1 DISTRIBUTION

$$F(z) = 1 - (1+z)^{-\eta}, \qquad \eta > 1$$

We have:

$$\begin{aligned}
f(z) &= \frac{\eta}{(1+z)^{1+\eta}} \\
\mu &= \frac{1}{\eta - 1} \\
\Omega(x) &= 1 - \left[\frac{1 + x\eta}{(1+x)^{\eta}}\right]
\end{aligned}$$

2 CALIBRATION

We set $k^h = 0$. We have the following parameters to be defined/calibrated:

$$\Theta = \left(\theta^d, \theta^a, k^d, k^a, \phi, \eta\right)$$

The model is calibrated to match data in 1984. First, we set $\theta^c = 1.01$, $\theta^d = 0.1$ and $\theta^a = 0.01$. So now we have the following set of parameters to be calibrated:

$$\Theta' = \left(k^d, k^a, \phi, \eta\right)$$

We have the following targets:

$T(1)$	$=$	0.15	ratio of cash to money holdings in 1984 (HP filtered)
$T(2)$	$=$	0.97	ratio of deposits to MMDAs in 1984 (HP filtered)
$T(3)$	$=$	0.01	resource cost as a fraction of GDP
$T(4)$	$=$	0.01	labor cost of trips to the bank

Mapping these moments to the model, we have:

$$\begin{aligned}
T(1) &= \frac{\theta^c \Omega(\gamma)}{n} \\
T(2) &= \frac{\Omega(\delta) - \Omega(\gamma)}{n} \\
T(3) &= k^d \left[F(\delta) - F(\gamma)\right] + k^a \left[1 - F(\delta)\right] \\
T(4) &= \frac{\phi n \left(1 + T(3)\right)}{1 - \phi n}
\end{aligned}$$

There are four equations. Together with the equilibrium conditions, we have a system of seven equations and seven unknowns (remember that $r = 0.095$):

$$\left(k^d, k^a, \phi, \eta, \gamma, \delta, n\right)$$

$$\begin{aligned}
T(1) &= \frac{\theta^c \Omega(\gamma)}{n} \\
T(2) &= \frac{\Omega(\delta) - \Omega(\gamma)}{n} \\
T(3) &= k^d \left[F(\delta) - F(\gamma)\right] + k^a \left[1 - F(\delta)\right] \\
T(4) &= \frac{\phi n (1 + T(3))}{1 - \phi n} \\
\frac{n^2 \phi}{1 - \phi n} &= \frac{(\theta^c - 1)\Omega(\gamma) + r\left[\theta^c \Omega(\gamma) + \theta^d (\Omega(\delta) - \Omega(\gamma)) + \theta^a (1 - \Omega(\delta))\right]}{1 + k^d \left[F(\delta) - F(\gamma)\right] + k^a \left[1 - F(\delta)\right]} \\
\delta &= \gamma \frac{\frac{\theta^c - 1}{r} + \theta^c - \theta^d}{\theta^d - \theta^a} \frac{k^a - k^d}{k^d} \equiv \kappa(r)\gamma \\
n &= \gamma \frac{1}{\mu} \frac{\theta^c - 1 + r\left(\theta^c - \theta^d\right)}{k^d}
\end{aligned}$$

2.1 ALGORITHM

We solve for η

Step 1: Guess a value for η.

Step 2: we solve for γ, k^d and k^a using:

$$\begin{aligned}
\frac{T(1)}{T(2)} &= \frac{\theta^c \Omega(\gamma)}{\Omega(\kappa(r)\gamma) - \Omega(\gamma)} \\
\kappa(r) &= \gamma \frac{\frac{\theta^c - 1}{r} + \theta^c - \theta^d}{\theta^d - \theta^a} \frac{k^a - k^d}{k^d} \\
T(3) &= k^d \left[F(\kappa(r)\gamma) - F(\gamma)\right] + k^a \left[1 - F(\kappa(r)\gamma)\right]
\end{aligned}$$

Step 2: we solve for γ given the equation:

$$\begin{aligned}
T(1) &= \frac{\theta^c \Omega(\gamma)}{(\theta^c - 1)\Omega(\gamma) + 1} \\
0 &= T(1)\left[(\theta^c - 1)\Omega(\gamma) + 1\right] - \theta^c \Omega(\gamma)
\end{aligned}$$

Step 3: given γ and η, we solve for δ using the equation:

$$\begin{aligned}
T(2) &= \frac{\Omega(\delta) - \Omega(\gamma)}{1 - \Omega(\delta)} \\
0 &= T(2)\left[1 - \Omega(\delta)\right] - \left[\Omega(\delta) - \Omega(\gamma)\right]
\end{aligned}$$

