Computing the Aiyagari (QJE94) Model

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Content

Roadmap

Roadmap for this topic:

1. Model
2. Stationary Equilibrium
3. Computational Approaches

Aiyagari's Model

- Aiyagari; production economy. H-A version of Brock-Mirman (1972, JET) stochastic growth model.
- Now, the aggregate of agents' assets must equal the total capital stock in the economy.
- Stationary equilibrium: (since $\exists$ only idiosyncratic shocks), the aggregate prices, i.e wage and capital rental rates w and r, are now constants.
- A neoclassical aggregate theory:

$$r = MPK - \delta, \qquad w = MPN$$

both depends on aggregate capital stock.

Agents: borrowing limit

- Labor endowment, $e \in E \subset \mathbb{R}$; Asset $a \in A = [-\phi, +\infty)$.
- Recall budget constraint:

$$c_t + a_{t+1} = w(r)e_t + (1+r)a_t$$

If we require $c_t \geq 0$ a.s., then solving the above forward, we have

$$a_t \geq -\frac{1}{1+r}\sum_{j=0}^{\infty}\frac{w(r)e_{t+j}}{(1+r)^j} \qquad \text{a.s. } \forall t \geq 0$$

- Let $\min(e) := \underline{e}$. Then we have also Aiyagari's natural debt limit:

$$a_t \geq -\frac{w \cdot \underline{e}}{r}$$

Agents: borrowing limit

We can allow for tighter borrowing limits. Aiyagari does so by defining the borrowing constraint:

$$a_t \geq -\phi$$

where

$$\phi = \begin{cases} \min\left\{b, \frac{w \cdot \underline{e}}{r}\right\} & \text{for } r > 0 \\ b & \text{for } r \leq 0. \end{cases}$$

where $b > 0$ is an *ad hoc* credit limit (*á la* Huggett's arbitrary $\underline{a}$).

Agents: individual state

- Define maximal disposable wealth at time t as

$$z_t := we_t + (1+r)a_t + \phi;$$

i.e. current labor income + current savings + maximal borrowings.

Agents: individual state

- Maximal disposable wealth at time t is $z \in Z \subset \mathbb{R}$:

$$z_t := we_t + (1+r)a_t + \phi;$$

- For convenience later, work with:

$$\hat{a}_t := a_t + \phi.$$

  So then we can write

$$z_t = we_t + (1+r)\hat{a}_t + r\phi$$

- Index an individual by individual state (z_t, e_t).

Agents: Feasible Sets

- Borel σ-algebra $\mathcal{B}(A)$ generated by A.
- Agent's feasible action correspondence, $\Gamma(r) : Z \times E \rightrightarrows \mathcal{B}(A)$:

$$\Gamma(z, e; r) = \left\{ \hat{a}' \in A : z - \hat{a}' \geq 0, \hat{a}' \geq 0 \right\}.$$

Agents: Markov decision process

- Let $v(z, e; r)$ be the optimal value to an agent (z, e) following his optimal strategy beginning from (z, e).
- The Bellman functional equation is:

$$v(z, e; r) = \max_{\hat{a}' \in \Gamma(z,e;r)} \left\{ u(z - \hat{a}') + \beta \int_E v(z', e'; r) d\pi(e'|e) : \right. \\ \left. z' = w(r)e' + (1 + r)\hat{a}' - r\phi \right\}.$$

- Relative price r is pinned down in a stationary equilibrium.

Solution is a savings policy function

$$\hat{a}' = A(z, e; r).$$

for a given b. This has properties proven in Theorem 1 of Huggett.

We left e in the state variable purely as an information variable for predicting

- future employment e',
- and therefore $z' = we' + (1+r)\hat{a}' - r\phi$, conditional on realized action $\hat{a}'$.
- In special case $e \sim$ i.i.d. π, e is not a vital statistic for predicting future e'.

