Solving the RCK example: Brute-force value function iteration

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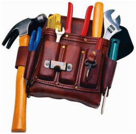
Outline

What you need to own



- Batteries not included:
  1. Optimization (Math A for Economists)
  2. High school calculus
  3. High school (elementary) algebra

Previously ...

Oh Joy! [Apologies to Roy Lichtenstein]

Amore a prima vista I

- Sequence problem was:

$$v(k_0) = \max_{\{k_{t+1} \in \Gamma(k_t)\}_{t \in \mathbb{N}}} \left\{ \sum_{t=0}^{\infty} \beta^t \ln(k_t^{\alpha} - k_{t+1}) : k_0 \text{ given} \right\}.$$

- Claimed (without basis): We could rewrite problem as the following *R*ecursive *F*unctional mapping:

$$v(k) = \max_{k_+ \in \Gamma(k)} \left\{ \ln(k^{\alpha} - k_+) + \beta v(k_+) : k \text{ given} \right\}.$$

Amore a prima vista II

- At first glance ...
- This seems an easy problem to solve, given that we no longer have to choose infinite list $\{k_{t+1}\}_{t \in \mathbb{N}}$.
- We now just choose a finite object k_+ (i.e. k_{t+1}) as some function, say g_t of the current decision state k (i.e. k_t).
- $k_{t+1} = g_t(k_t)$ is just a solution to the right-hand-side finite-dimensional maximization problem.
- Easy, no?

Amore a prima vista III

But we still have a **problem** to deal with. ...

What is the function v?

... We actually don't know what v looks like.

Amore a prima vista IV

- Is this an improvement? We couldn't solve the infinite sequence Lagrangean problem directly. So we re-wrote it as the RFM above. But now we don't know what is v.
- So now, our goal shifts to one of finding v first.
- If we can find v we can solve for the optimal decision function(s) g_t.
- Observe that v is a **function**. Mathematically, it is an element in an infinite dimensional space!
- Turns out we can solve for this v, at least "approximately". How good is the approximation is what we'll find out during this course.

Value function iteration I

Now we are ready to illustrate the technique—a.k.a. successive value function approximation using the Bellman functional equation (operator).

IDEA:

- Suppose we don't know what v is. Can we inductively find an approximation of v?
- Consider at any current k, we write

$v_{T+1}(k_{T+1}) = 0$
$\forall k_{T+1}$

$$\underline{\underline{v_{n+1}(k)}} = \max_{k_+ \in \Gamma(k)} \left\{ \underline{\ln(k^\alpha - k_+)} + \beta v_n(k_+) : k \text{ given} \right\}.$$

- Inductively construct a sequence of approximations to the mysterious function v, denoted by $(v_0, v_1, \ldots)$, where each function $v_n : X \to \mathbb{R}$, is indexed by $n = 0, 1, \ldots$.
- Is it true that $\lim_{n\to\infty} v_n \to v$?

Value function iteration II

The following steps will illustrate our proposed solution strategy.

Step $n = 0$. Guess that $v_0(k_+) = 0$ for all k_+. Then,

$$v_1(k) = \max_{k_+ \in \Gamma(k)} \left\{ \ln(k^\alpha - k_+) : k \text{ given} \right\}.$$

This is looking good. It's just a static maximum problem. Given k fixed, the maximum is attained by choosing $k_+ = g_0(k) = 0$.

So the value of the problem is

$$v_1(k) = \alpha \ln(k).$$

Value function iteration III

Step $n = 1$. Use the last result, i.e. $v_1(k_+) = \alpha(k)$ for all k_+, then,

$$\begin{aligned} v_2(k) &= \max_{k_+ \in \Gamma(k)} \left\{ \ln(k^\alpha - k_+) + \beta v_1(k_+) : k \text{ given} \right\} \\ &= \max_{k_+ \in \Gamma(k)} \left\{ \ln(k^\alpha - k_+) + \beta \left[\alpha \ln(k_+)\right] : k \text{ given} \right\}. \end{aligned}$$

Value function iteration IV

This is still looking good. It's just a two-period maximum problem. Given k fixed, the maximum is attained by choosing k_+ s.t.:

$$-\frac{1}{(k^\alpha - k_+)} + \beta \frac{\alpha}{k_+} = 0.$$

Solving for k_+ as a function of k we have

$$k_+ = \frac{\alpha\beta}{1+\alpha\beta} k^\alpha =: g_1(k).$$

The value of this problem is

$$v_2(k) = \ln\left(\frac{1}{1+\alpha\beta}\right) + \alpha\beta \ln\left(\frac{\alpha\beta}{1+\alpha\beta}\right) + \alpha(1+\alpha\beta)\ln(k).$$

Value function iteration V

Step $n = 2$. Now using the known v_2, find v_3. Try this as an exercise!