Step 4: given δ, γ and η, we can solve for k^a and k^d using the equations:

$$\begin{aligned} T\left(3\right) &= k^d\left[F\left(\delta\right)-F\left(\gamma\right)\right]+k^a\left[1-F\left(\delta\right)\right] \\ \delta &= \gamma\frac{\frac{\theta^c-1}{r}+\theta^c-\theta^d}{\theta^d-\theta^a}\frac{k^a-k^d}{k^d} \end{aligned}$$

Define:

$$C_1 \equiv \frac{\gamma}{\delta}\frac{\frac{\theta^c-1}{r}+\theta^c-\theta^d}{\theta^d-\theta^a}$$

So we have:

$$\begin{aligned} k^d &= C_1\left(k^a-k^d\right) \\ k^d &= \frac{C_1k^a}{1+C_1} \end{aligned}$$

And:

$$T\left(3\right)=k^a\left[\frac{C_1}{1+C_1}\left[F\left(\delta\right)-F\left(\gamma\right)\right]+1-F\left(\delta\right)\right]$$

Define:

$$C_2 \equiv \frac{C_1}{1+C_1}\left[F\left(\delta\right)-F\left(\gamma\right)\right]+1-F\left(\delta\right)$$

We have:

$$\begin{aligned} k^a &= \frac{T\left(3\right)}{C_2} \\ k^d &= \frac{C_1}{1+C_1}\frac{T\left(3\right)}{C_2} \end{aligned}$$

Step 5: given k^d, we solve for n

$$n=\frac{\gamma}{\mu}\frac{\theta^c-1+r\left(\theta^c-\theta^d\right)}{k^d}$$

Step 6: given δ, γ and η, we can solve for ϕ:

$$\begin{aligned} T\left(4\right) &= \frac{\phi n\left(1+T\left(3\right)\right)}{1-\phi n} \\ T\left(4\right)-\phi nT\left(4\right) &= \phi n\left(1+T\left(3\right)\right) \\ \phi &= \frac{1}{n}\frac{T\left(4\right)}{1+T\left(3\right)+T\left(4\right)} \end{aligned}$$

Step 7: we check if the following equation holds

$$\frac{\phi n^2}{1-\phi n} = \frac{(\theta^c - 1)\,\Omega(\gamma) + r\left[\theta^c \Omega(\gamma) + \theta^d (\Omega(\delta) - \Omega(\gamma)) + \theta^a (1 - \Omega(\delta))\right]}{1 + T(3)}$$

Easier if we define:

$$C_3 = (\theta^c - 1)\,\Omega(\gamma) + r\left[\theta^c \Omega(\gamma) + \theta^d (\Omega(\delta) - \Omega(\gamma)) + \theta^a (1 - \Omega(\delta))\right]$$

If not, update value for η.

3 SIMULATION

We want to check how the model behaves as we vary r.

So we first define a grid for r. For each value of r, we can solve for $\gamma(r)$ and $n(r)$.

$$\begin{aligned}
\frac{n^2\phi}{1-\phi n} &= \frac{(\theta^c - 1)\,\Omega(\gamma) + r\left[\theta^c \Omega(\gamma) + \theta^d (\Omega(\delta) - \Omega(\gamma)) + \theta^a (1 - \Omega(\delta))\right]}{1 + k^d\,[F(\delta) - F(\gamma)] + k^a\,[1 - F(\delta)]} \\
n &= \gamma \frac{1}{\mu} \frac{\theta^c - 1 + r\left(\theta^c - \theta^d\right)}{k^d} \\
\delta &= \gamma \frac{\frac{\theta^c - 1}{r} + \theta^c - \theta^d}{\theta^d - \theta^a} \frac{k^a - k^d}{k^d} \equiv \kappa(r)\,\gamma
\end{aligned}$$

First, we solve for γ such that:

$$\frac{\left(\gamma \frac{1}{\mu} \frac{\theta^c - 1 + r\left(\theta^c - \theta^d\right)}{k^d}\right)^2 \phi}{1 - \phi\gamma \frac{1}{\mu} \frac{\theta^c - 1 + r\left(\theta^c - \theta^d\right)}{k^d}} = \frac{(\theta^c - 1)\,\Omega(\gamma) + r\left[\theta^c \Omega(\gamma) + \theta^d (\Omega(\kappa(r)\,\gamma) - \Omega(\gamma)) + \theta^a (1 - \Omega(\kappa(r)\,\gamma))\right]}{1 + k^d\,[F(\kappa(r)\,\gamma) - F(\gamma)] + k^a\,[1 - F(\kappa(r)\,\gamma)]}$$

