Inflation, Inequality and Welfare in a Competitive Search Model*

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Abstract

We study long-run inflation in a competitive-search model with heterogeneous agents. Under competitive search, individuals' matching-probability (extensive) margins trade off against quantity (intensive) margins. With money and unfettered market participation, these trade-offs depend on inflation and individuals' heterogeneous money holdings. We find that welfare falls as inflation increases. However, money-holdings inequality is not monotonic in inflation. As inflation rises, liquid-wealth inequality first falls. For sufficiently high inflation, the overall extensive-margin effect dominates the intensive margin, and liquid-wealth inequality rises. The model also poses a new computational challenge to which we propose a novel solution method.

JEL Codes: E0; E4; E5; E6; C6

Keywords: Competitive Search; Inflation, Distributional Trade-offs; Computational Geometry.

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1 Introduction

We study the effects of long-run inflation in a model with competitive search and non-degenerate money distribution. Monetary search models provide a deeper microfoundation of money, where important market frictions or behavioral outcomes are not assumed but are consequences of environmental (e.g., informational and contractual) imperfections. The framework has been applied to answer many questions such as the existence and essentiality of money, financial intermediation, asset liquidity, and equilibrium price dispersion.[1]

In this paper, we follow and extend the model of Menzio, Shi and Sun (2013) to study non-zero inflation. The heterogeneity in our model is a result of equilibrium directed search and matching frictions. That is, we do not rely on exogenous shocks to individual states, and, as in Menzio et al. (2013), possible persistence in agent heterogeneity is induced by endogenous individual spending cycles.[2] However, a question remains open in this class of competitive-search models: How does inflation affect the trade-off underlying heterogenous individuals' matching and market participation rates, their corresponding terms of trade, and the resulting equilibrium distribution of agents' money holdings? This question is very relevant now as most advanced economies are facing higher inflation rates in recent times.[3]

In our model, individuals direct their search to sellers, and sellers post their trade terms in anticipation of different matching probabilities at different trading posts (or submarkets). These incentives and matching probabilities, in turn, depend on different levels of money holdings and inflation policy. This introduces a different channel of monetary policy that is muted in models without this endogenous mechanism. Consequently, we have equilibrium-determined distributions of market participation, market tightness, relative prices, and trade quantities.[4] Addressing our question is an important first step towards incorporating competitive search behavior into a more feature-laden and quantitative model.

With competitive search, it is well known that individuals face a trade-off between an extensive margin of trading probability and an intensive margin of trade quantity (see, e.g., Wright, Kircher, Julien and Guerrieri, 2021; Rocheteau and Wright, 2005; Moen, 1997). We emphasize that with a non-degenerate distribution of money balances in equilibrium, the effects of inflation on these trade-offs in our model depend on agents' money holdings. We provide a numerical comparative study of steady-

[1] Lagos et al. (2017) and Williamson and Wright (2010) provide comprehensive reviews on New Monetarist models. They show how many applications require the modeling of fundamental frictions that cannot be abstracted away using reduced-form or parametric assumptions. Wright et al. (2021) present an overview of the literature on competitive search models.

[2] This is an alternative to Bewley-style models where the fundamental source of agent heterogeneity is exogenous and comes from random income or taste shocks. Earlier Bewley-style monetary models use a cash-in-advance assumption (see, e.g., Imrohoroğlu and Prescott, 1991b,a). Extensions of Bewley-style models include the recent literature on HANK models. These are non-monetary models where sticky-price assumptions deliver monetary policy consequences (see, e.g., Kaplan et al., 2018). Recent models with random-matching markets (see, e.g., Bethune and Rocheteau, 2023; Bustamante, 2023; Rocheteau et al., 2019; Chiu and Molico, 2010) or one-shot, competitive-search markets (Sun and Zhou, 2018) also rely on Bewley-style idiosyncratic shocks to bolster ex-post agent heterogeneity. In such models, the shocks are assumed to be persistent. This induces additional, if not most of the, persistent heterogeneity in agents' responses (see, e.g., Wang, 2007).

[3] Cross-country CPI inflation comparisons showing higher inflation rates in recent years are available on the Australian Bureau of Statistics website: `https://www.abs.gov.au/articles/cpi-international-comparisons`.

[4] The competitive search mechanism in Menzio et al. (2013) is also useful for rationalizing price dispersion endogenously under agent heterogeneity. This aspect does not arise in heterogeneous-agent models with random search (see, e.g., Bethune and Rocheteau, 2023; Chiu and Molico, 2010; Rocheteau et al., 2019) or in standard Walrasian models like HANK (see, e.g., Kaplan et al., 2018). If agents must deterministically rebalance their wealth in a centralized market, and there is no wealth effect on agents' optimal portfolio decisions (see, e.g., Rocheteau and Wright, 2005), there will be no ex-post heterogeneity, even with competitive search. An example of using competitive search to generate price dispersion can be found in real models with housing market frictions (see Hedlund, 2016; Garriga and Hedlund, 2020).

state equilibria across two inflation regimes to illustrate the results (see Section 4).[5] In particular, we show that with higher long run inflation, while matching probabilities fall with inflation for all agents, agents with relatively lower money balances ("the poor") face a steeper decline in payments and matching probabilities than those with higher money holdings ("the rich").[6]This translates to an increase in dispersion of total payments and trading probabilities as inflation becomes higher.[7]

The difference in the steepness (with respect to higher inflation regimes) of the decline in matching probabilities and payment amounts between the "rich" and "poor" reflects the underlying tension between the intensive and extensive margin in the competitive search markets. This tension or trade-off varies depending on the agent's money holding. Our equilibrium comparisons show that the "rich", relative to the "poor" will also have a higher velocity of spending (i.e., a higher ratio of expected payments per dollar carried) due to the flatter decline in matching rates and payments. This enables the "rich" to replenish their liquidity faster to support better matching and trade outcomes in goods search markets as inflation rises, compared to the "poor".

Higher inflation has two effects on agents' money holdings. On one hand, higher inflation tends to compress dispersion in money holding. That comes from the intensive margin effect. This is typically known as the redistributive effect of the inflation tax and is a common feature in all heterogeneous-agent monetary models (see, for example, Erosa and Ventura, 2002). On the other hand, with higher inflation, the extensive margin aspect of competitive search gives us a rising dispersion in heterogeneous matching opportunities, spending amounts, and speeds of transactions. These gaps between the "rich" and "poor" widen with inflation and deliver an opposing force against the standard redistributive effect of inflation tax. When inflation is low, the intensive margin effect dominates, while the extensive margin effect becomes stronger at sufficiently high inflation rates. This leads to a novel non-monotonic relationship between inflation and money-holding inequality: For low long-run inflation rates, the inequality in money holdings tends to diminish with inflation. However, for sufficiently high inflation rates, the inequality in money holdings increases with inflation.[8]

As a consequence of this friction working against the compressing or redistributive effect of inflation, the welfare cost of inflation is nontrivial, especially when transitional dynamics are taken into account. In standard models, due to its redistributive property, the welfare cost of inflation is lowered relative to a representative-agent monetary model. With competitive search, this effect can be dominated by the effects of inflation on the extensive margin of trade. As a result, we find that the welfare cost of inflation can be as high as, or even higher than, the original representative-agent calculation of Lucas (2000).[9]

Finally, we also develop a novel way of solving the equilibrium computationally. There is a technical challenge posed by the theoretical work of Menzio et al. (2013): The agents' ex-ante value function

[5]We discipline our study by calibrating the benchmark setting to U.S. data.

[6]To economize on lengthy sentences, when we consider the distribution of agents' money holdings in a particular "decentralized market" or DM, we will use the short-hand terminology of "rich" and "poor" in lieu of the top 10th percentile and the bottom 10th percentile of a given distribution of money holdings (conditional on agents holding positive money balances) later. The ratio between these two numbers, the 90/10 ratio, is a common statistic used in studies of inequality (see, e.g., Meyer and Sullivan, 2023). When we consider the unconditional distribution of all agents, we summarize the inequality effect of higher inflation using the Gini coefficient (as it may be possible that a large mass of agents can hold zero balances in the unconditional distribution).

[7]In Section 4, we also show how these heterogeneous individuals' responses are associated with other distributional outcomes such as dispersions in pricing, spending, matching rates, market participation rate, and money wealth.

[8]In this paper, we focus our analysis on inflationary regimes within the standard range studied in the literature and experienced in most industrialized countries in the more recent decades. We do not consider inflation tending to the hyperinflationary direction. In such cases, the distribution will compress as the extreme tax will eventually make money valueless. We have verified this in our computations but do not focus on this in the paper.

[9]We present welfare cost comparisons between our model and some papers in the literature in Table 2 in Section 5.

(labelled as B later) induced by pure strategies with competitive search is typically non-concave/convex over multiple parts of its domain (money holdings). Thus, agents can be better off by choosing lotteries that convexify the graph of B. We propose a practical, efficient, and high-precision implementation of this idea using standard convex hull computational tools.[10] We explain this idea further in Section 3.1 and in more detail in Online Appendix E.

1.1 Related literature

Our paper uses a competitive search framework to study the effects of inflation. There is a vast literature analyzing the effects of inflation. We briefly review some of them here while paying more attention to papers more closely related to our approach. We then discuss a few papers which also feature non-degenerate distributions of money holdings and emphasize the difference between our model and theirs. Finally, we describe the essence of Menzio, Shi and Sun (2013), and discuss why it poses an open problem for us to study here in terms of inflation and its distribution and welfare effects.[11]

Inflation, heterogeneity and money distribution. Consider a taxonomy of the costs and benefits of inflation in standard Walrasian-market models (see, e.g., Erosa and Ventura, 2002). First, inflation acts as an intertemporal tax that distorts consumption. This feature raises the (welfare) cost of inflation in all monetary models (with or without heterogeneous agents). Second, inflation is costly since agents have to engage in precautionary liquidity management activities. Third, inflation may act as a redistributive tax that shifts resources from the "rich" to the "poor". This force tends to lower the welfare cost of inflation.

In most heterogeneous-agent models (see, e.g., Imrohoroğlu and Prescott, 1991a; Akyol, 2004; Boel and Camera, 2009; Meh et al., 2010), the redistributive-tax channel of inflation is strong. This is often because there is only an intensive margin through which inflation tax works.[12] That is, with higher inflation, agents would like to reduce their money holdings. Those with high money balances reduce their holdings more relative to those at the bottom end of the distribution. This tends to lower the average money balance. Hence, inflation acts as a progressive tax that reduces inequality of money holdings. This explains why in many heterogeneous-agent models, the welfare cost of inflation is often smaller than representative-agent models (Camera and Chien, 2014).[13]

In a random-matching, search-theoretic model of money, Molico (2006) shows that as inflation

[10] Menzio et al. (2013) focused on a purely analytical, zero-inflation equilibrium and did not offer a computational solution. Sun and Zhou (2018) did compute inflationary equilibria in their one-shot version of a competitive search market. However, they only have one lottery segment, or that the authors seemed to have only checked for or found just one in their setup. Generically, there may be more. Our method, which does not presume this and does not discretize the target value function, is more accurate and can find more than one possible lottery. As far as we know, this is a first in this literature, although similar computational strategies have been used in computational dynamic games (see Judd et al., 2003; Phelan and Stacchetti, 2001; Kam and Stauber, 2016).

[11] More detailed discussion of the model setup is in Section 2.

[12] One can also complicate Walrasian models with extensive margins to capture limited market participation, as in Alvarez et al. (2002). However, conditional on being in any market, agents within those markets are always trading. In competitive search models, some agents are participating and searching but may not find a match to trade.

[13] There are exceptions to this result due to the nature of idiosyncratic shocks, the financial structure and the sensitivity of labor supply to real wage changes. For example, Camera and Chien (2014) featured a reduced-form transaction technology that is free from inflationary effects, Erosa and Ventura (2002) showed that for sufficiently high returns to scale of the technology, inflation can become a regressive tax. In a different Bewley economy, where holding money is driven by precautionary motives, Wen (2015) showed that inflation can increase agents' consumption risks by tightening poorer agents' ad-hoc borrowing limits. There is also another class of models that combine these incomplete-markets and ex-post heterogeneity features with sticky price assumptions (see, e.g., Kaplan, Moll and Violante, 2018; Ravn and Sterk, 2020).

increases agents choose to pay more money in decentralized trades and a higher amount of money is paid per unit of the good.[14] This "real balance effect" can work against the redistributive effect of inflation. There is a similar effect in our model, but with a different underlying twist. In our setting with competitive search in decentralized trades, this is bolstered by the additional extensive margin effect: Higher inflation exacts a greater downside risk of not matching for agents by reducing the equilibrium matching probability for buyers. Although expected money carried in each decentralized trade will be lower per payment for goods, with lower equilibrium probability of matching, agents who get matched don't have to reduce consumption as much. This trade-off between matching probability and quantity of goods in the competitive search environment (see, e.g., Peters, 1984, 1991; Moen, 1997; Burdett et al., 2001; Julien et al., 2008; Shi, 2008) amplifies the speed at which agents expect to deplete their money in decentralized trades.

Chiu and Molico (2010) also have a notion of extensive margin, in the form of costly participation in centralized markets. In our setting, even without costly participation in markets, there is a non-trivial extensive margin. In Chiu and Molico (2010) and Rocheteau, Weill and Wong (2019), trading probabilities are fixed in decentralized-market meetings. This is due to their random matching assumption. In our setting, the extensive margin arises in the form of endogenous matching probabilities.

Jin and Zhu (2022) also consider the effect of long-run inflation in a random-matching model (Trejos and Wright, 1995; Shi, 1995). In their setting, agents can only hold indivisible amounts of money. Furthermore, the inflation policy in their framework is indirectly determined through an abstract fiscal tax-and-transfer scheme that is conditioned on individual wealth. In our setting, we take the anonymity of agents literally and do not presume that government policy has such superior informational advantage. Moreover, our setting is closer to the latest generation of monetary search models where goods and assets are divisible. This allows the models to be more amenable to empirical calibration and quantitative work.

Money and competitive search. In contrast to the Walrasian or random matching models discussed above, our Menzio et al. (2013) competitive search setup advances another channel to the standard taxonomy previously outlined.[15] The essence of their model is as follows: Suppose all agents begin in the economy as identical individuals. In one period, each agent has to make a decision whether to enter a centralized market (CM) to work and re-balance their money holdings, or to direct their search in a decentralized market (DM). In the search problem agents direct themselves to different trading posts with different terms of trade and matching probabilities. Firms anticipate that and create these trading posts accordingly. In equilibrium, there is an asymptotic distribution of heterogenous money holdings that must be consistent with the agents' different reponses in trading probabilities, payments and market participation rates. Menzio et al. (2013) characterize such an equilibrium in the special case of zero long-run inflation.

Our paper complements Menzio et al. (2013). We study how *inflation* drives agents' competitive search trade-offs, their endogenous market participations, and how this ties in with distributional outcomes. In our model, the responsiveness of agents in terms of their trading probabilities and quantities is endogenous. We show that because of the Menzio et al. (2013) heterogeneity in matching rates, there

[14] Chiu and Molico (2021) extend Molico (2006) to allow for aggregate shocks. The authors show, *inter alia*, that money has non-neutral and persistent real effects without imposing exogenous sticky-price assumptions.

[15] Another advantage of considering competitive search is that agents' decision problems are block recursive (as pointed out in Menzio et al., 2013). This means that agents' decision problems are recursively independent of the equilibrium distribution of assets. This has an accuracy benefit in practical computation—one does not have to *ad-hoc* parametrize aggregate wealth distribution as state variables when computing transitional dynamics. This point will be relevant to future extensions of this model which incorporate aggregate shocks.

is an opposing extensive margin effect that helps to mitigate the previously-discussed redistributive channel of inflation.[16] With higher inflation, agents are also spending faster in decentralized trades and entering the centralized market to rebalance their liquidity more frequently. Higher centralized-market participation implies that there are more agents with less money holding at the end of each period. These agents will enter the centralized market in the subsequent period. Also, there will be a smaller measure of agents at the upper end of the distribution since they top up with less liquidity in the centralized market and spend faster in the decentralized search market.

Endogenous matching probabilities via competitive search is not new (see, e.g., Rocheteau and Wright, 2009, 2005; Lagos and Rocheteau, 2005). What is different here is that the endogenous matching probabilities are heterogeneous in agent states and its distribution depends on inflation. This creates a nontrivial equilibrium, countervailing effect to what would be a traditional redistributive role of inflation. This is an important feature driving our non-monotone inequality results.[17] In addition to the introduction and study of inflation, we differ slightly from Menzio et al. (2013) by including quasi-linear utility of consumption and labor in the CM. This is done to enable a more flexible way to calibrate the model to data. It does not qualitatively alter the mechanism in Menzio et al. (2013).

Sun and Zhou (2018) also embed the competitive search market of Menzio et al. (2013) to study fiscal and monetary policy. However, there is a crucial difference between their model and Menzio et al. (2013), and thus our study of inflation here. They assumed away the endogenous duration of an agent's participation of the DM in the original Menzio et al. (2013) paper. Agents in their model can only stay for one period in the DM and must return to the CM (featuring quasilinear preferences) afterwards. Their model would be a version of the competitive search equilibrium in Rocheteau and Wright (2005) where there is a degenerate distribution of money and prices, if not for an assumption that agents in the CM draw an (i.i.d.) idiosyncratic income shock.[18] As a consequence, a one-shot and non-persistent dispersion in matching probabilities in Sun and Zhou (2018) is entirely buttressed by an assumption of exogenous heterogeneity in individual labor-supply productivities. In contrast, we follow Menzio et al. (2013) where ex-post heterogeneity arises in conjunction with equilibrium competitive-search dispersion in trading posts. This allows us to connect inflation policy to what we call the extensive margin underlying the distributional outcome, through agents' heterogeneous market participation and duration of such participation, and their transactions' speed.

The remainder of this paper is organized as follows. In Section 2, we set up and analyze a version of the model of Menzio et al. (2013) in a more general setting with non-zero inflation. In Section 3, we discuss our contribution in terms of a novel computational solution approach and also our calibration of the model. In Section 4, we conduct the main study on how inflation affects the equilibrium trade-offs that drive the model's distributional and welfare outcomes. We do so by comparing monetary equilibria under alternative long-run inflation policies. In section 5, we compute the welfare cost of inflation implied by this model. We conclude with Section 6.

[16] There is micro-level evidence of price dispersion driven by consumers directing their search to different stores and heterogeneity in the frequency of visiting different sellers (Kaplan and Menzio, 2015). In the competitive-search environment with heterogeneous agents, such observations are associated with the extensive margin of competitive search.

[17] In the Online Appendix A, we extend the model by introducing an exogenous probability α that ex-ante, each agent may go to the CM costlessly. With this additional feature, we show that our model can relate to two well-known models in the literature:the representative-agent random-matching model with competitive search DM of Rocheteau and Wright (2005), and the block-recursive ex-post heterogeneous agent model of Menzio et al. (2013). The main difference in one limit of our model (taking α to 1) to Rocheteau and Wright (2005) is that in Rocheteau and Wright (2005), some measure of households become sellers in the DM each period. In our setting, non-DM-buyer households are, in a sense, sellers only insofar as supplying labor to firms that create trading posts in the DM.

[18] Rocheteau et al. (2018) make the same remark on Sun and Zhou (2018).

2 The Model

The model builds on Menzio et al. (2013). Time is discrete and indexed by $t \in \mathbb{N}$. Hereinafter, we will denote $X := X_t$ and $X_{+1} := X_{t+1}$ for dynamic variables. There is one general good denoted by C. There are also I types of specific goods indexed by $i \in 1, 2, ..., I$, where $I \geqslant 3$. Agents in the economy consist of I types individuals, I types of firms, and a government that implements a (long-run) inflation target through controlling money-supply growth. There are measure one of each type i individuals, $i \in I$. An individual i consumes the general good C, the specific good i and produces good $i + 1$ (mod-$|I|$) as well as the general good C. For firms, each type i firm, $i \in I$, consists of a large number of firms. A type i firm produces type i good as well as the general good C. As in Menzio et al. (2013), firms are owned by the individuals through a balanced mutual fund.

There is a *centralized market* (CM) and a *decentralized market* (DM). The CM is a competitive Walrasian market where the individuals supply labor l, and, consume the general good C. As a result, they also manage their liquidity holding y to be carried into the following period. In the DM where the specific i goods are traded, we have a setting similar to Menzio et al. (2013). There is an information friction: Buyers of special DM goods are anonymous and cannot trade using private claims or contracts with selling firms. As a result, the only medium of exchange is money. For each type-i good, there is a continuum of submarkets indexed by the terms of trade $(x, q) \in \mathbb{R}^2_+$, where x is a real payment by a buyer and q is the quantity traded in exchange. Hereinafter, the explicit dependency on the type of good $i \in I$ will become unnecessary.[19] Each i-type firm commits to posted terms of trade in all submarkets it chooses to enter. Buyers of good i direct their search toward these submarkets that sell good i, by choosing the best terms of trade offered. However, as we will see, these buyers will have to balance their decision on terms of trades against the probability of getting matched. Since firms and buyers choose which submarket to participate in, a type i buyer will only participate in the submarkets where type i firms sell.

At any date, each individual decides which market—CM or DM—to participate in. An individual can only be in the CM or DM at a given time period. Firms operate in both CM and DM at the same time. Individuals demand money as a precaution against the need for liquidity in anonymous markets in the DM. A firm in the CM hires labor to produce the general CM good and the special DM goods. A type i firm hires labor service from type $i - 1$ (mod-$|I|$) individuals (in the CM spot labor market) and transforms it (linearly) into the same amount of DM good i.

Two features of the model give rise to market incompleteness: First, equilibrium matching in the DM (where money is essential) implies that agents face ex-ante uncertainty over being able to exchange and consume in those markets. Second, in the equilibria that emerge, there is endogenous limited participation in centralized markets. Since agents are anonymous in the DM, their individual risks are uninsurable: private state-contingent securities are not incentive feasible. Anonymity renders equilibrium value for money as a medium of exchange. Competitive search and matching with options to participate in the CM yields equilibrium-determined *ex-post* agent heterogeneity.

Figure 1 summarizes the timing of events and decisions between two arbitrary dates. Next, we detail the model primitives and decision problems.

[19]In Menzio et al. (2013), the CM is a spot labor market, and agents work to accumulate real money balances. Following Lagos and Wright (2005), we add the assumption that agents have a quasilinear preference in the CM—they have a strictly concave utility function over a CM good and a linear disutility of work. However, unlike Lagos and Wright (2005), or its variation with competitive search in Rocheteau and Wright (2005), we will have agent heterogeneity because agents are free to choose their participation in either the CM or DM. Our modification does not alter the theoretical aspect of the original model but renders it more useful for calibration later.

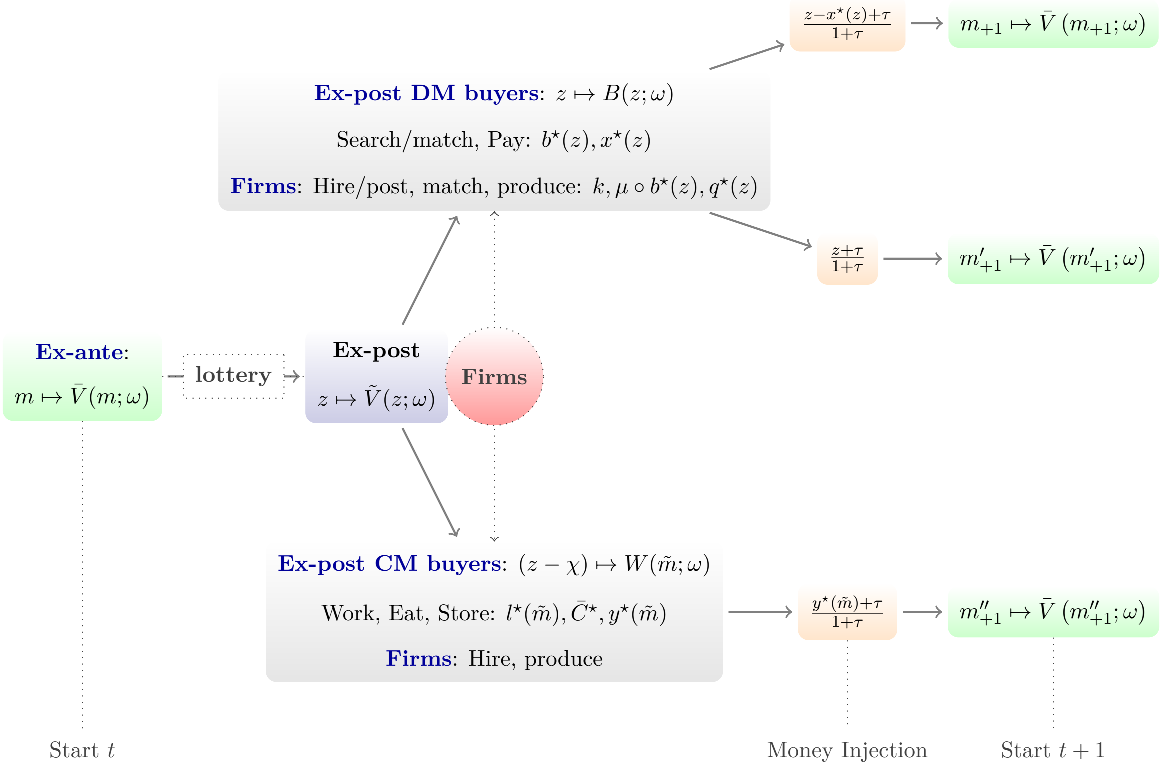

Figure 1: Timing, markets, outcomes

2.1 Individuals, matching and firms

2.1.1 Preference representation

The per-period utility function of an individual is

$$U(C) - h(l) + u(q). \tag{2.1}$$

where $U(C)$ is the utility of consuming the general good C, $h(l)$ is the disutility from supplying labor in the CM, and $u(q)$ is the utility of consuming the specific good in the DM. We assume that the functions U and u are continuously differentiable, strictly increasing, strictly concave, $U_1, u_1 > 0$, $U_{11}, u_{11} < 0$, and the following boundary conditions hold: $u(0) = U_1(\infty) = u_1(\infty) = 0$, and $u_1(0) < \infty$.[20] Also, we assume that $h(l) = l$. This simplifies the algebraic description of the CM decision problem and ensures that agents exiting the CM are identical.

2.1.2 Matching technology in the DM

Let $\theta \in \mathbb{R}_+$ denote the ratio of trading posts to buyers in a submarket—*i.e.*, its market tightness. In a submarket with tightness θ, the probability that a buyer is matched with a trading post is $b = \lambda(\theta)$. The probability a trading post is matched with a buyer is $s = \rho(\theta) := \lambda(\theta)/\theta$. We assume that the function $\lambda : \mathbb{R}_+ \to [0,1]$ is strictly increasing, with $\lambda(0) = 0$, and $\lambda(\infty) = 1$. The function $\rho(\theta)$ is strictly decreasing, with $\rho(0) = 1$, and $\rho(\infty) = 0$. We can re-write a trading post's matching probability $s = \rho(\theta) = \rho \circ \lambda^{-1}(b) \equiv \mu(b)$. Observe that the matching function μ is a decreasing function, and that $\mu(0) = 1$ and $\mu(1) = 0$. Assume that $1/\mu(b)$ is strictly convex in b.