As in Huggett,

- the equilibrium decision rule $A(z, e; r)$, and
- transition probability function for e, $\pi(e'|e)$,

can be used to construct an equilibrium transition probability function P. that maps a current probability measure over agent types $\psi(\mathcal{Z} \times \mathcal{E})$ into next period's probability measure $\psi'(B)$, where

- $\mathcal{Z} \times \mathcal{E} \in \mathcal{B}(Z) \times \mathcal{B}(E)$,
- $B \in \mathcal{B}(Z) \times \mathcal{B}(E)$

Optimal consumption can be calculated as

$$c = z - A(z, e),$$

and must satisfy

$$u_c(c) \geq \beta(1+r)\mathbb{E}\{u_c(c')|e\}, \qquad \text{“=” if } \hat{a}' > 0.$$

How does the mean of assets behave w.r.t. r in this model?

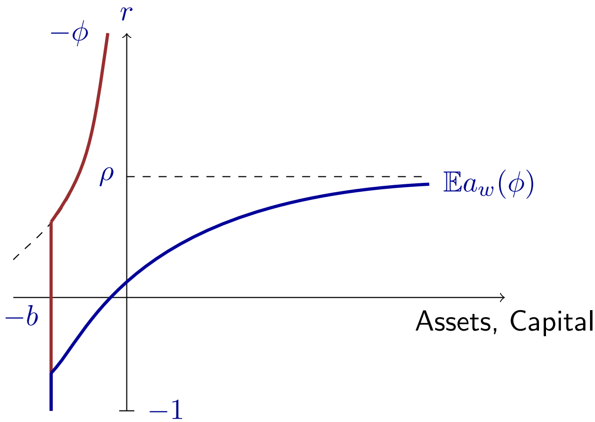

Figure: Typical shape of $\mathbb{E}_{w(r)}a(\phi)$.

Firm

Aggregate factors (K, N). Technology: $(K, N) \mapsto F(K, N)$.
Takes w and r as given. Profit:

$$F(K, N) - wN - (r + \delta)K$$

Neoclassical firm with profit maximizing conditions:

1. $r = F_K(K, N) - \delta$
2. $w = F_N(K, N)$

Technology s.t.:

- $F_K, F_N > 0$
- $\lim_{r \searrow -\delta} F_K(K, N) = +\infty$, $\lim_{r \nearrow \infty} F_K(K, N) = 0$.
- $\lim_{r \nearrow \infty} w(r) = 0$, $\lim_{r \searrow -\delta} w(r) = +\infty$.

Stationary Distributions

Candidate Markov kernel. At each r, define $Q_r : Z \times E \times \mathcal{B}(Z) \times \mathcal{B}(E) \rightarrow [0,1]$ by,

$$Q_r((z,e), \mathcal{Z} \times \mathcal{E}) = \sum_{e' \in \mathcal{E}} \mathbb{1}_{\{A(z,e) \in \mathcal{Z}\}} \pi(e'|e),$$

for all $\mathcal{Z} \times \mathcal{E} \in \mathcal{B}(Z) \times \mathcal{B}(E)$, where

$$\mathbb{1}_{\{A(z,e) \in \mathcal{Z}\}} = \begin{cases} 1 & \text{if } a' = A(z,e) \in \mathcal{Z} \\ 0 & \text{otherwise} \end{cases}.$$

- Let $M(Z \times E)$ be the set of probability measures on $(Z \times E, \mathcal{B}(Z) \times \mathcal{B}(E))$.
- Then Q_r is our candidate transition function with the associated operator $W_r : M(Z \times E) \to M(Z \times E)$, giving

$$\begin{aligned} \psi'(\mathcal{Z} \times \mathcal{E}) &= W_r(\psi)(\mathcal{Z} \times \mathcal{E}) \\ &= \int_{Z \times E} Q_r[(z,e), \mathcal{Z} \times \mathcal{E}] d\psi \end{aligned}$$

  for all $\mathcal{Z} \times \mathcal{E} \in \mathcal{B}(Z) \times \mathcal{B}(E)$.
- For a fixed r, a stationary distribution ψ^* satisfies both sides of this mapping.