We define:

$$\begin{aligned}
f_{lhs}(\gamma, r) &= \frac{\left(\gamma \frac{1}{\mu} \frac{\theta^c - 1 + r\left(\theta^c - \theta^d\right)}{k^d + k^h}\right)^2 \phi}{1 - \phi\gamma \frac{1}{\mu} \frac{\theta^c - 1 + r\left(\theta^c - \theta^d\right)}{k^d}} \\
f_{rhs}(\gamma, r) &= \frac{(\theta^c - 1)\,\Omega(\gamma) + r\left[\theta^c \Omega(\gamma) + \theta^d (\Omega(\kappa(r)\,\gamma) - \Omega(\gamma)) + \theta^a (1 - \Omega(\kappa(r)\,\gamma))\right]}{1 + k^d\,[F(\kappa(r)\,\gamma) - F(\gamma)] + k^a\,[1 - F(\kappa(r)\,\gamma)]}
\end{aligned}$$

We can define the function:

$$f_{aux}(r) = \frac{1}{\mu} \frac{\theta^c - 1 + r\left(\theta^c - \theta^d\right)}{k^d}$$

So we have:

$$\begin{aligned}
f_{lhs}(\gamma, r) &= \frac{(\gamma f_{aux}(r))^2 \phi}{1 - \phi\gamma f_{aux}(r)} \\
f_{rhs}^{up}(\gamma, r) &= (\theta^c - 1)\,\Omega(\gamma) + r\left[\theta^c \Omega(\gamma) + \theta^d\left(\Omega(\kappa(r)\gamma) - \Omega(\gamma)\right) + \theta^a\left(1 - \Omega(\kappa(r)\gamma)\right)\right] \\
f_{rhs}^{down}(\gamma, r) &= 1 + k^d\left[F(\kappa(r)\gamma) - F(\gamma)\right] + k^a\left[1 - F(\kappa(r)\gamma)\right]
\end{aligned}$$

UPPER BOUND
Note that:

$$\begin{aligned}
n &= \gamma \frac{1}{\mu} \frac{\theta^c - 1 + r\left(\theta^c - \theta^d\right)}{k^d} \\
\frac{\gamma}{\phi n} &= \frac{\mu}{\phi} \frac{k^d}{\theta^c - 1 + r\left(\theta^c - \theta^d\right)}
\end{aligned}$$

And since $\phi n < 1$:

$$\gamma < \frac{\mu}{\phi} \frac{k^d}{\theta^c - 1 + r\left(\theta^c - \theta^d\right)}$$

This is useful for settting an upper bound for γ in the code.
Then, we solve for n:

$$n = \gamma \frac{1}{\mu} \frac{\theta^c - 1 + r\left(\theta^c - \theta^d\right)}{k^d}$$

Next, we can compute the curve:

$$\frac{m_1}{GDP} = \frac{(\theta^c - 1)\,\Omega(\gamma) + 1}{n}$$

3.1 PARAMETER A

We choose A such that:

$$\begin{aligned}
\frac{m_1}{GDP} &= 0.25 = A\frac{(\theta^c - 1)\,\Omega(\gamma(0.06)) + 1}{n(0.06)} \\
A &= \frac{0.25 n(0.06)}{(\theta^c - 1)\,\Omega(\gamma(0.06)) + 1}
\end{aligned}$$

We have the data for the left-hand-side and we plot the right hand side based on the model.

With $\gamma(r)$ we compute $\delta(r)$:

$$\delta(r) = \kappa(r)\gamma(r)$$

So we can calculate the series for cash/deposit:

$$\frac{c}{d} = \frac{\Omega(\gamma(r))}{\Omega(\delta(r)) - \Omega(\gamma(r))}$$

Finally, we can calculate the series for deposit/(deposit+MMDA's):

$$\frac{d}{d+a} = \frac{\Omega(\delta(r)) - \Omega(\gamma(r))}{1 - \Omega(\gamma(r))}$$

4 Plotting $\gamma(r)$ and $n(r)$:

Here we assume we have r. We use a low value ($r = 2\%$), a medium value ($r = 5\%$) and high value ($r = 9\%$) for this exercise.

