponding to the non-negativity of the domestic (foreign) payments. Finally, μ (μ^f) represents the feasibility constraint for the local (foreign) fiat money, λ (λ^f) is the Lagrange multiplier corresponding to the liquidity constraint for the local (foreign) fiat money that arise because of the threat of counterfeiting.

It is important to highlight that the last two constraints are *endogenous* liquidity constraints in that they provide an upper bound on the quantities of *genuine* currencies that the seller will accept.[21] These upper bounds depend positively on the fixed cost of counterfeiting and negatively on the degree of matching efficiency σ. Note that a larger σ implies greater matching efficiency in the DM so that buyers and sellers are more likely to meet and trade. This creates a larger incentive for the buyer to produce counterfeits, thus increasing the information problem. Thus, in equilibrium, in order for sellers to accept buyers' offers, each buyer has a tighter upper-bound on his signal/offer of DM payment. The same logic applies to the effect of the fixed costs of counterfeiting, and, also to the effect of the aggregate returns on holding *genuine* currencies.

A critical feature of these liquidity constraints is that the marginal liquidity value of an additional unit of currency beyond the bound is zero. Thus, if the foreign currency has a

[21]Li, Rocheteau and Weill (2012) comment that Veronica Guerrieri has shown that competitive search generates the same liquidity constraint as the one obtained under TIOLI.

higher rate of return (lower inflation rate) but a lower counterfeiting cost, then the buyer will first pay with the foreign currency up to the bound and use domestic currency to pay for the remainder of the goods purchased.

3.5 Money supplies and seigniorage transfers

We assume that the supply of the fiat monies, M_t and M_t^f, respectively, grow at a constant rate of Π and Π^f. The resulting seigniorage revenues are transferred uniformly to the DM-buyers as $x_t = M_{t+1} - M_t = (\Pi - 1)\Pi^{t-1}M_0$ and $x_t^f = M_{t+1}^f - M_t^f = (\Pi^f - 1)(\Pi^f)^{t-1}M_0^f$, at the beginning of each CM. The initial stocks M_0 and M_0^f are known.

4 Monetary Equilibrium

We can now embed the equilibrium charaterization of the game into the overall monetary equilibrium of the model. Since preferences are quasilinear, the infinite history of past games between buyers and sellers does not matter for each current period agents' decision problems. This allows us to tractably incorporate the equilibrium characterization of the game previously described, into the overall dynamic general monetary setting. Before we do so, we return to describing the agents' dynamic decision problems.

4.1 Agents' Recursive Problems

DM-buyers' Problem As we previously saw, the beginning-of-CM value function for buyers $W^b(\cdot;\phi,e)$ is linear in the fiat currency portfolio (m, m^f). As a result, the buyer's intertemporal problem, conditional on an equilibrium of the private-information bargaining game, is given by

$$\max_{C_{-1},q,d,d^f,m,m^f} U^b(C_{-1},\omega,\eta(\omega),\hat{\pi}|\mathbf{s}_{-1};\phi,e) \qquad \text{s.t.}$$

$$\begin{aligned}
(\eta(\omega),\eta^f(\omega)) = \quad & (1,1), \qquad \hat{\pi} = 1, && \text{(13a)}\\
(\zeta): \quad & \phi\left(d + ed^f\right) - c(q) = 0, && \text{(13b)}\\
(\nu): \quad & 0 \leq d, && \text{(13c)}\\
(\mu): \quad & d \leq m, && \text{(13d)}\\
(\nu^f): \quad & 0 \leq d^f, && \text{(13e)}\\
(\mu^f): \quad & d^f \leq m^f, && \text{(13f)}\\
(\ \lambda): \quad & \phi d \leq \bar{\kappa}(\phi_{-1}/\phi), && \text{(13g)}\\
(\lambda^f): \quad & \phi e d^f \leq \bar{\kappa}^f(\phi_{-1}e_{-1}/\phi e). && \text{(13h)}
\end{aligned}$$

where the DM-buyer's lifetime expected payoff is given by

$$
\begin{aligned}
& U^b(C_{-1}, \omega, \eta(\omega), \eta^f(\omega), \hat{\pi}|\mathbf{s}_{-1}; \phi, e) = \\
= \quad & \mathcal{U}(C_{-1}) - C_{-1} + \phi_{-1}(y_{-1} + e_{-1}y^f_{-1}) - \left(\frac{\phi_{-1}}{\phi} - \beta\right)\phi m \\
& - \left(\frac{\phi_{-1}e_{-1}}{\phi e} - \beta\right)\phi e m^f + \beta\sigma\left[u(q) - \phi\left(d + ed^f\right)\right]. \qquad (14)
\end{aligned}
$$

In contrast to a full information setting, the threat of counterfeits which is private information to buyers, introduces additional endogenous state-dependent liquidity constraints (13g)-(13h) into a buyer's Bellman equation problem. These endogenous liquidity constraints are going to play an important role in determining the coexistence of the two currencies and the determinacy of nominal exchange rates.

The corresponding first order conditions of the DM-buyers' problem are given by

$$1 = \mathcal{U}'(C), \qquad (15)$$

$$0 = \beta\sigma u'(q) - \zeta c'(q), \qquad (16)$$

$$\beta\sigma = \zeta + \nu - \mu - \lambda, \qquad (17)$$

$$\beta\sigma = \zeta + \nu^f - \mu^f - \lambda^f, \qquad (18)$$

$$\mu = \frac{\phi_{-1}}{\phi} - \beta, \qquad (19)$$

$$\mu^f = \frac{\phi^f_{-1}}{\phi^f} - \beta. \qquad (20)$$

$$\zeta \geq 0, \nu \geq 0, \nu^f \geq 0, \mu \geq 0, \mu^f \geq 0, \lambda \geq 0, \lambda^f \geq 0. \qquad (21)$$

Note that Equation (15) describes the optimal within-period labor versus consumption trade-off in CM, where the marginal disutility of labor is -1 and the real-wage (marginal product of labor) is 1. Equation (16) corresponds to the first order condition for DM output which equates the marginal benefit of consuming and marginal value of the payment to the seller. Since the buyer offers a TIOLI, the payment is equal to the seller's DM production cost. Equations (17) and (18) summarize the optimal choice with respect domestic and foreign payment, respectively, and equate the value of holding a particular fiat currency from one CM to the next versus trading it in DM. Finally, equations (19) and (20) describe the optimal accumulation of local and foreign currency which of course depend on its implied rate of return. Equations (17) and (19) (or (18) and (20)) imply a sequence of intertemporal consumption Euler inequalities, where domestic (or foreign) currency is used as store of value.

DM-sellers' Problem A DM-seller's problem, embedding the game's equilibrium, is simpler as sellers cannot counterfeit. This is given by

$$\max_{C_{-1}} U^s(\breve{C}_{-1}, \omega, \eta(\omega), \eta^f(\omega), \hat{\pi}|\breve{\mathbf{s}}_{-1}; \phi, e) \quad \text{s.t.} \quad (\eta(\omega), \eta^f(\omega)) = (1,1) \text{ and } \hat{\pi} = 1;$$

where each seller's Bernoulli payoff is given by

$$\begin{aligned} U^s(\breve{C}_{-1}, \omega, &\eta(\omega), \eta^f(\omega), \hat{\pi}|\breve{\mathbf{s}}_{-1}; \phi, e) \\ &= \mathcal{U}(\breve{C}_{-1}) - \breve{C}_{-1} + \phi_{-1}(\breve{m}_{-1} + e_{-1}\breve{m}^f_{-1}) + \beta\sigma\left[\phi\left(d + ed^f\right) - c(q)\right]. \end{aligned} \tag{22}$$

4.2 Steady State Monetary Equilibrium

We will focus on steady-state monetary equilibria where the nominal exchange rate can grow at a constant rate. In fact, if we consider (for now) monetary equilibria where both monies circulate, we have the following intermediate observation:[22]

Proposition 6 *Assume the existence of a monetary equilibrium where both monies circulate. When there is no portfolio restriction on what currencies must serve as a medium of exchange in any country, the equilibrium nominal exchange rate growing in absolute terms at a constant and bounded rate γ_e, i.e.,*

$$\left|\frac{e_{t+1} - e_t}{e_t}\right| = \gamma_e \in [0, +\infty),$$

for all $t \geq 0$, is a (deterministic) monetary equilibrium property.

In the rest of the paper, we can thus focus on monetary equilibria in which the equilibrium exchange rate grows at some constant rate (possibly zero). Also, Proposition 6 will apply in the explicit two-country version of the world economy below. All that is required is to set $x_t = 0$ for every date t in the sequential CM budget constraints, as x_t will later appear in the DM feasibility constraint, since transfers will occur after each CM closes.

We study the implications of the endogenous liquidity constraints for the coexistence of multiple fiat currencies. This also allows us to understand under what conditions there is determinacy of the nominal exchange rate.

Define stationary variables by taking ratios of growing variables as follows $\frac{M}{M_{-1}} = \Pi = \frac{\phi_{-1}}{\phi}$; $\frac{M^f}{M^f_{-1}} = \Pi^f = \frac{\phi^f_{-1}}{\phi^f}$. In steady state, all real quantities are constant implying $\phi M = \phi_{-1}M_{-1}$, and, $e\phi M^f = e_{-1}\phi_{-1}M^f_{-1}$, which yields the steady state home currency (gross) depreciation/appreciation as

$$\frac{e}{e_{-1}} = \frac{\Pi}{\Pi^f}. \tag{23}$$

[22]We relegate the proof to Supplementary Appendix C.

In a monetary equilibrium where $d = m$ and $d^f = m^f$ (see Lemma 10 in the Supplementary Appendix) and money markets clear $m = M$ and $m^f = M^f$, a steady state the economy satisfies the following set of equations

$$
\begin{aligned}
1 &= \mathcal{U}'(C); && (24)\\
0 &= \beta\sigma u'(q) - \zeta c'(q), && (25)\\
\beta\sigma &= \zeta + \nu - \mu - \lambda, && (26)\\
\beta\sigma &= \zeta + \nu^f - \mu^f - \lambda^f && (27)\\
\mu &= \Pi - \beta \geq 0, && (28)\\
\mu^f &= \Pi^f - \beta \geq 0; && (29)\\
c(q) &= \phi M + e\phi M^f \Leftrightarrow \zeta > 0; && (30)\\
\phi M &\leq \frac{\kappa}{\Pi - \beta(1-\sigma)} \Leftrightarrow \lambda \geq 0; && (31)\\
e\phi M^f &\leq \frac{\kappa^f}{\Pi^f - \beta(1-\sigma)} \Leftrightarrow \lambda^f \geq 0; && (32)
\end{aligned}
$$

where Π (Π^f) correspond to the local (foreign) steady state inflation rate.

Some description of this monetary steady state is in order. Equation (24) describes the optimal consumption of the CM good. Equation (25) corresponds to the first order condition for DM output. Equations (26) and (27) summarize the optimal choice with respect domestic and foreign payment, respectively. Equations (28) and (29) describe the optimal accumulation of local and foreign currency. Finally, equations (30), (31) and (32) correspond to the multipliers $\zeta > 0$, $\lambda \geq 0$ and $\lambda^f \geq 0$. For the rest of the paper we focus on equilibria that satisfy $\zeta > 0$ (i.e. DM-seller's participation constraint (30) binds).

Intuitively, a determinate equilibrium arises if all conditions (24)-(32) hold with strict equality, with the multipliers (μ, μ^f) being strictly positive. However, this may not always hold. In particular, determinacy of the steady state equilibrium, and therefore its nominal exchange rate outcome, depends crucially on cross-country monetary policies (Π, Π^f). It also depends on the economic structure—in particular the counterfeiting costs (κ, κ^f) and matching friction σ. The following Proposition is the main result of the paper which provides sufficient conditions for the two currencies to coexist, and, for the nominal exchange rate to be determinate even if one of the currencies is dominated in rate of return.

Proposition 7 (Equilibria and Coexistence) *Depending on the relative inflation rates of the two fiat currencies, there are three cases charaterizing a steady-state monetary equilibrium:*

1. *When the domestic fiat money is dominated in rate of return* ($\Pi > \Pi^f$) *and*

   (a) *when neither liquidity constraints bind* ($\lambda = \lambda^f = 0$)*, or when only the liquidity* *constraint on domestic fiat money bind ($\lambda > 0, \lambda^f = 0$), then a monetary equilibrium exists with the unique outcome where only the foreign (or low inflation) currency circulates; or,*

(b) *the foreign liquidity constraint binds ($\lambda^f > \lambda = 0$), then there exists a monetary equilibrium with a unique outcome where the currencies coexist and the nominal exchange rate is determinate*

$$e = \frac{M}{M^f} \frac{\kappa^f}{c(q)\left[\Pi^f - \beta(1-\sigma)\right] - \kappa^f};$$

where q solves

$$\frac{\Pi - \beta}{\sigma\beta} = \frac{u'(q) - c'(q)}{c'(q)}.$$

(c) *both liquidity constraints bind ($\lambda^f > \lambda > 0$), then there exists a unique monetary equilibrium where the currencies coexist and the nominal exchange rate is determinate*

$$e = \frac{\kappa^f M}{\kappa M^f} \frac{\Pi - \beta(1-\sigma)}{\Pi^f - \beta(1-\sigma)}.$$

2. *When the domestic fiat money dominates in rate of return ($\Pi^f > \Pi$), the coexistence results are the symmetric opposite to those of Case 1.*

3. *When both domestic and foreign currencies have the same rate of return ($\Pi^f = \Pi$) and*

   (a) *neither liquidity constraints bind ($\lambda = \lambda^f = 0$), then the fiat monies coexist but the individual's currency portfolio composition and the nominal exchange rate are indeterminate;*

   (b) *both liquidity constraints bind ($\lambda = \lambda^f > 0$), then the currencies coexist. The individual's currency portfolio composition is unique, and thus there is a unique nominal exchange rate*

$$e = \frac{\kappa^f M}{\kappa M^f}.$$

The key point of this Proposition is that, although there is no counterfeiting in equilibrium, the threat of counterfeiting is all that is required to generate the coexistence and determinacy of the nominal exchange rate as long as one or both of the liquidity constraints, (31) and/or (32), bind. This is true even if the currencies are equivalent in all respects; i.e., the inflation rates and the counterfeiting costs are the same. Thus we have broken the curse of Kareken and Wallace.

Some comparative static results are interesting to note. In Cases 1 and 2, higher domestic inflation causes a depreciation of the domestic currency just as one would expect but for different reasons. In Case 1, the domestic counterfeiting constraint does not bind so it is unaffected by the higher domestic inflation rate. However, a higher domestic inflation rate causes agents to economize on real domestic balances which reduces q and thus increases e. In Case 2, the domestic currency constraint binds and becomes tighter with higher domestic inflation. Thus, the marginal liquidity value of an additional unit of domestic currency goes to zero sooner, hence it is less valuable relative to a unit of foreign currency.

What is actually shocking about Case 3b is that when the two currencies are identical in every respect, i.e., $\Pi = \Pi^f$ and $\kappa = \kappa^f$, and both can be used simultaneously as payment, then the nominal exchange rate is simply the ratio of the two money stocks. This is the standard exchange rate solution that comes from a symmetric, two country, domestic cash-in-advance (CIA) model as in Stockman (1980) and Lucas (1982). While one may be tempted to say that we have provided a 'micro-foundation' for the two country CIA model, this would be incorrect for two reasons. First, our result only holds for a set of parameter values. Second, in the standard CIA model only one of the currencies is used in a transaction, whereas here both currencies are used in the same transaction. Thus, equivalent exchange rate solutions should not be confused with equivalent results elsewhere. Nevertheless, it is interesting to note that it can be derived under certain conditions.

Proposition 7 also contemplates the possibility of just one currency circulating. It is important to highlight that this Proposition describes equilibria that have the property that when the liquidity constraint binds, the marginal value of an additional unit of this currency is zero since the seller will not accept it. At the margin DM-sellers will produce an extra unit of output only for the fiat currency that has a non-binding liquidity constraint. Under these circumstances, the buyer first offers the currency with the best rate of return. Once the endogenous liquidity constraint is binding, the buyer pays for additional units of the DM good with the lower return currency.

In addition to breaking the curse of Kareken and Wallace, Proposition 7 also implies the following property.

