Solow-Swan and convergence of living standards

Question 1 The Solow-Swan model predicts that countries that have lower initial per-worker capital stock will grow faster than their otherwise identical neighbors with higher initial per-worker capital stock. This is also supported empirically, at least for a subset of developed countries. However, in the long run, they converge to the same steady-state standard of living. Under the assumptions on the Solow-Swan model we want to prove that this is true. Let's do this in a few steps.

1. Assume a constant-returns-to-scale production technology given by $zF(K, N)$. Define per-worker capital stock as $k = K/N$ and the level of per-worker output at k is $y = zF(K/N, 1) := zf(k)$. Show that under this assumption, we have
$$MP_K := z\frac{\partial F(K, N)}{\partial K} = zf'(k).$$
and
$$MP_N := z\frac{\partial F(K, N)}{\partial N} = z[f(k) - f'(k)k].$$

2. Now using the previous results and assumptions show that the competitive equilibrium growth rate in the Solow model $\gamma_k := k'/k - 1$ is higher the further the initial state of per-worker capital stock is from the steady state. You may show this graphically by exploiting the expression of γ_k as a function of k.

3. Re-evaluate the results in the last two parts using the production function described by $Y = zF(K, N) = zK^{\alpha}N^{1-\alpha}$ where $\alpha \in (0, 1)$. Does this parametric example satisfy your more general findings previously under our assumptions on F?

Question 2 *Harrod-neutral technical progress.* Now consider a variation of the Swan-Solow model that has a deterministic trend growth in technology. The technology is *embodied* directly into labor. So often economists call this a model of labor-augmenting technical progress. Specifically, let x_t denote the level of labor-augmenting technology at time t, and assume that
$$x_{t+1} = (1 + g)x_t,$$
where $g > 0$ is the growth rate. So now, the only variation in the model is the production function:
$$Y_t = zF(K_t, x_tN_t).$$
where x is now embodied in the labor input.

1. Sketch the trend of the process $\{x_t\}_{t=0}^{\infty} := \{x_0, x_1, x_2, ...\}$ where x_0 is known. Sketch also the trend of the process $\{\ln(x_t)\}_{t=0}^{\infty}$.

2. Show that along the long run path, per-worker capital, k^*, *and* per-worker income, y^*, are no longer constant but they grow as a function of time.

3. However, show that now, if we define $\tilde{k}_t = K_t/x_tN_t = k_t/x_t$, interpreted as "capital per effective units of labor", the model has a well defined steady-state path $(\tilde{k}^*, \tilde{k}^*, \tilde{k}^*, ...)$ that is constant (i.e. does not grow with time).

4. Sketch in an appropriate diagram, the determination of the steady state in terms of capital per effective units of labor. What happens to this steady state if g is increased permanently?

5. Sketch the time path of log-income $\ln(Y)$ and log-consumption $\ln(C)$, before, during, and after the increase in g.