Aiyagari's Uninsured Idiosyncratic Risk and Aggregate Saving

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Content

Roadmap

Roadmap for this topic:

1. Empirical regularities or stylized facts
2. Quantitative and Theoretical explanation

1. Observed wealth inequality
2. Is observed capital accummulation due to precautionary saving?
3. Low risk free rate puzzle (Huggett model)

Theory

Idea:

- Incomplete asset markets model.
- For a given $q = 1/(1 + r)$, agents with idiosyncratic or individuals states optimize.
- Resulting aggregate distribution over agent assets-endowment pairs reflecting different histories of individual endowment shocks.
- This is what we learnt from Huggett (1993, JEDC).

- Aiyagari; production economy. H-A version of Brock-Mirman (1972, JET) stochastic growth model.
- Now, the aggregate of agents' assets must equal the total capital stock in the economy.
- Stationary equilibrium: (since $\exists$ only idiosyncratic shocks), the aggregate prices, i.e wage and capital rental rates w and r, are now constants.
- A neoclassical aggregate theory:

$$r = MPK - \delta, \qquad w = MPN$$

both depends on aggregate capital stock.

Agents

Each measure-zero agent's objective is to maximize

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t), \qquad \beta = \frac{1}{1+\rho} \in (0,1),$$

subject to

$$c_t + a_{t+1} = w(r)e_t + (1+r)a_t$$

and

$$c_t \geq 0, \qquad a_{t+1} \geq -\phi,$$

almost surely (a.s.).

An event happens almost surely (a.s.) if it happens with probability one. The analog of this is "almost everywhere (a.e.)" in measure theory.

Note:

- $e \in E \subset \mathbb{R}$
- $a \in A = [-\phi, +\infty)$

Warning

- Explicit $w = w(r)$!
- Equilibrium will require that r be a function of aggregate capital ...
- The outcome of r will also depend on aggregate saving or accumulation of capital.
- The outcome of r will also depend on the firm's demand for capital.

Aiyagari's borrowing limits. Recall budget constraint:

$$c_t + a_{t+1} = w(r)e_t + (1+r)a_t$$

If we require $c_t \geq 0$ a.s., then solving the above forward, we have

$$a_t \geq -\frac{1}{1+r}\sum_{j=0}^{\infty}\frac{w(r)e_{t+j}}{(1+r)^j} \qquad \text{a.s. } \forall t \geq 0$$

Let $\min(e) := \underline{e}$. Then we have also Aiyagari's natural debt limit:

$$a_t \geq -\frac{w \cdot \underline{e}}{r}$$

We can allow for tighter borrowing limits. Aiyagari does so by defining the borrowing constraint:

$$a_t \geq -\phi$$

where

$$\phi = \begin{cases} \min\left\{b, \frac{w \cdot \underline{e}}{r}\right\} & \text{for } r > 0 \\ b & \text{for } r \leq 0. \end{cases}$$

where $b > 0$ is an *ad hoc* credit limit (*á la* Huggett's arbitrary $\underline{a}$).

Individual state variables

- In Aiyagari, the presentation of the model assumes that $(e_t)_{t\in\mathbb{N}}$ is an i.i.d. stochastic process.
- More generally e_t may be an autocorrelated random variable, so then in an agent's problem there is an anticipated component of this process.
- Following Aiyagari, we do a change of state variables as follows ...
- Define maximal disposable wealth at time t as

$$z_t := we_t + (1+r)a_t + \phi;$$

i.e. current labor income + current savings + maximal borrowings.

Individual state variables

- Maximal disposable wealth at time t is $z \in Z \subset \mathbb{R}$:

$$z_t := we_t + (1+r)a_t + \phi;$$

- For convenience later, work with:

$$\hat{a}_t := a_t + \phi.$$

- So then we can write

$$z_t = we_t + (1+r)\hat{a}_t + r\phi$$

- Index an individual by individual state (z_t, e_t).

Recursive representation of budget constraint

Denote $\hat{a}' := \hat{a}_{t+1}$, $z' := z_{t+1}$. Agent's current claims on maximal disposable wealth in consumption or saving:

$$c + \hat{a}' \leq z.$$

Agent's next-period maximal disposable wealth becomes:

$$z' = we' + (1+r)\hat{a}' - r\phi.$$

- Borel σ-algebra $\mathcal{B}(A)$ generated by A.
- Agent's feasible action correspondence, $\Gamma(r) : Z \times E \rightrightarrows \mathcal{B}(A)$:

$$\Gamma(z, e; r) = \left\{ \hat{a}' \in A : z - \hat{a}' \geq 0, \hat{a}' \geq 0 \right\}.$$

Agent's recursive problem

- Let $v(z,e;r)$ be the optimal value to an agent (z,e) following his optimal strategy beginning from (z,e).
- The Bellman functional equation is:

$$v(z,e;r) = \max_{\hat{a}' \in \Gamma(z,s;r)} \Big\{ u(z - \hat{a}') + \beta \int_E v(z',e';r) d\pi(e') : \\ z' = we' + (1+r)\hat{a}' - r\phi \Big\}.$$

- Relative price r is pinned down in a stationary equilibrium.