Proposition 8 (First Best) *When both the domestic and foreign inflation rates are at the Friedman rule, $\Pi = \Pi^f = \beta$, the first best quantity q^* **may not** be attainable, Case (3b), and the nominal exchange rate may not be determinate, Case (3a).*

To demonstrate this result, it suffices to provide a counter-example to the claim that the Friedman rule is always optimal. Indeed we show that when there is the threat of counterfeiting, the Friedman Rule may no longer be able to achieve the first best as each DM-seller is not willing to produce more output than what can be afforded by a DM-buyer faced with binding endogenous liquidity constraints. Note that in environments without private information nor bargaining inefficiencies, resulting from Nash Bargaining, the Friedman rule is able to correct for the intertemporal distortion and achieve first best allocations as it does not distort the saving decisions of buyers.[23] However, in this environment having the highest rate of return on fiat currency is not enough as sellers have an upper bound of how much currency they are willing to accept. This liquidity constraint is a direct consequence of the counterfeiting problem they face. As a result, it is possible that sellers do not accept currencies consistent with first best production even at the Friedman rule.

4.3 Sensitivity of equilibrium coexistence

We briefly discuss how the fixed costs (κ and κ^f) consistent with coexistence equilibria vary with inflation (we consider Π^f), market/matching friction (σ), DM-buyers' intertemporal elasticity of substitution (paramtrized as $1/\theta := u'(q)/u''(q) \cdot q$) and the convexity of sellers' production cost (parametrized as α). We focus only on the more interesting example of $\Pi > \Pi^f > \beta$, where the domestic fiat money is dominated by its foreign counterpart in rate of return. The cases to consider are: (Case 1) $\lambda = \lambda^f = 0$, (Case 2) $\lambda > 0, \lambda^f = 0$, (Case 3) $\lambda = 0, \lambda^f > 0$, and (Case 4) $\lambda > 0, \lambda^f > 0$.

It was shown in Proposition 7 that in Cases 1 and 2, only the foreign fiat money survives as an equilibrium medium of exchange. However, in Case 3, we know that there is a unique $\hat{q}$ that solves $\beta\sigma[u'(q)/c'(q) - 1] = \Pi - \beta$. It is straightforward to arrive at the following set of inequalities that define feasible counterfeiting costs (κ, κ^f) consistent with equilibrium coexistence of the two monies in Case 3:

$$\kappa^f \cdot \frac{[\Pi - \beta(1-\sigma)]}{\Pi^f - \beta(1-\sigma)} \leq [\Pi - \beta(1-\sigma)]c(\hat{q}) < \kappa^f \cdot \frac{\Pi - \beta(1-\sigma)}{\Pi^f - \beta(1-\sigma)} + \kappa. \tag{33}$$

The first weak inequality comes about from combining the DM-seller's binding participation constraint, the binding liquidity constraint on the foreign fiat money, and the nonnegativity of domestic real money balance. The second strict inequality is due to the slack liquidity constraint on holding domestic money.

In Case 4, the coexistence equilibrium (given $\Pi > \Pi^f$) has the following restrictions on (κ, κ^f):

$$\kappa^f \cdot \frac{\Pi - \beta(1-\sigma)}{\Pi^f - \beta(1-\sigma)} + \kappa \leq [\Pi - \beta(1-\sigma)]c(\hat{q}), \qquad \kappa > 0, \qquad \kappa^f > 0. \tag{34}$$

These cases and their corresponding inequalities are derived in Supplementary Appendix F.

If we define $k^f(\hat{q}, \Pi^f, \beta, \sigma) := c(\hat{q})[\Pi^f - \beta(1-\sigma)]$ and $k(\hat{q}, \Pi, \beta, \sigma) := c(\hat{q})[\Pi - \beta(1-\sigma)]$, in the space of (κ, κ^f) pairs, then we can deduce the threshold levels of the pair (κ, κ^f) that are required to sustain coexistence equilibria of Case 3 and Case 4.

An example with $u(q) := [(q + \underline{q})^{1-\theta} - \underline{q}^{1-\theta}/(1-\theta)$, where $\underline{q} > 0$ and $\theta > 0$, and $c(q) = q^\alpha$,

[23] For more on the distortions induced by Nash bargaining and how monetary and fiscal policy can restore the first best we refer to Gomis-Porqueras et al (2010).

where $\alpha \geq 1$ is depicted in Figure 3 as the baseline setting.[24]

Consider raising Π^f, as shown in Figure 4, while holding all else equal. This has a tendency to reduce the opportunity cost of holding domestic money (which is dominated in rate of return). That is, DM-buyers now have relatively more incentive to counterfeit the domestic currency. At the same time, a higher Π^f also means that the liquidity constraint on holding foreign money is even tighter. These two imply that sellers will be less accepting of either currencies offered as genuine payments. Thus the coexistence equilibria of Case 4 can only be sustained with even higher costs of counterfeiting.

Consider next Figure 5, where all else the same, we increase the DM matching probability σ. A higher σ lowers the inefficiency wedge $[\Pi - \beta(1-\sigma)]/\beta\sigma$, between the monetary equilibrium outcome $\hat{q}$ in Case 3 (or $\tilde{q}$ in Case 4) and the first-best q^*. This means that both currencies will circulate more in either Case 3 or Case 4, to support a higher allocation of $\hat{q}$ (and $\tilde{q}$). That is buyers are more likely to use their portfolio of monies to spend in DM meetings and thus have more incentive to counterfeit both currencies, when we perturb their environment in terms of raising σ. Thus to ensure equilibrium acceptability and coexistence of both currencies, one needs to have relatively higher thresholds for (κ, κ^f).

Figure 6 illustrates the comparative static with respect to buyers' appetite to intertemporal consumption smoothing, θ. The larger θ is, the less tolerant they are of substitution in q across periods. In other words, DM-buyers would like to smooth out their consumption as much as possible over time, and the only means of intertemporal smoothing is money. Thus, there is a higher incentive to counterfeit both monies, the larger is θ. However, to ensure that there is offers of genuine monies in equilibrium where both monies coexist, then the cost of counterfeiting sustaining the equilibria must be higher.

Figure 7 depicts the comparative static with respect to the convexity of production cost of DM-sellers. The faster production cost rises at the margin for producers, the more willing they are to accept offers of payment in both currencies, so that lower counterfeiting costs are required to sustain truthful or genuine offers of both monies as payment for DM goods in the coexistence equilibria.

Put another way, consider an economy that has low enough inflation relative to its foreign counterpart, low market frictions, agents with large intertemporal insurance motives, or has sufficiently non-accelerating marginal costs of production. These comparative static analyses suggest that in such an economy, one must have an institution of high enough regulatory or technological costs to creating counterfeit media of exchange, so that multiple means of payments can co-exist, and so that there is no currency flight to quality. All these are ingredients that we can readily observe in pre-crisis European economies and even in the United States to date.

[24] `Python` code producing these comparisons are available from the authors' public repository at `https://github.com/phantomachine/_gkwcurse.git`.

5 Extensions

In this section we explore whether the threat of counterfeiting can help determine nominal exchange rates in a variety of environments. In particular, we consider the following modifications to the integrated world economy environment: (i) one where credit is possible, (ii) one where cash can be used as collateral, and (iii) one where foreign denominated assets exist and can be counterfeited. This allow us to determine the robustness of our proposed mechanism in determining the nominal exchange rate and have a broader interpretation of counterfeiting. We also investigate how fiscal policies can help determine nominal exchange rate when monetary policy alone and the fundamentals of the economy are not able to do so. Finally, we also consider a departure from the integrated world economy. We look at an explicit two country model—one where new monies are transferred after each CM closes (and before each DM begins) and where the explicit geographical separation of DMs matter.

5.1 Trade Credit

We consider the previous benchmark and allow agents to have access to credit. We can then analyze how this new payment system affects the determinacy of the nominal exchange rate.

As in Aruoba, Waller and Wright (2011), we consider the possibility that in some matches credit is possible as record-keeping and enforcement services are available. In particular, we assume that conditional on buyers being matched with a seller, the exogenous probability that a buyer or seller would engage in an exchange where trade credit is possible is $(1-\gamma) \in [0,1]$. That is, the event that a buyer can buy a good in the DM using credit occurs with probability $\sigma(1-\gamma)$. Since credit is assumed to be enforceable in such an event, a buyer is willing to take (and a seller is willing to give) out a loan, which we denote by l, in exchange for the DM good, say q_c. This loan is required to be repaid in full in the following CM. The particular terms of trade when credit is available are determined by a buyer take it or leave it offer.

The possibility that in some matches trade credit can be used reduces the *anonymity* frictions in DM which can be interpreted as an improvement in the record-keeping and enforcement powers, critical features of well functioning credit markets. In this new environment the payoff for the buyer that is offered a domestic nominal loan l is given by

$$
\begin{aligned}
Z(C_{-1}, \mathbf{s}_{-1}) - \left(\frac{\phi_{-1}}{\phi} - \beta\right)\phi m - \left(\frac{\phi_{-1}e_{-1}}{\phi e} - \beta\right)\phi e m^f - \kappa(1-\eta) - \kappa^f(1-\eta^f) \\
+ \beta\sigma\left(\gamma\hat{\pi}\left[u(q) - \phi\left(\eta d + \eta^f e d^f\right)\right] + (1-\gamma)\left[u(q_c) - \phi l\right]\right),
\end{aligned}
$$

while for the seller the payoff is given by

$$
Z(\check{C}_{-1}; \check{\mathbf{s}}_{-1}) + \beta\sigma\left(\gamma\pi(\omega)\left[\phi\left(\hat{\eta}d + \hat{\eta}^f e d^f\right) - c(q)\right] + (1-\gamma)\left[\phi l - c(q_c)\right]\right),
$$

where $Z(C_{-1}, \mathbf{s}_{-1})$ and $Z(\breve{C}_{-1}; \breve{\mathbf{s}}_{-1})$ are defined as in the benchmark model.

It is important to note that since the settlement of credit is done in the frictionless CM, allowing this additional medium of exchange does not really change the private information problem of counterfeiting currencies agents face in DM. Note that even when the loan is denominated in local or foreign currency, since the loan repayment in CM is done just with genuine fiat currencies. This is the case as any counterfeited fiat currency can be easily and immediately detected by government authorities. The seller then faces no private information problems when using credit as a means of payment. As a result, as in the previous section, the PBE re-ordering equilibrium of the counterfeiting game is such that each seller accepts with probability one and each buyer does not counterfeit.

It can be shown that the resulting monetary steady state equilibria in this new environment where the DM-seller's participation constraint binds ($\zeta > 0$) is given by

$$1 = \mathcal{U}'(C); \tag{35}$$

$$u'(q_c) = c'(q_c), \tag{36}$$

$$\phi l = c(q_c), \tag{37}$$

$$\beta\sigma\gamma u'(q) = \zeta c'(q), \tag{38}$$

$$\beta\sigma\gamma = \zeta - \mu - \lambda + \nu, \tag{39}$$

$$\beta\sigma\gamma = \zeta - \mu^f - \lambda^f + \nu^f, \tag{40}$$

$$\mu = \Pi - \beta \geq 0, \tag{41}$$

$$\mu^f = \Pi^f - \beta \geq 0; \tag{42}$$

$$c(q) = \phi M + e\phi M^f, \tag{43}$$

$$\phi M \leq \frac{\kappa}{\Pi - \beta(1 - \sigma\gamma)}, \tag{44}$$

$$e\phi M^f \leq \frac{\kappa^f}{\Pi^f - \beta(1 - \sigma\gamma)} \tag{45}$$

where the ζ represents the Lagrange multiplier associated with the seller's participation constraint, μ (μ^f) represents the feasibility constraint for the local (foreign) fiat money and λ (λ^f) is the Lagrange multiplier corresponding to the liquidity constraint for the local (foreign) fiat money that arise because of the threat of counterfeiting as in the benchmark model. Implicit in this steady state monetary equilibria is that money markets clear so that $m = M$ and $m^f = M^f$ where M and M^f denote the money supply of domestic and foreign currency, respectively.

As we can see relative to the benchmark economy without credit, given by equations (24)-(32), we now have two extra equilibrium conditions that specify the quantity and corresponding payment of the DM specialized good when credit is feasible. These are given by equations (36) and (37), respectively. This new credit environment also changes the matching probability for cash trades from σ to $\gamma\sigma$.

The insights of Proposition 7 regarding the coexistence of fiat currencies and determinacy of nominal exchange rates still hold when credit is available. In particular, in order to obtain determinacy of nominal exchange rates liquidity constraints, arising from the counterfeiting game, need to bind.[25]

Finally, we note that when agents have more access to credit then buyers and sellers are less likely to meet and trade with fiat currencies. This creates a smaller incentive for the buyer to produce counterfeits, thus decreasing the information problem. This implies that as credit markets develop the possibility of having nominal exchange rate indeterminacy increases inducing more volatility. This prediction is consistent with Keynes (1964) and Friedman (1956) who have suggested that interactions between the conduct of monetary policy and the financial system create considerable scope for endogenous volatility and indeterminacy.

5.2 Fiat Currencies As Collateral

When using cash as collateral, fiat money is also used as an asset. This type of trading arrangement is observed in several markets. For instance, the Moscow Exchange's Clearing Center accepts foreign currency as collateral when traders transact in this exchange.[26]

In order to contemplate this new trading arrangement, we follow Li, Rocheteau and Weill (2012) and consider a fiat retention requirement which can be also thought of as a overcollaterized loan. In this scenario a buyer who wishes to transfer d (d^f) units of domestic (foreign) fiat currency in the DM must hold $1+\rho$ ($1+\rho^f$) units of the domestic (foreign) currency.[27] Note that the fiat currency that is being collaterized or kept in retention is the same as the one transferred in a match. Thus if the transferred fiat currency is fraudulent, so is the one kept in retention. As in the benchmark model, buyers still face a fixed cost when producing counterfeited fiat currencies and sellers still face the private information problem.

We note that the introduction of a cash collateral does not change the nature of the counterfeiting game faced by agents. As in the benchmark model equilibrium offers are accepted with probability one.[28] It is easy to show that in this new environment the corresponding

[25]The particular values of the equilibrium observables are of course different from those of the benchmark model.

[26]In particular, in the Moscow Exchange, foreign currency registered as collateral is re-estimated in rubles by the Clearing Center on a daily basis during the evening clearing session, using the USD/RUB exchange rate determined as at 18:30 Moscow time.

[27]When the buyer is *keeping* an asset and is not using it all in payment is sending a signal regarding its quality. The collateral requirements in the Dodd-Frank Act can be thought as a way to mitigate the informational problems associated with the assets being traded.

[28]Li, Rocheteau and Weill (2012) show that when there is an asset retention policy and the cost of counterfeiting includes a variable cost, the RI-equilibrium is such that buyers do not counterfeit and sellers always accept to trade with buyers. The equilibrium presented here is a special case.

buyer's problem is then given by

$$\max_{q,d,d^f,m,m^f} \Bigg\{ -\left(\frac{\phi_{-1}}{\phi}-\beta\right)\phi m - \left(\frac{\phi_{-1}e_{-1}}{\phi e}-\beta\right)\phi e m^f + \beta\sigma\left[u(q)-\phi\left(d+ed^f\right)\right] :$$
$$\begin{aligned}
&(\zeta): \quad \phi\left(d+ed^f\right)-c(q)=0,\\
&(\nu): \quad 0 \leq d,\\
&(\mu): \quad (1+\rho)d \leq m,\\
&(\nu^f): \quad 0 \leq d^f,\\
&(\mu^f): \quad (1+\rho^f)d^f \leq m^f,\\
&(\lambda): \phi m \leq \frac{\kappa}{(\Pi-\beta)(1+\rho)+\beta\sigma};\\
&(\lambda^f): e\phi m^f \leq \frac{\kappa^f}{(\Pi^f-\beta)(1+\rho^f)+\beta\sigma} \qquad \Bigg\}.
\end{aligned}$$

It is easy to show that the resulting monetary steady state equilibria in this new environment is given by

$$1 = \mathcal{U}'(C), \tag{46}$$
$$\beta\sigma u'(q) = \zeta c'(q), \tag{47}$$
$$\beta\sigma = \zeta-\mu(1+\rho)-\lambda, \tag{48}$$
$$\beta\sigma = \zeta-\mu^f(1+\rho^f)-\lambda^f, \tag{49}$$
$$\mu = \Pi-\beta \geq 0, \tag{50}$$
$$\mu^f = \Pi^f-\beta \geq 0, \tag{51}$$
$$c(q) = \phi M + e\phi M^f, \tag{52}$$
$$\phi M \leq \frac{\kappa}{\Pi(1+\rho)-\beta(1+\rho-\sigma)}, \tag{53}$$
$$e\phi M^f \leq \frac{\kappa^f}{\Pi^f(1+\rho^f)-\beta(1+\rho^f-\sigma)}. \tag{54}$$

Implicit in this steady state monetary equilibria is that money markets clear so that $m = M$ and $m^f = M^f$ where M and M^f denote the money supply of domestic and foreign currency, respectively.