[20] We will use the notational convention, $f_i(x_1, ..., x_n) \equiv \partial f(x_1, ..., x_n)/\partial x_i$, to denote the value of the partial derivative of a function f with respect to its i-th variable. Likewise, f_{ij} will denote its cross-partial derivative function with respect to the j-th variable.

2.1.3 Firms

Consider a firm of type $i \in I$ that takes the CM good's relative price p (in units of labor) as given. Following Menzio et al. (2013), we define labor as the numéraire good. The firm hires labor on the spot market to produce the CM good and the DM good. One unit of labor is transformed into one unit of CM good linearly. In the DM, a firm takes the market tightness function θ as given, and chooses the measure of trading posts (viz., shops) $\mathrm{d}N(x,q)$ to open in each submarket.[21] If x is what a matched buyer is willing to pay for q and $s(x,q) := \rho(\theta(x,q))$, then $x \cdot s(x,q)$ is the firm's expected revenue in submarket (x,q). To produce q the firm must hire $c(q)$ units of labor. Hence $s(x,q)c(q)$ is its expected labor wage bill in submarket (x,q). We assume that $q \mapsto c(q)$ is a continuous convex function. The firm also pays a per-period fixed cost k of creating the trading post in submarket (x,q).

The firm's profit is:

$$\pi(p;k) = \max_{Y \in \mathbb{R}_+} \{pY - Y\} + \max_{\mathrm{d}N} \int_{\mathbb{R}^2_+} \{s(x,q)[x - c(q)] - k\} \,\mathrm{d}N(x,q), \tag{2.2}$$

where N is a positive measure on the Borel σ-algebra $\mathcal{B}(\mathbb{R}^2_+)$. The first term on the RHS is the firm's value from operating in the CM. The second is its DM total expected value across all submarkets it chooses to operate in.

Note that the firms' problem above (and also agents' decision problems to be discussed below) do not explicitly depend on the aggregate distribution of agents. This is because of the nature of competitive search in the DM: Firms and buyers take matching probabilities as given when making their respective price posting and directed search decisions. The observed terms of trade posts and matching probabilities suffice to condition their decision processes. Moreover, CM preferences are quasilinear such that agents are identical at the end of the CM. We discuss this further in Section 2.4.1.

2.2 Money supply

At the start of each date t, the total stock of money in the economy M is known. We assume that M grows at a constant rate τ:

$$\frac{M_{+1}}{M} = 1 + \tau, \tag{2.3}$$

We assume $\tau > \beta - 1$, where β is the discount factor. New money stock τM is injected lump sum to all agents at the end of date t.

Since we define labor as the numéraire good, if we denote ωM as the current nominal wage rate, where ω is normalized nominal wage (*i.e.*, nominal wage rate per units of M), then a dollar's worth of money is equivalent to $1/\omega M$ units of labor. The variable ω will be endogenously determined in a monetary equilibrium.[22] If M is the beginning-of-period aggregate stock of money in circulation, then $1/\omega = M \times 1/\omega M$ is the beginning-of-period real aggregate (per-capita) stock of money, measured in units of labor.

Denote (equilibrium) nominal wage growth as $\gamma(\tau) \equiv \omega_{+1}M_{+1}/(\omega M)$. Later, for a stationary

[21] This is equivalent to stating that the firms post and commit to their terms of trade in the particular submarket, taking the probability of being matched with a buyer as given.

[22] In Menzio et al. (2013) the unique good traded in the perfectly-competitive Walrasian spot market is labor. Hence the authors' decision to define labor as the numéraire. We maintain their definition for ease of comparison.

monetary equilibrium, we will require that equilibrium nominal wage grows at the same rate as money supply, *i.e.*, $\gamma(\tau)|_{(\omega_{+1}=\omega)} = M_{+1}/M$.

2.3 Individuals' decisions

An individual is identified by her current money balance, m (measured in units of labor). Given policy τ, her decisions also depend on the aggregate wage ω. Denote the relevant state vector as $\mathbf{s} := (m, \omega)$.[23] At the beginning of a period (ex ante), an individual decides either to work and consume in the CM or to be a buyer in the frictional DM.[24] Next, we describe the decision problems of agents who are ex-post CM or DM buyers. We then describe an agent's ex-ante decision problem of choosing which of CM or DM to go to.

2.3.1 Ex-post CM buyers

Suppose now we have an individual $\mathbf{s} := (m, \omega)$ who begins the current period in the CM. The individual takes policy, τ, and the sequence of aggregate prices, $(\omega, \omega_{+1}, ...)$, as given. Her value from optimally consuming C, supplying labor l, and accumulating end-of-period money balance y, is

$$W(\mathbf{s}) = \max_{(C,l,y)\in\mathbb{R}^3_+} \left\{ U(C) - h(l) + \beta \bar{V}\left(\mathbf{s}_{+1}\right) : pC + y \leq m + l, \quad m_{+1} = \frac{\omega y + \tau}{\omega_{+1}\left(1+\tau\right)} \right\}, \quad (2.4)$$

where $\bar{V} : S \to \mathbb{R}$ is her continuation value function (see Section 2.3.3 on the following page). This continuation value function yields her next-period expected total payoff from state m_{+1} . The continuation state for the individual, m_{+1}, is derived as follows: At the end of the CM, the individual would have accumulated balance y (measured in units of labor). In current units of nominal money, this is $\omega M \times y$. At the beginning of next period, each individual gets a nominal transfer of new money τM. In units of labor next period, the beginning-of-period balance would thus be $m_{+1} = (\omega M y + \tau M) / (\omega_{+1} M_{+1})$. Replacing for M/M_{+1} with the money supply process in (2.3) gives the expression for the individual's continuation state m_{+1} in (2.4).

2.3.2 Ex-post DM buyers

Now we focus on an individual who has just decided to be a DM buyer. The buyer chooses which submarket (or trading post) (x, q) to enter, taking the market tightness function $(x, q) \mapsto \theta(x, q)$ as

[23] In a steady state equilibrium, ω is a constant.

[24] Our assumption here is different to that of Sun and Zhou (2018). In Sun and Zhou (2018), from CM, individuals choose whether to go into the DM submarkets. However, all DM individuals can only stay in the DM market for one period. They must then go to the CM. Thus, in their model, at the end of every period (a period consists of two sub-periods), agents would be identical since in the CM agent preferences are quasilinear. To avoid degeneracy in the agent distribution, Sun and Zhou (2018) introduce Bewley-style idiosyncratic shocks. They do so in terms of preference shocks—*i.e.*, labor supply shocks. In contrast, our model preserves non-degeneracy of the money distribution without an additional assumption of exogenous idiosyncratic shocks.

given. The individual buyer, $\mathbf{s} := (m, \omega)$, has initial value:[25]

$$B(\mathbf{s}) = \max_{x\in[0,m], q\in\mathbb{R}_+} \left\{ \beta \left[1 - b\left(x, q\right)\right] \left[\bar{V}\left(\frac{\omega m + \tau}{\omega_{+1}\left(1+\tau\right)}, \omega_{+1}\right)\right] \right. \\ \left. + \, b\left(x, q\right) \left[u(q) + \beta \bar{V}\left(\frac{\omega\left(m - x\right) + \tau}{\omega_{+1}\left(1+\tau\right)}, \omega_{+1}\right)\right] \right\}. \quad (2.5)$$

Consider the first two terms on the RHS of Equation (2.5): With probability $1 - b\left(x, q\right) := 1 - \lambda\left(\theta\left(x, q\right)\right)$ the buyer fails to match with the trading post and must thus continue the next period with his initial money balance subject to inflationary transfer. With the complementary probability $b\left(x, q\right) := \lambda\left(\theta\left(x, q\right)\right)$ he matches with a trading post (x, q), pays the seller x in exchange for a flow payoff $u\left(q\right)$, and then continues into the next period with his net balance, also subject to inflationary transfers.

2.3.3 Ex-ante

Given a money balance z, the individual decides which markets to participate in, and her value becomes

$$\tilde{V}\left(z, \omega\right) = \max_{a\in\{0,1\}} \left\{ aW\left(z, \omega\right) + \left(1 - a\right) B\left(z, \omega\right) \right\}. \quad (2.6)$$

As shown in Menzio et al. (2013), the resulting value function B in Equation (2.5) may not be strictly concave in m.[26] This is the case even if primitive functions are strictly concave. As a result, the value function $\tilde{V}$ may not be concave either.[27] This implies that agents can be weakly better off by choosing a lottery over the pure participation outcomes. Suppose at the beginning of a period, an agent begins with money balance m. If there is a non-empty subset $[z_1, z_2]$ containing m such that any weighted average of the pure-action induced values $\tilde{V}(z_1, \omega)$ and $\tilde{V}(z_2, \omega)$ (weakly) dominates $\tilde{V}(m, \omega)$, then the agent will optimally play a fair lottery $(\pi_1, 1 - \pi_1)$ over the prizes $\{z_1, z_2\}$. This yields the *ex-ante* value

$$\bar{V}\left(\mathbf{s}\right) = \max_{\pi_1\in[0,1], z_1, z_2} \left\{ \pi_1 \tilde{V}(z_1, \omega) + \left(1 - \pi_1\right) \tilde{V}\left(z_2, \omega\right) : \pi_1 z_1 + (1 - \pi_1) z_2 = m \right\}. \quad (2.7)$$

2.4 Monetary equilibrium

In this paper, we restrict attention to the case of a monetary equilibrium. Hereinafter, whenever we refer to "monetary equilibrium", or "equilibrium", we mean a (Markovian) monetary equilibrium—one

[25] Implicit in the DM-buyer's problem here is that the buyer sees which trading posts—each indexed by its posted terms of trade (x, q)—are open. In equilibrium, each buyer's problem must be consistent with firms' price-posting strategies (to be further discussed in Section 2.4).

[26] This is due to the bilinear and non-concave interaction between $b\left(x, q\right)$ and $u\left(q\right)$ in the DM-buyer's objective function in Equation (2.5). These two terms, respectively, give rise to an extensive margin (*i.e.*, how likely a buyer gets to trade) and an intensive margin (*i.e.*, how much of q to consume given a match). Lagos and Rocheteau (2005) first noted a similar mathematical issue in their monetary model. In our setting, there is a contemporaneous complementarity effect between the choice of b (or x) and q. In Lagos and Rocheteau (2005) the decision is sequential: conditional on the real money balance choice in a preceding CM, an agent chooses a search intensity upon entering a DM. Like our B function, their agent's consolidated objective function need not be concave: see Equation (18) and corresponding Lemma 2 in Lagos and Rocheteau (2005). Lagos and Rocheteau (2005) first applied lattice-programming techniques from Topkis (1998) to establish monotone comparative statics for their set of optimizers with respect to average search intensity and money growth.

[27] In Online Appendix C (Lemma 5), we show that B is continuous and increasing. Hence $\tilde{V}$ also inherits these properties.

in which agent's decision functions are time-invariant maps. In what follows, we first characterize the equilibrium strategy of firms (section 2.4.1), the equilibrium value and decision functions of agents in the CM (section 2.4.2) and in the DM (section 2.4.3), and then we describe the market clearing conditions in equilibrium (section 2.4.4). At the end of this section, we define formally the notion of a stationary monetary equilibrium (SME).

2.4.1 Equilibrium strategy of firms

A firm's problem is static. We can characterize the equilibrium behavior of a firm given p (in the CM). Free entry in the CM will render zero profits to firms in equilibrium, and thus, $p = 1$. Likewise, free entry and zero-profit in the DM with competitive search imply that

$$r(x,q) := s\left(\theta(x,q)\right)[x - c(q)] - k \leq 0, \qquad \text{and,} \qquad \theta(x,q) \geq 0, \tag{2.8}$$

where the weak inequalities would hold with complementary slackness: For a submarket (x,q) such that $r(x,q) < 0$, the firm optimally chooses not to post in the submarket. If $r(x,q) = 0$, then the firm is indifferent to creating different numbers of trading posts in submarket (x,q). We can also deduce that $r(x,q) > 0$ cannot be an equilibrium: If expected profit is positive, then this implies $\theta(x,q) = +\infty$, and thus $s\left(\theta(x,q)\right) = 0$ which yields a contradiction to the case.[28] We will restrict attention to an equilibrium where Equation (2.8) also holds for submarkets not visited by any buyer.[29]

From (2.8), we can deduce that

$$s(x,q) \equiv \mu \circ b\,(x,q) = \begin{cases} \frac{k}{x-c(q)} & \iff x - c(q) > k \\ 1 & \iff x - c(q) \leq k \end{cases}. \tag{2.9}$$

Observe that the firm's probability of matching with a buyer, $s(x,q) := \rho\left(\theta\left(x,q\right)\right)$ depends only on the posted terms of trade (x,q). Likewise, the buyer's probability of matching with a firm is $b(x,q) := \lambda\left(\theta\left(x,q\right)\right)$, for a given the matching technology $\mu : [0,1] \to [0,1]$. Thus, in any submarket with positive measure of buyers, the market tightness, $\theta(x,q) \equiv b(x,q)/s(x,q)$, is neccessarily and sufficiently determined by free entry into the submarket. Moreover, the terms of trade of a submarket (x,q) is sufficient to identify the submarket. This implies that firms' and agents' optimal decision processes do not depend on the equilibrium distribution of agents. They only depend on the distribution through the aggregate statistic ω as a result of inflation. The equilibrium will be (partially) block recursive.

In equilibrium, there is a relation between q and (x,b). That is, in any equilibrium, each active trading post will produce and trade the quantity:

$$q = Q(x,b) \equiv c^{-1}\left[x - \frac{k}{\mu(b)}\right], \tag{2.10}$$

[28] If we let $(x,q) \mapsto N(x,q)$ denote the equilibrium distribution of trading post across submarkets, condition (2.8) implies that aggregate profit in the DM is zero: $\int \{s(x,q)\,[x - c(q)] - k\}\, \mathrm{d}N(x,q) = 0$. This implies that expected profit of each firm is zero. Following Menzio et al. (2013, p. 2275), we assume that actual profit of each firm is also zero. The number of firm is finite and each firm creates a large number of trading posts so that the law of large numbers applies to each firm to ensure dimishing revenues and cost for each firm.

[29] Justification for this off-equilibrium-path restriction can be rationalized via a "trembling-hand" argument: Suppose there is some exogenous perturbation that induces an infinitesimally small measure of buyers to show up in every submarket. Given a non-zero measure of buyers present in a submarket, if firms' expected profit is still negative in that submarket, *i.e.*, $r(x,q) < 0$, then the market will not be active. This restriction is commonly used in the directed search literature (see, e.g., Menzio et al., 2013; Acemoglu and Shimer, 1999; Moen, 1997).

given payment x and its matching probability $s = \mu(b)$. This relation allows us to perform a change of variables, and re-write the buyers' problems below in terms choices over (x, b), instead of over (x, q).

2.4.2 Equilibrium CM individual

Denote $\mathcal{C}[0, \bar{m}]$ as the set of continuous and increasing functions with domain $[0, \bar{m}]$. Let $\mathcal{V}[0, \bar{m}] \subset \mathcal{C}[0, \bar{m}]$ be the set of continuous, increasing and concave functions on the domain $[0, \bar{m}]$.

Proposition 1 (CM value and decision functions). Assume $\tau/\omega < \bar{m}$. For a given sequence of prices $\{\omega, \omega_{+1}, ...\}$, the value function of the individual beginning in the CM, $W(\cdot, \omega)$, has the following properties:

1. $W(\cdot, \omega) \in \mathcal{V}[0, \bar{m}]$, *i.e.*, it is continuous, increasing and concave on $[0, \bar{m}]$. Moreover, it is linear on $[0, \bar{m}]$.

2. The partial derivative functions $W_1(\cdot, \omega)$ and $\bar{V}_1(\cdot, \omega_{+1})$ exist and satisfy the first-order condition

$$\frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left(\frac{\omega y^{\star}(m,\omega)+\tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right)\begin{cases}\leq 1, & y^{\star}(m,\omega) \geq 0 \\ \geq 1, & y^{\star}(m,\omega) \leq y_{\max}(\omega;\tau)\end{cases}, \tag{2.11}$$

   and the envelope condition:

$$W_1(m, \omega) = 1, \tag{2.12}$$

   where $y^{\star}(m,\omega) = m + l^{\star}(m,\omega) - C^{\star}(m,\omega)$, $l^{\star}(m,\omega)$ and $C^{\star}(m,\omega)$, respectively, are the associated optimal choices on labor effort and consumption in the CM.

3. The stationary Markovian policy rules $y^{\star}(\cdot, \omega)$ and $l^{\star}(\cdot, \omega)$ are scalar-valued and continuous functions on $[0, \bar{m}]$.

   (a) The function $y^{\star}(\cdot, \omega)$, is constant valued on $[0, \bar{m}]$.

   (b) The optimizer $l^{\star}(\cdot, \omega)$ is an affine and decreasing function on $[0, \bar{m}]$.

   (c) Moreover, for every (m, ω), the optimal choice $l^{\star}(m, \omega)$ is interior: $0 < l_{\min} \leq l^{\star}(m) \leq l_{\max}(\omega;\tau) < +\infty$, where there is a very small $l_{\min} > 0$ and $l_{max}(\omega) := y_{\max}(\omega;\tau) + U^{-1}(1) < 2U^{-1}(1) \in (0, \infty)$.

In Proposition 1, we provide an extension of the results of Menzio et al. (2013) on CM individuals' value and policy functions to the case with non-zero inflation. (Its proof is relegated to Online Appendix B. The proposition says the following: First, $W(\cdot, \omega) \in \mathcal{V}[0, \bar{m}]$ is continuous, increasing and concave on $[0, \bar{m}]$, and it is linear on $[0, \bar{m}]$. Second, the partial derivative functions $W_1(\cdot, \omega)$ and $\bar{V}_1(\cdot, \omega_{+1})$ exist and satisfy a first-order condition. Third, agents' optimal money balance and labor decision rules, respectively, $y^{\star}(\cdot, \omega)$ and $l^{\star}(\cdot, \omega)$ are scalar-valued and continuous functions on $[0, \bar{m}]$, and their selections are always interior. Also, for a fixed ω, the graph of $y^{\star}(\cdot, \omega)$ is downward sloping and that of $l^{\star}(\cdot, \omega)$ is constant-valued or flat. We also derive the equilibrium decisions of the CM agent.

2.4.3 Equilibrium DM buyer

Observe that since $\bar{V}(\cdot,\omega), W(\cdot,\omega) \in \mathcal{V}[0,\bar{m}]$ (*i.e.*, are continuous, increasing and concave), then by (A.1), $\bar{V}(\cdot,\omega) \in \mathcal{V}[0,\bar{m}]$. In an equilibrium, the DM buyer's problem in (2.5) can be re-written as

$$B(\mathbf{s}) = \max_{x\in[0,m],b\in[0,1]} \left\{ \beta(1-b)\left[\bar{V}\left(\frac{\omega m+\tau}{\omega_{+1}(1+\omega)},\omega_{+1}\right)\right] \right.$$
$$\left. + b\Big[u(Q(x,b)) + \beta\bar{V}\left(\frac{\omega(m-x)+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)\Big]\right\}. \quad (2.13)$$

It appears as if the buyer is choosing his matching probability b along with payment x. This is comes from a change of variables utilizing the equilibrium relation (2.10) between quantity q and terms of trade (x,b). From this we can begin to see that there will be a trade-off to the buyer, in terms of an extensive margin (*i.e.*, trading opportunity b), and, an intensive margin (*i.e.*, trading quantity given payment x).

The operator defined by (2.13) does not preserve concavity: The objective function in (2.13) is not jointly concave in the decisions (x,b) and state m, since it is bilinear in the function b and the value function $\bar{V}$, or the flow payoff function u. However, using lattice programming arguments, we can still show that the resulting DM buyers' optimal choice functions for (x,b), denoted by $(x^\star,b^\star)$, are monotone, continuous, and have unique selections.

Proposition 2 (DM value and decision functions). For a given price sequence $\{\omega,\omega_{+1},...\}$, the following properties hold.

1. For any $\bar{V}(\cdot,\omega_{+1}) \in \mathcal{V}[0,\bar{m}]$, the DM buyer's value function is increasing and continuous in money balances, $B(\cdot,\omega) \in \mathcal{C}[0,\bar{m}]$.
2. For any $m \leq k$, DM buyers' optimal decisions are $b^\star(m,\omega) = x^\star(m,\omega) = q^\star(m,\omega) = 0$, and $B(m,\omega) = \beta\bar{V}[\phi(m,\omega),\omega_{+1}]$, where $\phi(m,\omega) := (\omega m+\tau)/[\omega_{+1}(1+\tau)]$.
3. At any (m,ω), where $m \in [k,\bar{m}]$ and the buyer matching probability is positive $b^\star(m,\omega) > 0$*:*
   (a) The optimal selections $(x^\star,b^\star,q^\star)(m,\omega)$ and $\phi^\star(m,\omega) := \phi[m-x^\star(m,\omega),\omega]$, are unique, continuous, and increasing in m.
   (b) The buyer's marginal valuation of money $B_1(m,\omega)$ exists if and only if the marginal ex-ante value of money $\bar{V}_1[\phi(m,\omega),\omega]$ exists.
   (c) $B(m,\omega)$ is strictly increasing in m.
   (d) the optimal policy functions $b^\star$and $x^\star$, respectively, satisfy the first-order conditions

$$u\circ Q[x^\star(m,\omega),b^\star(m,\omega)] + b^\star(m,\omega)(u\circ Q)_2[x^\star(m,\omega),b^\star(m,\omega)]$$
$$= \beta\left[\bar{V}(\phi(m,\omega),\omega_{+1}) - \bar{V}(\phi^\star(m,\omega),\omega_{+1})\right], \quad (2.14)$$

and,

$$(u\circ Q)_1[x^\star(m,\omega),b^\star(m,\omega)] = \frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1[\phi^\star(m,\omega),\omega_{+1}]. \quad (2.15)$$

Proposition 2 states the properties of a DM agent's value and policy functions. It extends the results from Menzio et al. (2013) to the setting with non-zero inflation. The proof is given in the appendix, we summarize the idea of the proof: Part 1 of the proposition states that $B(\cdot, \omega) \in \mathcal{C}[0, \bar{m}]$ is increasing and continuous in m. This observation uses standard results from optimization and is contained in Lemma 1 of the appendix. Part 2 is proven in Lemma 2 in the appendix, and its insight here is simple: If buyers do not carry enough money to at least pay for a trading post's fixed cost, no firm will want to set up that post in equilibrium, and so the buyers get nothing. Part 3(a) is proven in Lemma 3 using the fact that a log-transform of a DM buyer's objective function is jointly concave in the choice variables (x, b), and is continuous in m (fixing the aggregate state ω). It nevertheless satisfies an increasing difference—and therefore, supermodularity—property. Thus, by lattice programming arguments, we can show that the DM buyer's optimal policies are increasing in m. Lemmata 4 and 5 in the appendix, together establish the following properties (Parts 3(b) and 3(c) of Proposition 2): Whenever a buyer has a chance of matching, her value function is differentiable. As a result, we can also characterize her best response in terms of a matching probability (extensive margin) and spending level (intensive margin) via Euler equations—see Part 3(d) in Proposition 2—and this is proven in Lemma 6.

2.4.4 Market clearing

In equilibrium, the total production of CM good equals its demand, $Y = C$. Given equilibrium policy functions, $x^\star$ and $b^\star$, the equilibrium distribution of money G, and wage ω, Equation (2.10) pins down market clearing conditions for each submarket in the set of equilibrium submarkets. Money demanded must also equal money supplied:

$$\frac{1}{\omega} = \int m \mathrm{d}G(m; \omega) > 0. \tag{2.16}$$

Since M is the beginning of period aggregate stock of money in circulation, the LHS of (2.16), $1/\omega = M \times 1/\omega M$, is the beginning of period real aggregate stock of money, measured in units of labor. The RHS of (2.16) is beginning of period aggregate demand, or holdings, of real money balances measured in the same unit.

2.5 Existence of a SME with a unique distribution

For the rest of the paper, we focus on a stationary monetary equilibrium (SME), which comprises the characterizations from Section 2.4, where prices are constant over time: $\omega = \omega_{+1}$.

Definition 1. A *stationary monetary equilibrium* (SME), given an exogenous monetary policy τ, is: **(i.)** a list of value functions $\mathbf{s} \mapsto (W, B, \bar{V})(\mathbf{s})$, satisfying the Bellman equation, jointly in (2.4), (2.5), (2.6) and (2.7); **(ii.)** a list of corresponding decision rules $\mathbf{s} \mapsto (l^\star, y^\star, b^\star, x^\star, q^\star, z^\star, \pi^\star)(\mathbf{s})$ supporting the value functions; **(iii.)** a market tightness function $\mathbf{s} \mapsto \mu \circ b^\star(\mathbf{s})$ given a matching technology μ, satisfying firms' profit maximizing strategy (2.9) and (2.10) at all active trading posts; **(iv.)** a wage rate function $\mathbf{s} \mapsto \omega(\mathbf{s})$ satisfying the money stock adding up condition (2.16); and **(v.)** an ergodic distribution of real money balances $G(\mathbf{s})$ satisfying an equilibrium law of motion

$$G(E) = T(G)(E) := \int P(\mathbf{s}, E) \mathrm{d}G(\mathbf{s}), \qquad \forall E \in \mathcal{B}(S), \tag{2.17}$$

where $\mathcal{B}(S)$ is the Borel σ-algebra generated by open subsets of the product state space S, and, $\mathbf{s} \mapsto P(\mathbf{s}, \cdot)$ is a Markov kernel induced by agents' best responses $(l^\star, x^\star, q^\star, z^\star, \pi^\star)$ and equilibrium matching $\mu \circ b^\star$.

In Online Appendix D we show that a composite Bellman function for each agent satisfies Banach's fixed point theorem. Then, from Propositions 1 and 2, we know that agents' decision functions are monotone and continuous. This implies that for a fixed ω, the equilibrium Markov operator on a current distribution of agents G is a monotone map and satisfies measurability conditions. This implies a monotone mixing property as a result of the equilibrium self-map (2.17) on the space of distributions G. These allow us to conclude that there is a unique fixed point (in a weak-convergence sense). Finally, we also show that there is at least one fixed point in the space of ω satisfying the SME conditions by utilizing the intermediate value theorem.[30]

Theorem 1. *There is a SME with a unique nondegenerate distribution G.*

3 Calibrating the SME for Analyses

Finding a SME requires numerical computation. In this section, we briefly comment on our contribution in terms of a novel computational solution method. Then, we calibrate the model to the US economy and describe the equilibrium properties of the benchmark model.

