LM1

Neoclassical optimal growth / accumulation for finite T.

Assume:

u continuous	same for f
$u' > 0$	
$u'' < 0$	
$\lim_{c \to +\infty} u'(c) = 0$	
$\lim_{c \to 0} u'(c) = +\infty$	

$k_0 > 0$ given

Problem: Let $\tilde{f}(k_t) := \overbrace{f(k_t) + (1-\delta)k_t}^{\text{total resources at end of } t}$

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{T}} \left\{ \sum_{t=0}^{T} \beta^t u(c_t) : \begin{array}{l} k_{t+1} = \tilde{f}(k_t) - c_t \\ k_0 \text{ given} \\ c_t \geq 0,\ k_{T+1} \geq 0 \end{array} \right\}$$

Verbosely,

$$\max_{\substack{(k_1, k_2, \ldots, k_{T+1}) \\ (c_0, c_1, \ldots, c_T)}} \left\{ \begin{array}{ll} u(c_0) + \beta u(c_1) + \ldots + \beta^T u(c_T): & \\ \lambda_0: & k_1 = \tilde{f}(k_0) - c_0 \\ \lambda_1: & k_2 = \tilde{f}(k_1) - c_1 \\ & \vdots \\ \lambda_T: & k_{T+1} = \tilde{f}(k_T) - c_T \\ & k_0 \text{ given} \\ \mu_T: & c_t,\ k_{T+1} \geq 0. \end{array} \right\}$$

Assume $u(\cdot)$ s.t. c_t will always be interior

Lagrange function:

$$u(c_0) + \beta u(c_1) + \dots + \beta^T u(c_T)$$

$$- \lambda_0 [k_1 - \tilde{f}(k_0) + c_0]$$

$$- \lambda_1 [k_2 - \tilde{f}(k_1) + c_1]$$

$$\vdots$$

$$- \lambda_T [k_{T+1} - \tilde{f}(k_T) + c_T]$$

$$- \mu_T [0 - k_{T+1}]$$

More compact notation:

date-0 shadow price of wealth

$$\sum_{t=0}^{T} \beta^t u(c_t) - \sum \lambda_t [k_{t+1} - \tilde{f}(k_t) + c_t]$$

$$- \mu_T k_{T+1}$$

Karush-Kuhn-Tucker conditions:

$$
\left.\begin{aligned}
c_0 &: \quad u'(c_0) - \lambda_0 = 0 \\
c_1 &: \quad \beta u'(c_1) - \lambda_1 = 0 \\
&\quad\vdots \\
c_T &: \quad \beta^T u'(c_T) - \lambda_T = 0
\end{aligned}\right\} \quad (T+1) \text{ equations}
$$

$$
\left.\begin{aligned}
\lambda_0 &: \quad k_1 = \tilde{f}(k_0) - c_0 \\
&\quad\vdots \\
\lambda_T &: \quad k_{T+1} = \tilde{f}(k_T) - c_T
\end{aligned}\right\} \quad (T+1) \text{ equations}
$$

$$
\left.\begin{aligned}
k_1 &: \quad -\lambda_0 + \lambda_1 \tilde{f}'(k_1) = 0 \\
&\quad\vdots \\
k_{T+1} &: \quad -\lambda_T + \mu_T = 0 \\
\\
\mu_T k_{T+1} &= 0.
\end{aligned}\right\} \quad (T+1) \text{ equation}
$$

Simplify, we have:

(1) $\underbrace{\beta^t u'(c_t)}_{\lambda_t} = \underbrace{\beta^{t+1} u'(c_{t+1})}_{\lambda_{t+1}} \cdot \tilde{f}'(k_{t+1}),$ $\quad t = 0, \ldots, T,$

(2) $k_{t+1} = \tilde{f}(k_t) - c_t,$ $\quad t = 0, \ldots, T,$

and

(3) $\beta^T u'(c_T)\, k_{T+1} = 0 \Rightarrow k_{T+1} = 0$

since $u'(\cdot) > 0$ everywhere.

(4) $k_0 > 0$ given.

Note: 2^{nd} order difference equation with two boundary conditions.

$(1) \Rightarrow (T+1)$ equations $\quad \overbrace{c_0, c_1, \dots, c_T}^{(T+1)\text{ unknowns}}$

$(2) \Rightarrow (T+1)$ equations $\quad \underbrace{k_1, \dots, k_{T+1}}_{(T+1)\text{ unknowns}}$

$(3), (4) \Rightarrow 2$ restrictions $\quad \underbrace{k_0, \; k_{T+1}}$

disciplines optimal path $\{k_t\}_{t=0}^{T+1}$

BJ3 (Sundaram, Ch. 11) Note 7.2

IHDP RCK example

0.) Solution (stationary Markovian policy)

$$k_{t+1} = \pi(k_t)$$

$$c_t = h(k_t) = \tilde{f}(k_t) - \pi(k_t)$$

1) Both π, h are ~~monotone~~, nondecreasing functions.

2) So, π induces a monotone sequence on real numbers: $\{k_{t+1}(k_0)\}_{t=0}^{\infty}$.

Likewise, $\{c_t(k_0)\}_{t=0}^{\infty}$

These are bounded sequences. (Why?)

3) Property of monotone sequences that are bounded?

$\boxed{\text{MCT}}$: $X = \{(k_{t+1}, c_t)(c_0)\}_{t=0}^{\infty}$ monotone,

has a limit (k_∞, c_∞)

$\Updownarrow$

X bounded.

4) Show that

$$(k_\infty, c_\infty) = (k_{ss}, c_{ss})$$

$\uparrow$

unique.

(Use optimality conditions, eval at s.s.)

See similarity to Solow-Swan dynamics?