As in the previous environment where credit was posible in some trades, the main message of Proposition 7 that binding liquidity constraints can help us break the Kareken and Wallace indeterminacy result still holds in this cash collateral environment. The main changes relative to the benchmark model is the value of the Lagrange multipliers corresponding to the fiat currencies, μ and μ^f, and the upper bound on the liquidity constraints.

It is worth examining the case of the Moscow exchange and characterizing how the collateral requirement on foreign currency would affect the nominal exchange rate. The Moscow exchange can be modelled as one where there is no cash collateral for the domestic currency; i.e, $\rho = 0$ and the domestic inflation rate is higher than the US dollar so that $\Pi > \Pi^f$. Adapting the results of Proposition 7 to this new environment, it is easy to show that when both liquidity constraints bind the nominal exchange rate is given by

$$e = \frac{\kappa^f M}{\kappa M^f} \frac{\Pi - \beta(1-\sigma)}{\Pi^f(1+\rho^f) - \beta(1+\rho^f - \sigma)}.$$

which suggests that as the collateral requirement for the foreign currency increases, the nominal exchange rate appreciates. On the other hand, when only the foreign-asset liquidity constraint binds, we have that the nominal exchange rate is given by

$$e = \frac{M}{M^f} \frac{\kappa^f}{c(q)\left[\Pi^f(1+\rho^f) - \beta(1+\rho^f - \sigma)\right] - \kappa^f};$$

where q solves

$$\frac{\Pi - \beta}{\sigma\beta} = \frac{u'(q) - c'(q)}{c'(q)}.$$

Similarly, when the collateral requirement for the foreign currency increases the nominal exchange rate appreciates.

5.3 Nominal Bonds

In this section we consider an economy where domestic and foreign denominated bonds can be counterfeited rather than the fiat currencies themselves. This new environment allows us to think how counterfeiting of assets denominated in different currencies affect the determinacy of nominal exchange rates. The fraud in asset-backed securities has been argued to be an important aspect present in the global financial crises but so far the literature has not exploited the fact that these securities were denominated in different currencies. When explicitly taking this feature into account, we can study the impact of these potentially fraudulent securities on the nominal exchange rate.

Relative to the benchmark model, we now have two different nominal assets, one-period nominal bonds, that compete with each other as a medium of exchange. Nominal domestic (foreign) bonds, denoted by b (b^f), can be counterfeited at a positive fixed cost κ (κ^f) > 0. These nominal bonds are perfectly divisible, payable to the bearer and issued by the two governments in CM. The price of a domestic (foreign) bond is φ (φ^f) in terms of domestic (foreign) currency. These bonds when redeemed pay one unit of corresponding fiat money in the next CM. Both the supply of domestic (foreign) money and nominal bonds grow at the constant rate, which in principle can be potentially be different across the two monetary authorities.

Following the same logic as in the benchmark model, the equilibrium of the counterfeiting game is such that buyers do not counterfeit nominal bonds and sellers accept to trade with buyers. The DM-buyer's problem is

$$\max\left\{\left[-\left(\frac{\varphi\Pi-\beta}{\beta}\right)\phi b-\left(\frac{\varphi^f\Pi^f-\beta}{\beta}\right)\phi e b^f+\beta\sigma\left[u(q)-\phi\left(d+ed^f+d_b+ed_b^f\right)\right]\right]:\right.$$
$$(\zeta):\quad \phi\left(d+ed^f+d_b+ed_b^f\right)-c(q)\geq 0,$$
$$(\mu_b):\quad d_b\leq b,$$
$$(\mu_b^f):\quad d_b^f\leq b^f,$$
$$(\ \lambda):\quad \phi d_b\leq\frac{\kappa}{\varphi\Pi-\beta(1-\sigma)},$$
$$\left.(\lambda^f):\quad \phi e d_b^f\leq\frac{\kappa^f}{\varphi^f\Pi^f-\beta(1-\sigma)}\right\},$$

where d_b and d_b^f represent the payment by the buyer to the seller in nominal domestic and foreign denominated bonds, respectively.

From now on we focus on monetary equilibria where $\varphi\Pi>\beta$ and $\varphi^f\Pi^f>\beta$ so that buyers give all their bond holdings to the seller when trading in DM; i.e, $d_b=b$ and $d_b^f=b^f$. The resulting monetary steady state equilibria is then given by

$$1 = \mathcal{U}'(C), \tag{55}$$
$$\beta\sigma u'(q) = \zeta c'(q), \tag{56}$$
$$\beta\sigma = \zeta-\mu_b-\lambda, \tag{57}$$
$$\beta\sigma = \zeta-\mu_b^f-\lambda_f, \tag{58}$$
$$\mu_b = \varphi\Pi-\beta>0, \tag{59}$$
$$\mu_b^f = \varphi^f\Pi^f-\beta>0, \tag{60}$$
$$c(q) = \phi\left(B+eB^f\right)\Leftrightarrow\zeta>0, \tag{61}$$
$$\phi B \leq \frac{\kappa}{\varphi\Pi-\beta(1-\sigma)}\Leftrightarrow\lambda\geq 0, \tag{62}$$
$$\phi e B^f \leq \frac{\kappa^f}{\varphi^f\Pi^f-\beta(1-\sigma)}\Leftrightarrow\lambda^f\geq 0. \tag{63}$$

Implicit in this steady state monetary equilibria is that bond markets clear so that $b=B$ and $b^f=B^f$ where B and B^f denote the money supply of domestic and foreign bonds, respectively.

As in the previous environment where trade credit was available in some trades, the main message of Proposition 7 that binding liquidity constraints can help us break the Kareken and Wallace indeterminacy result still holds when the threat of counterfeiting in assets denominated in domestic and foreign currencies is possible.

5.4 Fiscal Policies

In this section we study how fiscal policies can restore determinacy of the nominal exchange rate in Case 3a of Proposition 7. Finding such policies is crucial for policy analysis as an environment with indeterminacy requires the selection of a specific allocation and prices consistent with equilibria. Establishing an appropriate selection rule is extremely difficult.

Let us now consider an environment where the local government is able to impose a tax on all domestic production generated in the centralized market and follows a constant money growth rate rule. For simplicity, we assume that the tax revenues fund wasteful government expenditures. In this new environment the buyer's sequential budget constraint for each CM is given by

$$C_t \leq (1-\tau)N_t - \phi_t(m_{t+1} - m_t) - \phi_t e_t(m^f_{t+1} - m^f_t),$$

where τ is the income tax rate. The same taxation assumption applies to the DM-seller's and their corresponding CM budget constraints. This labor tax does not give any of the currencies a distinct advantage over the other. In short, we are not specifying how taxes are paid.

It is straightforward to show that the resulting liquidity constraints for a stationary monetary equilibrium for both domestic and foreign currency are given by

$$\frac{\phi M}{1-\tau} \leq \frac{\kappa}{\Pi - \beta(1-\sigma)};$$

$$\frac{e\phi M^f}{1-\tau} \leq \frac{\kappa^f}{\Pi^f - \beta(1-\sigma)}.$$

As we can see from these new endogenous liquidity constraints, taxation affects the value of fiat currency which affects the incentives to counterfeit. Note that for a given monetary allocation, we can always find a tax rate $\bar{\tau}$ such that one of the liquidity constraints bind. Thus fiscal policies in coordination with monetary policy can increase the set of equilibria where the nominal exchange rate outcome is determinate. This parallels the analysis of fiscal-monetary policy interconnections for determinacy of equilibria of Leeper (1991) in frictional economies of another kind.

5.5 An explicit two-country setup

In the previous setup and its variations, we have seigniorage revenue transfers x_t occurring at the beginning of each CM(t), and the fact that preferences are quasilinear, we end up with an equilibrium property that it does not matter who receives the transfers (i.e., Home or Foreign agents), nor which transfers (i.e., Home or Foreign currency). This fact allowed us to abstract from modelling explicitly multiple country-specific decentralized markets. That is, we could focus on a single integrated world economy. We now consider a more explicit two-country variation which is closer to a standard two-country general equilibrium international macro-finance model (see e.g., Gomis-Porqueras, Kam and Lee, 2013). Looking ahead, we obtain a more general version of Proposition 7 on monetary coexistence and nominal exchange rate determinacy.

Some additional notation is required for this setting. Let a variable X (or X^f) denote an object produced in the Home (or Foreign) country, which is held by agents in the Home country. Denote $X_\star$ (or $X_\star^f$) as an outcome produced in the Home (or Foreign) country but destined for use by Foreign agents. We make two assumptions. First, we now assume that transfers of seigniorage happen at the beginning of each DM, after each CM portfolio decisions are made and all CM markets close. Second, only Home DM-agents receive a uniform transfer of seigniorage revenue $x := M_t - M_{t-1}$. Likewise, only Foreign DM-agents receive the uniform transfer $x_\star^f := M_{\star,t}^f - M_{\star,t-1}^f$. This is innocuous for the results we will obtain later.[29]

From the model described earlier, now the initial state of DM-buyers at the start of each CM(t) is $y_t \equiv m_t$ and $y_t^f \equiv m_t^f$ since transfers are no longer made at the beginning of CM. However, the version of the Dm-buyer's Bernoulli payoff from (7) is now:

$$\begin{aligned} U^b[C_{-1}, \omega, H(\omega), \hat{\pi}|\mathbf{s}_{-1}; \phi, e] = & \\ & Z(\mathbf{s}_{-1}) - \left(\frac{\phi_{-1}}{\phi} - \beta\right)\phi m - \left(\frac{\phi_{-1}e_{-1}}{\phi e} - \beta\right)\phi e m^f \\ & - \kappa(1-\eta) - \kappa^f(1-\eta^f) + \beta\sigma\hat{\pi}\left[u(q) - \phi\left(\eta d + \eta^f e d^f\right)\right] + \beta\phi x. \end{aligned} \tag{7'}$$

The last term $\beta\phi x$ is the result of the lump-sum transfer of seigniorage revenues (from the Home monetary authority) to DM-buyers in the Home country at the start of DM(t). Equivalently, if we write the DM-buyer's payoff function for the Foreign country, there will be an additional term $\beta\phi^f x_\star^f$. (There is an equivalent set of payoff functions for DM-buyers and DM-sellers for the Foreign country, but we will not duplicate them here.) The RI-equilibrium characterization will appear similar to that in Proposition 5, with the exception that the feasibility constraints on the monies will now say $d \leq m + x$ and $d^f \leq m^f$. That is, what is to be offered as payments in DM(t) exchange are now bounded above by what Home DM-buyers have accumulated at the end of each CM($t-1$), respectively m and m^f, including the nominal transfer from the Home government x.

Symmetrically, there is another characterization of an RI-equilibrium for the Foreign coun-

[29] If all Home agents receive the transfer uniformly then each DM-buyer gets $x/2$ and each DM-seller gets $x/2$, since each population is of measure one. DM-sellers have no use of the transfer $x/2$ and will only spend it in the ensuing CM, and decide to work less there. From the perspective of the agents, this will only alter the DM-buyers' initial wealth by $x/2$, which is taken as parametric by the agents, and thus will not affect their optimal decision margins.

try with constraints $d_\star^f \leq m_\star^f + x_\star^f$ and $d_\star \leq m_\star$. Note that since x and $x_\star^f$ are exogenous to DM-buyers in each country, they do not affect their marginal decisions, however, they will matter for the global adding-up conditions for monies supplied and demanded in equilibrium.

Given the familiarity with the simpler integrated world economy setting earlier, we can now jump ahead to a describing a steady-state monetary equilibrium for this explicit two-country variation. After some algebra, the necessary conditions for such an equilibrium can is given by the Home DM-buyers' Euler conditions and DM-sellers' participation constraint, respectively,

$$\beta\sigma \left[\frac{u'(q)}{c'(q)} - 1\right] = \lambda - \nu + (\Pi - \beta) = \lambda^f - \nu^f + (\Pi^f - \beta), \tag{64a}$$

and,

$$c(q) = \phi d + \phi^f d^f, \tag{64b}$$

where λ (λ^f) is the Home liquidity constraint on Home (Foreign) real money balance, and ν (ν^f) is the nonnegativity constraint on d (d^f). The Karush-Kuhn-Tucker (KKT) conditions are and the KKT conditions are:

$$\begin{aligned} \lambda \cdot [\bar{\kappa}(\Pi) - \phi d] &= 0, \qquad \lambda \geq 0, \qquad \phi d \leq \bar{\kappa}(\Pi), \\ \lambda^f \cdot \left[\bar{\kappa}^f(\Pi^f) - \phi^f d^f\right] &= 0, \qquad \lambda^f \geq 0, \qquad \phi^f d^f \leq \bar{\kappa}^f(\Pi^f), \\ -\nu \cdot d &= 0, \qquad d \geq 0, \qquad \nu \geq 0, \\ -\nu^f \cdot d^f &= 0, \qquad d^f \geq 0, \qquad \nu^f \geq 0. \end{aligned} \tag{64c}$$

where $\bar{\kappa}(\Pi) := \kappa/[\Pi - \beta(1-\sigma)]$ and $\bar{\kappa}^f(\Pi^f) := \kappa^f/[\Pi^f - \beta(1-\sigma)]$.

Similarly, we have the Foreign block as

$$\beta\sigma \left[\frac{u'(q_\star^f)}{c'(q_\star^f)} - 1\right] = \lambda_\star - \nu_\star + (\Pi - \beta) = \lambda_\star^f - \nu_\star^f + (\Pi^f - \beta), \tag{65a}$$

and,

$$c(q_\star^f) = \phi^f d_\star^f + \phi d_\star, \tag{65b}$$

where $\lambda_\star$ (or $\lambda_\star^f$) is the Foreign liquidity constraint on Home (Foreign) real money balance, and $\nu_\star$ (or $\nu_\star^f$) is the nonnegativity constraint on $d_\star$ (or $d_\star^f$). The KKT conditions are:

$$\begin{aligned} \lambda_\star \cdot [\bar{\kappa}_\star(\Pi) - \phi d_\star] &= 0, \qquad \lambda \geq 0, \qquad \phi d_\star \leq \bar{\kappa}_\star(\Pi), \\ \lambda_\star^f \cdot \left[\bar{\kappa}_\star^f(\Pi^f) - \phi^f d_\star^f\right] &= 0, \qquad \lambda_\star^f \geq 0, \qquad \phi^f d_\star^f \leq \bar{\kappa}_\star^f(\Pi^f), \\ -\nu_\star \cdot d_\star &= 0, \qquad d_\star \geq 0, \qquad \nu_\star \geq 0, \\ -\nu_\star^f \cdot d_\star^f &= 0, \qquad d_\star^f \geq 0, \qquad \nu_\star^f \geq 0. \end{aligned} \tag{65c}$$

where $\bar{\kappa}_\star(\Pi) := \kappa_\star/[\Pi - \beta(1-\sigma)]$ and $\bar{\kappa}_\star^f(\Pi^f) := \kappa_\star^f/[\Pi^f - \beta(1-\sigma)]$, with $\kappa_\star > 0$ and $\kappa_\star^f > 0$. In the explicit two-country setting, the following adding-up conditions for Home and Foreign monies, must hold between demand for real balances across countries and their supplies. Respectively, these are

$$\phi d_\star = \phi(M - d), \tag{66a}$$

and,

$$\phi d_\star^f = \phi^f(M^f - d^f). \tag{66b}$$

The following states a generalization of our main result on coexistence, nominal exchange rate determinacy where there is rate of return parity or even dominance between the fiat monies. (In the interest of space, we omit the symmetric case where the foreign fiat money is dominated in rate of return.)