3.1 A novel computational scheme

Recall that the directed search problem makes the value function $\tilde{V}(\cdot,\omega)$ non-concave. Since there may exist lotteries that make agents better off than playing pure strategies over participating in DM or CM, we have to devise a means for finding these lotteries that convexify the graph of the function $\tilde{V}(\cdot,\omega)$. A common way to do this is to discretize the function's original domain of $[0,\bar{m}]$. Then, around each finite element of the domain, one must check if there is a linear segment that *approximately* convexifies *graph* $\left[\tilde{V}(\cdot,\omega)\right]$. This is prone to compounded errors, especially if the grid is coarse.[31] This approximation scheme works fine when we only have a lottery where the lower bound in the domain $[0,\bar{m}]$ is included, *i.e.*, a lottery on a set like $\{0,z_2\}$, where $z_2<\bar{m}$, as is the case of Sun and Zhou (2018).[32] However, it becomes less accurate when lotteries may exist on upper segments of the function, *i.e.*, lotteries on sets like $\{z_1',z_2'\}$, where $0<z_1'<z_2'<\bar{m}$, but we have no prior knowledge of what the lower bound z_1' is. When there is inflation, multiple lottery sets may arise, which will be discussed further.

Our computational contribution exploits the insight that $\tilde{V}(\cdot,\omega)$ has a bounded and convex domain, thus there exists a smallest convex set that contains its graph. This set is indeed graph $\left[\bar{V}(\cdot,\omega)\right]$, where $\bar{V}(\cdot,\omega)$ was defined in (2.7). The rest of the work then can be done by using a standard and robust convex-hull algorithm to back out a finite set of coordinates representing the convex hull, *i.e.*, graph $\left[\bar{V}(\cdot,\omega)\right]$. Given these points, we approximate the function $\bar{V}(\cdot,\omega)$ by interpolation on a chosen continuous basis function. We use the family of shape-preserving, linear B-splines. With

[30]Whether a SME is unique remains elusive to us as the frequency function $dG(m;\omega)$ does not admit a closed form expression in terms of known functions, and in general, it will also depend on the equilibrium candidate ω. This statement is also true for the original Menzio et al. (2013) setting, if the authors' model had money supply growth. With money supply growth, the authors cannot derive analytically how long it will take for DM-unmatched buyers' balances to get eroded by inflation before they have to go to work again. This makes their version of the frequency function $dG(m;\omega)$ with money supply growth not tractable. In contrast, the variation in Sun and Zhou (2018) admits an analytical form for $dG(m;\omega)$ and as a result they can show that there is a unique SME. This special result arises from their assumption that all types of agents in the DM must deterministically enter the CM *after one round* of trade (or no trade) in the DM. In their model, without an exogenous distribution of CM preference shocks, given the market timing assumptions, there would be no distribution of agents since preferences are quasilinear in their CM.

[31]We thank Amy Sun for sharing her `MATLAB` code which confirms this usage.

[32]This is also the case in dynamic contracting models like Clementi and Hopenhayn (2006).

this algorithm, we can very quickly and directly determine the entire set of possible lotteries that exists with an arbitrarily high precision, for any given non-convex/concave function $\tilde{V}(\cdot,\omega)$. For more details and the full algorithm for computing a SME, please see our Online Appendix E.

3.2 Statistical calibration

The CM and DM preference functions are, respectively,

$$U(C) - h(l) = \frac{C^{1-\sigma_{CM}}}{1-\sigma_{CM}} - l, \quad \text{and,} \quad u(q) = \bar{U}_{DM}\left[\ln\left(q+\underline{q}\right)\text{-}\ln\left(\underline{q}\right)\right],$$

where $\underline{q} = 1 \times 10^{-8}$. Following Menzio et al. (2013), the matching function specifies that a trading post's matching probability as a function of a buyer's matching probability is $\mu(b) = 1-b$.[33] Note that there is no parameter required for the DM production technology, *i.e.*, we had assumed that $c(q) = q$.

In Table 1, we list the parameters of the model. The values of parameters τ and β are externally pinned down, while the remaining parameters $\left(\sigma_{CM}, k, \bar{U}_{DM}\right)$ are jointly calibrated to match an empirical money demand curve (including its shift and elasticity) and labor hours statistics. We will now provide an intuitive explaination of our identification and calibration strategy.

Table 1: Benchmark estimates

Parameter	Value	Empirical Targets	Description
$1+\tau$	$(1+0.0089)^{1/4}$	Inflation rate[a]	Inflation rate
$1+i$	$(1+0.0385)^{1/4}$	3-month T-bill rate[a]	Nominal interest rate
β	0.99879	-	Discount factor, $\frac{(1+i)}{(1+\tau)}$
σ_{CM}	2	Aux reg. $(i, M/PY)^b$	CM risk aversion
k	0.003997	Aux reg. $(i, M/PY)^b$	Price-posting cost
$\bar{U}_{DM}$	407.77	Mean Hours $(\frac{1}{3})$	Preference scale

[a] Mean nominal interest and inflation rates in the data are annual.
[b] The auxiliary statistics (data) are from a spline function fitted to the data on annual observations of the (3-month T-bill) nominal interest rate (i) and Lucas-Nicolini New-M1-to-GDP ratio (M/PY).

External calibrations. The benchmark SME inflation rate τ is estimated by the sample mean of long-run (1915-2007) CPI inflation data obtained from FRED (CPIAUCNS). Given the sample mean of the three-month Treasury Bill rate (i) (sourced from the dataset of Lucas and Nicolini, 2015), we can pin down an estimate of the discount factor β using Fisher's *ex-post* relation: $\beta = (1+i)/(1+\tau)$.

Money demand: Identification and internal calibrations. In the model, we can map the taste parameter σ_{CM} from the observed aggregate money demand relationship. The risk aversion parameter σ_{CM} affects money demand through the individual money demand condition (see Equation (B.2) in our Online Appendix for detail), its related envelope condition embedded in marginal continuation value function $\bar{V}_1$, and through aggregation in the overall SME. From Theorem 1, $\bar{V}_1$ depends on

[33] This particular function is used in models of labor matching (see, e.g., Petrosky-Nadeau and Zhang, 2017; Hagedorn and Manovskii, 2008; den Haan et al., 2000), which is derived from the well-known telegraph- or telephone-line matching function (see, e.g., Stevens, 2007; Cox and Miller, 1965). Let $\mathscr{B}$ denote the number of buyers, $\mathscr{S}$ the number of sellers at a trading post and v the number of matches that has occurred. A buyer's matching probability is thus $b = v/\mathscr{B}$, while a seller's matching probability is $s = v/\mathscr{S}$. This implies that $\mathscr{B}/\mathscr{S} = s/b$. The number of matches in a telephone-line matching function is given by $v = \mathscr{B}\mathscr{S}/\left(\mathscr{B}^\rho + \mathscr{S}^\rho\right)^{1/\rho}$, where $\rho > 0$. From these, we can derive the relationship: $s \equiv \mu(b) = (1-b^\rho)^{1/\rho}$. In our robustness checks, we found that our results do not change qualitatively with ρ. Hence we chose not to calibrate ρ and instead normalized it as $\rho = 1$.

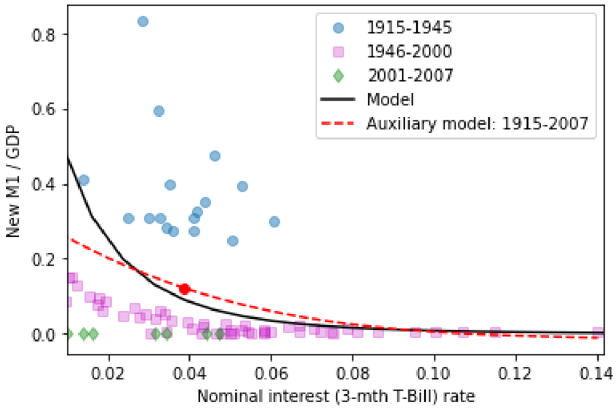

Figure 2: Lucas and Nicolini (2015) money demand annual data (1915-2007), model (green-dashed line) and auxiliary regression model target (red-dashed line). The red dot refers to the sample average for nominal interest.

probable ex-post DM or CM outcomes. Hence, *ex-ante* $\bar{V}_1$ depends on DM and CM preferences. That is, the CRRA parameter σ_{CM} influences equilibrium money demand.[34]

Likewise, from Equations (2.6), (2.10) and (2.13), the *ex-post* market participation problem depends on cost parameter k. In turn, this influences ex-post participation value function $\tilde{V}$. This feeds into $\bar{V}_1$ through the ex-ante lottery problem in Equation (2.7) and the optimal money demand condition (see Equation (B.2) in our Online Appendix for detail).

Since we focus on the long-run equilibrium, we calibrate the pair (σ_{CM}, k) to minimize the distance between the model-implied aggregate money demand relationship and an auxiliary (spline) money-demand model. The auxiliary model is fitted to long-run data from 1915 to 2007.[35] Our approach of using long-run data is similar in spirit to Lucas (2000).

Figure 2 depicts the model's aggregate money demand curve (solid black line), with a three-month Treasury-bill measure of the nominal interest rate (i) and the Lucas and Nicolini (2015) "New" M1-to-GDP ratio ($M1/PY$) on the horizontal and vertical axes, respectivly. The long-run data is shown as scatter points with various shapes: circles for pre-WWII observations, squares for post-WWII and pre-Great-Recesssion observations. The dashed line is the auxiliary, empirical money demand curve used as our target for indirectly estimating the model's money demand (solid curve). The scatter plots indicate that the empirical money demand has shifted in several regimes in the historical data (see also, Ireland, 2009). Following Lucas (2000), we can consider our approach as specifying a model-implied money demand curve that is a "halfway-house" between these different historical episodes. Indeed, from Figure 2, we can see that the solid curve (model) lies in between the various sub-samples and is close to the empirical (auxiliary) money demand curve.

Hours worked: Identification and internal calibrations. The preference scaling parameter $\bar{U}_{DM}$, which determines the relative size of DM and CM payoffs, is identified from empirically measured hours worked. In the model, $\bar{U}_{DM}$ is related to the marginal utility function U_1 via the individual

[34] Readers familiar with (Lagos and Wright, 2005)-type models might expect calibrations in terms of a corresponding CRRA parameter σ_{DM} instead of σ_{CM}. This is because in (Lagos and Wright, 2005)-type models, agents typically move in and out of the DM together and deterministically. In our setting, there is an individual choice on stochastic transitions between the two markets. Thus, there is a direct link from σ_{CM} to the characterization of money demand. We chose to normalize $\sigma_{DM} = 1$. In summary, it does not matter if we had freed up the DM utility-of-consumption CRRA σ_{DM} for calibration in lieu of σ_{CM}.

[35] The 2007 data measurement preceded the start of the Great Recession.

money demand and labor supply. (For details, see Equations (B.2) and (B.8) in the Online Appendix). Thus, $\bar{U}_{DM}$ influences individual optimal labor supply. Through SME, $\bar{U}_{DM}$ is identified from average labor hours, which is 0.33 of total available hours per person in the U.S. data.

It is worth noting that we do not target money or pricing distribution statistics in our calibration. However, our benchmark calibration implies an equilibrium price dispersion (standard deviation) statistic of 21.7 percent. A study by Kaplan and Menzio (2015), which uses price-scanner data in the U.S., found that their big-data sample of prices exhibits dispersion. Measured in terms of standard deviation, price dispersion in the data ranges from 19 percent (if goods are defined according to their universal product codes) to 36 percent (if goods are aggregated with different name brands and sizes). A generic-brand aggregation would imply a pricing distribution with about 21 percent in terms of standard deviation.[36]

3.3 Calibrated SME: Equilibrium functions

In Figure 3, we plot the SME value functions $\left(\tilde{V}, \bar{V}\right)$ in the benchmark economy. In the benchmark economy, our algorithm finds two lottery segments. We know that the graph of $W(\cdot, \omega)$ is that of an affine function. This is because the CM utility function is quasilinear, so that W is linear in m. The non-convex/concave value function for DM buyers is $B(\cdot, \omega)$. We do not plot these in the figure since the upper envelope of these two graphs give us $\tilde{V}(\cdot, \omega)$, the thick solid green line shown in the figure. (Note that due to relative scales, the lower segment or the affine part of $\tilde{V}(\cdot, \omega)$ attributable to $W(\cdot, \omega)$ may appear "flat" in the figure but it is actually an increasing affine graph.) Denote conv $\{\cdot\}$ as the convex-hull set operator. The solid magenta graph is the graph of $\bar{V}(\cdot, \omega)$ obtained through our convex-hull approximation scheme, once we have located all the intersecting coordinates between the set graph $\left[\tilde{V}(\cdot, \omega)\right]$ and the upper envelope of the set conv $\left\{\text{graph}\left[\tilde{V}(\cdot, \omega)\right], (0, 0), (\bar{m}, 0)\right\}$.

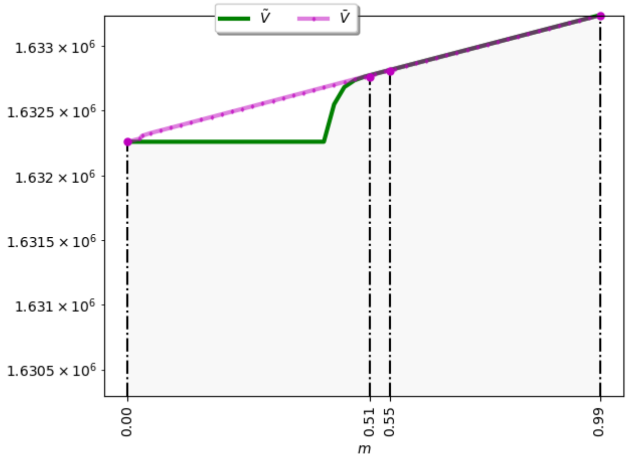

Figure 3: Value functions for benchmark economy.

Sustaining the equilibrium value functions are the policy functions $(l^\star, b^\star, x^\star, q^\star)$, and the lottery

[36] We do not calibrate the model to match the empirical distribution of consumers' money holdings. Our purpose in this paper is not to build a full-fledged quantitative model. Since the SME with inflation is not analytically tractable, we have to discipline the model by calibration to some relevant statistics. Pursuing the matching of statistics on pricing and money distributions may not be as relevant at this stage given the lack of bells and whistles—*i.e.*, additional exogenous shocks and (parametric) frictions—that are usually employed in the empirical pursuits of large-scale, quantitative heterogeneous-agent models. In this paper, the purpose is to take an intermediate step to study how the mechanism of Menzio et al. (2013)—*i.e.*, equilibrium agent behavior and distributions—responds under alternative inflationary settings. This question remains unanalyzed in the literature, and we need to do so before complicating the model in a quantitative direction.

policies $(\pi_1, 1-\pi_1)$ and $(\pi_1', 1-\pi_1')$ over the prize supports (z_1, z_2) and (z_1', z_2'), where $\pi_1(m,\omega) = (z_2 - m)/(z_2 - z_1)$ and $\pi_1'(m,\omega) = (z_2' - m)/(z_2' - z_1')$.

The other policy functions can be seen in Figure 4. Consider the panel depicting the graph of the CM labor supply function. As shown in Proposition 1, labor supply is affine and decreasing in money balance. There are three shaded patches in the Figure's panels. The darker (and narrowest) patch corresponds to the region where $m \in [0, k)$. In this region, an agent will never match nor trade in the DM. The orange patches (one of which overlaps the dark-red patch) are the regions of the agent's state space in which a lottery may be played—i.e, $[z_1, z_2]$ and $[z_1', z_2']$. What matters for each agent in the SME is then the loci of these policy functions outside of the orange patch, but including the points on its boundary. These will be consistent with the equilibrium's ergodic state space of agents. As shown in Proposition 2, the policy functions $(b^\star, x^\star, q^\star)$ are monotone in m in the relevant subspace where an agent can exist at any point in time. The relevant ergodic subspace of $[0, \bar{m}]$ in equilibrium is given by $\{z_1, [z_2, z_1'], [z_2', \bar{m}]\} = \{0, [0.52, 0.54], [0.98.., \bar{m}]\}$ in the benchmark economy in Figure 3 or Figure 4.

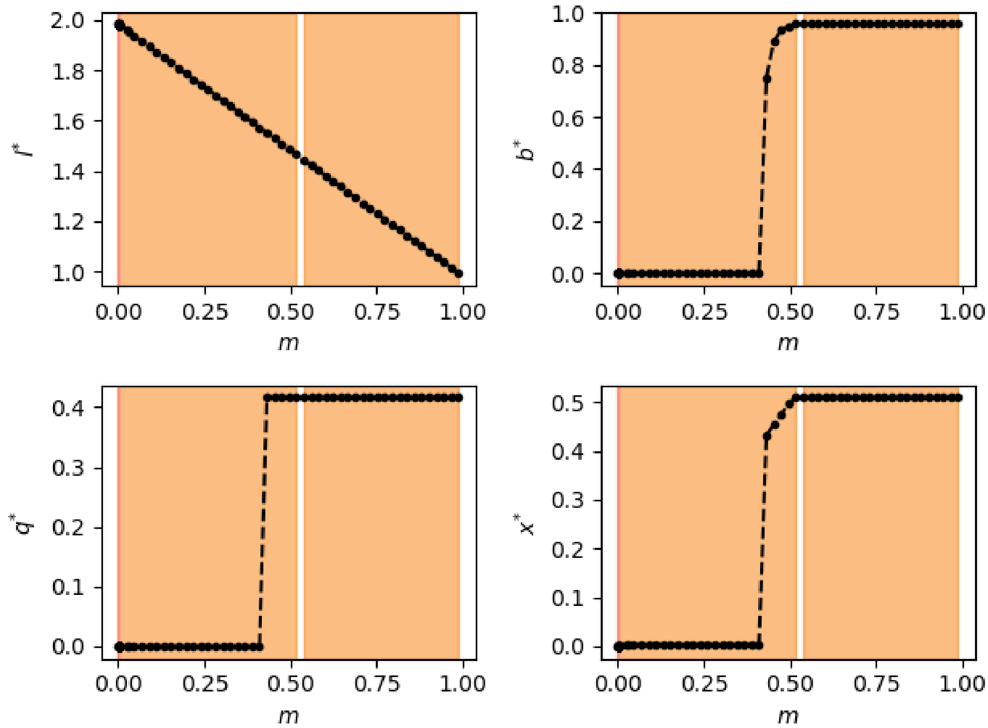

Figure 4: Markov policy functions in the benchmark economy.

Given the information about our benchmark SME's active or relevant agent state space and the corresponding policy functions, we can simulate an agent's outcomes and also compute the equilibrium distribution of real money holdings.[37] To do so, one may begin from any initial agent named (m, ω) and apply the decision rules computed earlier, as in Figure 4. Details of the algorithms for simulating the SME outcomes can be found in our Online Appendix F. We now proceed to discuss the equilibrium trade-offs faced by agents (*i.e.*, the model mechanism) in the next section.

[37] With competitive search, the domain of real money balances will be finite. Menzio et al. (2013) derive a unique closed-form for the graph of the distribution of money holdings, in the special case of zero inflation. When there is non-zero inflation, this becomes analytically intractable. We can numerically compute this, given agents' equilibrium policy functions.

4 Inflation, trade-offs and distribution

The results in our model are driven by a trade-off between the extensive margin of trading probability and the intensive margin of trade quantities. Consider the equilibrium description of firms' optimal DM production in (2.10). Given the firms' best responses in a SME, a potential DM buyer has the following trade-off: On one hand, faced with a given probability b of matching, the more a buyer is willing to pay, x, the more q she can consume. (This is the *intensive margin* of DM trade—*i.e.*, how much one can purchase.) On the other hand, given a required payment, x, a buyer who seeks to match with higher probability, b, tolerates consuming less q. (This is the *extensive margin* of DM trade—*i.e.*, trading opportunities.)

In this section, we discuss *how inflation affects individuals' equilibrium intensive- and extensive-margin trade-offs and in turn impacts on money holdings and pric*es. The effects of inflation on this well-known competitive search trade-off in heterogeneous-agent models are not yet well-understood in the literature. This question is what makes our adaptation of Menzio et al. (2013) to an inflationary setting interesting and worthwhile of study.

In Menzio et al. (2013) and this model, when there is non-zero inflation ($\tau \neq 0$), agents' decision rules will depend on τ through the equilibrium, aggregate statistic ω (per-dollar nominal wage). In turn, ω must be consistent with agents' decisions through a market clearing requirement in a SME. However, this also implies that there is no closed form characterization of the equilibrium distribution of money, nor its expression as some analytical function of τ. Unlike the purely block-recursive equilibrium feature under a zero-inflation setting in Menzio et al. (2013), with inflation there is now only a partially block recursive SME (see also a brief discussion in Menzio et al., 2013). We will develop our insights on the equilibrium behavior numerically, by disciplining our analysis around the calibrated economy.[38]

Consider an increase in the long-run inflation rate from 0% to 10% *per annum*. We have the following observations regarding behavior across the respective equilibria, denoted by SME($\tau = 0$) and SME($\tau = 10$).

CM labor supply (money demand) and inflation. First, we consider the effect of inflation on the intensive margin of decision in the CM. This is summarized by the CM agents' labor supply (which also implies money demand) response to inflation:

Observation 1. *Agents' optimal labor supply in the CM (demand for money) uniformly shifts down as inflation becomes higher. Average labor supply falls.*

[38] Partial block-recursivity under inflation also implies that we cannot perform an analytical comparative-static analysis across different τ-induced SMEs. This is also the case in Bewley-Aiyagari types of heterogeneous-agent models.

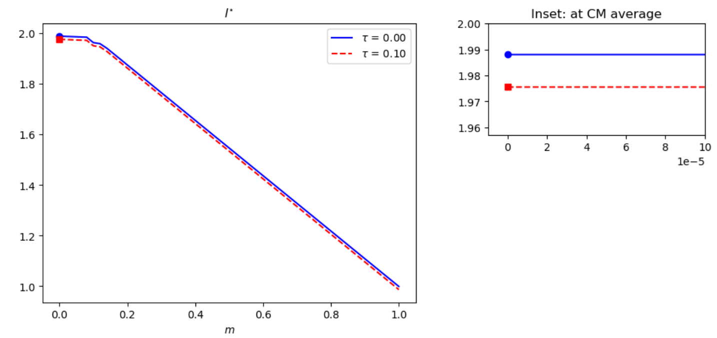

Figure 5: Labor supply falls with inflation. *Notes*: For reference, the circled-blue (squared-red) marker corresponds to the response of an agent with an average money balance under SME($\tau = 0$) (SME($\tau = 10$)). Those who work turn out to be the agents who have zero initial money balance. The inset figure zooms in to a subset of the graphs to emphasize the relative positions of the averages.

The solid-blue (dashed-red) graph in Figure 5 depicts the SME labor supply as a function of m when $\tau = 0\%$ ($\tau = 10\%$) *per annum*. For illustration, the circled-blue and squared-red markers, respectively, correspond to the labor supply responses of agents with an average money balance under SME($\tau = 0$) and SME($\tau = 10$). In both cases, the average is zero since the agents who work in the CM are those with zero initial money balances. The figure demonstrates that their labor supplies—and also their money demands—fall with inflation. That is, with higher inflation (tax) agents will tend to carry less money balances over time.

DM competitive-search extensive-intensive margins and inflation. Next we report the effects of inflation on the DM responses, where the extensive margin arises.

Observation 2. *As inflation becomes higher, DM-buyers' optimal matching probability response shifts down. The elasticity of matching probability with respect to money balance falls uniformly.*

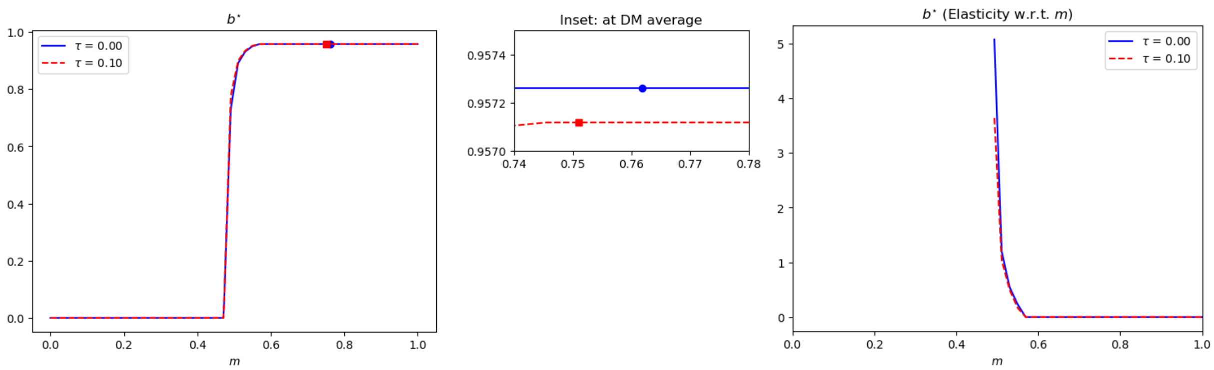

Figure 6: DM-buyers' matching probability and its elasticity with respect to money holdings shift down with higher inflation. *Notes*: For reference, the circled-blue (squared-red) marker corresponds to the response of an agent with an average money balance under SME($\tau = 0$) (SME($\tau = 10$)).

In both main panels of Figure 6, the solid-blue graphs correspond to SME($\tau = 0$) and the dashed-red ones are for SME($\tau = 10$). As in the previous figure, the circled-blue and squared-red markers (with inset figure) correspond to the matching-probability response of agents with an average money balance conditional on being in the DM, under SME($\tau = 0$) and SME($\tau = 10$), respectively. The uniform shift down in the matching probability function, in Figure 6 (*left panel*), is the *extensive margin* response to inflation: All else equal, higher inflation induces agents in equilibrium to face lower matching rates with DM trading posts. This margin, or force, also corresponds to a shift down in agents' m-wealth elasticity of matching probabilities in Figure 6 (*right panel*). That is, these agents would be optimally less m-wealth sensitive in their desired matching rates.

Now consider the *intensive margin* of competitive search:

Observation 3. *As inflation becomes higher, DM-buyers' optimal payments uniformly shift down and average payment* outcomes *fall.*

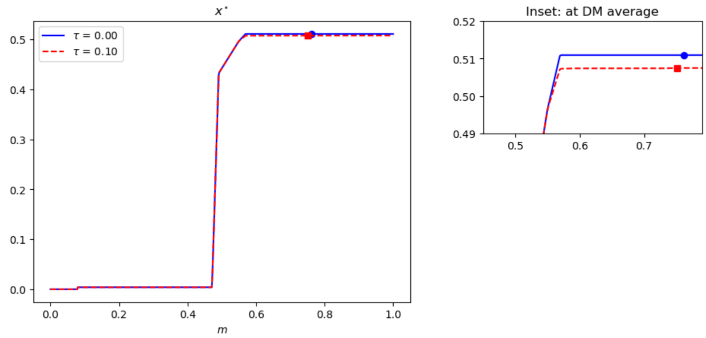

Figure 7: DM-buyers' payments schedule x and inflation. *Notes*: For reference, the circled-blue (squared-red) marker corresponds to the response of an agent with an average money balance under SME($\tau = 0$) (SME($\tau = 10$)).