We create a grid for values of γ, and for each value we calculate n for each curve:

$$\begin{aligned}
\frac{n^2\phi}{1-\phi n} &= \frac{(\theta^c - 1)\Omega(\gamma) + r\left[\theta^c\Omega(\gamma) + \theta^d(\Omega(\delta) - \Omega(\gamma)) + \theta^a(1 - \Omega(\delta))\right]}{1 + k^d[F(\delta) - F(\gamma)] + k^a[1 - F(\delta)]} \\
n &= \gamma\frac{1}{\mu}\frac{\theta^c - 1 + r\left(\theta^c - \theta^d\right)}{k^d}
\end{aligned}$$

For the quadratic equation, let's define:

$$A(\gamma, r) = \frac{(\theta^c - 1)\Omega(\gamma) + r\left[\theta^c\Omega(\gamma) + \theta^d(\Omega(\kappa(r)\gamma) - \Omega(\gamma)) + \theta^a(1 - \Omega(\kappa(r)\gamma))\right]}{1 + k^d[F(\kappa(r)\gamma) - F(\gamma)] + k^a[1 - F(\kappa(r)\gamma)]}$$

We have:

$$\begin{aligned}
n^2 &= \frac{1-\phi n}{\phi}A(\gamma, r) \\
n^2 &= \frac{1}{\phi}A(\gamma, r) - A(\gamma, r)n \\
n^2 + A(\gamma, r)n - \frac{1}{\phi}A(\gamma, r) &= 0
\end{aligned}$$

So we have:

$$n = \frac{-A(\gamma, r) \pm \sqrt{(A(\gamma, r))^2 + 4\frac{1}{\phi}A(\gamma, r)}}{2}$$

5 SIMULATION (decentralized economy)

We want to check how the model behaves as we vary r. Now we take k^h and r^* as given.

We first define a grid for r. For each value of r, we can solve for $\gamma(r)$ and $n(r)$ using the system:

$$\begin{aligned} \frac{n^2\phi}{1-\phi n} &= \frac{(\theta^c-1)\,\Omega(\gamma)+r\left[1+(\theta^c-1)\,\Omega(\gamma)\right]-r^*\left[1-\Omega(\gamma)\right]}{1+k(1-F(\gamma))} \\ n\max\left\{k-\left(r\left(1-\theta^d\right)-r^*\right)\frac{\gamma}{\mu}\frac{1}{n},k^h\right\} &= \left[(1+r)\,(\theta^c-1)+r^*\right]\frac{1}{\mu}\gamma \end{aligned}$$

5.1 Case 1: Assuming $k-\left(r\left(1-\theta^d\right)-r^*\right)\frac{\gamma}{\mu}\frac{1}{n}>k^h$

$$\begin{aligned} nk-\left(r\left(1-\theta^d\right)-r^*\right)\frac{\gamma}{\mu} &= \left[(1+r)\,(\theta^c-1)+r^*\right]\frac{\gamma}{\mu} \\ nk &= \frac{\gamma}{\mu}\left[(1+r)\,(\theta^c-1)+r\left(1-\theta^d\right)\right] \\ n &= \frac{\gamma}{\mu}\frac{\left[(\theta^c-1)+r\left(\theta^c-\theta^d\right)\right]}{k} \end{aligned}$$

Then we can solve for $\gamma(r)$.

$$\frac{\left(\frac{\gamma}{\mu}\frac{\left[(\theta^c-1)+r\left(\theta^c-\theta^d\right)\right]}{k}\right)^2}{1-\phi\frac{\gamma}{\mu}\frac{\left[(\theta^c-1)+r\left(\theta^c-\theta^d\right)\right]}{k}}\phi=\frac{(\theta^c-1)\,\Omega(\gamma)+r\left[1+(\theta^c-1)\,\Omega(\gamma)\right]-r^*\left[1-\Omega(\gamma)\right]}{1+k(1-F(\gamma))}$$

Steps:
Define:

$$case1\_faux(y)=\frac{\left[(\theta^c-1)+y\left(\theta^c-\theta^d\right)\right]}{\mu k}$$

Then we define:

$$case1\_lhs(x,y)=\frac{(x case1\_faux(y))^2}{1-\phi x case1\_faux(y)}\phi$$

And we define:

$$case1\_rhs(x,y)=\frac{(\theta^c-1)\,\Omega(x)+y\left[1+(\theta^c-1)\,\Omega(x)\right]-r^*\left[1-\Omega(x)\right]}{1+k(1-F(x))}$$

So for each r we solve for γ such that:

$$case1\_rhs(\gamma, r) = case1\_lhs(x, y)$$

Then we calculate n :

$$n = x.case1\_faux\,(y)$$

Limit:

$$\phi n = \phi \frac{\gamma}{\mu} \frac{\left[(\theta^c - 1) + r \left(\theta^c - \theta^d \right) \right]}{k}$$