Proposition 9 (Two-Country Monetary Coexistence and Exchange Rate) *Assume the two-country world economy with independent monetary policies* Π *and* Π^f *where injections/withdrawals of lump sum monetary take place after CM trades.*

Case 1 *When Home fiat money is dominated in rate of return* ($\Pi > \Pi^f$)*, there exist a monetary equilibrium with a determinate nominal exchange rate with*

- *(a) the two currencies coexisting in only one country when some liquidity constraints bind, or*
- *(b) both currencies circulating in the Home and Foreign countries when all liquidity constraints bind in both countries.*

Case 2 *When Home fiat money dominates in rate of return* ($\Pi^f > \Pi$)*, the coexistence results are the symmetric opposite to those of Case 1.*

Case 3 *When Home fiat money has the same rate of return as the Foreign currency* ($\Pi = \Pi^f$)*, there exist a monetary equilibrium with a determinate nominal exchange rate with both currencies circulating in the Home and Foreign countries when all liquidity constraints bind in both countries.*

The fact that lump sum monetary transfers/withdrawals take place after CM changes significantly the equilibrium properties of the economy. This is the case as in this new economy the operating procedure for monetary policy has wealth effects as it can help expand the set of feasible offers that buyers can make. As a result, even when both currencies coexist the nominal exchange rate is rather different from the one we obtain in the integrated economy. Nevertheless is worth pointing out that the key insight of the paper still holds though. In other words, the binding liquidity constraints induced by the threat of counterfeiting are key in pinning down the nominal exchange rate.

6 Conclusion

In this paper we present a search theoretic model of money with informational asymmetry to study nominal exchange rate determinacy. Agents in this economy trade sequentially in decentralized and Walrasian markets where they can use both currencies, domestic and foreign, to settle transactions. Buyers may counterfeit both fiat currencies at some fixed cost prior to decentralized-market exchanges. Counterfeiting is private information to buyers, which gives rise to endogenous liquidity constraints on the use of alternative currencies as media of payment.

An interesting feature of our results is that there is no counterfeiting in equilibrium. It is the threat of counterfeiting that pins down the nominal exchange rate. A critical feature of these liquidity constraints is that the marginal liquidity value of an additional unit of currency beyond the binding constraint is zero. Thus, if the foreign currency has a higher rate of return (lower inflation rate) but a lower counterfeiting cost, then the buyer will first pay with the foreign currency up to the bound and use domestic currency to pay for the remainder of the goods purchased. Because of this, both currencies can circulate even though one currency is dominated in the rate of return.

We show that the threat of counterfeiting mechanism is robust in determining nominal exchange rates. In particular, when liquidity constraints bind, due to the private information problem, nominal exchange rates can be determined: (i) when trade credit is possible, (ii) when currencies can be used as collateral and (ii) when nominal bonds denominated in domestic and foreign currency can be counterfeited. Finally, we show that in the case of nominal exchange rate indeterminacy one can find a fiscal policy that will make one of the limits bind, thus restoring equilibrium determinacy. This allows one to rationalize why currencies with similar rates of return remain in circulation (apart from obvious explanations in terms of legal restrictions) as media of exchange.

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Supplementary Appendix

A Degeneracy of asset portfolios

Given that we are interested in monetary equilibria from now on we restrict attention to economies where $\phi_{t-1}/\phi_t \geq \beta$; and $\phi^f_{t-1}/\phi^f_t \equiv e_{t-1}\phi_{t-1}/e_t\phi_t \geq \beta$. The following lemma allows us to simplify the Bernoulli payoff function given by equation (6).

Lemma 10 *Under any optimal measurable strategy $\tilde{\boldsymbol{\sigma}}^b$, genuine portfolio choices are always such that:*

$$m \begin{cases} = \chi d, & \text{if } \phi_{-1}/\phi > \beta \\ \geq \chi d, & \text{if } \phi_{-1}/\phi = \beta \end{cases}; \text{ and, } m^f \begin{cases} = \chi^f d^f, & \text{if } \phi_{-1}e_{-1}/\phi e > \beta \\ \geq \chi^f d^f, & \text{if } \phi_{-1}e_{-1}/\phi e = \beta \end{cases}.$$

Moreover, whenever $\phi_{-1}/\phi = \beta$ (or $\phi_{-1}e_{-1}/\phi e = \beta$), demanding $m > \chi d$ (or $m^f > \chi^f d^f$) is U^b-payoff equivalent for the buyer to demanding $m = \chi d$ (or $m^f = \chi^f d^f$).

Proof. First consider the cases where the returns of either (or both) assets are strictly dominated by β. Then, holding either (or both) assets beyond what is necessary for payments in the DM (i.e. d and d^f) is intertemporally costly since the price levels ϕ^{-1}, and, $(\phi^f)^{-1}$ are respectively growing at the rates $\gamma - 1$ and $\gamma^f - 1$. Thus holding only $m = (1-\chi)d$ or (and) $m^f = (1-\chi^f)d^f$ is optimal for the DM-buyer under any optimal strategy $\tilde{\boldsymbol{\sigma}}^b$.

Second, consider the cases where the returns of either (or both) assets are equal to β. Then any portfolio demand comprising $m \geq (1-\chi)d$ or (and) $m^f \geq (1-\chi^f)d^f$ is optimal. However, since W is linear, the only terms involving m and m^f in the buyer's payoff function U^b in (6) are the expected costs of holding unused genuine assets, given by the linear functions

$$\left(\frac{\phi_{-1}}{\phi} - \beta\right)\phi m - \left(\frac{\phi_{-1}e_{-1}}{\phi e} - \beta\right)\phi e m^f.$$

Observe that in the cases where the asset returns are equal to β, the value of these costs are zero. Therefore, the second statement in the Lemma is true. ■

This result stems from two observations: (i) if the returns on the two fiat currencies are strictly below the discount factor β, then, holding these assets are intertemporally costly; and (ii) if their returns are equal to β, the linearity of $W(\cdot)$ ensures that any excess asset demands beyond what is necessary for trade in the DM is inconsequential to the payoff $U^b(\cdot)$.

Lemma 10 also implies that each $G(\cdot|\omega)$ consistent with $\tilde{\boldsymbol{\sigma}}^b$ is degenerate, as far as characterizing the Bernoulli payoff function U^b is concerned. That is, given the realization of $\aleph(\omega) := (\chi(\omega), \chi^f(\omega))$, we have the following

$$G[a(\omega)|\omega] = \delta_{\{(1-\chi)d,(1-\chi^f)d^f\}}, \qquad \forall \aleph \in \{0,1\}^2, \tag{67}$$

where δ_E denotes the Dirac delta function defined to be everywhere zero-valued except on events E, on which the function has value 1. In short, we can characterize the buyer's mixed strategy $G(\cdot|\omega)$ (over portfolio accumulation) in the subgame following the buyer's finite history of play, $\langle \omega, \aleph(\omega) \rangle$, prior to comprehensively describing equilibrium in the game.

B Proof of Proposition 5 (RI-equilibrium)

Denote the maximum value of the program in (12), when $\hat{\pi} = \pi(\omega) = 1$ and $H = (\hat{\eta}, \hat{\eta^f}) = (\eta(\omega), \eta^f(\omega)) = (1,1)$, as $(U^b)^*$. The aim is to show that an equilibrium $\tilde{\boldsymbol{\sigma}}$ yields the same value as $(U^b)^*$, and it satisfies the characterization in Proposition 5 (Case 1); and that any other candidate strategy $\tilde{\boldsymbol{\sigma}}' := \langle \omega', H', \pi' \rangle$ such that $\hat{\pi}' = \pi'(\omega) \neq 1$ and/or $\hat{H}' \neq (1,1)$ will induce a buyer's valuation that is strictly less than $(U^b)^*$, and therefore cannot constitute an equilibrium (Cases 2-5).

Consider the subgame following offer ω. Let $\rho(\chi, \chi^f)$ denote the joint probability measure on events $\{(\chi, \chi^f)\}$, where the pure actions over counterfeiting are $(\chi, \chi^f) \in \{0,1\}^2$. Denote $P := 2^{\{0,1\}^2}$ as the power set of $\{0,1\}^2$. By the definition of probability measures, it must be that $\sum_{\{z\} \in P} \rho(z) = 1$.

The seller's problem in (9) is equivalent to:

$$\pi(\omega) \in \left\{ \arg \max_{\pi' \in [0,1]} \pi' \Bigg[\phi \Bigg([1 - \hat{\rho}(1,0) - \hat{\rho}(1,1)]d \right. \\ \left. + [1 - \hat{\rho}(0,1) - \hat{\rho}(1,1)]ed^f \Bigg) - c(q) \Bigg] \right\}. \tag{68}$$

This is a linear programming problem in π, given the seller's rational belief system $\hat{\rho}$ and buyer's offer ω. Thus the seller's best response satisfies:

$$\left(\phi \left([1 - \hat{\rho}(1,0) - \hat{\rho}(1,1)]d + [1 - \hat{\rho}(0,1) - \hat{\rho}(1,1)]ed^f \right) - c(q) \begin{cases} > 0 \\ < 0 \\ = 0 \end{cases} \right) \\ \Rightarrow \left(\pi(\omega) \begin{cases} = 1 \\ = 0 \\ \in [0,1] \end{cases} \right). \tag{69}$$

Let $U^b_{\{z\}} \equiv U^b[\omega, \{z\}, \hat{\pi} | \mathbf{s}_{-1}, \phi, e]$ denote the buyer's expected payoff from *realizing* pure actions (χ^h, χ^f), given offer ω and rational belief system $\hat{\pi} \in [0,1]$, where $\{z\} \in P$. We have the following possible payoffs following each event $\{z\}$:

$$U^b_{\{(0,0)\}} = -\left(\frac{\phi_{-1}}{\phi} - \beta\right)\phi d - \left(\frac{\phi_{-1}e_{-1}}{\phi e} - \beta\right)\phi e d^f + \beta\sigma\hat{\pi}\left[u(q) - \phi\left(d + ed^f\right)\right]; \tag{70}$$

$$U^b_{\{(0,1)\}} = -\kappa^f - \left(\frac{\phi_{-1}}{\phi} - \beta\right)\phi d + \beta\sigma\hat{\pi}\left[u(q) - \phi d\right]; \tag{71}$$

$$U^b_{\{(1,0)\}} = -\kappa - \left(\frac{\phi_{-1}e_{-1}}{\phi e} - \beta\right)\phi e d^f + \beta\sigma\hat{\pi}\left[u(q) - \phi e d^f\right]; \tag{72}$$

$$U^b_{\{(1,1)\}} = -\kappa^f - \kappa + \beta\sigma\hat{\pi}u(q). \tag{73}$$

Observe that

$$U^b_{\{(0,1)\}} + U^b_{\{(1,0)\}} = U^b_{\{(0,0)\}} + U^b_{\{(1,1)\}}. \tag{74}$$

There are five cases to consider.

Case 1. Suppose there is a set of candidate equilibria such that $\rho(0,0) = 1$ and $\rho(z) = 0$, for all $\{z\} \in P$ and $z \neq (0,0)$. Then, we have $U^b_{\{(0,0)\}} > \max\{U^b_{\{(1,0)\}}, U^b_{\{(0,1)\}}, U^b_{\{(1,1)\}}\}$. Since $U^b_{\{(0,0)\}} > U^b_{\{(1,0)\}}$ and $U^b_{\{(0,0)\}} > U^b_{\{(0,1)\}}$, then, from (70)-(73) we can derive that

$$\phi d < \frac{\kappa}{\frac{\phi_{-1}}{\phi} - \beta(1 - \sigma\hat{\pi})}, \tag{75}$$

and,

$$\phi e d^f < \frac{\kappa^f}{\frac{\phi_{-1}e_{-1}}{\phi e} - \beta(1 - \sigma\hat{\pi})}. \tag{76}$$

The interpretation from (75) and (76) is that the liquidity constraints on either monies are slack. Therefore the buyer's expected payoff in this case can be evaluated from (70). If $\hat{\pi} < 1$, then from the seller's decision rule (69) we can deduce $\omega \equiv (q, d, d^f)$ must be such that the seller's participation/incentive constraint binds:

$$c(q) = \phi(d + ed^f). \tag{77}$$

Since (77) holds, all we need to do is verify the buyer's payoff. Since, the buyer's liquidity constraints (75) and (76) do not bind at $\hat{\pi} < 1$, a small increment in either payment offered, d or d^f, relaxes (77) and this raises $\hat{\pi}$, and thus the buyer's payoff (70). The maximal payoff to the buyer, keeping the seller in participation, is when $\pi(\omega) = \hat{\pi} = 1$, and the offer $\overline{\omega}$ is such that

$$\overline{U}^b \equiv U^b_{\{(0,0)\}}[\overline{\omega}|\pi(\omega) = \hat{\pi} = 1] = \sup_{\omega} \Bigg\{ U^b_{\{(0,0)\}}[\omega|\pi(\omega) = \hat{\pi} = 1] : \\ \phi d \leq \frac{\kappa}{\frac{\phi_{-1}}{\phi} - \beta(1-\sigma)}, \\ \phi e d^f \leq \frac{\kappa^f}{\frac{\phi_{-1}e_{-1}}{\phi e} - \beta(1-\sigma)}, \\ c(q) \leq \phi(d + ed^f) \Bigg\}.$$

Then it is easily verified that this maximal value coincides with the maximum value of the program given in (12) in Proposition 5, i.e. $\overline{U}^b = (U^b)^*$, since the payoff function is continuous, and the constraints also define a nonempty, compact subset of the feasible set $\Omega(\phi, e) \ni \overline{\omega}$. Since the seller has no incentive to deviate from $\pi(\overline{\omega}) = 1$, then a behavior strategy $\tilde{\boldsymbol{\sigma}} = \langle \overline{\omega}, (1,1), 1 \rangle$ inducing the TIOLI payoff $\overline{U}^b$ is a PBE.

Case 2. Note that in any equilibrium, a seller will never accept an offer if $\rho(1,1) = 1$, and, a buyer will never counterfeit both assets with probability 1—counterfeiting for sure costs $\kappa + \kappa^f$ and the buyer gains nothing. Therefore, $\rho(1,1) < 1$ is a necessary condition for an equilibrium in the subgame following ω. Likewise, all unions of disjoint events with this event of counterfeiting all assets—i.e. $\{(\chi, \chi^f)\} \in \{(0,1)\} \cup \{(1,1)\}$ or $\{(\chi, \chi^f)\} \in \{(1,0)\} \cup \{(1,1)\}$—such that $\rho(0,1) + \rho(1,1) = 1$ or $\rho(1,0) + \rho(1,1) = 1$, respectively, cannot be on any equilibrium path.

Case 3. Suppose instead we have equilibria in which $\rho(0,0) + \rho(1,0) = 1$, $\rho(1,0) \neq 0$, and $\rho(1,1) + \rho(0,1) = 0$, so $U^b_{\{(1,0)\}} = U^b_{\{(0,0)\}} > \max\{U^b_{\{(0,1)\}}, U^b_{\{(1,1)\}}\}$.

Given this case, and from (74), we have $U^b_{\{(0,1)\}} = U^b_{\{(1,1)\}}$. From $U^b_{\{(1,0)\}} = U^b_{\{(0,0)\}}$, and (70) and (72), respectively, we have:

$$\hat{\pi} = \frac{\kappa - (\phi_{-1}/\phi - \beta)\phi d}{\beta\sigma\phi d}, \tag{78}$$

and,

$$\phi e d^f < \frac{\kappa^f}{\frac{\phi_{-1}e_{-1}}{\phi e} - \beta(1 - \sigma\hat{\pi})}. \tag{79}$$

If $\hat{\pi} < 1$, then from the seller's decision rule (69) we can deduce $\omega \equiv (q, d, d^f)$ must be such that the seller's participation/incentive constraint binds:

$$\begin{aligned} c(q) &= \phi[(1 - \rho(1,0) - \rho(1,1))d + (1 - \rho(0,1) - \rho(1,1))ed^*] \\ &= \phi[(1 - \rho(1,0))d + ed^*]. \end{aligned} \tag{80}$$

The buyer's payoff can be evaluated from (72). If $\hat{\pi} < 1$, then reducing d infinitesimally will increase $\hat{\pi}$ in (78), and this increase the buyer's payoff in (72). The buyer would like to attain $\hat{\pi} = 1$ since the seller's participation constraint will still be respected:

$$c(q) \leq \phi[(1 - \rho(1,0))d + ed^f]. \tag{81}$$

Let the maximum of the buyer's TIOLI value (72) such that the constraints (78), (79) and (81) are respected, in this case be $(U^b)^\dagger$. However, since $\rho(1,0) \neq 0$, it is easily verified that $(U^b)^\dagger < U^b_{\{(0,0)\}}[\overline{\omega}|\pi(\omega) = \hat{\pi} = 1; \rho(1,0) = 0] = \sup_{\omega,\rho(1,0)}\{U^b_{\{(1,0)\}}|(78),(79),(81)\} = (U^b)^*$, in which the last equality is attained when $\rho(1,0) = 0$. This contradicts the claim that $\rho(0,0) + \rho(1,0) = 1$ and $\rho(1,0) \neq 0$ is a component of a PBE.