However, from the previous observation, inflation also lowers money holdings for agents entering each DM. This turns out to be particularly the case for the average DM agents. The average DM agent ends up with an *outcome* of matching at a lower probability, offering less payment, and this corresponds to a less elastic matching probability with respect to money balance.

Consider the firms' side of the DM. In Figure 8 we compute the pricing function implied from the equilibrium behavioral functions for DM total payments in submarkets (x) and traded goods (q)—*i.e.*, $p := x/q$. Higher inflation causes the function p to uniformly shift up.

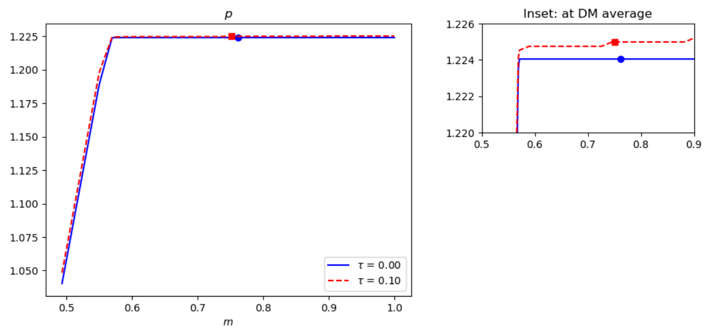

Figure 8: DM (submarkets) pricing function (p) and inflation.

DM speed of trading and CM liquidity-management participation. The last discussion is also connected to how fast agents expect to spend their monies in the DM. This will be tied to how often they re-enter the CM to manage their liquidity positions. To see this, we illustrate the implied expected transactions per dollar, $b \circ x/m$, under each inflationary equilbrium, SME($\tau = 0$) and SME($\tau = 10$).

Observation 4. *Agents trade faster: Individual per-dollar expenditure, bx/m, rises with inflation on average. High money balance agents are less sensitive in their speed of trading with respect to their money balance than low money balance agents.*

Although b and x uniformly fall with inflation (see the previous observation), each agent optimally would have a per-dollar expected expenditure response function $b \circ x/m$ that may rise or fall with inflation. This is shown in Figure 9 (*left panel*). However, on average they expect to be "trading faster" and offloading their money balance each time they expect to trade in the DM. Moreover, Figure 9 (*right panel*) shows that high-m agents are also less sensitive in their speed of trading with respect to m than low-m agents.

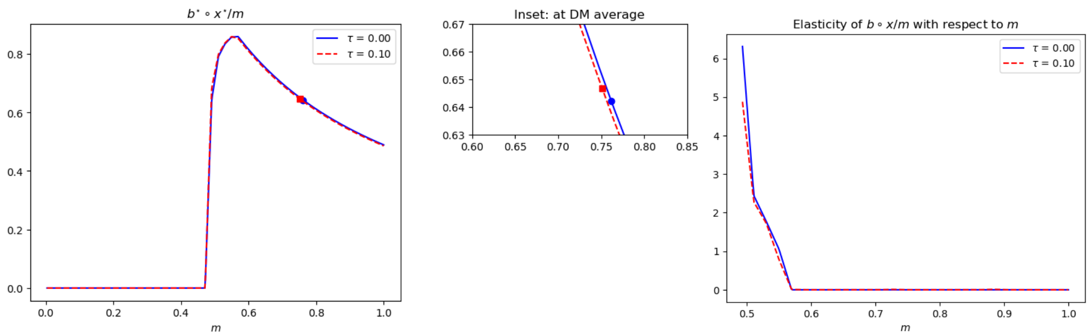

Figure 9: DM-buyers' (implied) expected transactions per dollar, $b \circ x/m$, and inflation. *Notes*: For reference, the circled-blue (squared-red) marker corresponds to the response or elasticity of an agent with an average money balance under SME($\tau = 0$) (SME($\tau = 10$)).

The equilibrium best responses above suggest to us a few key insights about agents at different money positions: At a given inflation rate, the higher money-balance ("rich") agents are less elastic with respect to money balance in terms of their matching probabilities and velocities of spending in the DM. With higher inflation, although average outcomes of DM buyer matching rates and payments become lower, the average outcome of DM speed of transactions become higher. (This finding is further reinforced by the observation that the top-10% of DM agents trade faster relative to the bottom-10% as inflation rise. See Figure 11.)

Intensive-extensive margin: Distributional effects of inflation. The previous discussion centers on equilibrium decision or allocation functions, as well as the responses of the average money-balance agents, conditional on them being in the CM or DM. We now consider agents across the entire distribution of money holding. In addition, instead of illustrating comparative equilibria using an equilibrium with 0% inflation and one with 10% inflation *per annum*, we now consider a set of economies ordered by different inflation at (just above) the Friedman rule to 10% *per* annum.

We now use the calibrated model to demonstrate how the intensive-versus-extensive-margin tension resolves, in the face of higher inflation. On the horizontal axes of the following figures, we increase the (quarterly) steady-state inflation rate, τ, within the set $(\beta - 1, 0.025]$. On the vertical axes, we measure relevant statistics for each corresponding economy under policy τ.

We begin with the distribution of agents in the DM (or the DM-conditional distribution of agents) since fundamental source of the trade-off arises in the DM. We shall see that the pattern induced by inflation on the distributions of outcomes here will also emerge in the aggregate distribution of agents' money holdings.

In Figure 10 (*top left and right panels*), the dashed and circled lines denote the bottom- and the top-ten percent of outcomes, of the respective SME(τ) distributions (conditional on agents in the DM), at each inflation rate. The "rich" face a slower decline in their total payments ($x^{\star}$) for DM goods relative to the "poor", with respect to higher inflation (Figure 10 (*top-left panel*)). Also, while matching probabilities of all buyers fall with inflation, the "rich" experience a slower decline in these probabilities relative to the "poor" (see Figure 10 (*bottom panel*)). Therefore, there is an increased dispersion in total payments and trading probabilities.

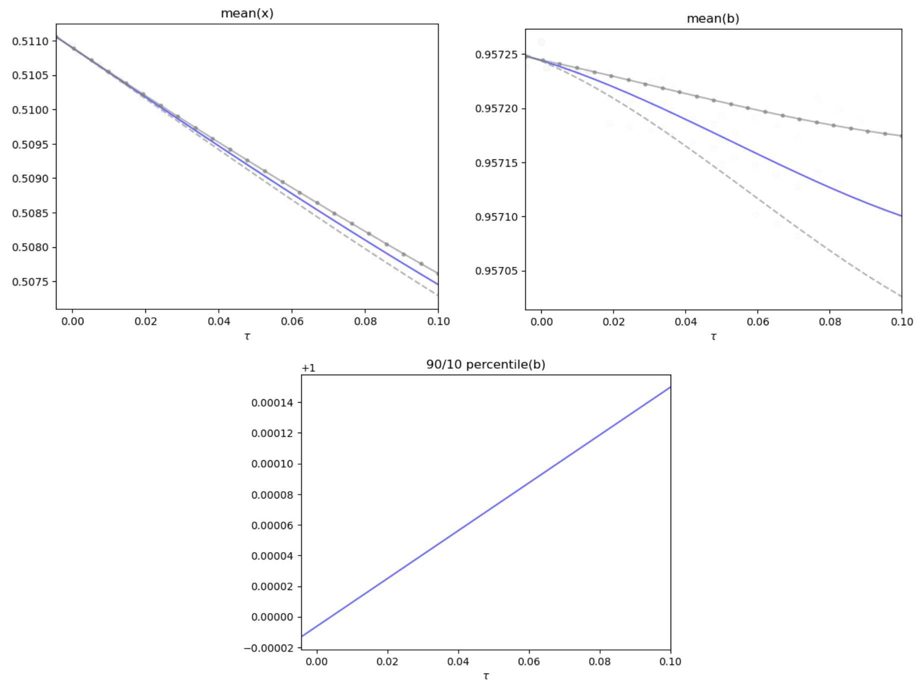

Figure 10: Buyer matching probabilities and quantities—Mean (top panels, solid), 90% (solid-dotted) and 10% (dashed) percentiles of DM-conditional distribution of agents.

Observation 5. *On average, agents end up paying less and matching at lower rates as inflation becomes higher. However, the decline of these outcomes with inflation is flatter for the "rich" agents than the "poor" agents. This induces the distribution of payments and matching risk in the DM to widen or be more dispersed.*

Another way to see the increased dominance of the extensive or trading-opportunity margin as inflation rises is as follows. Consider Figure 11. Since the probability of not getting matched, $1-b^{\star}(m)$, increases with inflation for all agents in the DM, this exacerbates the cost of holding money for DM buyers who are unmatched, especially those holding higher money balances. We observe that across the distribution of agents, matching probabilities $b^{\star}(m)$ decrease with inflation. However, what matters for DM agents is how quickly they can dispose of a given amount of liquidity they carry into each DM round to exchange for DM goods. Above, we introduced a useful summary statistic: the (average) payment in the DM across buyers, bx/m. This ratio increases with inflation. This means that within the DM-conditional distribution, the dispersions in matching rates, payments, and transaction velocities increase with inflation, with the “rich” agents facing a less steep decline in these outcomes than the “poor” agents. This symptom is the consequence of what we outlined above regarding the dominance of the DM extensive margin as inflation rises.

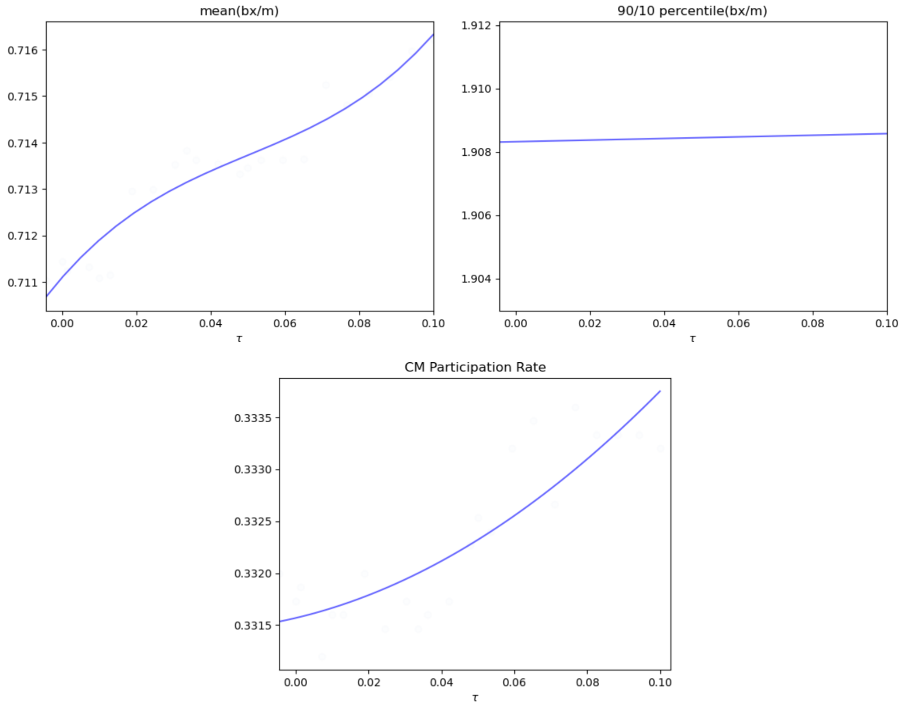

Figure 11: *Top-left*: On average, agents expect to have a higher per-dollar spending rate in the DM, *i.e.*, to “trade faster”. *Top-right:* Inflation tends to make the rich trade faster than the poor—the dispersion (90/10 ratio) in expected DM spending per-dollar rises with inflation. *Bottom*: Agents trade faster on average in DM and return to CM quicker.

At the distributional level, we again see the intensive-versus-extensive-margins tension through Figure 11 (*top-right panel*): as inflation rises, their average spending per dollar rises faster relative to the “poor”. A consequence of this is that agents would also go to the CM more often to manage their liquidity, as shown in Figure 11 (*bottom panel*). We summarize what we have thus far:

Observation 6. *On average, agents expect to have a higher per-dollar spending rate in the DM, i.e., to “trade faster”. The rich agents trade faster on average in the DM and return to the CM quicker as inflation rises.*

In summary, the extensive margin effect creates more dispersion in the matching, payment and speed-of-trading outcomes in the DM. This tends to work against the redistributive or compression effect of inflation. For low inflation, the latter dominates and for higher inflation, the former takes over. Figure 12 shows the effect of this tension on the DM-conditional distribution of money, across inflation regimes. We summarize this in the following observation:

Observation 7. *At low inflation rates there is non-monotonicity in the inequality (90/10 percentile-ratio) of money balances of DM agents as a function of inflation.*

Overall money and pricing distributions and inflation. We next show that the non-monotone inequality effect of inflation in the DM gets inherited in the overall (DM-and-CM) distribution of money and prices.

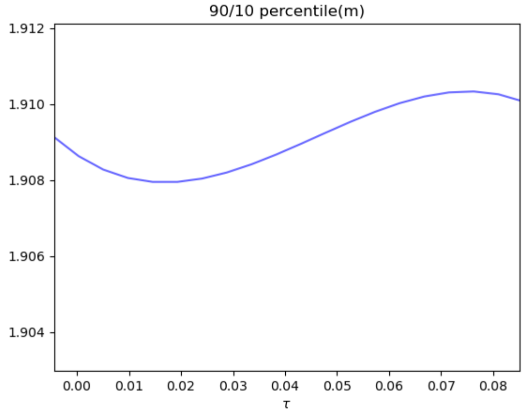

Figure 12: Inflation and DM-conditional money distributions' inequality statistic (ratio of 90-th to 10th percentile).

Figure 13 reports an alternative Gini measure for money holdings inequality for the overall distribution.[39] The green square marker in Figure 13 (*bottom panel*) denotes a reference SME at zero inflation, or at $\tau = 0$. The red diamond marker is at an SME with annual inflation of 10%.

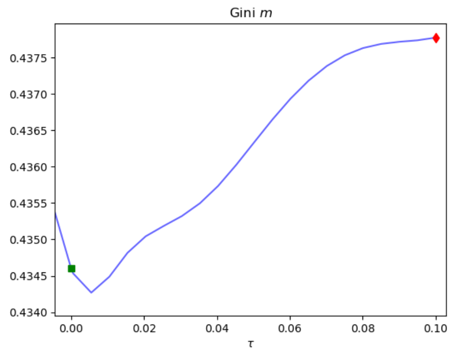

Figure 13: Inflation and the Gini coefficient for the overall money distribution.

Finally, a feature of competitive search in this Menzio et al. (2013)-type model is that there is potentially an equilibrium-determined dispersion in goods' terms-of-trade or pricing outcomes. This provides further motivation to study the effect of inflation in this Menzio et al. (2013) type of monetary, heterogeneous-agent model where agents are free to choose their participation in markets, which gives rise to endogenous trading opportunities. As illustrated in Figure 14, both the average price and price dispersion increase with each inflationary equilibrium.

[39]We use the Gini measure here since the overall distribution contains agents in the CM, and quantitatively, there are at least 10% of agents that hold zero money balances. (See Online Appendix H for two examples of the overall distribution.) As such, the 90/10 ratio is not well defined. Using the Gini measure also further reinforces the point that the U-shaped inequality feature from the DM gets inherited in the overall distribution of agents. We thank an anonymous referee for clarifying this detail.

We also note that we have experimented with arbitrarily high inflation beyond 10\% per annum. In such cases, the dispersion in money wealth will begin to fall as the economy tends towards a non-monetary equilibrium. At some point, the SME will break down: One would expect that as monetary equilibrium should disappear with hyperinflation. However, that is not the focus of this paper.

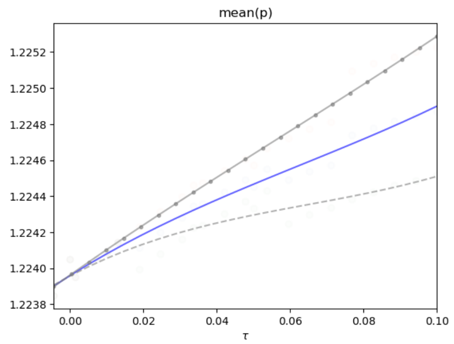

Figure 14: Inflation and price dispersion. Mean (solid), 90% (solid-dotted) and 10% (dashed) percentiles of prices.

Observation 8. *There is also a non-monotone inequality (Gini index) in money balances of all agents as a function of inflation. Prices increase and price dispersion rises with inflation.*

In summary, the extensive margin of search is an additional conduit for inflation to impact on the cross section of money holdings by affecting their heterogeneous matching probabilties. Unlike the results in earlier heterogeneous-agent monetary models (see, e.g., Imrohoroğlu and Prescott, 1991b; Erosa and Ventura, 2002), inflation may not necessarily be a redistributive tax that reduces (money) wealth inequality. With a trade-off between inflation tax on the intensive margin of allocations and inflation incentivizing agents to trade faster on the extensive margin, we get non-monotone distributional consequences—a U-shaped inflation-money-inequality relationship.

5 Inflation and welfare

We now turn to the traditional question of the welfare cost of inflation, from the calibrated model's perspective. We measure welfare as how much consumption equivalent variation (CEV) an ex-ante agent is willing to give up in order to move from a zero-inflation economy to a higher-inflation one. This CEV measure falls with inflation.[40]

Figure 15 shows that the welfare cost of inflation rises with inflation, for both average agents and other agents across the respective distributions. Consider the solid line in Figure 15: The *mean* welfare cost of moving the economy from zero (green-square marker) to ten percent (red-diamond marker) inflation *per annum* is about 0.83 percent of consumption loss (relative to the zero-inflation SME mean consumption outcome.

Cross-model welfare cost comparisons. In Table 2, we compare our model's welfare cost of inflation with some well-known studies in the literature. In representative-agent models such as Lucas (2000) and Lagos and Wright (2005) (which has additional bargaining frictions), the comparative-steady-state welfare cost of inflation can be quite high. However, this welfare cost tends to be lower

[40] An individual's *ex-ante* welfare is naturally measured as $Z_\tau := \int \bar{V}(m, \omega(\tau))\, \mathrm{d}G(m, \omega(\tau))$. Consider a reference equilibrium, SME(τ_0). Its corresponding individuals' *ex-ante* value is Z_{τ_0}. In an alternative economy SME(τ), the consumption equivalent variation (CEV) required to move the individual from the reference τ_0-economy to the τ-economy will be defined as

$$CEV(\tau) = \left[\frac{U^{-1}(Z_\tau)}{U^{-1}(Z_{\tau_0})} - 1\right] \times 100\%.$$

This individual-specific variation is measured in units of the CM good (*i.e.*, labor). In the comparisons, we set $\tau_0 = 0$, *i.e.*, the zero-inflation economy.

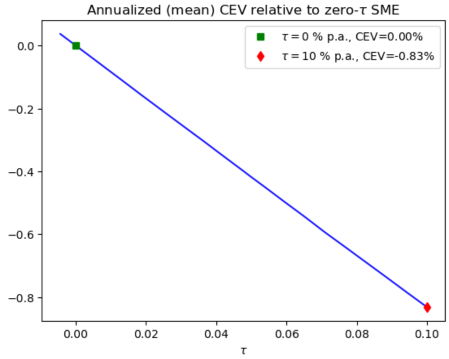

Figure 15: Mean welfare (CEV) falls for all types (0% to 10% inflation *p.a.*).

when one revisits the question in a heterogeneous-agent version of the models. It is well known that the redistributive margin of inflation tax is always present in heterogeneous-agent models. This margin tends to reduce the inefficiency of holding money in the presence of inflation (see, e.g., Camera and Chien, 2014; Kocherlakota, 2005; Erosa and Ventura, 2002). This is also the case in random-matching versions of such models (see, e.g., Chiu and Molico, 2010; Molico, 2006). In contrast, in heterogeneous-agent models such as Imrohoroğlu and Prescott (1991b), with more free parameters to govern frictions, one could obtain a welfare cost of inflation as high as 0.9 percent *per annum* in CEV terms.

Table 2: Welfare cost (CEV) from 0% to 10% (p.a.) inflation economy.

Economy	**Welfare Cost** (%)[a]	**Remarks**
Benchmark	**0.83 / 1.31**	static / transition
Imhoroğlu-Prescott (1991)	0.90	Bewley-CIA[b]-HA[d]
Chiu-Molico (2010)	0.41	RM[c]-HA[d]
Lagos-Wright (2005)	1.32	RM[c]-RA[e]-TIOLI[f]
Lucas (2000)	0.87	CIA[b]-RA[e]

[a] Annualized CEV cost (relative to zero-inflation economy)
[b] CIA: Cash-in-advance model
[c] RM: Random matching model
[d] HA: Heterogeneous agent model
[e] RA: Representative agent model
[f] TIOLI: Take-it-or-leave-it bargaining

We also calculate the (mean) welfare cost of inflation between a zero-inflation and a ten-percent-inflation SME, taking into account the effects of *transitional dynamics*. Figure 16 shows the transition of the aggregate state variable in terms of total money holdings (*left panel*) and its inverse statistic which is the model's real wage rate, ω (*right panel*). The vertical axes are measure in percentage deviations from the respective outcomes in the new or terminal SME. The economy is assumed to be in the initial SME(τ) where money supply growth rate is $\tau_{-1} = 0$ percent. At date $t = -1$, money supply growth rate jumps to $\tau'_{-1} = \tau' = 10$ percent *per annum*. The economy reacts in date $t = 0$ and takes some time to transit to the new SME(τ'). We use a standard shooting algorithm to compute the transition. Total welfare cost of inflation, along the transition is 1.31 percent of the initial SME's consumption, as summarized in Table 2 for our benchmark economy.

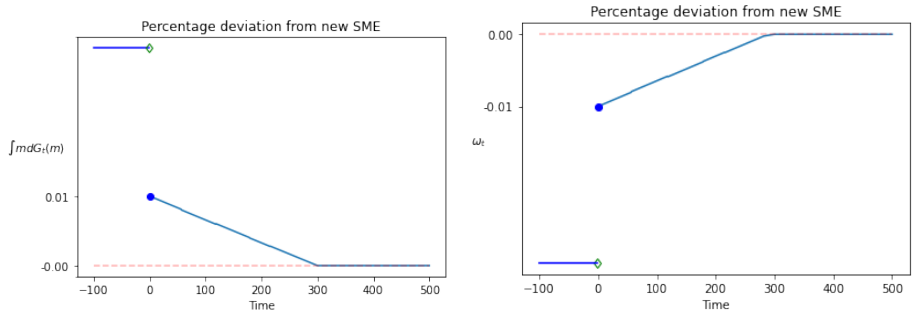

Figure 16: Transition from zero- to ten-percent-inflation SME. *Left*: Aggregate money. *Right*: Real wage rate.

6 Conclusion

How inflation drives trading-probability and spending intensity trade-off of individuals in heterogeneous-agent search models remained an open question. In such a setting due to Menzio, Shi and Sun (2013), we study the effect of long-run inflation on welfare and money-holdings inequality.

We show that the endogenous trade-off between these intensive and extensive margins culminate in a non-monotone effect of inflation on money-holdings inequality. The effect of inflation tax on liquid-wealth inequality is also non-monotone. Thus, the welfare cost of inflation in our model is sizable, despite the redistributive effect of inflation that tends to induce heterogenous-agent monetary models to produce lower costs of inflation, relative to their representative-agent counterparts.

In this paper, we focus on a single-asset, pure-currency economy in order to have a simple and clear understanding of the the effect of inflation on equilibrium extensive- and intensive-margins of trade-off of individuals and its distributional consequence. We think that if we allowed agents to hold additional illiquid assets (say, in the centralized markets), this may further exarcerbate the inequality result in our model. We are currently exploring this conjecture in an expanded setting with liquid and illiquid assets, and, further with aggregate dynamics.[41]

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[41] Since the framework renders agents' Markov decision processes independent from the aggregate distribution, but for an aggregate scalar statistic, we will have an exact Krusell and Smith (1998) sort of algorithm for computing equilibria. This is especially pertinent for the extended setting with aggregate dynamics, as the competitive search refinement means that the solution method can be made more efficient and more precise, than models where one has to brute-force approximate distributions when solving agent problems.

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Online Appendix

Inflation, Inequality, and Welfare in a Competitive Search Heterogeneous-agent Model

Omitted Proofs and Other Results

This document is also available from https://github.com/phantomachine/csm.

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A Extension and special cases

Consider a variation on the benchmark setting in the paper. In particular, suppose that each agent $\mathbf{s} := (m, \omega)$ has an initial value each period as

$$\bar{V}(\mathbf{s}) = \alpha W(\mathbf{s}) + (1-\alpha) V(\mathbf{s}), \tag{A.1}$$

where $V(\mathbf{s})$ is the value of playing a fair lottery $(\pi_1, 1-\pi_1)$ over the prizes $\{z_1, z_2\}$, *i.e.*,

$$V(\mathbf{s}) = \max_{\pi_1 \in [0,1], z_1, z_2} \left\{ \pi_1 \tilde{V}(z_1, \omega) + (1-\pi_1) \tilde{V}(z_2, \omega) : \pi_1 z_1 + (1-\pi_1) z_2 = m \right\}; \tag{A.2}$$

is a natural upper bound on CM saving (in real money balances).

The difference between (A.1) and (A.2) and their respective counterparts in (2.7) and (2.6) in Section 2.3.3 on page 11 of the paper, is that there is a measure α of agents who will participate in the CM for sure each period.

We have the following cases:

1. When $\alpha = 0$, we recover the model presented in the main paper.
2. When $\alpha = 0$, $U(C) = 0$ for all C, and the labor utility function $h(l)$ is strictly convex, we recover the original Menzio et al. (2013) model as a special case.
3. Note that when $\alpha = 1$ (*i.e.*, agents get to enter the CM deterministically), and the continuation value from CM is a convexification of $B(\cdot, \omega)$, our model becomes a version of Rocheteau and Wright (2005) with competitive search markets without agent and price dispersion.

All proofs (to results in the paper) below are written with the more general case of $\alpha \in [0, 1)$ in mind.

B CM individual's problem

Preliminaries. Consider the feasible choice set for CM saving, y: If $\bar{m}$ is an upper bound on end-of-period balance plus transfer (measured in units of labor), then this gives the bounds on end-of-period money balance plus transfer, in current money value, as:

$$\tau M \leq \omega M y + \tau M \leq \omega M \bar{m},$$

where ωM is current nominal wage. Since there is inflation in nominal wage, then next-period initial balance is current end-of-period nominal money balance normalized by the next period nominal wage $M_{+1}\omega_{+1}$, *i.e.*,

$$\frac{\tau M}{\omega_{+1} M_{+1}} \leq m_{+1} \equiv \frac{\omega M y + \tau M}{\omega_{+1} M_{+1}} \leq \frac{\omega M \bar{m}}{\omega_{+1} M_{+1}}.$$

Using (2.3), we can re-write the above bounds as

$$0 \leq y \leq y_{\max}(\omega; \tau) := \bar{m} - \frac{\tau}{\omega},$$

which applies in the pair of KKT complementary slackness conditions B.2 on page OA.B-3.