General Equilibrium

Market clearing requires

$$K = \int_{Z\times E} a(z,e;r)d\psi(r) =: \mathbb{E}a(r)$$

and

$$N = \int_{Z\times E} ed\psi(r)$$

ψ is the fixed point of the operator W at the equilibrium interest rate r:

$$\psi(B) = W_r(\psi)(B), \quad \forall B \in \mathcal{B}(Z) \times \mathcal{B}(E).$$

Capital market clearing:

- Intersection of aggregate supply $\mathbb{E}_{w(r)}(a(r)) =: \mathbb{E}a(r)$ schedule with the capital demand schedule implicit in $r = F_K(K, N) - \delta$.
- Previously, we took w as a parameter. Now, $w = F_N(K, N)$ will be pinned down in equilibrium.
- It can be shown that $\mathbb{E}_w a$ shifts right with $w = w(r)$. Why?

Stationary Equilibrium: Intuitive

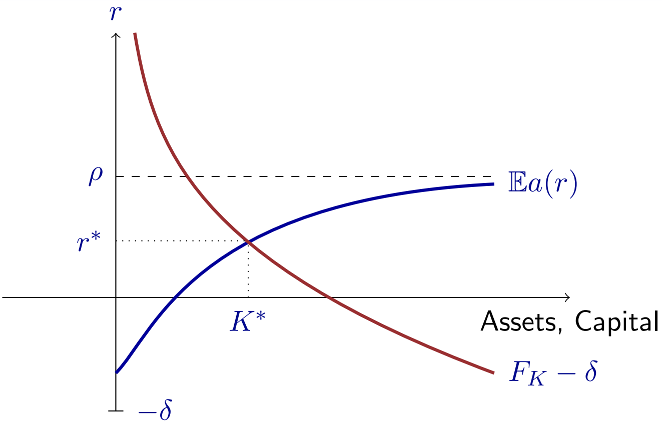

Figure: Stationary equilibrium: $\mathbb{E}a(r)$ and $F_K(K, N) - \delta$ determines equilibrium capital stock $K(r^*)$ and r^*. Note; since w is decreasing in r, $\mathbb{E}a(r)$ may well be non-monotonic. Equilibrium may not be unique. No analytical results – in general need to check numerically. ▸ Example SE

▸ Existence

Exercise

Argue that if there were full insurance on labor risk (i.e. certainty), then the risk-free rate is not as low and equilibrium capital stock is lower.

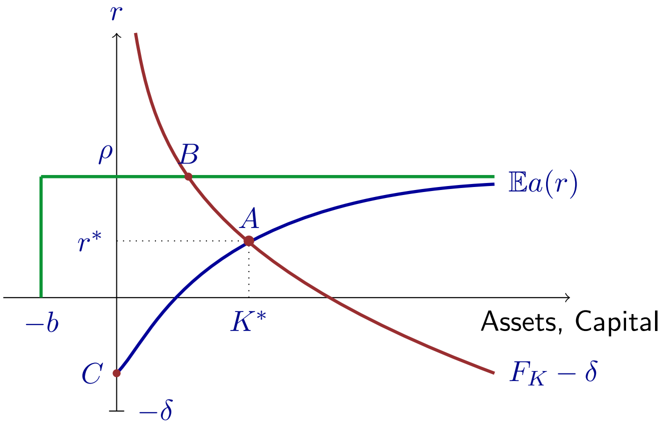

Figure: Comparing allocations between model equilibrium with idiosyncratic risk (A), and model equilibrium with full insurance (B).

Remark:

- At A: Equilibrium of economy with uninsurable idiosyncratic risk and no insurance (NI). Equilibrium (K^*, r^*).
- With full insurance (FI), representative agent result. Receive a constant earnings $w \cdot 1$ each period. If $r < \rho$, representative agent would always be borrowing constrained, $a = -\phi$. If $r = \phi$, agent's asset equals whatever initial asset he has. If $r > \rho$, $a \to \infty$.
- So FI asset supply schedule (desired asset holding) is given by green right-angled line.