Case 4. Suppose there are equilibria consisting of $\rho(0,0)+\rho(0,1) = 1$ with $\rho(0,1) \neq 0$, and $\rho(1,0) = \rho(1,1) = 0$. The buyer's payoff is such that $U^b_{\{(0,1)\}} = U^b_{\{(0,0)\}} > \max\{U^b_{\{(1,0)\}}, U^b_{\{(1,1)\}}\}$. Given this assumption, we have from (74) that $U^b_{\{(1,0)\}} = U^b_{\{(1,1)\}}$. From (70) and (71), we can derive

$$\hat{\pi} = \frac{\kappa^f - (\phi_{-1}e_{-1}/\phi e - \beta)\phi ed^f}{\beta\sigma\phi ed^f}. \tag{82}$$

From the case that $U^b_{\{(0,0)\}} > U^b_{\{(1,0)\}}$ and (70)-(72), we have:

$$\phi d < \frac{\kappa}{\frac{\phi_{-1}}{\phi} - \beta(1 - \sigma\hat{\pi})}. \tag{83}$$

The buyer's payoff can be evaluated from (71). If $\hat{\pi} < 1$, from (69), we can deduce that the seller's participation constraint is binding. If $\hat{\pi} < 1$, then reducing d^f infinitesimally will increase $\hat{\pi}$ in (82), and this increase the buyer's payoff in (71). The buyer would like to attain $\hat{\pi} = 1$ since the seller's participation constraint will still be respected at that point:

$$c(q) \leq \phi[d + (1 - \rho(0,1))ed^f]. \tag{84}$$

Let the maximum of the buyer's TIOLI value (71) such that the constraints (82), (83) and (84) are respected, in this case be $(U^b)^{\dagger\dagger}$. However, since $\rho(1,0) \neq 0$, it is easily verified that $(U^b)^{\dagger\dagger} < U^b_{\{(0,0)\}}[\overline{\omega}|\pi(\omega) = \hat{\pi} = 1; \rho(0,1) = 0] = \sup_{\omega}\{U^b_{\{(0,1)\}}|(82),(83),(84)\} = (U^b)^*$, in which the last equality is attained when $\rho(0,1) = 0$. This contradicts the claim that $\rho(0,0) + \rho(0,1) = 1$ and $\rho(0,1) \neq 0$ is a component of a PBE.

Case 5. Suppose a candidate equilibrium is such that $\sum_{\{z\}\in P}\rho(z) = 1$, $\rho(z) \neq 0$ for all $\{z\} \in P$, and that $U^b_{\{(0,1)\}} = U^b_{\{(0,0)\}} = U^b_{\{(1,0)\}} = U^b_{\{(1,1)\}}$. Then from (71) and (72), we can derive

$$\hat{\pi} = \frac{\kappa^f - (\phi_{-1}e_{-1}/\phi e - \beta)\phi ed^f}{\beta\sigma\phi ed^f} = \frac{\kappa - (\phi_{-1}/\phi - \beta)\phi d}{\beta\sigma\phi d}. \tag{85}$$

If the payment offered (d, d^f) are such that $\hat{\pi} < 1$, then from the seller's decision rule (69) we can deduce $\omega \equiv (q, d, d^f)$ must be such that the seller's participation/incentive constraint binds:

$$c(q) = \phi[(1 - \rho(1,0) - \rho(1,1))d + (1 - \rho(0,1) - \rho(1,1))ed^f]. \tag{86}$$

However, the buyer can increase his expected payoff in (73) by reducing both (d, d^*), thus raising $\hat{\pi}$ in (85) while still ensuring that the seller participates, until $\hat{\pi} = 1$, where

$$c(q) \leq \phi[(1 - \rho(1,0) - \rho(1,1))d + (1 - \rho(0,1) - \rho(1,1))ed^f]. \tag{87}$$

Let the maximum of the buyer's TIOLI value (73) such that the constraints (85) and (87) are respected, in this case be $(U^b)^{\ddagger}$. However, since $\rho(1,0), \rho(0,1), \rho(1,1) \neq 0$, it is easily verified that $(U^b)^{\ddagger} < U^b_{\{(1,1)\}}[\overline{\omega}|\pi(\omega) = \hat{\pi} = 1; \rho(0,0) = 1] = \sup_{\omega}\{U^b_{\{(1,1)\}}|(85),(84)\} = (U^b)^*$, in which the last equality is attained when $\rho(0,0) = 1$. This contradicts the claim that $\sum_{\{z\}\in P} \rho(z) = 1$, $\rho(z) \neq 0$ for all $\{z\} \in P$, is a component of a PBE.

Summary. From Cases 1 to 5, we have shown that the only mixed-strategy Nash equilibrium in the subgame following an offer ω must be one such that $\langle \rho(0,0), \pi \rangle = \langle 1, 1 \rangle$, and that the offer ω satisfies the program in (12) in Proposition 5.

Finally, since $u(.)$ and $-c(.)$ are strictly quasiconcave functions and the inequality constraints in program (12) define a convex feasible set, the program (12) has a unique solution.

C Proof of Proposition 6 (Constant exchange rate growth)

A monetary equilibrium implies that the general goods market, asset (money) markets and labor markets must clear in every country. This also implies that each DM-buyer's sequential (and therefore intertemporal) budget constraint must hold. We will evaluate these budget constraints at a monetary equilibrium. For the treatment below, we assume that seigniorage revenue x_t is transferred to the DM-buyers uniformly at the beginning of each CM.

First, we re-write the date-t budget constraint of a DM-buyer for end-of-period change in domestic real money holding:

$$\phi_t(m_{t+1} - m_t) = N_t - C_t + \phi_t x_t - \phi_t^f \left(m_{t+1}^f - m_t^f \right), \qquad \forall t \geq 0, \tag{88}$$

where m_t (or m_t^f) is the initial stock of domestic (or foreign) currency held by the DM-buyer. Also, ϕ_t (or $\phi_t^f = e_t \phi_t$), given an equilibrium nominal exchange rate e_t, is the value of a unit of domestic (or foreign) currency in units of the domestic (or foreign) CM consumption good C_t (or C_t^f). In equilibrium N_t the amount of labor supplied is also the labor income to the DM-buyer.

Summing these sequential budget constraints up to some finite date $T > 0$, multiplying both sides by a constant $\beta^T \in (0,1)$, and taking the limit of $T \nearrow +\infty$, we have:

$$\lim_{T\nearrow+\infty} \beta^T \phi_{T+1} m_{T+1} = \lim_{T\nearrow+\infty} \beta^T \left(\frac{\phi_T}{\phi_0}\right) \left\{ \phi_0(m_0 + e_0 m_0^f) + \sum_{t=0}^{T} \left(\frac{\phi_0}{\phi_t}\right) [\phi_t x_t + N_t - C_t] + \phi_0 \sum_{t=0}^{T} \left(\frac{e_{t+1} - e_t}{e_t}\right) e_t m_{t+1}^f \right\}, \tag{89}$$

where $(e_{t+1} - e_t)/e_t \equiv (\phi_{t+1}^f/\phi_{t+1} - \phi_t^f/\phi_t)/(\phi_t^f/\phi_t)$ is the one-period growth rate in the nominal exchange rate.

The first term on the LHS of (89) equals zero by the transversality condition on assets. That is, in the infinite-horizon limit, the discounted value of accumulated domestic real money balances must be zero in a monetary equilibrium. The first term on the RHS also goes to zero, since in any monetary equilibrium the T-period return on domestic money ϕ_T/ϕ_0 must be finite-valued, and, the given initial real balances on both monies $\phi_0(m_0 + e_0 m_0^f)$ are also finite. The second term on the RHS must equal zero since the infinite series of real wealth (including real seigniorage transfers) net of CM consumption expenditure must be finite in any well-defined equilibrium. Since $\beta \in (0,1)$, then this term must be zero in the infinite-horizon limiting economy. Therefore, that leaves one with the final term on the RHS which must then equal zero to satisfy the conditions for any monetary general equilibrium. We thus arrive at the following observation:

Lemma 11 *Assuming monetary equilibria with coexistence of both monies, then in any infinite-horizon monetary equilibrium, the discounted total changes in the real value of foreign money holdings of any DM-buyer must be zero,*

$$\lim_{T\nearrow+\infty} \left\{ \beta^T \phi_T \sum_{t=0}^{T} \left(\frac{e_{t+1} - e_t}{e_t}\right) e_t m_{t+1}^f \right\} = 0.$$

This fact will allow us to deduce an admissible property of the *nominal exchange rate* path consistent with the existence of a monetary equilibrium (with coexistence of both monies). In particular we have the following result summarized in Proposition 6.

Proof of Proposition 6. Lemma 11 and the fact that $\beta^T \in (0,1)$ for any $T = 0, 1, 2, \ldots$, implies that for a monetary equilibrium to exist, it is necessary to have

$$\lim_{T\nearrow+\infty} \left| \sum_{t=0}^{T} \left(\frac{e_{t+1}}{e_t} - 1\right) \left(\frac{\phi_T}{\phi_t}\right) \phi_t^f m_{t+1} \right| < +\infty.$$

Let the one-period growth rate in the nominal exchange rate and the end of date-t CM holding of foreign currency real balance (in units of date-T CM good), respectively, be denoted as $a_t := (e_{t+1}/e_t - 1)$, and, $b_t := (\phi_T/\phi_t)\, \phi_t^f m_{t+1}^f$.

Now let the partial sum $\sum_{s=0}^{t} b_t =: B_t$. By Abel's Lemma (summation by parts), the date-T series can be transformed as follows:

$$\begin{aligned} S_T \equiv \sum_{t=0}^{T} a_t b_t &= a_0 b_0 - a_0 B_0 + a_T B_T + \sum_{t=0}^{T-1} B_t (a_t - a_{t+1}) \\ &= a_T B_T - \sum_{t=0}^{T-1} B_t (a_{t+1} - a_t). \end{aligned}$$

For any integer $t \geq 0$, any integer $k \geq 0$, and some finite real number a, we can write

$$S_{t+k} - S_t = (a_{t+k} - a) B_{t+k} + (a_t - a) B_t + a(B_{t+k} - B_t) + \sum_{s=0}^{t+k-1} B_s (a_s - a_{s+1}).$$

Suppose the partial sums B_t do not converge: $|B_t| \to +\infty$. Then $\lim_{t \nearrow +\infty} |S_{t+k} - S_t| \neq 0$, and, by the Cauchy convergence criterion, this implies that

$$\lim_{T \nearrow +\infty} |S_T| \equiv \lim_{T \nearrow +\infty} \left| \sum_{t=0}^{T} \left(\frac{e_{t+1}}{e_t} - 1 \right) \left(\frac{\phi_T}{\phi_t} \right) \phi_t^f m_{t+1} \right| = +\infty.$$

But this violates Lemma 11 in a monetary equilibrium. Thus, the B_t partial sums must be convergent in any monetary equilibrium.

Therefore, we may assume some upper bound $0 \leq B < +\infty$ for the sequences of absolute partial sums, $\{|B_t|\}_{t \geq 0}$. Note that

$$\begin{aligned} |S_{t+k} - S_t| &= \left| (a_{t+k} - a) B_{t+k} + (a_t - a) B_t + a(B_{t+k} - B_t) + \sum_{s=0}^{t+k-1} B_s (a_s - a_{s+1}) \right| \\ &\leq |a_{t+k} - a||B_{t+k}| + |a_t - a||B_t| + \left| \sum_{s=0}^{t+k-1} B_s (a_s - a_{s+1}) \right| \\ &\leq |a_{t+k} - a| B + |a_t - a| B + |a_{t+k} - a_t| B. \end{aligned}$$

Thus, by applying the Cauchy convergence criterion again, the nominal exchange rate's absolute growth rate converging to a constant $\gamma_e \equiv a \in [0, +\infty)$ in the infinite horizon economy, i.e. $(a_t \to a) \Leftrightarrow |a_{t+k} - a_t| \to 0$, then $|S_{t+k} - S_t| \to 0 \Leftrightarrow \lim_{T \nearrow +\infty} S_T < +\infty$, satisfies the requirement of Lemma 11. That is, it satisfies CM market clearing and agents' intertemporal budget constraints in any monetary equilibrium. ■

Remark 12 *In the rest of the paper, we can thus focus on monetary equilibria in which the* *equilibrium exchange rate grows at some constant rate. Also, Proposition 6 will apply in the explicit two-country version of the world economy below. All that is required is to set $x_t = 0$ for every date t in the sequential CM budget constraints, since x_t will later appear in the DM feasibility constraint, since transfers will occur after each CM closes.*

D Proof of Proposition 7 (Equilibria and Coexistence)

Recall from (12), we denoted

$$\frac{\kappa}{\phi_{-1}/\phi - \beta(1-\sigma)} =: \bar{\kappa}(\phi_{-1}/\phi); \qquad \text{and,} \qquad \frac{\kappa^f}{\phi_{-1}e_{-1}/\phi e - \beta(1-\sigma)} =: \bar{\kappa}^f(\phi_{-1}e_{-1}/\phi e).$$

In steady state we have these, respectively, as $\bar{\kappa}(\Pi)$ and $\bar{\kappa}^f(\Pi^f)$.

After some elementary algebra, the necessary first-order conditions for a (steady-state) monetary equilibrium together with their Karush-Kuhn-Tucker (KKT) conditions are as follows: The set of Euler operators evaluated at steady state are

$$\beta\sigma\left[\frac{u'(q)}{c'(q)} - 1\right] = \lambda - \nu + (\Pi - \beta) = \lambda^f - \nu^f + (\Pi^f - \beta). \tag{90a}$$

The DM-sellers' participation constraint is binding ($\zeta > 0$), so that

$$c(q) = \phi d + \phi^f d^f, \tag{90b}$$

and the KKT conditions are:

$$\begin{aligned}
\lambda \cdot [\bar{\kappa}(\Pi) - \phi d] = 0, &\qquad \lambda \geq 0, \qquad \phi d \leq \bar{\kappa}(\Pi), \\
\lambda^f \cdot \left[\bar{\kappa}^f(\Pi^f) - \phi^f d^f\right] = 0, &\qquad \lambda^f \geq 0, \qquad \phi^f d^f \leq \bar{\kappa}^f(\Pi^f), \\
-\nu \cdot d = 0, &\qquad d \geq 0, \qquad \nu \geq 0, \\
-\nu^f \cdot d^f = 0, &\qquad d^f \geq 0, \qquad \nu^f \geq 0.
\end{aligned} \tag{90c}$$

We must consider different cases depending on when the endogenous liquidity constraint associated with the local and foreign currency bind or not, given domestic (foreign) inflation rate, money supply, counterfeiting costs and the matching probability. We focus on equilibria that satisfy $\zeta > 0$, and at least one of ν and ν^f is zero, so that buyers and sellers trade in DM, and $\mu, \mu^f > 0$.

Case 1(a) $\Pi - \Pi^f > 0$ **and**

(i) $\lambda = 0, \lambda^f = 0$: This case is trivial to check. When both liquidity constraints are not binding, and the foreign currency dominates in rate of return, buyers demand only the foreign fiat money, so that only M^f is in circulation.

(ii) $\lambda > 0, \lambda^f = 0$. Since $\lambda > 0$ and $\lambda^f = 0$ and $\Pi^f < \Pi$, all agents optimally demand zero domestic fiat money, so that $d = 0$ or by complementary slackness $\nu > 0$, and, $d^f = M^f$ with $\nu^f = 0$.

In both subcases, we have the following characterization of a unique equilibrium outcome. First, from the equilibrium Euler condition (90a), q solves

$$\beta\sigma \left[\frac{u'(q)}{c'(q)} - 1 \right] - (\Pi^f - \beta) = 0.$$

Second, since u' (c') is a continuous and monotone decreasing (non-decreasing) function on a compact set $[0, q^*]$, and since it is optimal to consume $q > 0$ and given that $\Pi^f - \beta > 0$, then there exists a unique solution $q \in (0, q^*)$ where q^* satisfies the first-best solution $u'(q) = c'(q)$. Therefore, there is a unique price level, $1/\phi^f$, determined from the DM-sellers' participation constraint (90b): $\phi^f = c(q)/M^f$.