Notice that as you work along each iteration n,

- the sequence of value functions have a particular form: $v_n(k) = A_n + B_n \ln(k)$.
- Corresponding to each known v_n, we have an optimal decision rule of the form: $k_+ = C_n k^\alpha =: g_n(k)$.
- the coefficients (A_n, B_n, C_n) are functions of the model's underlying microeconomic parameters (α, β).

Value function iteration VI

Why don't we try for $n \geq 3$?

Value function iteration VII

Exercise

Show inductively that, at each $n \geq 2$ using the value function derived from the last $(n-1)$-th iteration, we can derive the corresponding optimal decision rule as

$$k_+ = \alpha\beta \left[\frac{1-(\alpha\beta)^n}{1-(\alpha\beta)^{n+1}} \right] k^\alpha =: g_n(k).$$

And we can derive the updated $(n+1)$-th value function as

$$\begin{aligned} v_{n+1}(k) = & \underline{\sum_{m=0}^{n} \beta^m \ln \left(\frac{1}{1+\alpha\beta+\cdots+(\alpha\beta)^{n-m}} \right)} \\ & + \underline{\sum_{m=0}^{n-1} \Big\{ \beta^m \alpha\beta (1+\alpha\beta+\cdots+(\alpha\beta)^{n-1-m})} \\ & \qquad \times \left[\ln \left(\alpha\beta \frac{1+\alpha\beta+\cdots+(\alpha\beta)^{n-1-m}}{\underline{1+\alpha\beta+\cdots+(\alpha\beta)^{n-m}}} \right) \right] \Big\} \\ & + \alpha \left[\sum_{m=0}^{n} (\alpha\beta)^m \right] \ln(k). \qquad (\dagger) \end{aligned}$$

Value function iteration VIII

- Consider the sequences of the three terms (functions) indexed by n, that make up v_{n+1}.
- As we take $n \to \infty$, the first term on the right of $(\dagger)$, has the limit

$$\frac{\ln(1-\alpha\beta)}{1-\beta},$$

  since $\beta \in (0,1)$ and $\alpha \in (0,1)$, so that $\alpha\beta \in (0,1)$.
- As we take $n \to \infty$, the second term on the right of $(\dagger)$, has the limit

$$\frac{\alpha\beta}{(1-\alpha\beta)(1-\beta)} \ln(\alpha\beta).$$

Value function iteration IX

- Finally, as we take $n \to \infty$, the last term on the right of (†), has the limit

$$\lim_{n\to\infty} \alpha \left[\sum_{m=0}^{n} (\alpha\beta)^m \right] \ln(k) = \frac{\alpha}{1-\alpha\beta} \ln(k).$$

- These three terms — (a sum of two) constant and slope terms, respectively A_n and B_n, of v_n — all converge monotonically as $n \to \infty$.
- Moreover, each successive function is converging geometrically fast.

Value function iteration X

Punchline:

- We have illustrated: from a naïve guess of the value function v_n,
- Using the Bellman operator, approximate v as a limit of a sequence of updated value function approximations:

  $$v(k) = \lim_{n \to \infty} v_n,$$

  and
- in this analytical example,

$$v(k) = \frac{1}{1-\beta}\left\{\ln(1-\alpha\beta) + \frac{\alpha\beta}{(1-\alpha\beta)}\ln(\alpha\beta)\right\} + \frac{\alpha}{1-\alpha\beta}\ln(k).$$

Policy Function I

Now that we have found v we can solve the finite-dimensional problem:

$$v(k) = \max_{k_+ \in \Gamma(k)} \left\{ \ln(k^\alpha - k_+) + \beta v(k_+) : k \text{ given } \right\}.$$

The first order condition yields the decision rule

$$k_+ = \alpha\beta k^\alpha =: g(k).$$

If we take the limit as $n \to \infty$, the sequence of approximate decision rules $\{g_n\}$ associated with $\{v_n\}$ has the limit

$$\lim_{n\to\infty} g_n(k) = \lim_{n\to\infty} \alpha\beta \left[\frac{1-(\alpha\beta)^n}{1-(\alpha\beta)^{n+1}}\right] k^\alpha = \alpha\beta k^\alpha =: g(k).$$

So the sequence of approximate decision rules $\{g_n\}$ also converges monotonically and the rate of convergence is geometric.

Policy Function II

Exercise

In this example, can you characterize the optimal trajectory of the economy $\{k_{t+1}(k_0)\}_{t\in\mathbb{N}}$*, given* k_0*?*

Qualitatively, what does it look like? Does it look qualitatively similar to the Solow-Swan model you studied as undergraduates?

In what way does it differ from that model?

Remarks and Lookahead I

- So the solution method above:
    - gave us a form for the value function v.
    - Once we know v the problem appears to be a simple two-period optimization problem.
    - The solution to that problem is a decision function g.
    - The optimal decision function induces the optimal trajectory of the economy (by recursion), given an initial state: $k_+ = g(k)$.