Endogenizing Savings in an OLG Model

The Big Picture. We have seen that in the Solow-Swan model, equilibrium is pinned down as a solution to a single difference equation in k_t: $k_{t+1} = g(k_t)$ with the initial condition k_0 given. This gave us a recursive map that allows us to generate any sequence of equilibrium capital stocks $\{k_{t+1}(k_0)\}_{t=0}^{\infty}$ that satisfies the equilibrium condition for the Solow-Swan model. The equilibrium conditions then allowed us to back out the equilibrium trajectory of per capita consumption $\{c_t(k_0)\}_{t=0}^{\infty}$ and output $\{y_t(k_0)\}_{t=0}^{\infty}$ as well. This week we see that the OLG model, which embeds a more detailed description of consumption behavior—i.e., it has a model for decision problems as the foundation for demand and supply in markets—will also produce a similar equilibrium map g describing the equilibrium trajectory of the economy. So mechanically, we have not much to worry about, if we are familiar with the Solow-Swan model, and once we have obtained this equilibrium function g.

The key differences to note this week is the following: (i) The OLG model's equilibrium g mapping depends on deeper descriptions of agents' preferences and firms' production technologies. In contrast, the Solow-Swan model, goes only as far as a black-box description of behaviour, as encapsulated in the reduced-form behavioural parameter s. On the supply/technology side of things, it still shares common ground with the OLG model. We shall see what the OLG model says about the long-run and transitions to it, *vis-á-vis* the Solow-Swan model. (ii) Modeling preference and production problems more deeply allows us to also talk about welfare implications of government policy change. We shall see that later in a long-term intergenerational fiscal policy example. (iii) Finally, having a more microfounded model underpinning aggregate demand and supply allows us to also deal with understanding economies where differences in agents' preferences or technologies may account for different dynamic and long-run equilibrium allocations of resources.

Please prepare you answers before attending tutorials. You will be expected to contribute to class discussions.

Homework 0. Attempt all the exercises listed on the lecture slides.

GruntWork 1. Consider the OLG model from lectures with logarithmic per-period utility of consumption, such that lifetime utility is $U(c, c') = \ln(c) + \beta \ln(c')$, and with Cobb-Douglas aggregate production, $F(K, N) = K^{\alpha} N^{1-\alpha}$, where $0 < \alpha < 1$. Let ρ denote the subjective rate of time preference, such that $\rho = 1/\beta - 1$, and $\beta \in (0, 1)$ is the subjective discount factor. The population of young agents N_t grows at a constant rate $n > -1$. The initial states K_0 and N_0 are given.

1. Show that the equilibrium in pinned down by a scalar difference equation in per-worker (young person) capital stock:
$$k_{t+1} = \frac{1-\alpha}{(1+n)(2+\rho)} k_t^{\alpha} =: g(k_t); \qquad t = 0, 1, 2, ...; \ k_0 \text{ given.}$$
2. Now solve for the steady state per-worker capital stock, denoted by k_{ss}.
3. Take a first-order Taylor expansion of the equilibrium map in Part 1 above, about the steady state point k_{ss}.
4. Calculate the rate of convergence to the steady state in this model — i.e. how much does k change each period? What does this rate depend on?
5. Also, calculate the half-life of an initial per-worker income, assuming the economy began with income zk_0^{α} where $k_0 > k_{ss}$.

6. Do the same as in Part 2 above, with the Solow-Swan model. Compare your finding with the last part. Comment on your results using the economics underpinning these two models.

Homework 2. Reconsider the last problem. Now assume that the preference function is

$$u(c) + \beta u(c'),$$

where

$$u(x) := \lim_{\tilde{\sigma} \to \sigma} \left\{ \frac{x^{1-\sigma} - 1}{1 - \sigma} \right\}.$$

Show that now, the marginal propensity to consume out of current income is no longer a constant. What does it depend on? Can you solve for an equilibrium by hand? If not, explain why. Suggest how you may proceed to solve the problem numerically.