The upper bound on real, end-of-period money holdings, $\bar{m} \in (0,\infty)$, can be derived as:

$$0 < \bar{m} < (U_1)^{-1}(1) \iff 1 < U_1(\bar{m}) < U_1(0). \tag{B.1}$$

Later, in Part 3 of this section, we show that in any equilibrium $U'(C) = 1$. Assuming $U'(C) = 1 < U_1(\bar{m})$ suffices. Intuitively, this permits an agent to accumulate real balances at the end of each period beyond the level of optimal CM consumption $C^\star = U'^{-1}(1)$. Below, we show that having $\bar{m} \in (0,\infty)$ will ensure that there is always an optimal, largest labor effort that is always finite and that in all dates, money balances are bounded.

The following gives the proof of Proposition 1 in the paper.

Proposition 1. *Assume $\tau/\omega < \bar{m}$. For a given sequence of prices $\{\omega, \omega_{+1}, ...\}$, the value function of the individual beginning in the CM, $W(\cdot,\omega)$, has the following properties:*

1. $W(\cdot,\omega) \in \mathcal{V}[0,\bar{m}]$, i.e., *it is continuous, increasing and concave on $[0,\bar{m}]$. Moreover, it is linear on $[0,\bar{m}]$.*

2. *The partial derivative functions $W_1(\cdot,\omega)$ and $\bar{V}_1(\cdot,\omega_{+1})$ exist and satisfy the first-order condition*

$$\frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left(\frac{\omega y^\star(m,\omega)+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)\begin{cases}\leq 1, & y^\star(m,\omega) \geq 0\\ \geq 1, & y^\star(m,\omega) \leq y_{max}(\omega;\tau)\end{cases}, \tag{B.2}$$

and the envelope condition:

$$W_1(m,\omega) = 1, \tag{B.3}$$

where $y^\star(m,\omega) = m + l^\star(m,\omega) - C^\star(m,\omega)$, $l^\star(m,\omega)$ and $C^\star(m,\omega)$, respectively, are the associated optimal choices of labor effort and consumption in the CM.

3. *The stationary Markovian policy rules $y^\star(\cdot,\omega)$ and $l^\star(\cdot,\omega)$ are scalar-valued and continuous functions on $[0,\bar{m}]$.*

   (a) *The function $y^\star(\cdot,\omega)$ is constant valued on $[0,\bar{m}]$.*

   (b) *The optimizer $l^\star(\cdot,\omega)$ is an affine and decreasing function on $[0,\bar{m}]$.*

   (c) *Moreover, for every (m,ω), the optimal choice $l^\star(m,\omega)$ is interior: $0 < l_{\min} \leq l^\star(m) \leq l_{\max}(\omega;\tau) < +\infty$, where there is a very small $l_{min} > 0$ and $l_{max}(\omega) := y_{max}(\omega;\tau) + U^{-1}(1) < 2U^{-1}(1) \in (0,\infty)$.*

Proof. (*Part* 1). Consider the individual's problem beginning in the CM (2.4). Since $U_1(C) > 0$ for all C, the budget constraint always binds. Thus we can re-write (2.4) as

$$W(\mathbf{s}) = \max_{(C,y)\in\mathbb{R}_+\times[0,\bar{m}]}\left\{U(C) - [pC + y - m] + \beta\bar{V}\left(\frac{\omega y+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)\right\}. \tag{B.4}$$

Let

$$(C^{\star}, y_c^{\star})(m,\omega) \in \underset{(C,y)\in\mathbb{R}_+\times[0,\bar{m}]}{\arg\max} \left\{ U(C) - [pC + y - m] + \beta \bar{V}\left(\frac{\omega y + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right)\right\}. \tag{B.5}$$

From (B.4), it is clear that $W_1(\cdot,\omega)$ exists on $[0,\bar{m}]$, and moreover, we have the envelope condition $W_1(\cdot,\omega) = 1 > 0$. This implies that the value function $W(\cdot,\omega)$ is continuous, increasing and concave in m. Moreover it is affine in m.

(*Part* 2). First, we make the following observations: Since U is strictly concave in C, the objective function is strictly concave in C. Moreover, the objective function on the RHS of (B.4) is continuously differentiable with respect to C. The optimal decision, $C^{\star}(m,\omega)$ satisfies the following Karush-Kuhn-Tucker (KKT) conditions:

$$U_1(C) \begin{cases} = p, & C > 0 \\ < p, & C = 0 \end{cases}. \tag{B.6}$$

In an equilibrium, $p > 0$ will be finite—in fact, $p = 1$. Therefore, $C^{\star}(m,\omega) \equiv \bar{C}^{\star} = (U_1)^{-1}(1)$ is a finite and non-negative constant. Thus, we only have to verify that the optimal decision correspondence, given by $l_c^{\star}(m,\omega) \equiv p\bar{C}^{\star} + y_c^{\star}(m,\omega) - m$ at each (m,ω), exists and is at least a compact-valued and upper-semicontinuous (*usc*) correspondence: Fixing $C = \bar{C}^{\star}$, the objective function on the RHS of (B.4) is continuous and concave on the compact choice set $[0,\bar{m}] \ni y$. By Berge's Maximum Theorem, the maximizer $y_c^{\star}(m,\omega)$, or $l_c^{\star}(m,\omega)$, is compact-valued and *usc* on $[0,\bar{m}]$. (After we further establish that the derivative $V_1(\cdot,\omega_{+1})$ exists, we show below that it is constant and single-valued with respect to m.)

Second, we take a detour and show that the derivative $\bar{V}_1(\cdot,\omega_{+1})$ exists. This will be used later to characterize a first-order condition with the respect to y. The results below relies on the observation that since $V(\cdot,\omega_{+1})$ is a concave, real-valued function on $[0,\bar{m}]$, it has right- and left-hand derivatives (see, e.g., Rockafellar, 1970, Theorem 24.1, pp.227-228). Fix $C^{\star}(m,\omega) \equiv \bar{C}^{\star}$. Since $y_c^{\star}(m,\omega)$ is *usc* on $[0,\bar{m}]$, then for all $\varepsilon \in [0,\delta]$, and taking $\delta \searrow 0$, there exists a selection $y^{\star}(m-\varepsilon,\omega) \in y_c^{\star}(m-\varepsilon,\omega)$ feasible to a CM agent m. Similarly, there is a $y^{\star}(m,\omega) \in y_c^{\star}(m,\omega)$ that is feasible to a CM agent $m - \varepsilon$. Moreover, if $l^{\star}(m,\omega) \in l_c^{\star}(m,\omega)$ is an optimal selection associated with $y^{\star}(m,\omega)$, then for an agent at m,

$$\begin{aligned} W(m,\omega) &= \underbrace{U\left(\bar{C}^{\star}\right) - l^{\star}(m,\omega) + \beta\bar{V}\left[\frac{\omega\left[m + l^{\star}(m,\omega) - \bar{C}^{\star}\right] + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right]}_{\equiv Z[m,y^{\star}(m,\omega)]} \\ &\geq \underbrace{U\left(\bar{C}^{\star}\right) - l^{\star}(m-\varepsilon,\omega) + \beta\bar{V}\left[\frac{\omega\left[m + l^{\star}(m-\varepsilon,\omega) - \bar{C}^{\star}\right] + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right]}_{\equiv Z[m,y^{\star}(m-\varepsilon,\omega)]}; \end{aligned}$$

and, for an agent at $m-\varepsilon$,

$$\begin{aligned}
&W\left(m-\varepsilon,\omega\right)\\
&\underbrace{=U\left(\bar{C}^{\star}\right)-l^{\star}\left(m-\varepsilon,\omega\right)+\beta\bar{V}\left[\frac{\omega\left[\left(m-\varepsilon\right)+l^{\star}\left(m-\varepsilon,\omega\right)-\bar{C}^{\star}\right]+\tau}{\omega_{+1}\left(1+\tau\right)},\omega_{+1}\right]}_{\equiv Z\left[m-\varepsilon,y^{\star}\left(m-\varepsilon,\omega\right)\right]}\\
&\qquad\qquad\underbrace{\geq U\left(\bar{C}^{\star}\right)-l^{\star}\left(m,\omega\right)+\beta\bar{V}\left[\frac{\omega\left[\left(m-\varepsilon\right)+l^{\star}\left(m,\omega\right)-\bar{C}^{\star}\right]+\tau}{\omega_{+1}\left(1+\tau\right)},\omega_{+1}\right]}_{\equiv Z\left[m-\varepsilon,y^{\star}\left(m,\omega\right)\right]}.
\end{aligned}$$

Rearranging these inequalities, we have the following fact:

$$\begin{aligned}
&\frac{Z\left[m,y^{\star}\left(m-\varepsilon,\omega\right)\right]-Z\left[m-\varepsilon,y^{\star}\left(m-\varepsilon,\omega\right)\right]}{m-\left(m-\varepsilon\right)}\\
&\qquad\leq\frac{W\left(m,\omega\right)-W\left(m-\varepsilon,\omega\right)}{m-\left(m-\varepsilon\right)}\leq\frac{Z\left[m,y^{\star}\left(m,\omega\right)\right]-Z\left[m-\varepsilon,y^{\star}\left(m,\omega\right)\right]}{m-\left(m-\varepsilon\right)},
\end{aligned}$$

which, after simplifying the denominator and taking limits, yields:

$$\begin{aligned}
&\lim_{\varepsilon\searrow0}\left\{\frac{Z\left[m,y^{\star}\left(m-\varepsilon,\omega\right)\right]-Z\left[m-\varepsilon,y^{\star}\left(m-\varepsilon,\omega\right)\right]}{\varepsilon}\right\}\\
&\leq\lim_{\varepsilon\searrow0}\left\{\frac{W\left(m,\omega\right)-W\left(m-\varepsilon,\omega\right)}{\varepsilon}\right\}\leq\lim_{\varepsilon\searrow0}\left\{\frac{Z\left[m,y^{\star}\left(m,\omega\right)\right]-Z\left[m-\varepsilon,y^{\star}\left(m,\omega\right)\right]}{\varepsilon}\right\}\\
&\Longleftrightarrow\\
&\beta\lim_{\varepsilon\searrow0}\left\{\frac{\bar{V}\left[\frac{\omega\left(m+l^{\star}\left(m-\varepsilon,\omega\right)-\bar{C}^{\star}\right)+\tau}{\omega_{+1}\left(1+\tau\right)},\omega_{+1}\right]-\bar{V}\left[\frac{\omega\left(m-\varepsilon+l^{\star}\left(m-\varepsilon,\omega\right)-\bar{C}^{\star}\right)+\tau}{\omega_{+1}\left(1+\tau\right)},\omega_{+1}\right]}{\varepsilon}\right\}\leq W_1\left(m,\omega\right)\\
&\leq\beta\lim_{\varepsilon\searrow0}\left\{\frac{\bar{V}\left[\frac{\omega\left(m+l^{\star}\left(m,\omega\right)-\bar{C}^{\star}\right)+\tau}{\omega_{+1}\left(1+\tau\right)},\omega_{+1}\right]-\bar{V}\left[\frac{\omega\left(m-\varepsilon+l^{\star}\left(m,\omega\right)-\bar{C}^{\star}\right)+\tau}{\omega_{+1}\left(1+\tau\right)},\omega_{+1}\right]}{\varepsilon}\right\}.
\end{aligned}$$

Since, from (B.4), $W\left(\cdot,\omega\right)$ is clearly differentiable with respect to m, the second term in the inequalities above is equal to the partial derivative $W_1\left(m,\omega\right)$, which is constant. As $\varepsilon\searrow0$, there is a selection $l^{\star}\left(m-\varepsilon,\omega\right)\to l^{\star}\left(m,\omega\right)$, and, by Rockafellar (1970, Theorem 24.1) the first is the left derivative of $\bar{V}\left(\cdot,\omega_{+1}\right)$. Moreover, the last term is identical to the first, *i.e.*,

$$\begin{aligned}
&\frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left[\frac{\omega\left(m^{-}+l^{\star}\left(m,\omega\right)-\bar{C}^{\star}\right)+\tau}{\omega_{+1}\left(1+\tau\right)},\omega_{+1}\right]\\
&\qquad\leq W_1\left(m,\omega\right)\leq\frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left[\frac{\omega\left(m^{-}+l^{\star}\left(m,\omega\right)-\bar{C}^{\star}\right)+\tau}{\omega_{+1}\left(1+\tau\right)},\omega_{+1}\right].
\end{aligned}$$

Therefore, if the optimal selection is interior, these weak inequalities must hold with equality, so we have the left derivative of $\bar{V}$ with respect to the agent's decision variable y as:

$$\frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left[\frac{\omega y^{\star-}(m,\omega)+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right]=W_1(m,\omega).$$

where $y^{\star-}(m,\omega)\equiv m^{-}+l^{\star}(m,\omega)-\bar{C}^{\star}$.

By similar arguments, we can also prove that the right directional derivative of $\bar{V}(\cdot,\omega_{+1})$ exists, and show that the right derivative of $\bar{V}$ with respect to the agents decision y as:

$$\frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left[\frac{\omega y^{\star+}(m,\omega)+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right]=W_1(m,\omega),$$

where $y^{\star+}(m,\omega)\equiv m^{+}+l^{\star}(m,\omega)-\bar{C}^{\star}$. From the last two equations, we can conclude that the right and left directional derivatives must agree, and thus, we have the first-order KKT condition as

$$\frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left(\frac{\omega y^{\star}(m,\omega)+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)\begin{cases}\leq 1, & y^{\star}(m,\omega)\geq 0\\ \geq 1, & y^{\star}(m,\omega)\leq y_{\max}(\omega;\tau)\end{cases},$$

where the weak inequalities apply with complementary slackness. Since $\bar{V}$ is strictly concave, the condition above ensures a unique selection $y^{\star}(m,\omega)$ at each state. Also, note that in the previous proof of Part 1), we have established the envelope condition:

$$W_1(m,\omega)=1.$$

(*Part* 3.) Observe that given the assumption in (B.1), we have (B.6) always binding: $U'(C)=p=1$. Also, observe from (B.6) and (B.2) that an individual's current money holding m and the aggregate state ω have no influence on his optimal decision on consumption, $C^{\star}(m,\omega)=\bar{C}^{\star}$. However, equilibrium CM asset decision will depend on the aggregate state ω, i.e.,

$$y^{\star}(m,\omega)=\bar{y}^{\star}(\omega) \tag{B.7}$$

and this satisfies the first-order condition (B.2).

However, from the budget constraint, m clearly does affect the optimal labor decision,

$$\begin{aligned}l^{\star}(m,\omega)&=pC^{\star}(m,\omega)+y^{\star}(m,\omega)-m\\ &\underset{(p=1)}{\equiv}\bar{C}^{\star}+\bar{y}^{\star}(\omega)-m.\end{aligned} \tag{B.8}$$

Clearly, $l^{\star}(m,\omega)$ is single-valued, continuous, affine and decreasing in m.

Finally, we show that the optimal choice of l will always be interior. Evaluating the budget constraint in terms of optimal choices at the current individual state m,

$$l^{\star}(m,\omega)=\bar{y}^{\star}(\omega)-m+\bar{C}^{\star}.$$

Since $m\in[0,\bar{m}]$, then, the minimal l attains when m is maximal at $\bar{m}$, and, $\bar{y}^{\star}(\bar{m},\omega)=0$:

$$l_{\min}:=\check{l}^{\star}(\bar{m},\omega)\equiv 0-\bar{m}+\bar{C}^{\star}>0.$$

The last inequality obtains from (B.1) which requires $\bar{m}<U^{-1}(1)$, and from optimal CM consumption

(B.9) which yields

$$C^{\star}\left(m,\omega\right)\equiv\bar{C}^{\star}=\left(U_{1}\right)^{-1}\left(1\right), \tag{B.9}$$

a finite and non-negative constant.

The maximal l attains when $m=0$ and $\bar{y}^{\star}\left(0,\omega\right)=y_{\max}\ \left(\omega;\tau\right)$:

$$l_{\max}\left(\omega,\tau\right):=y_{\max}\ \left(\omega;\tau\right)-0+\bar{C}^{\star}=y_{\max}\left(\omega;\tau\right)+U^{-1}\left(1\right)<2U^{-1}\left(1\right). \tag{B.10}$$

Clearly, $l_{\max}\left(\omega,\tau\right) < +\infty$. If we do not have hyperinflation, or, if transfers are not excessively large—*i.e.*, if $\tau/\omega<\bar{m}$—then, $l_{\max}\left(\omega,\tau\right)>0$ will be well-defined. So if $\tau/\omega<\bar{m}$, then we will have an interior optimizer for all m: $0<l_{\min}\leq l^{\star}\left(m\right)\leq l_{\max}\left(\omega;\tau\right)<+\infty$. $\square$

C DM agent's problem

In the main paper, we provided a verbal summary of the properties of DM agents' equilibrium value function (B) and decision rules (b,x,q). Here we state these results precisely:

Proposition 2 (DM value and policy functions)**.** *For a given sequence of prices $\{\omega, \omega_{+1}, ...\}$, the following properties hold.*

1. *For any $\bar{V}(\cdot, \omega_{+1}) \in \mathcal{V}[0, \bar{m}]$, the DM buyer's value function is increasing and continuous in money balances, $B(\cdot, \omega) \in \mathcal{C}[0, \bar{m}]$.*
2. *For any $m \leq k$, DM buyers' optimal decisions are $b^\star(m, \omega) = x^\star(m, \omega) = q^\star(m, \omega) = 0$, and $B(m, \omega) = \beta\bar{V}[\phi(m, \omega), \omega_{+1}]$, where $\phi(m, \omega) := (\omega m + \tau)/[\omega_{+1}(1+\tau)]$.*
3. *At any (m, ω), where $m \in [k, \bar{m}]$ and the buyer matching probability is positive $b^\star(m, \omega) > 0$:*
   (a) *The optimal selections $(x^\star, b^\star, q^\star)(m, \omega)$ and $\phi^\star(m, \omega) := \phi[m - x^\star(m, \omega), \omega]$, are unique, continuous, and increasing in m.*
   (b) *The buyer's marginal valuation of money $B_1(m, \omega)$ exists if and only if $\bar{V}_1[\phi(m, \omega), \omega]$ exists.*
   (c) *$B(m, \omega)$ is strictly increasing in m.*
   (d) *the optimal policy functions $b^\star$ and $x^\star$, respectively, satisfy the first-order conditions*

$$u \circ Q[x^\star(m, \omega), b^\star(m, \omega)] + b^\star(m, \omega)(u \circ Q)_2[x^\star(m, \omega), b^\star(m, \omega)]$$
$$= \beta\left[\bar{V}(\phi(m, \omega), \omega_{+1}) - \bar{V}(\phi^\star(m, \omega), \omega_{+1})\right], \quad \text{(C.1)}$$

and,

$$(u \circ Q)_1[x^\star(m, \omega), b^\star(m, \omega)] = \frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1[\phi^\star(m, \omega), \omega_{+1}]. \quad \text{(C.2)}$$

In the rest of this section, we prove the results leading up to Proposition 2 in the paper. Part 1 of Proposition 2 is obtained in Lemma 1, Part 2 is proven as Lemma 2. Part 3(a) is proven as Lemma 3. Lemmata 4 and 5 together establish Parts 3(b) and 3(c) of Proposition 2. Finally, Lemma 6 establishes Part 3(d) of Proposition 2.

C.1 DM buyer optimal policies

Recall the DM buyer's problem from (2.13):

$$B(\mathbf{s}) = \max_{x\in[0,m], b\in[0,1]} \{f(x, b; m, \omega)\},$$

where

$$f(x, b; m, \omega) := \beta(1-b)\left[\bar{V}\left(\frac{\omega m + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right)\right] + b\left[u^Q(x, b) + \beta\bar{V}\left(\frac{\omega(m-x)+\tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right)\right],$$

and, we have re-defined the composite function $u \circ Q$ as u^Q. Note that we have not explicitly written $f(x, b; m, \omega)$ as depending on ω_{+1} which is taken as parametric. In an equilibrium, ω_{+1} will be recursively dependent on ω, thus our small sleight of hand here in writing $f(x, b; m, \omega)$.

The following Lemmata 1, 2, 3, 4, 5, and 6 make up Proposition 2. Also, these results will rely on the following statements and notations:

1. Assume $\{\omega, \omega_{+1}, \omega_{+2}...\}$ is a given sequence of prices.

2. Let

$$\phi(m, \omega) := \frac{\omega m + \tau}{\omega_{+1}(1+\tau)},$$

and,

$$\phi^\star(m, \omega) = \phi\left[m - x^\star(m, \omega), \omega\right].$$

3. Equivalently define the objective function $f(\cdot, \cdot; m, \omega)$ in the DM buyer's problem (2.13) as follows:

$$\begin{aligned} f(x, b; m, \omega) = \beta \bar{V}\left(\frac{\omega m + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right) \\ + b\left[u^Q(x, b) + \beta \bar{V}\left(\phi^\star(m, \omega), \omega_{+1}\right) - \beta \bar{V}\left(\phi(m, \omega), \omega_{+1}\right)\right]. \\ \equiv \beta \bar{V}\left(\phi(m, \omega), \omega_{+1}\right) + bR(x, b; m, \omega). \end{aligned} \tag{C.3}$$

Remark. Observe that maximizing the value of the objective function $f(x, b; m, \omega)$ in the DM buyer's problem (2.13) is equivalent to maximizing the second term, $bR(x, b; m, \omega)$. Note that the function $R(x, b; m, \omega)$ has the interpretation of the DM buyer's surplus from trading with a particular trading post named (x, b), by offering to pay x in exchange for quantity $Q(x, b)$.

Lemma 1. *For any $\bar{V}(\cdot, \omega_{+1}) \in \mathcal{V}[0, \bar{m}]$, the DM buyer's value function is increasing and continuous in money balances, $B(\cdot; \omega) \in \mathcal{C}[0, \bar{m}]$.*

Proof. Since the functions $W(\cdot, \omega_{+1}), V(\cdot, \omega_{+1}) \in \mathcal{C}[0, \bar{m}]$, *i.e.*, are continuous and increasing on $[0, \bar{m}]$, and $\bar{V} := \alpha W + (1 - \alpha V)$, then $\bar{V}(\cdot, \omega_{+1}) \in \mathcal{C}[0, \bar{m}]$. The feasible choice set $\Phi(m) := [0, m] \times [0, 1]$ is compact, and it expands with m at each $m \in [0, \bar{m}]$. By Berge's Maximum Theorem, the maximizing selections $(x^\star, b^\star)(m, \omega) \in \Phi(m)$ exist for every fixed $m \in [0, \bar{m}]$, since the objective function is continuous on a compact choice set (Berge, 1963). Evaluating the Bellman operator (2.13), we have that the value function $B(\cdot, \omega) \in \mathcal{C}[0, \bar{m}]$. □

Lemma 2. *For any $m \leq k$, DM buyers' optimal decisions are such that $b^\star(m, \omega) = 0$ and $B(m, \omega) = \beta \bar{V}[\phi(m, \omega), \omega_{+1}]$, where $\phi(m, \omega) := \frac{\omega m + \tau}{\omega_{+1}(1+\tau)}$.*

Proof. Since a buyer's payment x is always constrained above by her initial money balance m in the DM, it will never be optimal for any firm to trade with such a buyer whose $m \leq k$, as the firm will be making an economic loss. In equilibrium it is thus optimal for a buyer $m \leq k$ to optimally not trade and exit the DM with end-of-period balance as m (*i.e.*, with beginning-of-next-period balance $\phi(m,\omega)$ when inflationary transfers are accounted for). As a result, the continuation value is $\bar{V}\left[\phi(m,\omega),\omega_{+1}\right]$, and thus, $B\left(m,\omega\right) = \beta\bar{V}\left[\phi(m,\omega),\omega_{+1}\right]$, if $m \leq k$. $\square$

Lemma 3. *For any (m,ω), where $m \in [k,\bar{m}]$ and the buyer matching probability is positive $b^{\star}\left(m,\omega\right) > 0$, the optimal selections $(x^{\star},b^{\star},q^{\star})\left(m,\omega\right)$ and $\phi^{\star}(m,\omega) := \phi\left[m - x^{\star}\left(m,\omega\right),\omega\right]$ are unique, continuous, and increasing in m.*

Observe that the DM buyer's problem has a general structure similar to that of Menzio et al. (2013). The main difference is in the details underlying the buyer's continuation value function, which in our setting is denoted by $\bar{V}\left(\cdot,\omega\right)$. Nevertheless, we still have that $\bar{V}\left(\cdot,\omega\right) \in \mathcal{V}\left[0,\bar{m}\right]$. As a consequence the proof of Lemma 3.3 in Menzio et al. (2013) can be adapted to our setting. For the reader's convenience, we repeat the proof strategy of Menzio et al. (2013) below for our model setting in a few steps:

Proof. The DM buyer's problem (2.13) can be re-written as

$$B(\mathbf{s}) = \beta\bar{V}\left(\phi(m,\omega),\omega_{+1}\right) + \exp\left\{\max_{x\in[0,m],b\in[0,1]}\left\{\ln\left(b\right) + \ln\left[R\left(x,b;m,\omega\right)\right]\right\}\right\}.$$

The optimizers thus must satisfy

$$(x^{\star},b^{\star})(m,\omega) \in \left\{\arg\max_{x\in[0,m],b\in[0,1]}\left\{\ln\left(b\right) + \ln\left[R\left(x,b;m,\omega,\omega_{+}\right)\right]\right\}\right\}. \tag{C.4}$$

(*Uniqueness and continuity of policies.*) First we establish that the policy functions are continuous, and, at every state, there is a unique optimal selection: Since $u^{Q}\left(x,b\right)$ is continuous, jointly and strictly concave in (x,b), and by assumption, $\bar{V}\left(\cdot,\omega\right) \in \mathcal{V}\left[0,\bar{m}\right]$, then

$$R\left(x,b;m,\omega\right) \equiv u^{Q}(x,b) + \beta\bar{V}\left(\phi^{\star}(m,\omega),\omega_{+1}\right) - \beta\bar{V}\left(\phi(m,\omega),\omega_{+1}\right)$$

is continuous, jointly and strictly concave in the choice variables (x,b). Also, $\ln(b)$ is strictly increasing and strictly concave in b. Thus the maximand is jointly and strictly concave in (x,b). By Berge's Maximum Theorem, the optimal selections $(x^{\star},b^{\star})(m,\omega)$ are continuous and unique at any m. Since $c \mapsto c(q)$ is bijective, then

$$q^{\star}(m,\omega) = c^{-1}\left[x^{\star}(m,\omega) - k/\mu\left(b^{\star}(m,\omega)\right)\right]$$

is continuous in m; and so is $\phi^{\star}(m,\omega)$.

