How to derive Aiyagari's NBL

Let $w := w(e)$ and $R := 1+r$

Budget constraint:

$$c_t + a_{t+1} \leq w e_t + R a_t$$

Question to ask: What is the most one could borrow sustainably (i.e., in a P.V. budget feasibility sense)?

Let this maximal borrowing at date t be denoted by $\underline{a}_t$.

Intuitively, if I hit the "shortest" asset position $\underline{a}_t$ now, the best I need to do is not to consume any of my current and future endowments:

$$\boxed{c_t = 0 \quad \text{for all } t \geq 0}$$

The P.V. budget constraint is then:

$$0 + \underline{a}_{t+1} = we_t + R\underline{a}_t$$

$$\Rightarrow \quad \underline{a}_t = \frac{1}{R}\left[\underline{a}_{t+1} - we_t\right]$$

$$= \frac{1}{R}\left[\frac{1}{R}\left(\underline{a}_{t+2} - we_{t+1}\right) - we_t\right]$$

$$= \frac{1}{R^2}\left[\frac{1}{R}\left(\underline{a}_{t+3} - we_{t+2}\right) - we_{t+1}\right] - \frac{we_t}{R}$$

$$\vdots$$

$$\Rightarrow \quad \boxed{\underline{a}_t = -\frac{1}{R}\sum_{j=0}^{\infty}\frac{we_{t+j}}{R^j}}$$

NBL ≡ Natural borrowing limit

If $c_t = 0$ for $t \geq 0 \Leftrightarrow a_t = \underline{a}_t$

Then $c_t > 0$ for $t \geq 0 \Leftrightarrow a_t > \underline{a}_t$

So, for $c_t \geq 0$:

$$a_t \geq \underline{a}_t \equiv -\frac{1}{1+r} \sum_{j=0}^{\infty} \frac{w(e)\, e_{t+j}}{(1+r)^j}$$

as in the slides, p. 10.

A Karush - Kuhn - Tucker problem:

Given r:

$$v(z,e) = \max_{\hat{a}'} \left\{ \begin{array}{l} u(z-\hat{a}') + \beta \mathbb{E}\{v(z',e')|e\} \\ \\ \text{s.t.} \\ (1)\ \ z' = we' + (1+r)\hat{a}' - r\phi \\ (2)\ \ \hat{a}' \leq z \\ (3)\ \ 0 \leq \hat{a}' \end{array} \right\}$$

$\Gamma(z,s)$

def: (on p.15)

We can write this as a KKT problem:

$$v(z,e) = \max_{\hat{a}'} \Big\{ u(z-\hat{a}') + \beta \mathbb{E}\{ v[\overbrace{we' + (1+r)\hat{a}' - r\phi}^{\overset{(1)}{\equiv}}, e'] | e\} - \bar{\lambda}[\hat{a}' - z] - \underline{\lambda}[0 - \hat{a}'] \Big\}$$

Envelop BS:

$$v_z(z,e) = u_c(z-\hat{a}') \equiv u_c(c)$$

$$\Rightarrow v_z(z',e') = u_c(z'-\hat{a}'') \equiv u_c(c') \quad (4)$$

FOCs

$$-u_c(c) + \beta \mathbb{E}\left\{ v_z(z',e')\cdot(1+r) \,\middle|\, e \right\} - \bar{\lambda} + \underline{\lambda} = 0 \quad (5)$$

KKT Complementary slackness:

$$\bar{\lambda} \geq 0, \quad \bar{\lambda}(\hat{a}'-z) = 0 \quad (6)$$

$$\underline{\lambda} \geq 0 \quad \underline{\lambda}\,\hat{a}' = 0 \quad (7)$$

Assume $u(\cdot)$ s.t. we always have $\hat{a}' < z$ $\Leftrightarrow \bar{\lambda} = 0$. [i.e., one never optimal choose $c=0$!]

So we focus on $\underline{\lambda} \geq 0$.

- $\underline{\lambda} > 0 \Leftrightarrow \hat{a}' = 0$ (ad-hoc BL binding)

- $\underline{\lambda} = 0 \Leftrightarrow \hat{a}' > 0$ (not)

(4), (5) and (7) together imply:

$$u_c(c) = \beta(1+r)\,\mathbb{E}\{u_c(c')\,|\,e\} \quad \text{if} \left(\underline{\lambda} = 0 \;\overset{\hat{a}' > 0}{\Updownarrow}\right)$$

$$u_c(c) > \beta(1+r)\,\mathbb{E}\{u_c(c')\,|\,e\} \quad \text{if} \left(\underline{\lambda} > 0 \;\underset{\hat{a}' = 0}{\Updownarrow}\right)$$

As seen on slide 19