Case 1(b): $\lambda = 0, \lambda^f > 0$ and $\Pi - \Pi^f > 0$. From the result in Lemma 10, we have in any monetary equilibrium, a buyer at the end of every CM will make offers of payments $\left(d, d^f\right)$ up to the respective limits of their portfolio components $\left(m, m^f\right)$—i.e. $d = m \geq 0$ and $d^f = m^f \geq 0$—which implies that the multipliers on payment upper-bounds are strictly positive:

$$\mu = \Pi - \beta > 0,$$

$$\mu^f = \Pi^f - \beta > 0.$$

In this case, the Euler conditions (90a) reduce to

$$\beta\sigma \left[\frac{u'(q)}{c'(q)} - 1 \right] - (\Pi - \beta) = 0.$$

which solve for a unique q, following similar arguments in Case 1(a), except that now we also have $\Pi - \beta > 0$.

Next, we show that there is coexistence of the home currency with the foreign currency, in spite of the former being dominated in its return, $\Pi > \Pi^f$. By construction the highest sustainable allocation of q is a $q^* > 0$ satisfying the first best trade-off: $u'(q^*) = c'(q^*)$. Comparing the first-best condition with the monetary equilibrium condition for q above, we can easily deduce that $q < q^*$ since $\Pi - \beta > 0$. Since the liquidity constraint on the foreign currency payment is binding, then, from the seller's participation constraint we can re-write as:

$$\phi m = c(q) - \bar{\kappa}^f(\Pi^f) \geq 0.$$

Suppose to the contrary that the demand for home currency were zero, $m = 0$. Then we have $q = c^{-1}\left(\bar{\kappa}^f(\Pi^f)\right) < q^*$. Since each buyer can increase his lifetime payoff by accumulating more domestic money ($\lambda = 0$) and offering it to the seller in the DM to consume more q; and the seller would willingly accept it by producing more q while ensuring that her participation constraint is still binding, then we have in this equilibrium positive demand for home real currency, $\phi m = c(q) - \bar{\kappa}^f > 0$. In equilibrium in the integrated economy, $m = M$ and $m^f = M^f$.

From the DM-sellers' participation constraint (90b), there is a unique value for domestic fiat money:

$$\phi = \frac{c(q) - \bar{\kappa}^f(\Pi^f)}{M}.$$

Since $\lambda^f > 0$ or that the foreign money liquidity constraint binds (90c), then there is a unique value for the foreign fiat money:

$$\phi^f = \frac{\bar{\kappa}^f(\Pi^f)}{M^f}.$$

Therefore, there is a unique monetary equilibrium outcome in this case, with the nominal exchange determined as

$$e := \frac{\phi^f}{\phi} = \frac{M}{M^f} \cdot \frac{\bar{\kappa}^f(\Pi^f)}{c(q) - \bar{\kappa}^f(\Pi^f)}.$$

Case 1(c): $\lambda > 0, \lambda^f > 0$ and $\Pi - \Pi^f > 0$ It is easy to show that the resulting system of equations has a unique solution for $\{\mu, \mu^f, \phi, e, q, \lambda, \zeta\}$. For a given domestic and foreign inflation rates, money supplies, counterfeited costs and matching probability, the relevant block of the steady state equilibrium conditions is given as follows. First, as in the previous cases,

$$\mu = \Pi - \beta > 0$$

$$\mu^f = \Pi^f - \beta > 0.$$

Second, since the DM-buyer is liquidity constrained in both currencies, then in real terms, he would demand and offer payments up to the limits of both constraints: $\phi M = \bar{\kappa}(\Pi) > 0$ and $\phi e M^f = \bar{\kappa}^f(\Pi^f) > 0$ as measured in units of the home CM good. From the home currency liquidity constraint, we can solve for

$$\phi = \frac{\bar{\kappa}(\Pi)}{M};$$

and then using this in the foreign currency liquidity constraint, we can derive a unique equilibrium nominal exchange rate

$$e = \frac{M}{M^f}\frac{\bar{\kappa}^f(\Pi^f)}{\bar{\kappa}(\Pi)}.$$

Finally, the other relevant equilibrium conditions:

$$c(q) = \bar{\kappa}^f(\Pi^f) + \bar{\kappa}(\Pi);$$

$$\sigma\beta\frac{u'(q) - c'(q)}{c'(q)} = \mu + \lambda;$$

$$\zeta = \beta\sigma + \Pi - \beta + \lambda;$$

$$\lambda^f = \Pi - \Pi^* + \lambda,$$

pin down a unique q, λ, ζ and λ^f, respectively. Therefore, in this case, there is a determinate monetary equilibrium, with a unique nominal exchange rate, and coexistence of both currencies.

Case 2: $\Pi - \Pi^f < 0$ This is the symmetric opposite to the analyses in Case 1. Therefore there can exist a unique steady state e and coexistence of the two currencies, in spite of $\Pi < \Pi^f$.

Case 3(a): $\lambda = \lambda^f = 0$ **and** $\Pi - \Pi^f = 0$**.** This case corresponds to the indeterminacy result in Kareken and Wallace. Since both liquidity constraints are not binding, and both currencies yield equal rates of return, then buyers are indifferent as to which currency to hold and sellers' participation constraint binds for any composition of payments offered.

Case 3(b): $\lambda \neq \lambda^f > 0$ **and** $\Pi - \Pi^f = 0$**.** In this case, when both liquidity constraints bind, the analysis is similar to Case 1(c) above. Therefore we have coexistence of the two currencies and determinacy of the equilibrium nominal exchange rate.

Case 3(c): $\Pi - \Pi^f = 0$ **and** $(\lambda > 0$ **and** $\lambda^f = 0)$ **or** $(\lambda = 0$ **and** $\lambda^f > 0)$**.** We can rule out these two cases as equilibria. Since one liquidity constraint binds in either subcases, then $\nu = \nu^f = 0$, but this implies that either $\lambda + \Pi - \beta > \lambda^f + \Pi^f - \beta$ in subcase (i), or, $\lambda + \Pi - \beta < \lambda^f + \Pi^f - \beta$, which respectively, is an impossibility since $\Pi = \Pi^f$ so that this violates the Euler conditions (90a). Therefore these two cases cannot be an equilibrium.

E Proof of Proposition 8 (Non First Best)

It suffices to construct a counterexample. Consider Case 3 of Proposition Proposition 7. The relevant block characterizing steady-state monetary equilibrium is

$$\mu = \Pi - \beta,$$

$$\mu^f = \Pi^f - \beta,$$

$$c(q) = \phi M + e\phi M^f,$$

$$\phi M \leq \frac{\bar{\kappa}}{\beta\sigma},$$

$$e\phi M^f \leq \frac{\bar{\kappa}^f}{\beta\sigma},$$

$$\sigma\beta\frac{u'(q) - c'(q)}{c'(q)} = \mu + \lambda,$$

$$\zeta = \beta\sigma + \Pi - \beta + \lambda,$$

$$\lambda^f = \Pi - \Pi^* + \lambda.$$

When $\Pi = \beta$ and $\Pi^f = \beta$ it implies that $\mu = 0$ so that

$$\sigma\beta\frac{u'(q) - c'(q)}{c'(q)} = \lambda.$$

Notice that the DM first best q^*, which satisfies $u'(q^*) = c'(q^*)$, can only occur if $\lambda = 0$. However, in order for the first best to be a monetary equilibrium, the participation constraint for the seller has to be satisfied and the nominal exchange rate has to be positive. These two conditions are respectively given by

$$c(q^*) \leq \frac{\bar{\kappa}}{\beta\sigma} + \frac{\bar{\kappa}^f}{\beta\sigma},$$

$$c(q^*) \geq \frac{\bar{\kappa}}{\beta\sigma}.$$

Thus, even when both the domestic and foreign inflation rates converge to the Friedman rule, the DM first best may not be attainable and the nominal exchange rate may not be determinate.

F Coexistence, Return Dominance and Counterfeiting Costs

Here we derive the inequalities shown in (33) and (34) in Section 4.3. Recall Cases 1 and 2 imply no coexistence with only the foreign (high return) money in equilibrium circulation. These two cases will be complements to the following two.

Case 3: $\Pi > \Pi^f > \beta$ and $\lambda^f > \lambda = 0$. From the Euler conditions at steady state, the equilibrium allocation $\hat{q}$ satisfies

$$u'(q) = \frac{\Pi - \beta(1-\sigma)}{\beta\sigma} c'(q).$$

Since the inefficiency wedge, $[\Pi - \beta(1-\sigma)]/\beta\sigma$, is strictly positive due to $\Pi > \beta(1-\sigma)$, and since the derivative function c' is continuous and nondecreasing, and u' is continuous and monotonically decreasing, then the monetary equilibrium solution $\hat{q} < q^*$ where q^* is the efficient solution to $u'(q) = c'(q)$.

Since the foreign money liquidity constraint binds, then the seller's participation constraint in this equilibrium is

$$c(\hat{q}) = \phi M + \frac{\kappa^f}{\Pi^f - \beta(1-\sigma)}.$$

This condition also used the equilibrium requirement in the integrated world economy of $d = m = M$.

Since real balance ϕM have to be non-negative, and since the domestic liquidity constraint is slack, then we have the two respective inequalities

$$0 \leq \phi M = c(\hat{q}) - \frac{\kappa^f}{\Pi^f - \beta(1-\sigma)} < \frac{\kappa}{\Pi - \beta(1-\sigma)}.$$

Tidying up yields the expression in (33).

Case 4: $\Pi > \Pi^f > \beta$, $\lambda^f > 0$, and $\lambda > 0$. Since both liquidity constraints are binding in this case, then the seller's participation constraint implies a unique solution $\tilde{q}$ such that

$$c(\tilde{q}) = \frac{\kappa}{\Pi - \beta(1-\sigma)} + \frac{\kappa^f}{\Pi^f - \beta(1-\sigma)}.$$

In Case 4, since both liquidity constraints are binding, and the seller's participation constraint holds with equality, the equilibrium outcome is $\tilde{q} \leq \hat{q}$, where $\hat{q}$ was defined in the solution to Case 3's equilibrium. Therefore,

$$c(\tilde{q}) = \frac{\kappa}{\Pi - \beta(1-\sigma)} + \frac{\kappa^f}{\Pi^f - \beta(1-\sigma)} \leq c(\hat{q}),$$

given that $\kappa > 0$ and $\kappa^f > 0$. Rearranging, we have the inequalities restricting (κ, κ^f) in (34).

G Proof of Proposition 9 (Two-country Variation)

We will only prove the two cases of $\Pi > \Pi^f$ (Case 1) and $\Pi = \Pi^f$ (Case 3). Given each case, there are $\binom{4}{4} + \binom{4}{3} + \binom{4}{2} + \binom{4}{1} + \binom{4}{0} = 16$ subcases (i.e., combinations of the total of four liquidity constraints) to consider. However, depending on the magnitude of Π relative to Π^f, we can easily show that some of the subcases cannot exist as an equilibrium.

Case 1(a): $\Pi > \Pi^f$ **with the following subcases.**

1. $\lambda = \lambda^f = \lambda^f_\star = \lambda_\star = 0$;

2. $\lambda > 0$ and $\lambda^f = \lambda^f_\star = \lambda_\star = 0$;

3. $\lambda = \lambda^f = 0, \lambda^f_\star = 0, \lambda_\star > 0$; or

4. $\lambda > 0$, $\lambda^f = \lambda^f_\star = 0$ and $\lambda_\star > 0$.

The characterization is similar to Case 1(a) of Proposition 7. Since $\Pi > \Pi^f$ and the liquidity constraint on holding Foreign money in both countries are slack, then all agents in all countries will only demand the Foreign fiat money. Thus, there is a unique monetary equilibrium with only the low inflation (Foreign) fiat money in circulation in both countries.

Case 1(b): $\Pi > \Pi^f$ **with all the following subcases.** All these exist as a unique monetary equilibrium exhibiting coexistence of both fiat monies.

5. $\lambda > 0, \lambda^f > 0, \lambda^f_\star = \lambda_\star = 0$:

   Since both liquidity constraints for Home DM-buyers bind, then from the Home DM-seller's participation constraint (64b) and the first two KKT conditions in (64c), we have a unique solution to $q = c^{-1}[\bar{\kappa}(\Pi) + \bar{\kappa}(\Pi^f)]$. Also, from the global money market clearing condition (66a) on Home fiat currency we have that $\phi d_\star = \phi M - \bar{\kappa}(\Pi)$. However, since the Foreign fiat money dominates in rate of return, and Foreign DM-buyers are not liquidity constrained ($\lambda^f_\star = 0$), then $d_\star = 0$, and we can deduce that $\phi = \bar{\kappa}(\Pi)/M$.

   Also, since Foreign DM-buyers demand all of their national fiat money, $d^f_\star > 0$ (or $\nu^f_\star = 0$ by complementary slackness), then there is a unique $q^f_\star \in (0, q^*)$ solving the Euler condition (65a): $\beta\sigma[u'(q)/c'(q) - 1] = \Pi^f - \beta$. From the Foreign DM-seller's participation constraint, we have $\phi^f d^f = \phi^f M^f - c(q^f_\star)$. Since $\lambda^f > 0$, we have $\phi^f d^f = \bar{\kappa}^f(\Pi^f) = \phi^f M^f - c(q^f_\star)$. Rearranging, we have $\phi^f = [\bar{\kappa}^f(\Pi^f) + c(q^f_\star)]/M^f$.

   Therefore there is a unique nominal exchange rate

$$e := \frac{\phi^f}{\phi} = \left[\frac{\bar{\kappa}^f(\Pi^f) + c(q^f_\star)}{\bar{\kappa}(\Pi)}\right] \cdot \frac{M}{M^f}.$$

There is only coexistence of both monies in the Home country's DM. Only the Foreign currency circulates in the Foreign DM.

6. $\lambda = 0, \lambda^f > 0, \lambda^f_\star = \lambda_\star = 0$:

   From the Home DM-buyers' Euler conditions (64a), we have that $\nu = \Pi - (\Pi^f + \lambda^f) = 0$ since the DM-buyer is constrained in holding the Foreign fiat money. Thus, q solves $\beta\sigma[u'(q)/c'(q) - 1] = \Pi - \beta$ uniquely. From the Home DM-seller's participation constraint (64b) and the Home DM-buyers' binding liquidity constraint on Foreign real money balance, we have $c(q) - \bar{\kappa}^f(\Pi^f) = \phi d$. Substituting this into the global clearing condition on Home fiat money (66a), and since no Foreign DM-buyers will optimally demand Home fiat money ($d_\star = 0$), we have $\phi = [c(q) - \bar{\kappa}^f(\Pi^f)]/M]$. Also since Foreign DM-buyers demand only Foreign fiat money, then $q^f_\star$ uniquely solves the Foreign Euler condition (65a): $\beta\sigma[u'(q)/c'(q) - 1] = \Pi^f - \beta$. Using this fact in the Foreign DM-sellers' participation constraint (65b), combined with the global adding-up condition on Foreign fiat money (66b) and the fact that $\phi^f d^f = \bar{\kappa}^f(\Pi^f)$, we have a determination of the inverse Foreign price level as $\phi^f = [c(q^f_\star) - \bar{\kappa}^f(\Pi^f)]/M^f$. Therefore there is a unique equilibrium nominal exchange rate

   $$e := \frac{\phi^f}{\phi} = \left[\frac{c(q^f_\star) - \bar{\kappa}^f(\Pi^f)}{c(q) - \bar{\kappa}^f(\Pi^f)}\right] \cdot \frac{M}{M^f}.$$

   There is only coexistence of both monies in the Home country's DM. The Home currency does not circulate in the Foreign DM.