We have:

$$0 < \phi n < 1$$

So:

$$\begin{aligned} 1 &> \phi \frac{\gamma}{\mu} \frac{\left[(\theta^c - 1) + r \left(\theta^c - \theta^d \right) \right]}{k} \\ \gamma &< \frac{\mu k}{\phi} \frac{1}{(\theta^c - 1) + r \left(\theta^c - \theta^d \right)} \end{aligned}$$

5.2 Case 2: Assuming $k - \left(r \left(1 - \theta^d \right) - r^* \right) \frac{\gamma}{\mu} \frac{1}{n} < k^h$

$$n = \frac{\gamma}{\mu} \left[\frac{(1 + r)\,(\theta^c - 1) + r^*}{k^h} \right]$$

Then we can solve for $\gamma\,(r)$.

$$\frac{\left(\frac{\gamma}{\mu} \frac{(1+r)(\theta^c - 1) + r^*}{k^h} \right)^2}{1 - \phi \frac{\gamma}{\mu} \frac{(1+r)(\theta^c - 1) + r^*}{k^h}} \phi = \frac{(\theta^c - 1)\,\Omega\,(\gamma) + r\,[1 + (\theta^c - 1)\,\Omega\,(\gamma)] - r^*\,[1 - \Omega\,(\gamma)]}{1 + k(1 - F(\gamma))}$$

Steps:
define:

$$case2\_faux\,(x) = \frac{(1 + x)\,(\theta^c - 1) + r^*}{\mu k^h}$$

Then we define:

$$case2\_lhs(x, y) = \frac{(x case2\_faux\,(y))^2}{1 - \phi x case2\_faux\,(y)} \phi$$

And we define:

$$case2\_rhs(x,y) = \frac{(\theta^c - 1)\,\Omega\,(x) + y\,[1 + (\theta^c - 1)\,\Omega\,(x)] - r^*\,[1 - \Omega\,(x)]}{1 + k(1 - F(x))}$$

So for each r we solve for γ such that:

$$case2\_rhs(\gamma, r) = case2\_lhs(x, y)$$

Then we calculate n:

$$n = x.case2\_faux\,(y)$$

Limit:

$$\phi n = \phi \frac{\gamma}{\mu} \frac{(1+r)\,(\theta^c - 1) + r^*}{k^h}$$

$$\begin{aligned}
1 &> \phi \frac{\gamma}{\mu} \frac{(1+r)\,(\theta^c - 1) + r^*}{k^h} \\
\gamma &< \frac{\mu k^h}{\phi} \frac{1}{(1+r)\,(\theta^c - 1) + r^*}
\end{aligned}$$

5.2.1 Simulation

M1/GDP

$$\begin{aligned}
m_1 &= \theta^c c + d \\
&= \theta^c \frac{px\Omega\,(\gamma)}{n} + \frac{px\,(1 - \Omega\,(\gamma))}{n} \\
&= \frac{px}{n}\,[(\theta^c - 1)\,\Omega\,(\gamma) + 1]
\end{aligned}$$

$$GDP = px$$

So we have:

$$\frac{m_1}{GDP} = \frac{(\theta^c - 1)\,\Omega\,(\gamma) + 1}{n}$$

C/M1

$$\frac{c}{m_1} = \frac{\Omega\,(\gamma)}{(\theta^c - 1)\,\Omega\,(\gamma) + 1}$$

5.3 Fixing γ

We have:

$$\frac{n^2\phi}{1-\phi n} = \frac{(\theta^c-1)\,\Omega(\gamma^*) + r\left[1+(\theta^c-1)\,\Omega(\gamma^*)\right] - r^*\left[1-\Omega(\gamma^*)\right]}{1+k(1-F(\gamma^*))}$$

$$A(r) = \frac{(\theta^c-1)\,\Omega(\gamma^*) + r\left[1+(\theta^c-1)\,\Omega(\gamma^*)\right] - r^*\left[1-\Omega(\gamma^*)\right]}{1+k(1-F(\gamma^*))}$$

We have:

$$\begin{aligned} n^2 &= \frac{1-\phi n}{\phi}A(r) \\ n^2 &= \frac{1}{\phi}A(r) - A(r)\,n \\ n^2 + A(r)\,n - \frac{1}{\phi}A(r) &= 0 \end{aligned}$$

So we have:

$$n = \frac{-A(r) \pm \sqrt{(A(r))^2 + 4\frac{1}{\phi}A(r)}}{2}$$