Solution is a savings policy function

$$\hat{a}' = A(z, e; r).$$

for a given b. This has properties proven in Theorem 1 of Huggett.

We left e in the state variable purely as an information variable for predicting

- future employment e',
- and therefore $z' = we' + (1+r)\hat{a}' - r\phi$, conditional on realized action $\hat{a}'$.
- In special case $e \sim$ i.i.d. π, e is not a vital statistic for predicting future e'.

As in Huggett,

- the equilibrium decision rule $A(z, e; r)$, and
- transition probability function for e, $\pi(e'|e)$,

can be used to construct an equilibrium transition probability function P. that maps a current probability measure over agent types $\psi(\mathcal{Z} \times \mathcal{E})$ into next period's probability measure $\psi'(B)$, where

- $\mathcal{Z} \times \mathcal{E} \in \mathcal{B}(Z) \times \mathcal{B}(E)$,
- $B \in \mathcal{B}(Z) \times \mathcal{B}(E)$

Optimal consumption can be calculated as

$$c = z - A(z, e),$$

and must satisfy

$$u_c(c) \geq \beta(1+r)\mathbb{E}\{u_c(c')|e\}, \qquad \text{“=” if } \hat{a}' > 0.$$

How does the mean of assets behave w.r.t. r in this model?

How does the mean of assets behave w.r.t. r in this model? Trick:

- Define $M_t = \beta^t (1+r)^t u'(c_t) \geq 0$.
- Then,

$$M_{t+1} - M_t = \beta^t (1+r)^t [\beta(1+r) u'(c_{t+1}) - u'(c_t)].$$

- Re-write the Euler inequality as:

$$\mathbb{E}_t(M_{t+1}) \leq M_t.$$

- So M_t is a supermartingale relative to $(\{\mathcal{F}_t\}, \psi)$, where $\mathcal{F}_t \in \mathcal{B}(E)$ is the time-t information set.

Trick:

- Applying Doob's martingale convergence theorem, M_t converges almost surely (w.p.1) to a nonnegative random variable, $\overline{M}$.
- Note: $\overline{M}$ itself is an integrable function.

Three cases:

1. $\beta(1+r) > 1$
2. $\beta(1+r) < 1$
3. $\beta(1+r) = 1$

Case 1 and 3: $\beta(1+r) \geq 1$ or $r \geq \rho$

- $M_t \to \overline{M}$ a.s., implies that $u'(c_t)$ converges a.s. to zero.
- If u unbounded, then $c_t \to \infty$ and $a_t \to \infty$.
- Similar for Case 3: Chamberlain and Wilson (2000).

Intuition:

- An artifact of infinite horizon decision process.
- If $r > \rho$, then each agent will want to delay consumption to future, and be current lender. Consumption profile is upward sloping, and to finance this, accumulate an infinitely large quantity of assets, so then, $\Rightarrow \mathbb{E}_w a \nearrow \infty$.
- Borderline case, $r = \rho$, agents wants to smooth marginal utility of consumption. At the margin, costless to obtain extra unit of asset. But there is a positive probability of a sequence of bad shocks (e), so to maintain consumption smoothing, need arbitrarily large quantity of assets, so then, $\Rightarrow \mathbb{E}_w a \nearrow \infty$.

Case 2: $\beta(1+r) < 1$ or $r < \rho$

- This leaves open possibility that $u'(c_t)$ does not converge a.s..
- $u'(c_t)$ remains finite.
- So does c_t and a_t.

These imply an aggregate asset supply schedule, $\mathbb{E}_{w(r)}(a(r))$, for fixed w ...

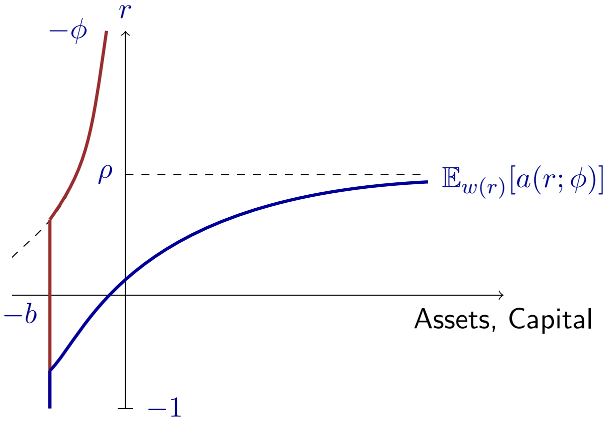

Figure: Illustration of $\mathbb{E}_{w(r)}a(r;\phi)$ for a fixed w although $w = w(r)$.