(*Monotonicity of policies.*) The remainder of this proof establishes that the policy functions are increasing. The key idea of the proof is in showing that the choice set is a lattice equipped with a partial order, that the choice set is increasing in m, and, has increasing differences on the choice set, and the slices of the buyer's objective is supermodular in each given direction of his choice set. By Theorem 2.6.2 of Topkis (1998), these properties are sufficient to ensure that the buyer's objective function is supermodular. Together, these properties suffice, by Theorem 2.8.1 of Topkis (1998), for showing that the buyer's optimal policies are increasing functions in m.

1. The function $R(\cdot,\cdot,\cdot,\omega)$ in (C.4) has *increasing difference* in (x,b,m) and is therefore supermodular:

   Fix an $m \in [k,\bar{m}]$ and $b \in (0,1]$. (The case of $b=0$ is trivially uninteresting.) It suffices to optimize over the function $\ln[R(\cdot,b,m,\omega)]$ in (C.4). Then the optimizer

   $$\tilde{x}(b,m,\omega) \in \left\{ \arg \max_{x\in[k,\bar{m}]} \{\ln [R(x,b,m,\omega)]\} \right\}$$

   is unique for each (m,b,ω), since the objective functions is strictly concave.

   Next we show how the value of the objective function has increasing differences in (x,b,m), throughout taking the sequence $\{\omega,\omega_{+1},...\}$ as fixed. Thus we will now write $R(x,b,m) \equiv R(x,b,m,\omega)$ to temporarily ease the notation. First, the feasible choice set

   $$\mathcal{F}_m := \{(x,b,m) : x \in [0,m], b \in [0,1], m \in [k,\bar{m}]\},$$

   is a partially ordered set with relation $\leq$, and it has least-upper and greatest-lower bounds. It is therefore a sublattice in $\mathbb{R}^3_+$. Observe that $\mathcal{F}_m$ is increasing in m. Second, pick any $m' > m$, $b' > b$, and $x' > x$ in $\mathcal{F}_m$:

   (a) Fix x, consider $m' > m$ and $b' > b$. Then, we can write

   $$\begin{aligned} R(x,b',m') - R(x,b,m) &= \left[u^Q(x,b') - u^Q(x,b)\right] \\ &\quad + \beta\left[\bar{V}\left(\frac{\omega(m'-x)+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right) - \bar{V}\left(\frac{\omega(m-x)+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)\right] \\ &\quad - \beta\left[\bar{V}\left(\frac{\omega m'+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right) - \bar{V}\left(\frac{\omega m+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)\right]. \end{aligned}$$

   Observe that the RHS is separable in b and m: The first term on the right, $u^Q(x,b') - u^Q(x,b) < 0$, shows increasing difference in b. Likewise the remainder two difference terms on the RHS show increasing differences in m. Overall $R(x,b,m)$ has increasing differences on the lattice $[0,1]\times[0,\bar{m}] \ni (b,m)$.

   (b) Fix m, consider $x' > x$ and $b' > b$. Observe that

   $$\begin{aligned} R(x,b,m) - R(x',b,m) &= \left[u^Q(x,b) - u^Q(x',b)\right] \\ &\quad + \beta\left[\bar{V}\left(\frac{\omega(m-x)+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right) - \bar{V}\left(\frac{\omega(m-x')+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)\right]. \end{aligned} \tag{C.5}$$

Now, using the expression (C.5) twice below, we have that

$$
\begin{aligned}
&\left[R\left(x', b', m\right)-R\left(x, b', m\right)\right]-\left[R\left(x', b, m\right)-R\left(x, b, m\right)\right] \\
&=\left[u^{Q}(x', b')-u^{Q}(x, b')\right] \\
&\qquad+\beta\left[\bar{V}\left(\frac{\omega\left(m-x'\right)+\tau}{\omega_{+1}\left(1+\tau\right)}, \omega_{+1}\right)-\bar{V}\left(\frac{\omega\left(m-x\right)+\tau}{\omega_{+1}\left(1+\tau\right)}, \omega_{+1}\right)\right] \\
&\qquad-\left[u^{Q}(x', b)-u^{Q}(x, b)\right] \\
&\qquad-\beta\left[\bar{V}\left(\frac{\omega\left(m-x'\right)+\tau}{\omega_{+1}\left(1+\tau\right)}, \omega_{+1}\right)-\bar{V}\left(\frac{\omega\left(m-x\right)+\tau}{\omega_{+1}\left(1+\tau\right)}, \omega_{+1}\right)\right] \\
&\qquad\qquad=\left[u^{Q}(x', b')-u^{Q}(x, b')\right]-\left[u^{Q}(x', b)-u^{Q}(x, b)\right]>0,
\end{aligned}
$$

where the last inequality is implied by the fact that $\left(u^{Q}\right)_{12}(x, b)>0$. Therefore $R\left(x, b, m\right)$ has increasing differences on the lattice $[0, m] \times [0, 1] \ni (x, b)$.

(c) For fixed b, consider $x' > x$ and $m' > m$. Observe that

$$
\begin{aligned}
&\left[R\left(x', b, m'\right)-R\left(x, b, m'\right)\right]-\left[R\left(x', b, m\right)-R\left(x, b, m\right)\right] \\
&=\left[u^{Q}(x', b)-u^{Q}(x, b)\right] \\
&\qquad+\beta\left[\bar{V}\left(\frac{\omega\left(m'-x'\right)+\tau}{\omega_{+1}\left(1+\tau\right)}, \omega_{+1}\right)-\bar{V}\left(\frac{\omega\left(m'-x\right)+\tau}{\omega_{+1}\left(1+\tau\right)}, \omega_{+1}\right)\right] \\
&\qquad-\left[u^{Q}(x', b)-u^{Q}(x, b)\right] \\
&\qquad-\beta\left[\bar{V}\left(\frac{\omega\left(m-x'\right)+\tau}{\omega_{+1}\left(1+\tau\right)}, \omega_{+1}\right)-\bar{V}\left(\frac{\omega\left(m-x\right)+\tau}{\omega_{+1}\left(1+\tau\right)}, \omega_{+1}\right)\right] \\
&=\beta\left[\bar{V}\left(\frac{\omega\left(m'-x'\right)+\tau}{\omega_{+1}\left(1+\tau\right)}, \omega_{+1}\right)-\bar{V}\left(\frac{\omega\left(m'-x\right)+\tau}{\omega_{+1}\left(1+\tau\right)}, \omega_{+1}\right)\right] \\
&\qquad\qquad-\beta\left[\bar{V}\left(\frac{\omega\left(m-x'\right)+\tau}{\omega_{+1}\left(1+\tau\right)}, \omega_{+1}\right)-\bar{V}\left(\frac{m-x+\tau}{1+\tau\omega}, \omega_{+1}\right)\right] \geq 0,
\end{aligned}
$$

where the last weak inequality obtains from the property that $\bar{V}\left(\cdot, \omega_{+1}\right) \in \mathcal{V}[0, \bar{m}]$, and $\bar{V}\left(\cdot, \omega_{+1}\right)$ is therefore weakly concave. Therefore $R\left(x, b, m\right)$ has increasing differences on the lattice $[0, m] \times [0, \bar{m}] \ni (x, m)$.

From parts (1a), (1b), and (1c), we can conclude that the objective function $R\left(\cdot, \cdot, \cdot, \omega\right)$ has increasing differences on $\mathcal{F}_m$. This suffices to prove that the objective function $R\left(\cdot, \cdot, \cdot, \omega\right)$ is supermodular (see Topkis, 1998, Corollary 2.6.1), since the domain of the function is a direct product of a finite set of chains (partially ordered sets with no unordered pair of elements), and the objective function is real valued (see Topkis, 1978).

2. Since $R\left(\cdot, b, m\right)$ is supermodular, for fixed choice b, the optimizer $\tilde{x}\left(b, m, \omega\right)$ is increasing in (b, m), for given ω:

Let $\tilde{x}\left(b, m, \omega\right) = \arg\max_{x \in [0, m]} R\left(x, b, m\right)$. From part (1a) above, we can deduce that for fixed $\tilde{x}\left(b, m\right)$, $\tilde{R}\left(b, m\right) \equiv R\left[\tilde{x}\left(b, m, \omega\right), b, m\right]$ is supermodular on the lattice $[0, 1] \times [0, \bar{m}] \ni (b, m)$. Since $R(x, b, m)$ is strictly decreasing in b, then

$$
\tilde{R}\left(b, m\right) \equiv R\left[\tilde{x}\left(b, m, \omega\right), b, m\right]
$$

is strictly decreasing in b. Observe that for any $m' \geq m$, where $m', m \in [k, \bar{m}]$, we have

$$\begin{aligned}
& R\left(x, b, m'\right) - R\left(x, b, m\right) \\
& = \beta\left[\bar{V}\left(\frac{\omega m + \tau}{\omega_{+1}\left(1+\tau\right)}, \omega_{+1}\right) - \bar{V}\left(\frac{\omega\left(m - x\right) + \tau}{\omega_{+1}\left(1+\tau\right)}, \omega_{+1}\right)\right] \\
& \qquad - \beta\left[\bar{V}\left(\frac{\omega m' + \tau}{\omega_{+1}\left(1+\tau\right)}, \omega_{+1}\right) - \bar{V}\left(\frac{\omega\left(m' - x\right) + \tau}{\omega_{+1}\left(1+\tau\right)}, \omega_{+1}\right)\right] \geq 0,
\end{aligned}$$

since $\bar{V}\left(\cdot, \omega\right)$ is concave. Since this inequality holds at each fixed pair(x, b), then,

$$\begin{aligned}
& \tilde{R}\left(b, m\right) \equiv R\left[\tilde{x}\left(b, m, \omega\right), b, m\right] \\
& \qquad \leq R\left[\tilde{x}\left(b, m, \omega\right), b, m'\right] \leq R\left[\tilde{x}\left(b, m', \omega\right), b, m'\right] \equiv \tilde{R}\left(b, m'\right).
\end{aligned}$$

The last weak inequality obtains because the choice set is increasing in m, and so $\tilde{x}\left(b, m, \omega\right)$ is a feasible selection for the more relaxed problem whose value is

$$R\left[\tilde{x}\left(b, m', \omega\right), b, m'\right] = \max_{x \in [0, m']} R\left(x, b, m'\right).$$

From these weak inequalities, we can conclude that $\tilde{R}\left(b, m\right)$ is increasing in m.

Now we are ready to apply Theorem 2.8.1 of Topkis (1998) to show that $b^{\star}\left(m, \omega\right)$ increases with m: Let

$$b^{\star}\left(m, \omega\right) = \arg\max_{b \in [0,1]} r\left(b, m\right)$$

where $r\left(b, m\right) = b \cdot \tilde{R}\left(b, m\right)$ and $\tilde{R}\left(b, m\right) \equiv R\left(\tilde{x}\left(b, m, \omega\right), b, m, \omega\right)$. Observe the following identity:

$$\begin{aligned}
& \left[r\left(b', m'\right) - r\left(b, m'\right)\right] - \left[r\left(b', m\right) - r\left(b, m\right)\right] = \\
& \qquad b'\left\{\tilde{R}\left(b', m'\right) - \tilde{R}\left(b, m'\right) - \left[\tilde{R}\left(b', m\right) - \tilde{R}\left(b, m\right)\right]\right\} \\
& \qquad\qquad + \left(b' - b\right)\left[\tilde{R}\left(b, m'\right) - \tilde{R}\left(b, m\right)\right],
\end{aligned}$$

for any $b, b' \in (0, 1], m, m' \in [k, \bar{m}]$ where $b' > b$ and $m' > m$. The first term on the RHS is positive, since $b' > 0$ and since $\tilde{R}\left(b, m\right)$ is supermodular in (b, m), then Topkis (1998, Theorem 2.6.1) applies, so that $\tilde{R}\left(b, m\right)$ has increasing differences on $[0, 1] \times [0, \bar{m}]$ (*i.e.*, the terms in the curly braces are positive). Since we have previously established that $\tilde{R}\left(b, m\right)$ is increasing in m, and $b' - b > 0$, then the second term on the RHS is also positive. Thus the objective $r(b, m)$ is supermodular on $[0, 1] \times [k, \bar{m}] \ni (b, m)$. (Note that the choice set of b does not depend on m.)

Therefore, by Theorem 2.8.1 of Topkis (1998), the optimal selection $b^{\star}\left(m, \omega\right)$ is increasing in m. Since $\tilde{x}\left(b, m, \omega\right)$ is increasing in (b, m), for given ω, then we can conclude that the optimal payment choice $x^{\star}\left(m, \omega\right) = \tilde{x}\left(b^{\star}\left(m, \omega\right), m, \omega\right)$ is also increasing in m.

3. The decision $q^{\star}\left(m, \omega\right)$ is monotone in m:

   We perform a change of decision variables. Denote $a \equiv \varphi + c(q)$, where, $\varphi \equiv m - x$. Then we have a change of the DM buyer's decision variables from (x, q) to (a, q). From (2.9), we can re-write $m - x = a - c(q)$ and $b = \mu^{-1}\left[k / \left(m - a\right)\right]$. Since $b \in [0, 1]$, the domain of a is $[0, m - k]$, and the domain for q is $[0,a]$. The buyer's problem from (2.13) is thus equivalent to

$$B(m,\omega)-\beta\bar{V}\left(\frac{\omega m+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)=\max_{a\in[0,m-k],q\in[0,a]}\left\{\mu^{-1}\left(\frac{k}{m-a}\right)\left[u(q)\right.\right.$$
$$\left.\left.+\beta\bar{V}\left(\frac{\omega(a-q)+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)-\beta\bar{V}\left(\frac{\omega m+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)\right]\right\}. \quad \text{(C.6)}$$

Recall we take the sequence $(\omega,\omega_{+1},...)$ as parametric here. This problem can be broken down into two steps: Fix (a,ω). Find the optimal q for any a, to be denoted by $\tilde{q}(a,\omega)$, and then, find the optimal a given (a,ω), to be denoted by $a^\star(m,\omega)$. Then we can deduce the optimal $q^\star(m,\omega)\equiv\tilde{q}[a^\star(m,\omega),m,\omega]$. We details these steps below:

(a) For any fixed outcome a and (m,ω), $\tilde{q}(a,\omega)$ induces the value

$$J(a,\omega)=\max_{q\in[0,a]}\left\{u(q)+\beta\bar{V}\left(\frac{\omega(a-q)+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)\right\}. \quad \text{(C.7)}$$

Observe that q and J do not depend on m, given a fixed a. The objective function on the RHS is clearly supermodular on the lattice $[0,m-k]\times[0,a]\ni(a,q)$. Since the objective function is strictly concave, the selection $\tilde{q}(a,\omega)$ in unique for every a, given ω. Also, the choice set $[0,a]$ increases with a, and, the objective function is increasing. Therefore, respectively by Theorems 2.8.1 (increasing optimal solutions) and 2.7.6 (preservation of supermodularity) of Topkis (1998), we have that $\tilde{q}(a,\omega)$ and $J(a,\omega)$ are increasing in a.

(b) Given best response $\tilde{q}(a,\omega)$, the optimal $a^\star(m,\omega)$ choice satisfies

$$a^\star(m,\omega)=\arg\max_{a\in[0,m-k]}g(a,m,\omega),$$

where

$$g(a,m,\omega)=\mu^{-1}\left(\frac{k}{m-a}\right)\left[J(a,\omega)-\beta\bar{V}\left(\frac{\omega m+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)\right].$$

(Again, note that we have suppressed dependencies on ω_{+1} since this is taken as parametric by the agent, and, in equilibrium ω_{+1} recursively depends on ω.)
Consider the case $J(a,\omega)-\beta\bar{V}\left(\frac{\omega m+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)\geq 0$. Since $\mu(b)$ is strictly decreasing in b, and $1/\mu(b)$ is strictly convex in b, then $\mu^{-1}\left(\frac{k}{m-a}\right)$ is strictly increasing in m, strictly decreasing in a, and is strictly supermodular in (a,m). Pick any $a',a\in[0,m-k]$, and any

$m', m \in [k, \bar{m}]$, such that $a' > a$ and $m' > m$. We have the identity:

$$\begin{aligned}
&[g(a', m', \omega) - g(a, m', \omega)] - [g(a', m, \omega) - g(a, m, \omega)] = \\
&\left[\mu^{-1}\left(\frac{k}{m'-a'}\right) - \mu^{-1}\left(\frac{k}{m-a'}\right)\right][J(a', \omega) - J(a, \omega)] \\
&+ \left[\mu^{-1}\left(\frac{k}{m'-a}\right) - \mu^{-1}\left(\frac{k}{m'-a'}\right)\right] \\
&\quad \times \left[\beta\bar{V}\left(\frac{\omega m' + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right) - \beta\bar{V}\left(\frac{\omega m + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right)\right] \\
&+ \left[\mu^{-1}\left(\frac{k}{m'-a'}\right) - \mu^{-1}\left(\frac{k}{m'-a}\right) - \mu^{-1}\left(\frac{k}{m-a'}\right) - \mu^{-1}\left(\frac{k}{m-a}\right)\right] \\
&\qquad\qquad\qquad\qquad \times \left[J(a, \omega) - \beta\bar{V}\left(\frac{\omega m + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right)\right]
\end{aligned}$$

The first term on the RHS is positive since $\mu^{-1}\left(\frac{k}{m-a}\right)$ is strictly increasing in m, and we have previously shown that $J(a, \omega)$ is increasing in a. The second term on the RHS is positive since $\mu^{-1}\left(\frac{k}{m-a}\right)$ is strictly decreasing in a, and, $\tilde{V}(\cdot, \omega) \in \mathcal{V}[0, \bar{m}]$. The last term on the RHS is positive since $\mu^{-1}\left(\frac{k}{m-a}\right)$ is supermodular, and therefore its first term in the product shows increasing differences Topkis (1998, Theorem 2.6.1). Its last term in the product is positive under the case we are considering. Therefore the LHS is positive, and this suffices to establish that $g(a, m, \omega)$ is strictly supermodular (Topkis, 1998, Theorem 2.8.1).

Finally, since the choice set $[0, m-k]$ is increasing in m, the solution $a^\star(m, \omega)$ is also increasing in m Topkis (1998, Theorem 2.6.1). Since we have established in part (3a) that $\tilde{q}(m, \omega)$ is increasing in a, then, $q^\star(m, \omega) \equiv \tilde{q}[a^\star(m, \omega), \omega]$ is also increasing in m.

4. The decision $\phi^\star(m, \omega)$ is monotone in m:

   Similar to the procedure in the last part, we perform a change of decision variables via $a \equiv \varphi + c(q)$, where, $\varphi \equiv m - x$. The domain for φ is $[0, \min\{m, a\}]$. However, an optimal choice under $b > 0$ means that we will have $\varphi < m$ (the end of period residual balance is less than the beginning of period money balance). This is because, if $\varphi = m$ then it must be that $x = 0$, *i.e.*, the buyer pays nothing; but this is not optimal for the buyer if the buyer faces a positive probability of matching $b > 0$. Moreover, $\varphi < a$, if $u'(0)$ is sufficiently large—*i.e.*, the buyer can always increase utility by raising x (thus lowering φ such that $\varphi < a$ attains). Thus the upper bound on φ will never be binding. As such, the buyer's problem from (2.13) can be re-written as

$$\begin{aligned}
B(m, \omega) - \beta\bar{V}\left(\frac{\omega m + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right) = \max_{a \in [0, m-k], \varphi \geq 0} &\left\{\mu^{-1}\left(\frac{k}{m-a}\right)\left[u^C(a - \varphi)\right.\right. \\
&\left.\left. + \beta\bar{V}\left(\frac{\omega\varphi + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right) - \beta\bar{V}\left(\frac{\omega m + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right)\right]\right\}, \quad \text{(C.8)}
\end{aligned}$$

   where $u^C(q) := u \circ c^{-1}(q)$, which is continuously differentiable with respect to $q \geq 0$. For fixed $a \in [0, m-k]$, denote the value

$$J(a, \omega) = \max_{\varphi \geq 0}\left\{u^C(a - \varphi) + \beta\bar{V}\left(\frac{\omega\varphi + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right)\right\}, \quad \text{(C.9)}$$

and the optimizer,

$$\tilde{\varphi}(a,\omega) = \arg\max_{\varphi\geq 0}\left\{u^C\left(a-\varphi\right)+\beta\bar{V}\left(\frac{\omega\varphi+\tau}{\omega_{+1}\left(1+\tau\right)},\omega_{+1}\right)\right\}. \tag{C.10}$$

Denote also $\tilde{q}(a,\omega) = c^{-1}\left[a-\tilde{\varphi}(a,\omega)\right]$.

Given $\tilde{\varphi}(a,\omega)$, the optimal choice over a, *i.e.*, $a^\star\left(m,\omega\right)$, solves

$$B(m,\omega)-\beta\bar{V}\left(\frac{\omega m+\tau}{\omega_{+1}\left(1+\tau\right)},\omega_{+1}\right) = \max_{a\in[0,m-k]}\left\{\mu^{-1}\left(\frac{k}{m-a}\right)\left[J\left(a,\omega\right)-\beta\bar{V}\left(\frac{\omega m+\tau}{\omega_{+1}\left(1+\tau\right)},\omega_{+1}\right)\right]\right\}.$$

Applying the similar logic in the proof in part 3 on page OA.C-13, we can show that $\tilde{\varphi}(a,\omega)$ is increasing in a; that $a^\star\left(m,\omega\right)$ is increasing in m, and therefore, $\varphi^\star\left(m,\omega\right)\equiv\tilde{\varphi}\left[a^\star\left(m,\omega\right),\omega\right]$ is increasing in m. Finally, since

$$\phi^\star\left(m,\omega\right) := \left[\omega\varphi^\star\left(m,\omega\right)+\tau\right]/\left[\omega_{+1}\left(1+\tau\right)\right],$$

which is a linear transform of $\varphi^\star\left(m,\omega\right)$, then $\phi^\star\left(m,\omega\right)$ is increasing with m, since $\omega/\left[\omega_{+1}\left(1+\tau\right)\right]>0$.

□

C.2 DM buyer value function and first-order conditions

Let us return to the DM buyer's problem re-written as (C.8) in part (4) of the proof of Lemma 3 on page OA.C-10. The buyer's decision problem over $\varphi\equiv m-x$, for any fixed decision $a\equiv\varphi+c(q)$, yields the value $J\left(a,\omega\right)$ as defined in Equation (C.9) of that proof. The following intermediate results show that the value function $J\left(\cdot,\omega\right)$ is differentiable with respect to a and its marginal value can be related to primitives, *i.e.*:

Lemma 4. *The marginal value of $J\left(\cdot,\omega\right)$ agrees with the flow DM marginal utility with respect to the buyer's payment x,*

$$J_1\left(a,\omega\right) = u'\left[\tilde{q}(a,\omega)\right]\equiv\left(u^Q\right)_1\left[x^\star(m,\omega),b^\star(m,\omega)\right]>0. \tag{C.11}$$

Proof. Consider the problem described in Equations (C.8) and (C.9). Observe that since $\varphi\equiv a-c\left(q\right)$, then

$$\tilde{\varphi}(a,\omega) = \arg\max_{\varphi\geq 0}\left\{u^C\left(a-\varphi\right)+\beta\bar{V}\left(\frac{\omega\varphi+\tau}{\omega_{+1}\left(1+\tau\right)},\omega_{+1}\right)\right\} \tag{C.12}$$

is continuous with respect to a: There is some $\delta'>0$, such that for all $\varepsilon\in[0,\delta']$, the choices $\tilde{\varphi}(a+\varepsilon,\omega)$ and $\tilde{\varphi}(a-\varepsilon,\omega)$ exist. Moreover the optimal selection $\tilde{\varphi}(a,\omega)$ is unique since the objective function in (C.12) is strictly concave by virtue of u^C being strictly concave and $\bar{V}$ being concave. Denote also $\tilde{q}(a,\omega)=c^{-1}\left[a-\tilde{\varphi}(a,\omega)\right]$. The choices $\tilde{q}(a+\varepsilon,\omega)$ and $\tilde{q}(a-\varepsilon,\omega)$ also exist, by continuity of c^{-1} in its argument.