7. $\lambda = 0, \lambda^f > 0, \lambda^f_\star > 0, \lambda_\star = 0$:

   Since the Home DM-buyers' liquidity constraint on Foreign money is binding ($\lambda^f > 0$), and that on Home money is not ($\lambda = 0$), from the Home DM-buyers' Euler conditions (64a), we have that $\nu = \Pi - (\Pi^f + \lambda^f) = 0$. Thus, q solves $\beta\sigma[u'(q)/c'(q) - 1] = \Pi - \beta$ uniquely. From the Home DM-seller's participation constraint (64b) and the Home DM-buyers' binding liquidity constraint on Foreign real money balance, we have $c(q) - \bar{\kappa}^f(\Pi^f) = \phi d$. Substituting this into the global clearing condition on Home fiat money (66a), then Foreign DM-buyers' real demand for Home fiat money is $\phi d_\star = \phi M - [c(q) - \bar{\kappa}^f(\Pi^f)]$. Since the Foreign agents' demand for Foreign fiat money faces a binding constraint ($\lambda^f_\star > 0$), then from the Foreign Euler conditions (65a) under $\nu_\star = \lambda_\star = 0$, we have $q^f_\star$ solving $\beta\sigma[u'(q^f_\star)/c'(q^f_\star) - 1] = \Pi - \beta$ uniquely. From the Foreign DM-sellers' participation constraint, together with $c(q) - \bar{\kappa}^f(\Pi^f) = \phi d$, we can derive the inverse Home price level as $\phi = [c(q^f_\star - \bar{\kappa}^f_\star(\Pi^f) + c(q) - \bar{\kappa}^f(\Pi^f)]/M$. From the global Foreign money adding-up condition, we have $\bar{\kappa}^f_\star(\Pi^f) + \bar{\kappa}^f(\Pi^f) = \phi^f M^f$, which implies that $\phi^f = [\bar{\kappa}^f_\star(\Pi^f) + \bar{\kappa}^f(\Pi^f)]/M^f$. So then there is a unique nominal exchange rate

$$e := \frac{\phi^f}{\phi} = \left[\frac{\bar{\kappa}_\star^f(\Pi^f) + \bar{\kappa}^f(\Pi^f)}{c(q_\star^f) + c(q) - [\bar{\kappa}_\star^f(\Pi^f) + \bar{\kappa}^f(\Pi^f)]}\right] \cdot \frac{M}{M^f},$$

and the coexistence of both monies in both countries' DM.

8. $\lambda = 0, \lambda^f > 0, \lambda_\star^f = 0, \lambda_\star > 0$:

   Since Foreign DM-buyers are not liquidity constrained in their demand for the high return money ($\lambda_\star^f = 0$), so that $\nu_\star^f = 0$ and their optimal portfolio consists of this asset (i.e., $d_\star^f > 0$ and $d_\star = 0$), then from their Euler conditions (65a), there is a unique $q_\star^f$ satisfying $\beta\sigma[u'(q_\star^f)/c'(q_\star^f) - 1] = \Pi^f - \beta$. Since $d_\star = 0$, from the Foreign DM-sellers' participation constraint (65b), we have $\phi^f d_\star^f = c(q_\star^f)$. Using this result, together with $\phi^f d^f = \bar{\kappa}^f(\Pi^f)$ in the global clearing condition on Foreign fiat money (66b), we have $\phi^f = [c(q_\star^f) + \bar{\kappa}^f(\Pi^f)]/M^f$. Also, from the global clearing condition on Home fiat money (66a) together with the fact that $d_\star = 0$, we have $\phi = [c(q) - \bar{\kappa}^f(\Pi^f)]/M$. Therefore, there is coexistence (only in the Home country), and only the Foreign currency circulates in the Foreign DM, and the nominal exchange rate is

$$e := \frac{\phi^f}{\phi} = \left[\frac{c(q_\star^f) + \bar{\kappa}^f(\Pi^f)}{c(q) - \bar{\kappa}^f(\Pi^f)}\right] \cdot \frac{M}{M^f}.$$

9. $\lambda > 0, \lambda^f = 0, \lambda_\star^f > 0, \lambda_\star = 0$:

   Since DM-buyers in Home are not liquidity constrained on the higher return money, they will only demand the Foreign fiat money, and $d = 0$. From their Euler conditions, (64a), q solves $\beta\sigma[u'(q)/c'(q) - 1] = \Pi^f - \beta$, and from the Home DM-sellers' participation constraint (64b), we have Home DM-buyers' demand for real Foreign fiat money balance as $\phi^f d^f = c(q)$. Since the Foreign DM-buyers are liquidity constrained on the Foreign money, $\phi^f d_\star^f = \bar{\kappa}_\star^f(\Pi^f)$, then the global clearing condition on the Foreign fiat money (66b) implies a unique value for the Foreign fiat money $\phi^f = [\bar{\kappa}_\star^f(\Pi^f) - c(q)]/M^f$. Also, since $\lambda_\star^f > 0$ and $\lambda_\star = 0$, then $q_\star^f$ uniquely satisfies $\beta\sigma[u'(q_\star^f)/c'(q_\star^f) - 1] = \Pi - \beta$ in the Foreign DM-buyers' Euler equation in (65a). From the global clearing condition on the Home fiat money (66a), we thus have $\phi = [c(q_\star^f) - \bar{\kappa}_\star^f(\Pi^f)]/M$. In this case, there is a unique equilibrium nominal exchange rate

$$e := \frac{\phi^f}{\phi} = \left[\frac{\bar{\kappa}^f(\Pi^f) - c(q)}{c(q_\star^f) - \bar{\kappa}^f(\Pi^f)}\right] \cdot \frac{M}{M^f},$$

   and there is coexistence of both monies in the Foreign DM, while only the Foreign money circulates in the Home DM.

10. $\lambda > 0, \lambda^f > 0, \lambda_\star^f = 0, \lambda_\star > 0$:

    Since Home DM-buyers are liquidity constrained on both monies, then from the Home DM-sellers' participation constraint (64b), we can uniquely deduce q such that $c(q) = \bar{\kappa}(\Pi) + \bar{\kappa}^f(\Pi^f)$. Since Foreign DM-buyers will only demand Foreign money so that $d_\star = 0$, then from the global clearing condition on the Home fiat money (66a), we thus have $\phi = \bar{\kappa}(\Pi)/M$. In the Foreign DM-seller's participation constraint (65b), we have $\phi^f d_\star^f = c(q_*^f)$, where $q_\star^f$ uniquely solves $\beta\sigma[u'(q_\star^f)/c'(q_\star^f) - 1] = \Pi^f - \beta$ in their Euler condition (65a). Using this—and the fact that Home DM-agents are also liquidity constrained on holding the Foreign fiat money—the global clearing condition on the Foreign fiat money (66b), implies $\phi^f = [c(q_*^f) + \bar{\kappa}^f(\Pi)]/M^f$. Therefore, there is a unique nominal exchange rate

    $$e := \frac{\phi^f}{\phi} = \left[\frac{c(q_\star^f) + \bar{\kappa}^f(\Pi^f)}{\bar{\kappa}(\Pi)}\right] \cdot \frac{M}{M^f},$$

    and both monies circulate in the Home DM, but only the Foreign money circulates in the Foreign DM.

11. $\lambda > 0, \lambda^f > 0, \lambda_\star^f > 0, \lambda_\star = 0$:

    Here, Home DM-buyers are liquidity constrained on both monies, so that $\phi d = \bar{\kappa}(\Pi)$ and $\phi^f d^f = \bar{\kappa}^f(\Pi^f)$. For Foreign DM-buyer, since $\lambda_\star = 0$ and $\lambda_\star^f > 0$, then $q_\star^f$ can be uniquely determined as $\beta\sigma[u'(q_\star^f)/c'(q_\star^f) - 1] = \Pi - \beta$ in their Euler condition (65a). Then from the Foreign DM-sellers' participation constraint (65b), we have $\phi d_\star = c(q_*^f) - \bar{\kappa}_\star^f(\Pi^f)$, since DM-buyers are liquidity constrained on Foreign money, i.e. $\lambda_\star^f > 0$ or that $\phi^f d_\star^f = \bar{\kappa}_\star^f(\Pi^f)$. Using this in the global clearing condition on the Home fiat money (66a), we have $\phi = [c(q_*^f) - \bar{\kappa}_\star^f(\Pi^f) + \bar{\kappa}(\Pi)]/M$. Since the Foreign fiat money has a liquidity constraint binding in each country, then from the the global clearing condition on the Foreign fiat money (66b), we have $\phi^f = [\bar{\kappa}_\star^f(\Pi^f) + \bar{\kappa}^f(\Pi^f)]/M^f$. Thus, there is a unique nominal exchange rate

    $$e := \frac{\phi^f}{\phi} = \left[\frac{\bar{\kappa}_\star^f(\Pi^f) + \bar{\kappa}^f(\Pi^f)}{c(q_*^f) - \bar{\kappa}_\star^f(\Pi^f) + \bar{\kappa}(\Pi)}\right] \cdot \frac{M}{M^f},$$

    and both monies circulate in both Home and Foreign DMs.

12. $\lambda = 0, \lambda^f > 0, \lambda_\star^f > 0, \lambda_\star > 0$:

    The proof here is similar to that of the previous Case 1(b)(11). In this case, there is a unique nominal exchange rate

$$e := \frac{\phi^f}{\phi} = \left[\frac{\bar{\kappa}_\star^f(\Pi^f) + \bar{\kappa}^f(\Pi^f)}{c(q) + \bar{\kappa}_\star^f(\Pi^f) - \bar{\kappa}(\Pi)}\right] \cdot \frac{M}{M^f},$$

and both monies circulate in both Home and Foreign DMs.

13. $\lambda > 0, \lambda^f = 0, \lambda_\star^f > 0, \lambda_\star > 0$:

    The proof here is opposite to that of the previous Case 1(b)(10). We can easily show that there is a unique nominal exchange rate

    $$e := \frac{\phi^f}{\phi} = \left[\frac{c(q) + \bar{\kappa}_\star^f(\Pi^f)}{\bar{\kappa}_\star(\Pi)}\right] \cdot \frac{M}{M^f},$$

    and both monies circulate in the Foreign DM, but only the Foreign money circulates in the Home DM.

14. $\lambda = \lambda^f = 0, \lambda_\star^f > 0, \lambda_\star > 0$:

    The proof here is opposite to that of the previous Case 1(b)(5). Therefore there is a unique nominal exchange rate

    $$e := \frac{\phi^f}{\phi} = \left[\frac{\bar{\kappa}_\star^f(\Pi^f) + c(q)}{\bar{\kappa}_\star(\Pi)}\right] \cdot \frac{M}{M^f}.$$

    There is only coexistence of both monies in the Foreign country's DM. Only the Foreign currency circulates in the Home DM.

15. $\lambda = \lambda^f = 0, \lambda_\star^f > 0, \lambda_\star = 0$:

    Since Home DM-buyers are not liquidity constrained in their demand for the high return money ($\lambda^f = 0$), so that $\nu^f = 0$ and their optimal portfolio consists of this asset (i.e., $d^f > 0$ and $d = 0$), then from their Euler conditions (64a), there is a unique q satisfying $\beta\sigma[u'(q)/c'(q) - 1] = \Pi^f - \beta$. Since $d = 0$, from the Home DM-sellers' participation constraint (64b), we have $\phi^f d^f = c(q)$. Using this result, together with $\phi_\star^f d_\star^f = \bar{\kappa}_\star^f(\Pi^f)$ in the global clearing condition on Foreign fiat money (66b), we have $\phi^f = [c() + \bar{\kappa}_\star^f(\Pi^f)]/M^f$. Since Foreign DM-buyers are constrained on holding Foreign money ($\lambda_\star^f > 0, \lambda_\star = 0$), then $q_\star^f$ can be uniquely determined as $\beta\sigma[u'(q_\star^f)/c'(q_\star^f) - 1] = \Pi - \beta$ in their Euler condition (65a). Also, from the global clearing condition on Home fiat money (66a) together with the fact that $d = 0$, we have $\phi = [c(q_\star^f) - \bar{\kappa}_\star^f(\Pi^f)]/M$. Therefore, there is coexistence (only in the Home country), and only the Foreign currency circulates in the Foreign DM, and the nominal exchange rate is

$$e := \frac{\phi^f}{\phi} = \left[\frac{c(q) + \bar{\kappa}_\star^f(\Pi^f)}{c(q_\star^f) - \bar{\kappa}_\star^f(\Pi^f)} \right] \cdot \frac{M}{M^f}.$$

16. $\lambda > 0, \lambda^f > 0, \lambda_\star^f > 0, \lambda_\star > 0$:

    This case is obvious. When all liquidity constraints are binding in all countries, we have from the two adding up conditions on global monies, (66a) and (66b), a determination of relative prices as $\phi^f = [\bar{\kappa}^f(\Pi^f) + \bar{\kappa}_\star^f(\Pi^f)]/M^f$ and $\phi = [\bar{\kappa}(\Pi) + \bar{\kappa}_\star(\Pi)]/M$, respectively. The nominal exchange rate is

    $$e := \frac{\phi^f}{\phi} = \left[\frac{\bar{\kappa}^f(\Pi^f) + \bar{\kappa}_\star^f(\Pi^f)}{\bar{\kappa}(\Pi) + \bar{\kappa}_\star(\Pi)} \right] \cdot \frac{M}{M^f},$$

    and both monies circulate in both DMs.

Case 3(a): $\Pi = \Pi^f$ **with all the following subcases.**

1. $\lambda > 0, \lambda^f > 0, \lambda_\star^f > 0, \lambda_\star > 0$:

    This case is obvious. When all liquidity constraints are binding in all countries, we have from the two adding up conditions on global monies, (66a) and (66b), a determination of relative prices as $\phi^f = [\bar{\kappa}^f(\Pi^f) + \bar{\kappa}_\star^f(\Pi^f)]/M^f$ and $\phi = [\bar{\kappa}(\Pi) + \bar{\kappa}_\star(\Pi)]/M$, respectively. The nominal exchange rate is

    $$e := \frac{\phi^f}{\phi} = \left[\frac{\bar{\kappa}^f(\Pi^f) + \bar{\kappa}_\star^f(\Pi^f)}{\bar{\kappa}(\Pi) + \bar{\kappa}_\star(\Pi)} \right] \cdot \frac{M}{M^f},$$

    and both monies circulate in both DMs.

Case 3(b): $\Pi = \Pi^f$ **with all the following subcases.** Any of the following configurations, in which at least one country's liquidity constraint is active (inactive) while all of its other liquidity constraints are inactive (active), i.e.,

2. $\lambda > 0$, $\lambda^f = \lambda_\star^f = 0$ and $\lambda_\star > 0$;

3. $\lambda = 0, \lambda^f > 0, \lambda_\star^f = \lambda_\star = 0$;

4. $\lambda = 0, \lambda^f > 0, \lambda_\star^f > 0, \lambda_\star = 0$;

5. $\lambda = 0, \lambda^f > 0, \lambda_\star^f = 0, \lambda_\star > 0$;

6. $\lambda > 0$ and $\lambda^f = \lambda_\star^f = \lambda_\star = 0$;

7. $\lambda = \lambda^f = 0, \lambda_\star^f = 0, \lambda_\star > 0$;

8. $\lambda > 0, \lambda^f = 0, \lambda_\star^f > 0, \lambda_\star = 0$;

9. $\lambda > 0, \lambda^f > 0, \lambda_\star^f = 0, \lambda_\star > 0$;

10. $\lambda > 0, \lambda^f > 0, \lambda_\star^f > 0, \lambda_\star = 0$;

11. $\lambda = 0, \lambda^f > 0, \lambda_\star^f > 0, \lambda_\star > 0$;

12. $\lambda > 0, \lambda^f = 0, \lambda_\star^f > 0, \lambda_\star > 0$; or

13. $\lambda = \lambda^f = 0, \lambda_\star^f > 0, \lambda_\star = 0$,

cannot exist as equilibrium.

Consider the equilibrium Euler conditions (64a) and (65a), which would reduce to the respective conditions $\lambda - \nu = \lambda^f - \nu^f$ and $\lambda_\star^f - \nu_\star^f = \lambda_\star - \nu_\star$. For example, suppose Case 3(b)(1) were an equilibrium. Then, since agents are indifferent between either currency, then we may assume $\nu = \nu^f = \nu_\star^f = \nu_\star = 0$. If so, we have

$$\begin{aligned}\beta\sigma[u'(q)/c'(q) - 1] &= \lambda - \nu + \Pi - \beta \\ &> \lambda^f - \nu^f + \Pi^f - \beta = \beta\sigma[u'(q)/c'(q) - 1],\end{aligned}$$

which is a contradiction. A similar argument applies to all the other configurations.

Case 3(c): $\Pi = \Pi^f$ with the following subcases.

14. $\lambda = \lambda^f = \lambda_\star^f = \lambda_\star = 0$;

15. $\lambda > 0, \lambda^f > 0, \lambda_\star^f = \lambda_\star = 0$ or

16. $\lambda = \lambda^f = 0, \lambda_\star^f > 0, \lambda_\star > 0$,

    In all of these cases, there is a monetary equilibrium where both monies can coexist in both countries' DMs, but the nominal exchange rate is indeterminate.