Technology: $(K, N) \mapsto F(K, N)$. Takes w and r as given. Profit:

$$F(K, N) - wN - (r + \delta)K$$

Neoclassical firm with profit maximizing conditions:

1. $r = F_K(K, N) - \delta$
2. $w = F_N(K, N)$

General Equilibrium

Market clearing requires

$$K = \int_{Z \times E} a(z, e; r) d\psi(r)$$

and

$$N = \int_{Z \times E} e d\psi(r)$$

Capital market clearing intersects our supply $\mathbb{E}_{w(r)}(a(r)) =: \mathbb{E}a(r)$ schedule with the capital demand schedule implicit in $r = F_K(K, N) - \delta$. Also $w = F_N(K, N)$ will be pinned down. It can be shown that $\mathbb{E}_w a$ shifts right with $w = w(r)$. Why?

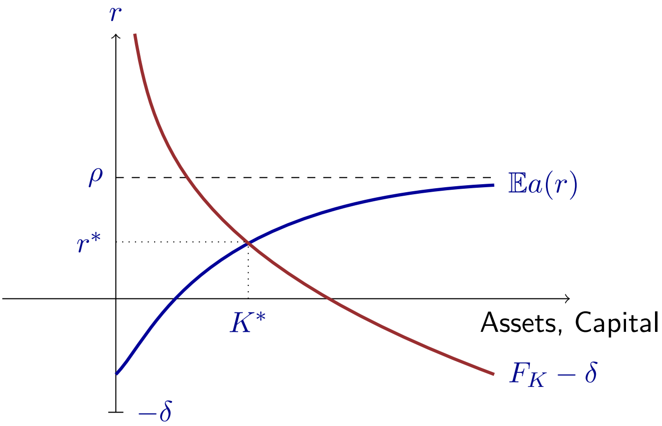

Figure: Stationary equilibrium: $\mathbb{E}a(r) := \mathbb{E}_{w(r^*)}a(r)$ and $F_K(K,N) - \delta$ determines equilibrium capital stock $K(r^*)$ and r^*. Note; since w is decreasing in r, $\mathbb{E}a(r)$ may well be non-monotonic. Equilibrium may not be unique. No analytical results – need to check numerically.

Exercise

Argue that if there were full insurance on labor risk (i.e. certainty), then the risk-free rate is not as low and equilibrium capital stock is lower.

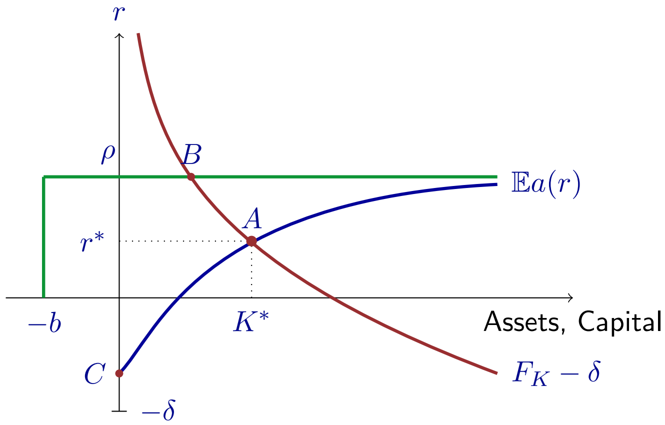

Figure: Comparing allocations between model equilibrium with idiosyncratic risk (A), and model equilibrium with full insurance (B).

Remark:

- At A: Equilibrium of economy with uninsurable idiosyncratic risk and no insurance (NI). Equilibrium (K^*, r^*).
- With full insurance (FI), representative agent result. Receive a constant earnings $w \cdot 1$ each period. If $r < \rho$, representative agent would always be borrowing constrained, $a = -\phi$. If $r = \rho$, agent's asset equals whatever initial asset he has. If $r > \rho$, $a \to \infty$.
- So FI asset supply schedule (desired asset holding) is given by green right-angled line.

- At B: FI equilibrium $K_{FI} < K^*$.
- Economy with uninsurable idiosyncratic risk and borrowing constraint produces precautionary saving motive. Too much saving and capital.
- Note: Equilibrium at C consistent with a partial equilibrium model such as Huggett.

Reading List

1. Aiyagari (1994, QJE)
2. Aiyagari (1992, FRB Minneapolis Working Paper)
3. Ljungqvist and Sargent (2004) 17.3 - 17.7