To verify (C.11), we can use the perturbed choices, $\tilde{\varphi}(a+\varepsilon,\omega)$ and $\tilde{\varphi}(a-\varepsilon,\omega)$, for evaluating the right- and left-derivatives of the functions u^C, $\bar{V}$ and J, in order to "sandwich" the derivative function $J_1(\cdot,\omega)$ and arrive at the claimed result. For notational convenience, we define the following function

$$K_\omega\left[a,\tilde{\varphi}(a,\omega)\right] \equiv u^C\left(a-\tilde{\varphi}(a,\omega)\right)+\beta\bar{V}\left(\frac{\omega\tilde{\varphi}(a,\omega)+\tau}{\omega_{+1}\left(1+\tau\right)},\omega_{+1}\right).$$

Consider first the right derivatives: Take $\delta' \searrow 0$ such that for all $\varepsilon \in [0,\delta']$, the choice $\tilde{\varphi}(a+\varepsilon,\omega)$ is affordable for a buyer a. Since $\tilde{\varphi}(\cdot,\omega)$ is an optimal policy satisfying (C.10), then under action $\tilde{\varphi}(a,\omega)$ we must have that

$$\begin{aligned} J\left(a,\omega\right) &= u^C\left(a-\tilde{\varphi}(a,\omega)\right)+\beta\bar{V}\left(\frac{\omega\tilde{\varphi}(a,\omega)+\tau}{\omega_{+1}\left(1+\tau\right)},\omega_{+1}\right) \\ &\geq u^C\left(a-\tilde{\varphi}(a+\varepsilon,\omega)\right)+\beta\bar{V}\left(\frac{\omega\tilde{\varphi}(a+\varepsilon,\omega)+\tau}{\omega_{+1}\left(1+\tau\right)},\omega_{+1}\right) \\ \Leftrightarrow J\left(a,\omega\right) &= K_\omega\left[a,\tilde{\varphi}(a,\omega)\right] \geq K_\omega\left[a,\tilde{\varphi}(a+\varepsilon,\omega)\right]. \end{aligned}$$

Again, take $\delta' \searrow 0$ such that $\forall \varepsilon \in [0,\delta']$, the choice $\tilde{\varphi}(a,\omega)$ is affordable for buyer $a+\varepsilon$. Since $\tilde{\varphi}(\cdot,\omega)$ is an optimal policy satisfying (C.10), then under $\tilde{\varphi}(a+\varepsilon,\omega)$ we must have that

$$\begin{aligned} J\left(a+\varepsilon,\omega\right) &= u^C\left(a+\varepsilon-\tilde{\varphi}(a+\varepsilon,\omega)\right)+\beta\bar{V}\left(\frac{\omega\tilde{\varphi}(a+\varepsilon,\omega)+\tau}{\omega_{+1}\left(1+\tau\right)},\omega_{+1}\right) \\ &\geq u^C\left(a+\varepsilon-\tilde{\varphi}(a,\omega)\right)+\beta\bar{V}\left(\frac{\omega\tilde{\varphi}(a,\omega)+\tau}{\omega_{+1}\left(1+\tau\right)},\omega_{+1}\right) \\ \Leftrightarrow J\left(a+\varepsilon,\omega\right) &= K_\omega\left[a+\varepsilon,\tilde{\varphi}(a+\varepsilon,\omega)\right] \geq K_\omega\left[a+\varepsilon,\tilde{\varphi}(a,\omega)\right]. \end{aligned}$$

Re-write the two inequalities above as

$$\begin{aligned} \frac{K_\omega\left[a+\varepsilon,\tilde{\varphi}(a,\omega)\right]-K_\omega\left[a,\tilde{\varphi}(a,\omega)\right]}{\varepsilon} &\leq \frac{J\left(a+\varepsilon,\omega\right)-J\left(a,\omega\right)}{\varepsilon} \\ &\leq \frac{K_\omega\left[a+\varepsilon,\tilde{\varphi}(a+\varepsilon,\omega)\right]-K_\omega\left[a,\tilde{\varphi}(a+\varepsilon,\omega)\right]}{\varepsilon}. \end{aligned}$$

Since the composite function u^C—and therefore the objective function in (C.9)—is differentiable with respect to a, $J_1\left(\cdot,\omega\right)$ clearly exists. Therefore, the right derivative of this value function must agree with its partial derivative: $\lim_{\varepsilon\searrow 0} J\left(a+\varepsilon,\omega\right) = J_1\left(a,\omega\right)$. Using this fact, the inequalities above imply

$$\begin{aligned} \lim_{\varepsilon\searrow 0}\frac{K_\omega\left[a+\varepsilon,\tilde{\varphi}(a,\omega)\right]-K_\omega\left[a,\tilde{\varphi}(a,\omega)\right]}{\varepsilon} &\leq J_1\left(a,\omega\right) \\ &\leq \lim_{\varepsilon\searrow 0}\frac{K_\omega\left[a+\varepsilon,\tilde{\varphi}(a+\varepsilon,\omega)\right]-K_\omega\left[a,\tilde{\varphi}(a+\varepsilon,\omega)\right]}{\varepsilon}. \end{aligned}$$

Moreover, by continuity of $\tilde{\varphi}(\cdot,\omega)$, we have that $\lim_{\varepsilon\searrow 0}\tilde{\varphi}(a+\varepsilon,\omega)=\tilde{\varphi}(a,\omega)$, so the inequalities above collapse to

$$\begin{aligned} u'\left[\tilde{q}(a^+,\omega)\right] := \lim_{\varepsilon\searrow 0}\frac{u^C\left(a+\varepsilon-\tilde{\varphi}(a,\omega)\right)-u^C\left(a-\tilde{\varphi}(a,\omega)\right)}{\varepsilon} &\leq J_1\left(a,\omega\right) \\ \leq \lim_{\varepsilon\searrow 0}\frac{u^C\left(a+\varepsilon-\tilde{\varphi}(a,\omega)\right)-u^C\left(a-\tilde{\varphi}(a,\omega)\right)}{\varepsilon} &=: u'\left[\tilde{q}(a^+,\omega)\right]. \end{aligned}$$

However, the first and the last term in the inequalities above are identical, and they are the same as the right derivative of u with respect to $q := \tilde{q}(a,\omega)$, *i.e.*, $u'\left[\tilde{q}(a^{+},\omega)\right]$. Thus, it must be that $u'\left[\tilde{q}(a^{+},\omega)\right] = J_1\left(a,\omega\right)$.

Using similar arguments as above, we can also consider the left-hand-side perturbation about a, to evaluate $\tilde{\varphi}(a-\varepsilon,\omega)$. It can be shown that

$$
\begin{aligned}
u'\left[\tilde{q}(a^{-},\omega)\right] := \lim_{\varepsilon\searrow 0} \frac{u^{C}\left(a-\varepsilon-\tilde{\varphi}(a,\omega)\right) - u^{C}\left(a-\tilde{\varphi}(a,\omega)\right)}{\varepsilon} &\leq J_1\left(a,\omega\right) \\
\leq \lim_{\varepsilon\searrow 0} \frac{u^{C}\left(a-\varepsilon-\tilde{\varphi}(a,\omega)\right) - u^{C}\left(a-\tilde{\varphi}(a,\omega)\right)}{\varepsilon} &=: u'\left[\tilde{q}(a^{-},\omega)\right],
\end{aligned}
$$

so that $u'\left[\tilde{q}(a^{-},\omega)\right] = J_1\left(a,\omega\right)$.

Combining the two arguments above, we have that

$$
u'\left[\tilde{q}(a,\omega)\right] = u'\left[\tilde{q}(a^{+},\omega)\right] = u'\left[\tilde{q}(a^{-},\omega)\right] = J_1\left(a,\omega\right) > 0.
$$

Finally, the equivalence $u'\left[\tilde{q}(a,\omega)\right] = \left(u^{Q}\right)_1\left[x^{\star}(m,\omega), b^{\star}(m,\omega)\right]$ can be derived using standard calculus, since the composite function $u^{Q} \equiv u \circ Q$ is a known continuously differentiable function in its arguments (x,b). The assumption on u that marginal utility is everywhere positive renders $u'\left[\tilde{q}(a,\omega)\right] > 0$. This completes the proof of the claim. $\square$

Lemma 5. *At any (m,ω), where $m \in [k,\bar{m}]$ and the buyer matching probability is positive $b^{\star}\left(m,\omega\right) > 0$,*

1. the buyer's marginal valuation of money $B_1(m,\omega)$ exists if and only if $\bar{V}_1\left[\frac{\omega m+\tau}{\omega_{+1}(1+\tau)},\omega\right]$ exists; and
2. $B(m,\omega)$ is strictly increasing in m.

Proof. Lemma 3 implies that $\tilde{q}(a,\omega)$ is increasing in a. Since we have shown that $u'\left[\tilde{q}(a,\omega)\right] = J_1\left(a,\omega\right) > 0$, then $J_1\left(a,\omega\right)$ is also decreasing in a. Since $J\left(a,\omega\right)$ is clearly increasing in a, then we conclude that it is also concave in a. The term $\mu^{-1}\left(\frac{k}{m-a}\right)$ is strictly decreasing and strictly concave in a. Therefore the objective function in (C.8) is strictly concave in a. Thus maximizing (C.8) over a yields a unique optimal selection $a^{\star}(m,\omega)$. Moreover, the objective function in (C.8) is continuously differentiable with respect to a; and using (C.11) we can show that $a^{\star}(m,\omega)$ satisfies the first-order condition:[42]

$$
\begin{aligned}
&J\left(a^{\star}(m,\omega),\omega\right)-\beta\bar{V}\left(\frac{\omega m+\tau}{\omega_{+1}\left(1+\tau\right)},\omega_{+1}\right)\\
&\qquad+u'\left[\tilde{q}\left(a^{\star}(m,\omega),\omega\right)\right]\cdot\frac{k\cdot\mu'\left[b^{\star}\left(m,\omega\right)\right]b^{\star}\left(m,\omega\right)}{\mu\left[b^{\star}\left(m,\omega\right)\right]^{2}}\begin{cases}=0, & \text{if } a^{\star}(m,\omega)<m-k\\ <0, & \text{if } a^{\star}(m,\omega)=m-k\end{cases}.
\end{aligned}\tag{C.13}
$$

Observe that $b^{\star}\left(m,\omega\right)>0$ implies the buyer has more than enough initial balance for purchasing $q^{\star}\left(m,\omega\right)$, *i.e.*,

$$
m-\varphi^{\star}\left(m,\omega\right)>c\left[q^{\star}(m,\omega)\right]+k\implies a(m,\omega)\equiv\varphi^{\star}\left(m,\omega\right)+c\left[q^{\star}(m,\omega)\right]<m-k.
$$

Since $a^{\star}(m,\omega)<m-k$, and $a^{\star}(m,\omega)$ is continuous in m, then there is an $\epsilon>0$ such that the following selections are also feasible: $a^{\star}(m+\epsilon,\omega)<m-k$, and, $a^{\star}(m,\omega)<(m-\epsilon)-k$. Define the open ball $\mathbf{N}_{\epsilon}\left(m\right):=\left(m-\epsilon,m+\epsilon\right)$. Note that for any $m'\in\mathbf{N}_{\epsilon}\left(m\right)$, the selection $a^{\star}\left(m',\omega\right)$ is feasible for an agent m; and $a^{\star}\left(m,\omega\right)$ is feasible for agent m'.

Given that $a^{\star}\left(m,\omega\right)$ is optimal for agent m, and since $\varphi^{\star}(m,\omega)=\tilde{\varphi}\left[a^{\star}\left(m,\omega\right)\right]$, then we have the buyer's optimal value as

$$
\begin{aligned}
B(m,\omega)&=\beta\bar{V}\left(\frac{\omega m+\tau}{\omega_{+1}\left(1+\tau\right)},\omega_{+1}\right)+\max_{a\in[0,m-k],\varphi\geq0}\left\{\mu^{-1}\left(\frac{k}{m-a}\right)\right.\\
&\qquad\left.\times\left[u\circ c^{-1}\left(a-\varphi\right)+\beta\bar{V}\left(\frac{\omega\tilde{\varphi}+\tau}{\omega_{+1}\left(1+\tau\right)},\omega_{+1}\right)-\beta\bar{V}\left(\frac{\omega m+\tau}{\omega_{+1}\left(1+\tau\right)},\omega_{+1}\right)\right\}\right.\\
&=F\left(a^{\star}\left(m,\omega\right),m\right)\geq F\left(a^{\star}\left(m+\epsilon,\omega\right),m\right).
\end{aligned}
$$

where $F\left(a,m\right):=\beta\bar{V}\left(\frac{\omega m+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)+\mu^{-1}\left(\frac{k}{m-a}\right)\left[J\left(a,\omega\right)-\beta\bar{V}\left(\frac{\omega m+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)\right]$. Similarly, for agent $m+\epsilon$, it must be that

$$
B\left(m+\epsilon,\omega\right)=F\left(a^{\star}\left(m+\epsilon,\omega\right),m+\epsilon\right)\geq F\left(a^{\star}\left(m,\omega\right),m+\epsilon\right).
$$

Clearly,

$$
\begin{aligned}
\frac{F\left(a^{\star}\left(m,\omega\right),m+\epsilon\right)-F\left(a^{\star}\left(m,\omega\right),m\right)}{\epsilon}&\leq\frac{B\left(m+\epsilon,\omega\right)-B\left(m,\omega\right)}{\epsilon}\\
&\leq\frac{F\left(a^{\star}\left(m+\epsilon,\omega\right),m+\epsilon\right)-F\left(a^{\star}\left(m+\epsilon,\omega\right),m\right)}{\epsilon}.
\end{aligned}
$$

Since $F\left(a,m\right)$ is continuous and concave in a, and, $a^{\star}\left(m,\omega\right)$ is continuous in m, the following limits exist (Rockafellar, 1970, Theorem 24.1, pp.227-228), and the inequality ordering is preserved in the

[42] Note that $b=\mu^{-1}\left(\frac{k}{m-a}\right)$. The term $\mathrm{d}b/\mathrm{d}a=k/\left(m-a\right)^{2}\times1/\mu'\left[b\right]$ can be derived using the implicit function theorem: Define $H(a,b)=k/\left(m-a\right)-\mu\left[b\right]=0$. Then $\mathrm{d}b/\mathrm{d}a=-H_{a}(a,b)/H_{b}(a,b)$, which yields the result. The first-order condition is thus derived as

$$
\begin{aligned}
&\left[J\left(a^{\star}(m,\omega),\omega\right)-\beta\bar{V}\left(\frac{\omega m+\tau}{\omega_{+1}\left(1+\tau\right)},\omega_{+1}\right)\right]\frac{k}{\left(m-a^{\star}(m,\omega)\right)^{2}}\frac{1}{\mu'\left[b^{\star}\left(m,\omega\right)\right]}\\
&\qquad+J_{1}\left(a^{\star}(m,\omega),\omega\right)\mu^{-1}\left(\frac{k}{m-a^{\star}(m,\omega)}\right)\begin{cases}=0, & \text{if } a^{\star}(m,\omega)<m-k\\ <0, & \text{if } a^{\star}(m,\omega)=m-k\end{cases}.
\end{aligned}
$$

Moreover, since $k/(m-a)=\mu(b)$, we can write $\mathrm{d}b/\mathrm{d}a=k/\left(m-a\right)^{2}\times1/\mu'\left[b\right]\equiv\left[\mu\left(b\right)\right]^{2}/k\times1/\mu'\left[b\right]$, and using the relation (C.11), the first-order condition can be further simplified to (C.13).

limit:

$$\lim_{\epsilon\searrow 0}\frac{F\left(a^{\star}\left(m,\omega\right),m+\epsilon\right)-F\left(a^{\star}\left(m,\omega\right),m\right)}{\epsilon}\leq\lim_{\epsilon\searrow 0}\frac{B\left(m+\epsilon,\omega\right)-B\left(m,\omega\right)}{\epsilon}$$
$$\leq\lim_{\epsilon\searrow 0}\frac{F\left(a^{\star}\left(m+\epsilon,\omega\right),m+\epsilon\right)-F\left(a^{\star}\left(m+\epsilon,\omega\right),m\right)}{\epsilon}.$$

Since $\lim_{\epsilon\searrow 0}a^{\star}\left(m+\epsilon,\omega\right)=a^{\star}\left(m,\omega\right)$, the inequalities above are equivalent to

$$\begin{aligned}
&b^{\star}\left(m,\omega\right)\left[J_1\left(a^{\star}\left(m,\omega\right),\omega\right)-\frac{\beta}{1+\tau}\bar{V}_1\left(\frac{\omega m^{+}+\tau}{\omega_{+1}\left(1+\tau\right)},\omega_{+1}\right)\right]\\
&+\frac{\beta}{1+\tau}\bar{V}_1\left(\frac{\omega m^{+}+\tau}{\omega_{+1}\left(1+\tau\right)},\omega_{+1}\right)\\
&\quad\leq B_1\left(m^{+},\omega\right)\\
&\qquad\leq b^{\star}\left(m,\omega\right)\left[J_1\left(a^{\star}\left(m,\omega\right),\omega\right)-\frac{\beta}{1+\tau}\bar{V}_1\left(\frac{\omega m^{+}+\tau}{\omega_{+1}\left(1+\tau\right)},\omega_{+1}\right)\right]\\
&\qquad\qquad+\frac{\beta}{1+\tau}\bar{V}_1\left(\frac{\omega m^{+}+\tau}{\omega_{+1}\left(1+\tau\right)},\omega_{+1}\right),
\end{aligned}$$

where

$$\bar{V}_1\left(\frac{\omega m^{+}+\tau}{\omega_{+1}\left(1+\tau\right)},\omega_{+1}\right)$$
$$:=\lim_{\epsilon\searrow 0}\left(1+\tau\right)\left(\frac{\omega_{+1}}{\omega}\right)\left[\bar{V}\left(\frac{\omega\left(m+\epsilon\right)+\tau}{\omega_{+1}\left(1+\tau\right)},\omega_{+1}\right)-\bar{V}\left(\frac{\omega m+\tau}{\omega_{+1}\left(1+\tau\right)},\omega_{+1}\right)\right]/\epsilon.$$

However, observe that the first and the last terms in the inequalities are identical. Thus we must have that the right derivative of $B\left(\cdot,\omega\right)$ satisfies

$$B_1\left(m^{+},\omega\right)=b^{\star}\left(m,\omega\right)\left[J_1\left(a^{\star}\left(m,\omega\right),\omega\right)-\frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left(\frac{\omega m^{+}+\tau}{\omega_{+1}\left(1+\tau\right)},\omega_{+1}\right)\right]$$
$$+\frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left(\frac{\omega m^{+}+\tau}{\omega_{+1}\left(1+\tau\right)},\omega_{+1}\right).$$

By a similar process to arrive at the left derivative of $B\left(\cdot,\omega\right)$, we have

$$B_1\left(m^{-},\omega\right)=b^{\star}\left(m,\omega\right)\left[J_1\left(a^{\star}\left(m,\omega\right),\omega\right)-\frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left(\frac{\omega m^{-}+\tau}{\omega_{+1}\left(1+\tau\right)},\omega_{+1}\right)\right]$$
$$+\frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left(\frac{\omega m^{-}+\tau}{\omega_{+1}\left(1+\tau\right)},\omega_{+1}\right),$$

where

$$\bar{V}_1\left(\frac{\omega m^{-}+\tau}{\omega_{+1}\left(1+\tau\right)},\omega_{+1}\right)$$
$$:=\left(1+\tau\right)\left(\frac{\omega_{+1}}{\omega}\right)\lim_{\epsilon\searrow 0}\left\{\frac{1}{\epsilon}\left[\bar{V}\left(\frac{\omega\left(m-\epsilon\right)+\tau}{\omega_{+1}\left(1+\tau\right)},\omega_{+1}\right)-\bar{V}\left(\frac{\omega m+\tau}{\omega_{+1}\left(1+\tau\right)},\omega_{+1}\right)\right]\right\}.$$

Using the result from (C.11) in Lemma 4 on page OA.C-16, we can re-write these right- and left-derivative functions, respectively, as

$$B_1\left(m^+,\omega\right) = b^\star(m,\omega)\left(u^Q\right)_1\left[x^\star(m,\omega), b^\star(m,\omega)\right] \\ + \frac{\beta\left[1-b^\star(m,\omega)\right]}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left(\frac{\omega m^+ + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right), \tag{C.14}$$

and,

$$B_1\left(m^-,\omega\right) = b^\star(m,\omega)\left(u^Q\right)_1\left[x^\star(m,\omega), b^\star(m,\omega)\right] \\ + \frac{\beta\left[1-b^\star(m,\omega)\right]}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left(\frac{\omega m^- + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right). \tag{C.15}$$

From (C.14) and (C.15), it is apparent that $B_1(m,\omega)$ exists if and only if the left- and right-derivatives of $\bar{V}(\cdot,\omega_{+1})$ exist and they agree at the continuation state from m, *i.e.*, if

$$\bar{V}_1\left(\frac{\omega m^- + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right) = \bar{V}_1\left(\frac{\omega m^+ + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right) = \bar{V}_1\left(\frac{\omega m + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right).$$

This proves the first part of the statement in the Lemma.

Since $\bar{V}(\cdot,\omega_{+1}) \in \mathcal{V}[0,\bar{m}]$, it is concave and increasing in m, and therefore,

$$\bar{V}_1\left(\frac{\omega m^- + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right) \geq \bar{V}_1\left(\frac{\omega m^+ + \tau}{\omega_{+1}(1+\tau)}, \omega_{+1}\right) \geq 0.$$

Since we assumed $b^\star(m,\omega) \in (0,1]$, and by Lemma 4, we have

$$J_1\left(a^\star(m,\omega),\omega\right) \equiv \left(u^Q\right)_1\left[x^\star(m,\omega), b^\star(m,\omega)\right] > 0,$$

then (C.14) and (C.15) imply that the first-order left and right derivatives of $B(\cdot,\omega_{+1})$ satisfy:

$$B_1\left(m^-,\omega\right) \geq B_1\left(m^+,\omega\right) \geq b^\star(m,\omega)\left(u^Q\right)_1\left[x^\star(m,\omega), b^\star(m,\omega)\right] > 0.$$

From this ordering, we can conclude that if $b^\star(m,\omega) > 0$, the buyer's valuation $B(m,\omega_{+1})$ is *strictly* increasing with his money balance, m. This proves the last part of the statement in the Lemma. □

Lemma 6. *For any (m,ω), where $m \in [k,\bar{m}]$ and the buyer matching probability is positive $b^\star(m,\omega) > 0$, the optimal policy functions $b^\star$and $x^\star$, respectively, satisfy the first-order conditions* (C.1) *and* (C.2).

Proof. We want to show that the first order conditions characterizing the optimal policy functions $b^\star$and $x^\star$, are indeed (C.1) and (C.2). It is immediate that the objective function (2.13) is continuously differentiable with respect to the choice $b \in [0,1]$. Holding fixed x, if the optimal choice for b is interior, $b^\star(m,\omega) \in (0,1)$, then it must satisfies the first order condition (C.1) with respect to b:

$$u^Q\left[x^\star(m,\omega), b^\star(m,\omega)\right] + b^\star(m,\omega)\left(u^Q\right)_2\left[x^\star(m,\omega), b^\star(m,\omega)\right] \\ = \beta\left[\bar{V}\left(\phi(m,\omega),\omega_{+1}\right) - \bar{V}\left(\phi^\star(m,\omega),\omega_{+1}\right)\right].$$

The first order condition with respect to x is more subtle. We can derive it by first defining one-sided derivatives of $B(\cdot,\omega)$. Assume beginning-of-next-period residual balance after current DM trade is positive—*i.e.*,

$$\phi^{\star}(m,\omega)=\frac{\omega\left[m-x^{\star}(m,\omega)\right]+\tau}{\omega_{+1}(1+\tau)}>0. \tag{C.16}$$

Since (C.16) holds, and since we have shown in Lemma 3 that $x^{\star}(m,\omega)$ and $\phi^{\star}(m,\omega)$ are continuous in $m\in[k,\bar{m}]$, then

$$(\phi^{\star})^{+}(m,\omega):=\frac{\omega\left[m+\varepsilon-x^{\star}(m,\omega)\right]+\tau}{\omega_{+1}(1+\tau)},$$

and,

$$(\phi^{\star})^{-}(m,\omega):=\frac{\omega\left[m-\varepsilon-x^{\star}(m,\omega)\right]+\tau}{\omega_{+1}(1+\tau)},$$

exist and are feasible (or affordable). From (2.13), the DM buyer's one-sided derivatives of $B(\cdot,\omega)$—*i.e.*, its left- or right-marginal valuation of initial money balance—are, respectively,

$$\begin{aligned}B_1\left(m^{+},\omega\right)=&\frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\\&\times\left\{\left[1-b^{\star}(m,\omega)\right]\bar{V}_1\left(\frac{\omega m^{+}+\tau}{\omega_{+1}(1+\tau)},\omega\right)+b^{\star}(m,\omega)\,\bar{V}_1\left[(\phi^{\star})^{+}(m,\omega),\omega_{+1}\right]\right\},\end{aligned} \tag{C.17}$$

and,

$$\begin{aligned}B_1\left(m^{-},\omega\right)=&\frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\\&\times\left\{\left[1-b^{\star}(m,\omega)\right]\bar{V}_1\left(\frac{\omega m^{-}+\tau}{\omega_{+1}(1+\tau)},\omega\right)+b^{\star}(m,\omega)\,\bar{V}_1\left[(\phi^{\star})^{-}(m,\omega),\omega_{+1}\right]\right\},\end{aligned} \tag{C.18}$$

where

$$\begin{aligned}&\bar{V}_1\left(\frac{\omega m^{\pm}+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)\\&\qquad:=(1+\tau)\left(\frac{\omega_{+1}}{\omega}\right)\lim_{\varepsilon\searrow 0}\left\{\frac{1}{\varepsilon}\left[\bar{V}\left(\frac{\omega(m\pm\epsilon)+\tau}{\omega_{+1}(1+\tau)},\omega_{+1}\right)-\bar{V}\left(\frac{\omega m+\tau}{\omega_{+1}(1+\tau)},\omega\right)\right]\right\}.\end{aligned}$$

From Lemma 5, we have shown by change of variable, that the one-sided derivatives of $B(\cdot,\omega)$ also satisfy (C.17) and (C.18). These are repeated here for convenience as the following equations:

$$\begin{aligned}B_1\left(m^{+},\omega\right)=&\frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\left[1-b^{\star}(m,\omega)\right]\bar{V}_1\left(\frac{\omega m^{+}+\tau}{\omega_{+1}(1+\tau)},\omega\right)\\&+b^{\star}(m,\omega)\left(u^{Q}\right)_1\left[x^{\star}(m,\omega),b^{\star}(m,\omega)\right],\end{aligned} \tag{C.19}$$

and,

$$B_1\left(m^-,\omega\right)=\frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\left[1-b^\star\left(m,\omega\right)\right]\bar{V}_1\left(\frac{\omega m^-+\tau}{\omega_{+1}\left(1+\tau\right)},\omega\right)$$
$$+\,b^\star\left(m,\omega\right)\left(u^Q\right)_1\left[x^\star\left(m,\omega\right),b^\star\left(m,\omega\right)\right]. \quad \text{(C.20)}$$

From the last term on the RHS of each of Equations (C.17), (C.18), (C.19), and, (C.20), we have the observation that

$$\frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left[\left(\phi^\star\right)^+\left(m,\omega\right),\omega_{+1}\right]=\frac{\beta}{1+\tau\omega}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left[\left(\phi^\star\right)^-\left(m,\omega\right),\omega_{+1}\right]$$
$$=\left(u^Q\right)_1\left[x^\star\left(m,\omega\right),b^\star\left(m,\omega\right)\right].$$

Since these marginal valuation functions are evaluated at the DM buyer's optimal choice, it must be that

$$\frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left[\left(\phi^\star\right)^+\left(m,\omega\right),\omega_{+1}\right]=\frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left[\left(\phi^\star\right)^-\left(m,\omega\right),\omega_{+1}\right]$$
$$=\frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left[\phi^\star(m,\omega),\omega_{+1}\right],$$

and, that this satisfies the first order condition (C.2), which is

$$\left(u^Q\right)_1\left[x^\star(m,\omega),b^\star(m,\omega)\right]=\frac{\beta}{1+\tau}\left(\frac{\omega}{\omega_{+1}}\right)\bar{V}_1\left[\phi^\star(m,\omega),\omega_{+1}\right].$$

□

D Proof of Theorem 1

Proof. First, we show that the value functions listed in the definition of a SME are unique given ω. For given ω, The CM agent's problem in (2.4) clearly defines a self-map $T_\omega^{CM}:\mathcal{V}\left[0,\bar{m}\right]\to\mathcal{V}\left[0,\bar{m}\right]$, which preserves monotonicity, continuity and concavity (see Proposition 1). However, for fixed ω, the DM buyer's problem in 2.13 defines an operator $T_\omega^{DM}:\mathcal{V}\left[0,\bar{m}\right]\to\mathcal{C}\left[0,\bar{m}\right]$, where $\mathcal{C}\left[0,\bar{m}\right]\supset\mathcal{V}\left[0,\bar{m}\right]$ is the set of continuous and increasing functions on the domain $[0,\bar{m}]$. This operator does not preserve concavity. Note that $\bar{V}(\cdot,\omega)\in\mathcal{V}\left[0,\bar{m}\right]$ as previously defined. Now we show that the ex-ante problem in (2.6) and (2.7) defines an operator that maps the CM agent's and the DM buyer's value functions, respectively, $W(\cdot,\omega)=T_\omega^{DM}\bar{V}(\cdot,\omega)$ and $B(\cdot,\omega)=T_\omega^{DM}\bar{V}(\cdot,\omega)$, back into the set of continuous, increasing and concave functions: $T_\omega:\mathcal{V}\left[0,\bar{m}\right]\to\mathcal{V}\left[0,\bar{m}\right]$. Since T_ω^{CM} and T_ω^{DM} are monotone functional operators that satisfy discounting with factor $0<\beta<1$, then the ex-ante problem in (2.6) and (2.7), which defines the composite operator $T_\omega:\mathcal{V}\left[0,\bar{m}\right]\to\mathcal{V}\left[0,\bar{m}\right]$, clearly preserves these two properties. (The convexification of the graph of T_ω via lotteries in (2.7) preserves concavity of the image of the operator, thus making it a self-map on $\mathcal{V}\left[0,\bar{m}\right]$.) It can be shown that $\mathcal{V}\left[0,\bar{m}\right]$ is a complete metric space. Thus $T_\omega:\mathcal{V}\left[0,\bar{m}\right]\to\mathcal{V}\left[0,\bar{m}\right]$ satisfies Blackwell's conditions, and has a unique fixed point, $\bar{V}\left(\cdot,\omega\right)=T_\omega\bar{V}\left(\cdot,\omega\right)$, by Banach's fixed point theorem.