In case 3(c)(14), since no liquidity constraints bind, and since all monies have the same rate of return, then $q = q_\star^f$ solves the same Euler conditions (64a) and (65a), so that $\beta\sigma[u'(q)/c'(q) - 1] = \Pi - \beta = \Pi^f - \beta$. Given $q = q_\star^f$, then the equilibrium conditions reduce to the global clearing conditions on Foreign and Home fiat money, respectively (66b) and (66a), and the seller's constraints in both contries, (64b) and (65b), determining $(d, d^f, d_\star^f, d_\star)$ for any arbitrary ratio $e := \phi^f/\phi \in [0, \infty)$.

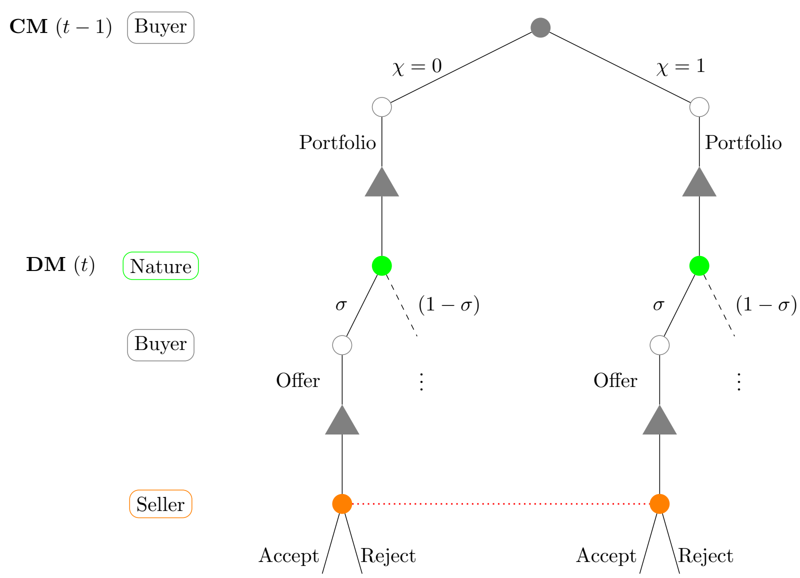
In case 3(c)(15), since only the Home DM-buyers' liquidity constraints bind, we can show that any positive and finite (ϕ, ϕ^{*}), such that there is a continuum of $e := \phi^{f}/\phi \in [0, \infty)$ that satisfies one equilibrium condition:

$$c(q^{f}_{\star}) = \phi M + \phi^{f} M^{f} - [\bar{\kappa}(\Pi) + \bar{\kappa}^{f}(\Pi^{f})].$$

In case 3(c)(16), since only the Foreign DM-buyers' liquidity constraints bind, we can show that any positive and finite (ϕ, ϕ^{*}), such that there is a continuum of $e := \phi^{f}/\phi \in [0, \infty)$ that satisfies one equilibrium condition:

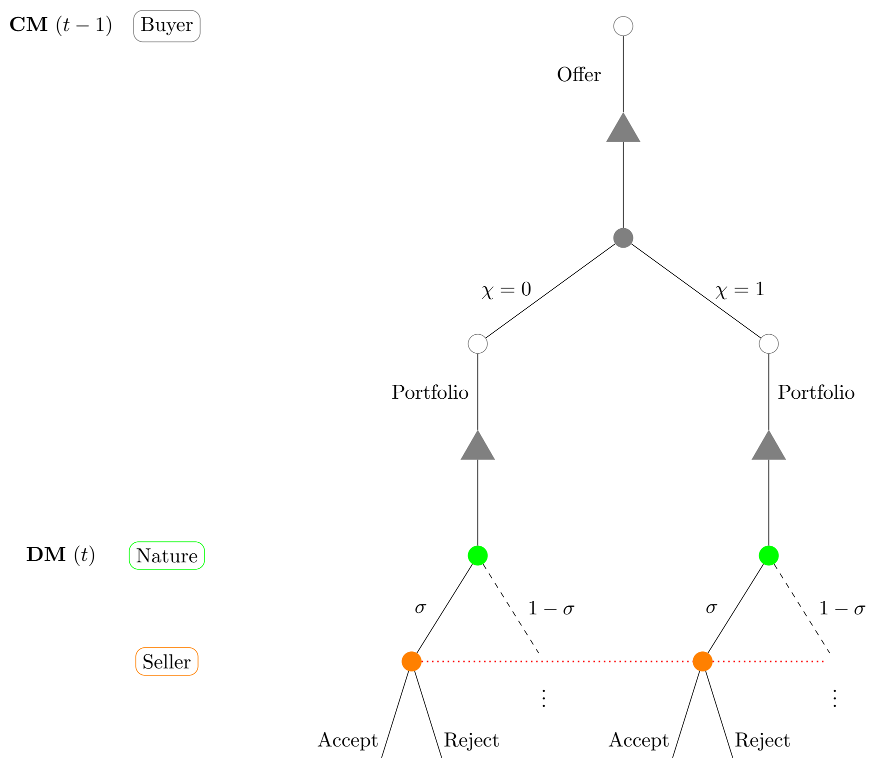
$$c(q) = \phi M + \phi^{f} M^{f} - [\bar{\kappa}^{f}_{\star}(\Pi^{f}) + \bar{\kappa}_{\star}(\Pi)].$$

Figure 1: Original extensive-form game.



Note: ● Buyer's discrete decision node; ○ Buyer's continuation to next decision node; ▲ Buyer's continuous decision node; ● Nature's discrete decision node; ● Seller's discrete decision node; ··· Information set.

Figure 2: Reverse-order extensive-form game.



Note: ● Buyer's discrete decision node; ○ Buyer's continuation to next decision node; ▲ Buyer's continuous decision node; ● Nature's discrete decision node; ● Seller's discrete decision node; ··· Information set.

Figure 3: Equilibrium coexistence or non-coexistence of monies when $\Pi > \Pi^f$ and counterfeiting costs—baseline setting.

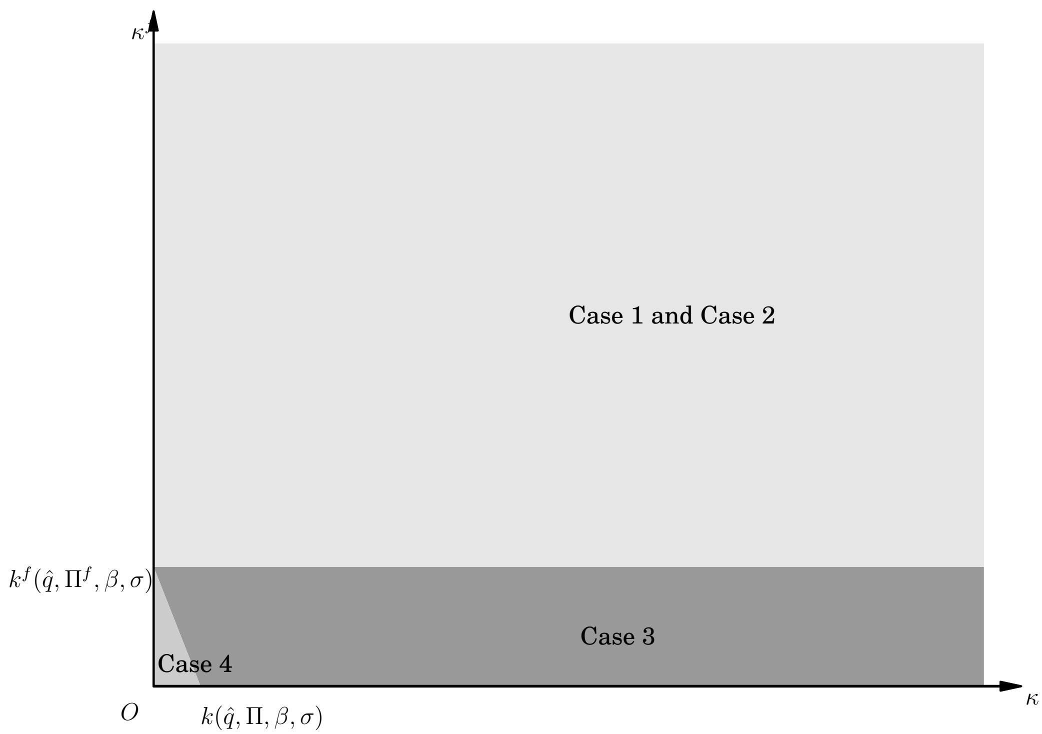


Figure 4: Equilibrium coexistence or non-coexistence of monies when $\Pi > \Pi^f$ and counterfeiting costs. Comparative static with higher Π^f.

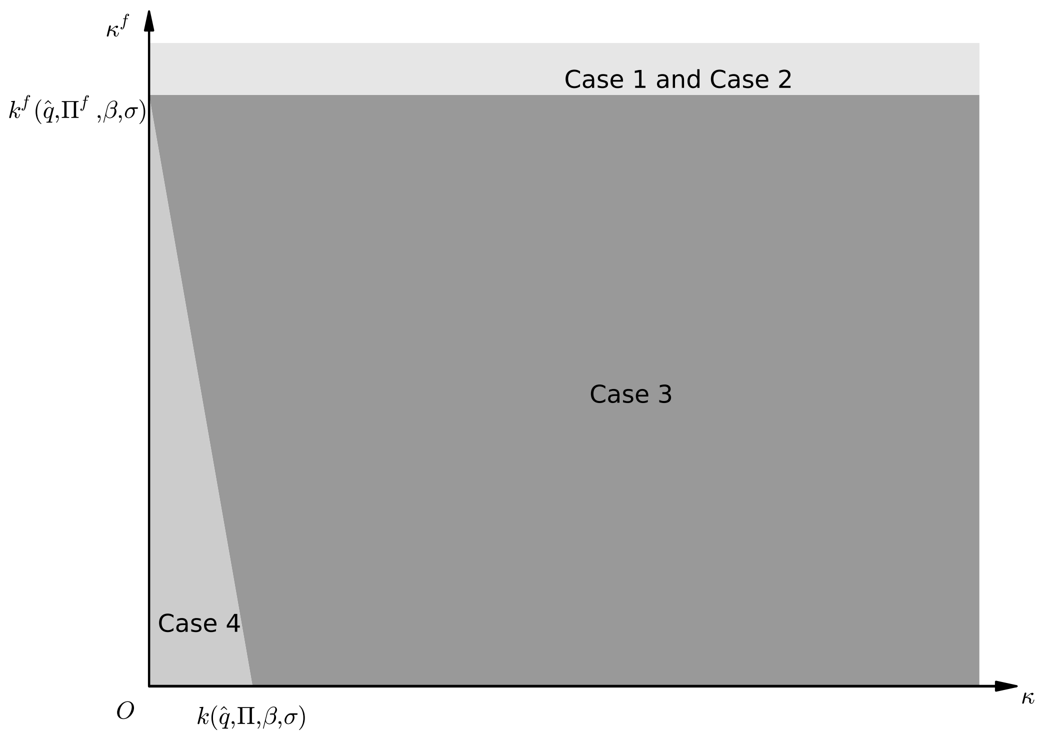


Figure 5: Equilibrium coexistence or non-coexistence of monies when $\Pi > \Pi^f$ and counterfeiting costs. Comparative static with higher matching probability σ.

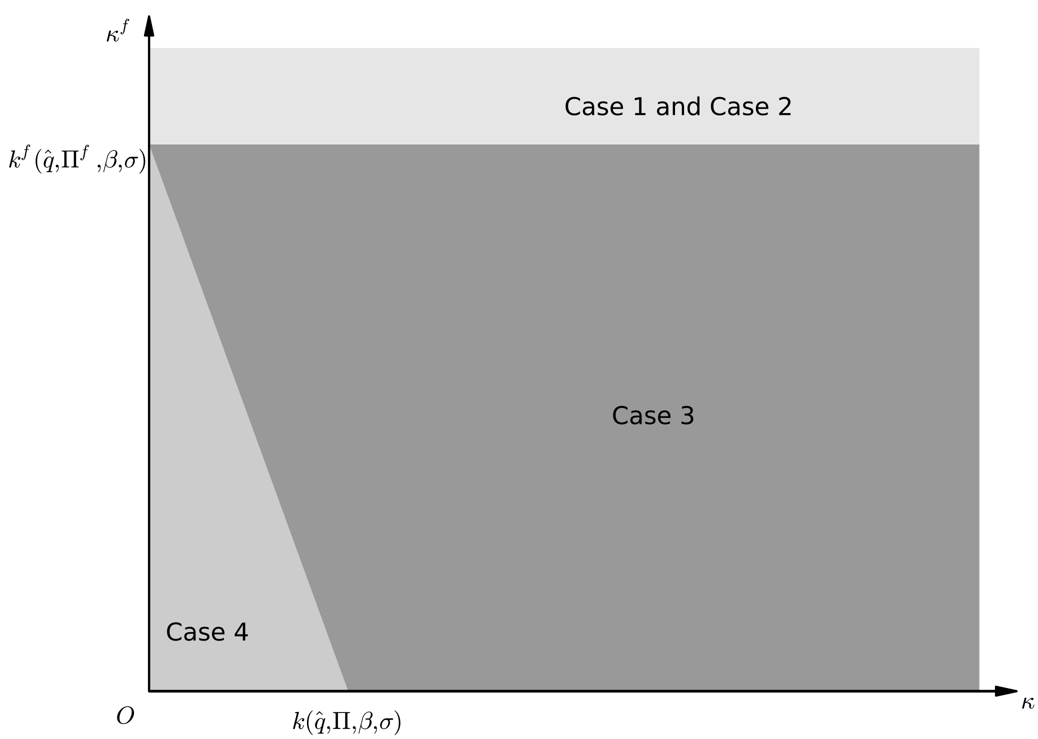


Figure 6: Equilibrium coexistence or non-coexistence of monies when $\Pi > \Pi^f$ and counterfeiting costs. Comparative static with higher risk aversion θ.

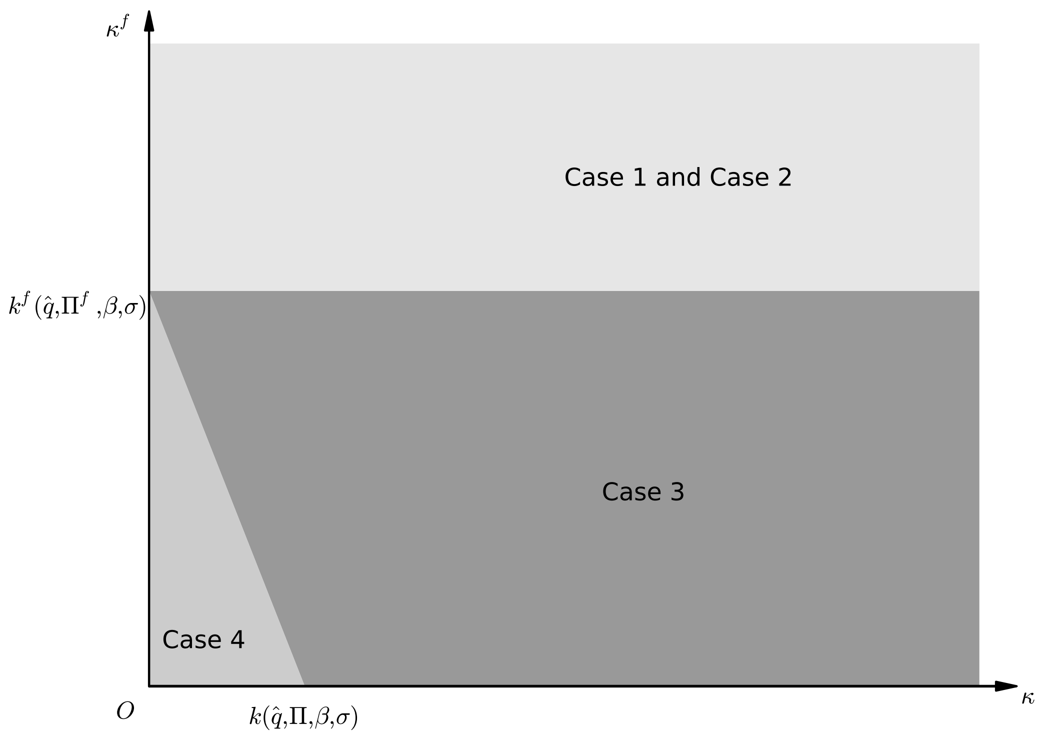


Figure 7: Equilibrium coexistence or non-coexistence of monies when $\Pi > \Pi^f$ and counterfeiting costs. Comparative static with more convex cost of DM production α.