Second, we verify the following three properties: (1) By Propositions 1 and 2, the agent's optimal policies are continuous, single-valued and monotone functions. This implies, for fixed ω, that the Markov kernel $P(\mathbf{s},\cdot)$ in the distributional operator (2.17) is a probability measure, and, $P(\cdot,E)$ for all Borel subsets $E \in \mathcal{B}([0, \bar{m}])$ is a measurable function. (2) Since each agent's policies are monotone, then $P(\mathbf{s}, \cdot)$ is increasing on $[0, \bar{m}]$. Thus the Markov kernel is a transition probability function. (3) The equilibrium policies clearly dictate that the monotone mixing conditions of Hopenhayn and Prescott (1992) are satisfied: Consider a DM buyer who has zero money balance. With non-zero probability either by pure luck (α) or by choosing a lottery that induces such an outcome, he will enter the CM to work and to accumulate some positive money balance. Likewise, consider an agent, either in the DM or CM with the highest possible initial balance of $\bar{m}$. Again, with non-zero probability, she will decumulate that balance, either by matching and spending that balance down in the DM, or, by working less and consuming more in the CM. These conditions, are sufficient, by Theorem 2 of Hopenhayn and Prescott (1992), for the Markov operator (2.17) to have a unique fixed point. Thus, regardless of an initial distribution of agents, the recursive operation on the initial distribution converges (in the weak* topology) to the same long run distribution G.[43]

Third, the LHS of (2.16), viz. $(1/\omega)$, is clearly continuous in $\omega \in (0 + \infty)$ and is a downward sloping parabola in the interior of $(0, +\infty)$. The market clearing condition (2.16) is continuous on the RHS: (1) The integrand is clearly continuous in m; and, (2) by Propositions 1 and 2, agents policy functions are continuous in m. By Theorem 1 via Equations (B.2) and (B.7) (evaluated at a stationary state $\omega = \omega_{+1}$), demand for real money balance is continuous in ω. Thus, the distribution $G(\cdot; \omega)$ is continuous in ω in the sense of convergence in the weak* topology (Stokey and Lucas, 1989, Theorem 12.13)—*i.e.*, if $\omega_n \to \omega^\star$, then for each $\omega_n \in \{\omega_n\}_{n \in \mathbb{N}}$, the Markov operator (2.17) defines a (weakly) convergent sequence of distributions: $G(\cdot; \omega_n) \to G(\cdot; \omega^\star)$. The RHS is strictly positive valued for all $\omega \in (0 + \infty)$ since agents' policy functions are non-negative valued and $G(\cdot; \omega)$ is a non-degenerate probability measure. Since the RHS is continuous, finite and positive in ω, and the LHS is a downward-sloping parabola with values in $(0 + \infty)$, then there must be at least one intersection point $\omega^\star \in (0 + \infty)$.

The three parts above establish that a SME exists. □

E Algorithm for finding a SME

The following algorithm presumes the more general setting from Section A, which allowed for a new parameter $\alpha \in [0, 1]$. We compute a SME as follows.

1. Fix a guess ω and guess $\bar{V}(\cdot, \omega) \in \mathcal{V}[0, \bar{m}]$.
2. Solve for CM policy and value functions:
   - We know $C^\star(m, \omega) = \bar{C}^\star$ already using Equation (B.9).
   - For fixed $\bar{C}^\star$, and, given guess of $\bar{V}(\cdot, \omega)$, iterate on Bellman Equation (B.4) solving a one-dimensional (1D) optimization problem over choice $y^\star(\cdot, \omega)$.
     - Note: By Equation (B.7), the solution $y^\star(\cdot, \omega) = \bar{y}^*(\omega)$ should be a constant with respect to m.
   - Back out $l^\star(m, \omega)$ using the binding budget constraint in Equation (B.8).
   - Store value function $W^\star(\cdot, \omega)$.
3. Solve for DM policy and value functions:

[43] Alternatively, one could check the more relaxed set of necessary and sufficient conditions of Kamihigashi and Stachurski (2014, Theorem 2) to guarantee that there is a unique distribution for a given ω, in a steady state SME.

- For each $m \leq k$, set
  - $b^\star(m,\omega) = x^\star(m,\omega) = q^\star(m,\omega) = 0$
  - $B(m,\omega) = \beta\bar{V}[\phi(m,\omega), \omega_{+1}]$,

  where $\phi(m,\omega) := (m+\tau)/(1+\tau\omega)$.
- For each $m \in [k, \bar{m}]$,
  - Invert first-order condition (C.2) to obtain implicit $b[m, x(m,\omega), \omega]$.
  - Plug the implicit expression for $b[m, x(x,\omega), \omega]$ into Bellman Equation (2.13), and do a 1D optimization over choices $x(m,\omega)$.
  - Get optimizer $x^\star(m,\omega)$ and corresponding value $B^\star(m,\omega)$.
  - Use previous step to now back out $b^\star(m,\omega)$.

4. Solve ex-ante decision problem:
   - Given approximants $W^\star(m,\omega)$ and $B^\star(m,\omega)$, solve the lottery problem (2.6) and (2.7).
   - Get policies $\left\{\pi_1^{j,\star}(m,\omega)\right\}_{j\in J}$ and $\left\{z_1^{j,\star}(m,\omega), z_2^{j,\star}(m,\omega)\right\}_{j\in J}$, where J is endogenous to the solution of (2.6) and (2.7).
   - Get value of the problem (2.6) and (2.7) as $V^\star(\cdot,\omega)$.
5. Construct the approximant of the ex-ante value function, $\bar{V}^\star(\cdot,\omega) = (1-\alpha)V^\star(\cdot,\omega) + \alpha W^\star(\cdot,\omega)$.
6. Given policy functions from Steps 2-4, construct limiting distribution $G(\cdot,\omega)$ solving the implicit Markov map (2.17). (See details in Section F on page OA.F-27.)
   - Check if market clearing condition (2.16) holds.
   - If not,
     - generate new guess and set $\omega \leftarrow \omega_{new}$;
     - set $\bar{V}(\cdot,\omega) \leftarrow \bar{V}^\star(\cdot,\omega)$; and
     - repeat Steps 2-6 again until (2.16) holds.

Algorithm 1 on page OA.F-26 summarizes the steps above with reference to function names in our actual `Python` implementation. Algorithm 1 is called `SolveSteadyState` in our `Python` class file `cssegmod.py`.

E.1 A novel computational scheme

We solve for a SME following the pseudocode E. Recall that the directed search problem makes the value function $\tilde{V}(\cdot,\omega)$ non-concave. Since there may exist lotteries that make agents better off than playing pure strategies over participating in DM (as buyer) or CM (as consumer/worker), we have to devise a means for finding these lotteries that convexify the graph of the function $\tilde{V}(\cdot,\omega)$.[44]

An existing way to do this in the literature is to use a grid $\mathrm{M}_g := \{0 < \cdots < \bar{m}\}$ to approximate the function's original domain of $[0,\bar{m}]$. Then, around each finite element of M_g, one must check if

[44]Interestingly, there is parallel similarity between our problem here and those in computational dynamic games. In the latter, non-convexities may sometimes arise in equilibrium payoff sets, and convexification of these payoff correspondence images are rationalized through a public randomization (sunspot) device, instead of lotteries or behavior strategies (see, e.g., Kam and Stauber, 2016).

Algorithm 1 Solving for an SME

Require: $\alpha \in [0, 1)$, $\omega > 0$, $\bar{V}(\cdot, \omega) \in \mathcal{V}[0, \bar{m}]$, $N_{\max} > 0$

for $n \leq N_{\max}$ **do**
  $\left(W^\star, \bar{C}^\star, l^\star, y^\star\right) \leftarrow$ `WorkerProblem`$(\bar{V}, \omega)$
  $(B^\star, b^\star, x^\star, q^\star) \leftarrow$ `BuyerProblem`$(\bar{V}, \omega)$
  $\tilde{V} \leftarrow \max\{B^\star(\cdot, \omega), W^\star(\cdot, \omega)\}$
  $\left(V^\star, \{z^{\star,j}, \pi^{\star,j}\}_{j \in J}\right) \leftarrow$ `ConvexHull` $\left[\text{graph}\left(\tilde{V}\right)\right]$
  $\bar{V}^\star \leftarrow \alpha W^\star + (1 - \alpha) V^\star$
  $\mathbf{v} \leftarrow (\bar{V}, B, W)$
  $\mathbf{p} \leftarrow \langle \{\pi_1^{j,\star}, z_1^{j,\star}, z_2^{j,\star}\}_{j \in J}, (b^\star, x^\star, y^\star, \bar{C}^\star) \rangle$
  $G \leftarrow$ `Distribution`$(\mathbf{p}, \mathbf{v})$
  $\omega^\star \leftarrow$ `MarketClearing`(G)
  $e \leftarrow \max\left\{\|\bar{V}^\star - \bar{V}\|, \|\omega^\star - \omega\|\right\}$
  if $e < \varepsilon$ **then**
    STOP
  else
    $(\bar{V}, \omega) \leftarrow (\bar{V}^\star, \omega^\star)$
    CONTINUE
  end if
end for
  return $\mathbf{p}, \mathbf{v}, G, \omega^\star$

there is a linear segment that *approximately* convexifies graph $\left[\tilde{V}(\cdot, \omega)\right]$.[45] This approximation scheme works fine when we only have a lottery where the lower bound in M_g is included, *i.e.*, a lottery on a set like $\{z_1, z_2\}$, where $z_1 = 0$, and, $z_2 < \bar{m}$. It becomes less accurate when lotteries may exist on upper segments of the function, *i.e.*, lotteries on sets like $\{z_1', z_2'\}$, where $0 < z_1' < z_2' < \bar{m}$, but we have no prior knowlege of what z_1' is. This is because in practice (on the computer) it is not feasible to implement this check which is meant to be done at every element on the domain $[0, \bar{m}]$, not its approximant M_g. As a result, its implementation on M_g may be prone to introducing non-negligible approximation errors, especially when the mesh of M_g is coarse. Thus, one would have to make M_g very fine, but, this will come at the cost of the overall SME solution time.

Instead, we propose a novel alternative here. We can exploit the property that $\tilde{V}(\cdot, \omega)$ has a bounded and convex domain, so then there exists a smallest convex set that contains $g\tilde{V} := \text{graph}\left[\tilde{V}(\cdot, \omega)\right]$, *i.e.*, conv $\left(g\tilde{V}\right)$. This set is indeed graph $\left[\bar{V}(\cdot, \omega)\right]$. We utilize `SciPy`'s interface to the fast `QHULL` algorithm to back out a finite set of coordinates representing the convex hull, *i.e.*, graph $\left[\bar{V}(\cdot, \omega)\right]$. Given these points, we approximate the function $\bar{V}(\cdot, \omega)$ by interpolation on a chosen continuous basis function. We use the family of linear B-splines available from `SciPy`'s `interpolate` class for this purpose. As a residual of this exercise, we can very quickly and directly determine the entire set of possible lotteries that exists with minimal loss of precision, for any given non-convex/concave function $\tilde{V}(\cdot, \omega)$.[46]

[45] See part (v) of the proof of Theorem 3.5 in Menzio et al. (2013) for an exact theoretical underpinning of this scheme. We thank Amy Sun for sharing her `MATLAB` code for Menzio et al. (2013) which confirms this usage.

[46] Detailed comments on how this is done can be found in the method `V` in our `Python` class `cssegmod.py`. We implement our solution in pure `Python` 3.10 (with OpenMPI parallelization of the agent decision problems on 24 logical cores). We have only tested our solutions on a Dell Precision T7810 workstation (with Intel Xeon E5-2680 v3, 2.50GHz, processors) running on the Centos 7.2 GNU/Linux operating system. In all our experiments, we have monotone convergence towards a unique SME solution, regardless of initial guesses on ω and $\bar{V}(\cdot, \omega)$. The average time taken to find the SME is between 90 to 120 seconds, given our hardware setting.

F Monte Carlo simulation of stationary distribution

We use a Monte Carlo method to approximate the steady-state distribution of agents at each fixed value of the aggregate state ω, in the `Distribution` step in Algorithm 1 on page OA.F-26. Again, the following algorithm presumes the more general setting from Section A, which allowed for a new parameter $\alpha \in [0,1]$.

For any current outcome of an agent named (m, ω) we can evaluate her ex-post optimal choices in either the CM (2.4), or the DM (2.5). The outcomes of the decision at each current state for an agent is summarized in Algorithm 2. In words, these go as follows: First, we must identify where the agent is currently in (DM or CM). Second, we evaluate the corresponding decisions and record the agent's end-of-period money balance as m'. Associated with each realized identity m we would also have a record of the agent's actions in that period, e.g., $y^\star(m,\omega)$ and $l^\star(m,\omega)$ if the agent was in the CM, or, $x^\star(m,\omega)$ and $b^\star(m,\omega)$ if she was in the DM.

Algorithm 2 `ExPostDecisions( )`

Require: ω, $(B, W) \leftarrow \mathbf{v}$, $(b^\star, x^\star, y^\star) \leftarrow \mathbf{p}$

if $W(m,\omega) \geq B(m,\omega)$ **then**
    $m' \leftarrow y^\star(m,\omega)$
else
    Get $u \sim \mathbf{U}[0,1]$
    if $u \in [0, b^\star(m,\omega)]$ **then**
        Get $x^\star(m,\omega) > 0$
        Get $b^\star(m,\omega) > 0$
        $m' \leftarrow m - x^\star(m,\omega)$
    else
        $x^\star(m,\omega) \leftarrow 0$
        $b^\star(m,\omega) \leftarrow 0$
        $m' \leftarrow m$
    end if
end if
      return m'

Algorithm 2 is then embedded in Algorithm 3 (the Monte Carlo approximation of the steady state distribution at ω) below. We begin, without loss, from an agent who had just accumulated money balances at the end of a CM, and track the evolution of the agent's money balances over the horizon $T \to +\infty$. Theorem 1 implies that if ω is any candidate equilibrium price, and $G(\cdot,\omega)$ is the unique limiting distribution of agents associated with the candidate equilibrium, then the agent will visit each of all possible states $(m,\omega) \in \mathrm{supp}G(\cdot,\omega)$ with frequency $\mathrm{d}G(m,\omega)$, as $T \to +\infty$.

Algorithm 3 does the following:

1. Begin with an arbitrary agent m.
2. At the start of each date $t \leq T$:
   (a) The agent realizes the shock $z \sim (\alpha, 1-\alpha)$.
   (b) Conditional on the shock z, the agent goes to the CM for sure (and costlessly), or, makes the ex-ante lottery decision.
   (c) If the agent has to solve the ex-ante decision problem, then we evaluate the corresponding ex-post decisions of the agent.

The main output of Algorithm 3 is the list m^T, which stores the stochastic realization of an agent's money balances each period. The long run distribution of the sample m^T is used to approximate $G(\cdot, \omega)$. Algorithms 2 and 3 can be found in the `Python` class `cssegmod.py,` respectively, as methods `ExPostDecisions` and `Distribution`.

Note that the function `Distribution` will be called each time we have an updated guess of ω. Because the Monte-Carlo problem is serially dependent, the only way to speed up the evaluations at this point is to compile it to machine code and execute it on the fly. The user will have the option to exploit `Numba` (a `Python` interface to the `LLVM` just-in-time compiler).

Algorithm 3 `Distribution( )`

Require: $\mathbf{v} \leftarrow (\bar{V}, B, W)$, $\mathbf{p} \leftarrow \langle \{\pi_1^{j,\star}, z_1^{j,\star}, z_2^{j,\star}\}_{j\in J}, (b^\star, x^\star, y^\star, \bar{C}^\star) \rangle$, T, ω

Get $\phi(m,\omega) \leftarrow \frac{m+\tau}{(1+\tau\omega)(1-\delta)}$
Set $m^T \leftarrow \emptyset$
$m \leftarrow y^\star(0,\omega)$
for $t \leq T$ **do**
  $m \leftarrow \phi(m,\omega)$
  Get $u \sim \mathbf{U}[0,1]$
  if $u \in [0,\alpha]$ **then**
    $m' \leftarrow y^\star(m,\omega)$
  else
    if $\exists j \in J$ and $m \in \left[z_1^{j,\star}(m,\omega), z_2^{j,\star}(m,\omega)\right]$ **then**
      Get $u \sim \mathbf{U}[0,1]$
      if $u \in \left[0, \pi_1^{j,\star}(m,\omega)\right]$ **then**
        $m \leftarrow z_1^{j,\star}(m,\omega)$
      else
        $m \leftarrow z_2^{j,\star}(m,\omega)$
      end if
    end if
    $m' \leftarrow$ `ExPostDecisions`$(m,\omega,\mathbf{p},\mathbf{v})$
  end if
  Set $m^T \leftarrow m^T \cup \{m\}$
  Set $m \leftarrow m'$
end for
  return m^T

G Sample SME outcome for an agent

Figure 17 on page OA.G-29 shows a subsample of an agent's outcome for the baseline economy. Corresponding to the DM/CM patterns of spending, we can also observe the subsample's evolution of money balances, in the panel with its vertical axis labelled m, in Figure 17. Here, we can see that at $t=0$, the agent has his initial real balance as some m. He decides to be in the DM, succeeds in matching with a trading post, and spends a fraction of the balance to consume some positive q. In the following period $t=1$, he begins with some positive balance—because of transfer $\tau/(\omega(1+\tau)) > 0$ combined with his residual balance—but this amount land in the lottery region; and so the agents plays the lottery. He realizes the high prize of $z_2 = 0.52$ in $t=1$, and so his money balance is z_2. He matches and gets to consume $q > 0$. (Hence, the record $q_1, x_1, b_1 > 0$.) A similar event realizes again in $t=2$, so the agent again gets to consume in the DM. In $t=3$, having spent his balance on consuming in the DM in the previous period, the agent realizes a low lottery payoff ($z_1 = 0$), and his initial balance is thus zero. The agent enters the CM, works, repays the entry cost, consumes, and saves some money balance. That is why we see a record of -1 for the figure panel labelled "match status" for $t = 3$. Subsequently in $t = 4$, he begins again with positive balance from the last CM trade. At this point, he decides to go shopping in the DM and again, spends it all in one round. he wins the high prize in the lottery, and finds it optimal to enter the CM to work.

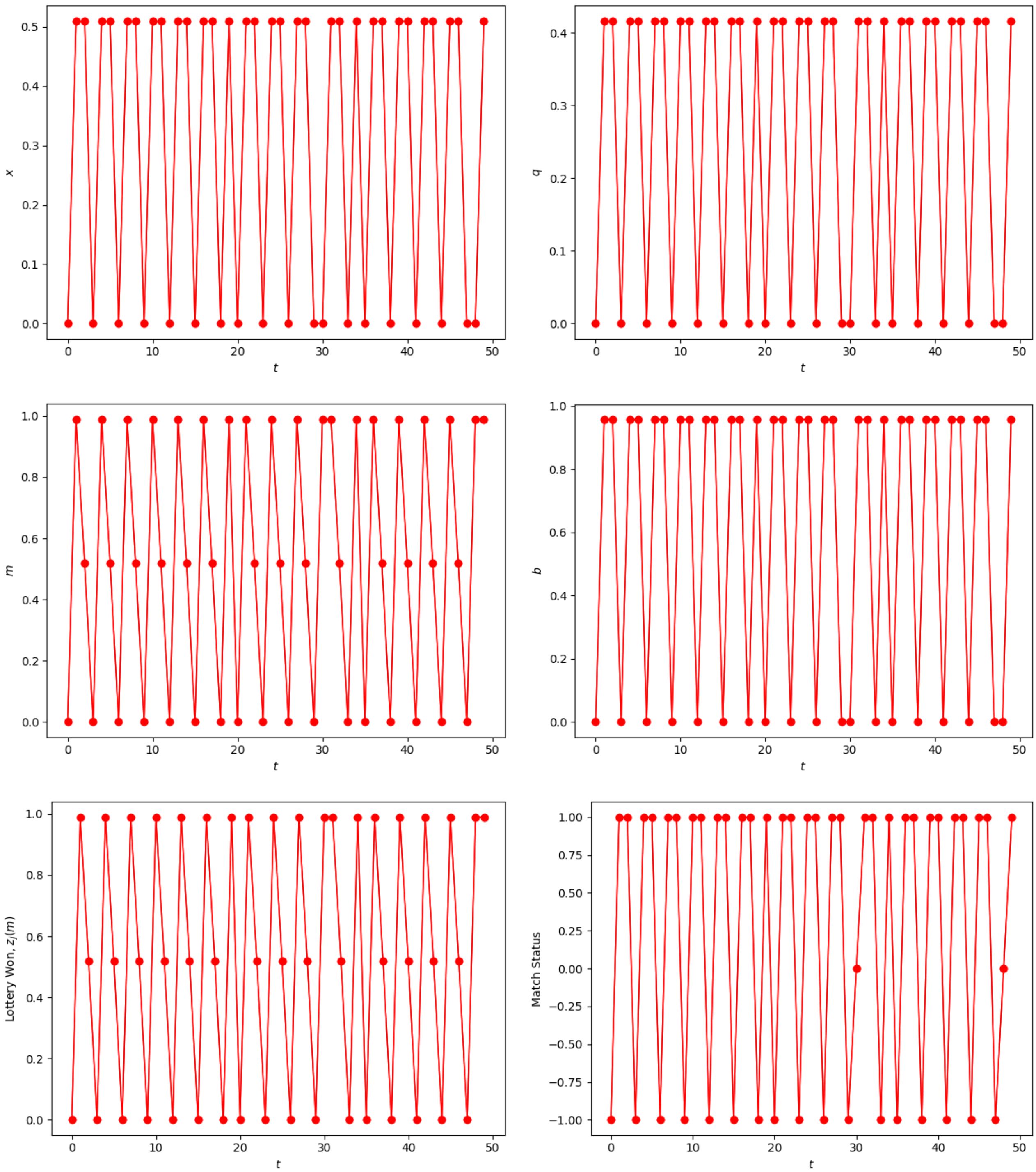

Figure 17: Agent sample path (Benchmark economy). Match Status: 0 (No Match in DM), 1 (Match in DM), -1 (in CM).

In summary, we can observe the following from our simulation: Agents can trade more than once in the DM sometimes. This depends on their lottery outcomes. Agents must also pay a fixed cost to enter the CM to load up on money balances. Depending on their money balance, they may sometimes find it worthwhile to borrow against their CM income to pay the fixed cost of CM entry. Thus, we have an equilibrium Baumol-Tobin type of money spending cycle in the model. Since agents endogenously do not have complete consumption insurance, the pattern of consumption in the DM, q in Figure 17, is not completely smooth.

H Inflation and overall distribution of money and prices

To illustrate this, we compare just the two stationary-equilibrium regimes, SME($\tau = 0$) and SME($\tau = 10$). In Figures 18 and 19 below, the distributions' support contain real money balances at the start of each period t.

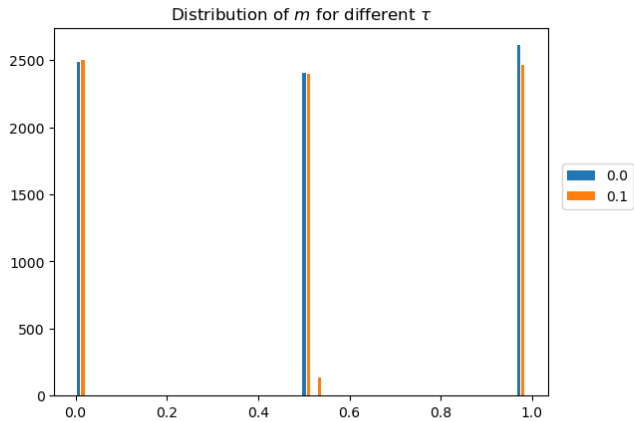

Figure 18: Money distribution and inflation for under SME($\tau = 0$) and SME($\tau = 10$).

We observe that intermediate-level money holders spread out in their masses in Figure 18 as inflation goes from 0% (blue/darker bars) to 10% (orange/lighter bars) *per annum*. This explains the additional masses in the intermediate region of the money distribution support. The measure of the highest-balance agents go down with inflation as agents work less. Additional lotteries can arise as agents in DM expect to trade faster. There is less mass in the intermediate m levels as agents trade faster out of the DM, and for agents who choose to enter the CM, the additional lottery outcomes push them toward the extreme edges of the high/low prize outcomes. This observation is associated with the extensive margin effect so that with higher inflation, there can be a higher inequality in m-wealth distribution.

At 0% inflation, despite the possibility of different submarkets in the DM, it turns out that there is no price dispersion. However, at 10% inflation, there is a marked distribution of pricing outcomes.

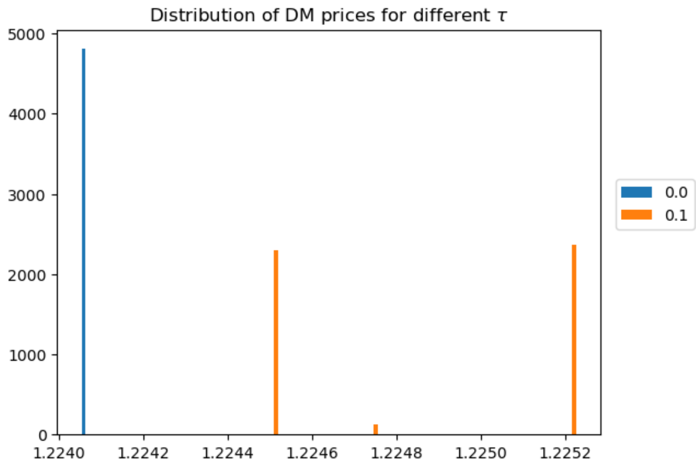

Figure 19: DM (submarkets) pricing function (p) and inflation. Distribution of DM pricing outcomes $p(m)$ for under SME($\tau = 0$) and SME($\tau = 10$).