- At B: FI equilibrium $K_{FI} < K^*$.
- Economy with uninsurable idiosyncratic risk and borrowing constraint produces precautionary saving motive. Too much saving and capital.
- Note: Equilibrium at C consistent with a partial equilibrium model such as Huggett.
- Equilibrium $r^* \in (-\delta, \rho)$. This will be useful to discipline computational guesses.

Stationary Equilibrium

As we illustrated in the diagram previously ...

Existence comes down to showing that excess demand as a function of price r is

- continuous, and
- crosses zero.

Uniqueness depends on showing that excess demand is strictly monotone.

Stationary Equilibrium: Existence

If we can show that aggregate supply of capital

$$\mathbb{E}a(r) := \int_{Z \times E} A(z, e; r) d\psi(z, e; r)$$

is continuous in r and crosses the aggregate capital demand function, then we have existence.

Let's check this ...

First we consider the limits.

1. (Previous lecture) Aiyagari showed:
    - $\mathbb{E}a(\rho) = +\infty$. In this case, there is no stationary distribution here, since $\mathbb{E}a(\rho)$ not defined.
    - $\lim_{r \nearrow \rho} \mathbb{E}a(r) = +\infty$.

   Also, $\mathbb{E}a(-1) = -\phi$.
2. Technology s.t.:
    - $F_K, F_N > 0$
    - $\lim_{r \searrow -\delta} F_K(K, N) = +\infty$, $\lim_{r \nearrow \infty} F_K(K, N) = 0$.
    - $\lim_{r \nearrow \infty} w(r) = 0$, $\lim_{r \searrow -\delta} w(r) = +\infty$.
    - Need: $K(-1) < \phi$, and, $\lim_{r \nearrow \rho} K(r) > -\infty$

   Guarantees:

$$\mathbb{E}a(-1) < K(-1), \qquad \lim_{r \nearrow \rho} \mathbb{E}a(r) + K(r) > 0,$$

so there is at least one r such that $\mathbb{E}a(r) + K(r) = 0$, if we can show continuity of $\mathbb{E}a(r) + K(r)$.

Example (Huggett 1993)

$K(r) = 0$. So continuity of $\mathbb{E}a(r)$ suffices to establish existence of a stationary equilibrium.

Example (Aiyagari 1994)

Cobb-Douglas technology. Firm solves

$$\max_{K,N} K^{\alpha} N^{1-\alpha} + (1-\delta)K - wN - (1+r)K.$$

Optimal capital demand function in this case is

$$K(r) = \left(\frac{\alpha N^{1-\alpha}}{r+\delta} \right)^{\frac{1}{1-\alpha}}.$$

Check: $\lim_{r \searrow -\delta} K(r) = +\infty$, and, $\lim_{r \nearrow +\infty} K(r) = 0$. So then verifying continuity of $\mathbb{E}a(r)$ establishes existence of stationary equilibrium. ▸ Figure SE

Stationary Equilibrium: Existence

If $K(r)$ continuous, now we just have to check:

- Existence of $\mathbb{E}a(r)$ – i.e. existence of a stationary probability measure ψ; and then,
- Continuity of $\mathbb{E}a(r)$ w.r.t. r.

There is a unique ψ s.t.:

$$\psi(\mathcal{Z} \times \mathcal{E}) = W_r(\psi)(\mathcal{Z} \times \mathcal{E}) = \int_{Z \times E} Q_r[(z, e), \mathcal{Z} \times \mathcal{E}] d\psi$$

for all $\mathcal{Z} \times \mathcal{E} \in \mathcal{B}(Z) \times \mathcal{B}(E)$, if

- Q_r is a transition probability function:
  - $Q_r((z, e), \cdot)$ is a probability measure on $(Z \times E, \mathcal{B}(Z) \times \mathcal{B}(E))$
  - $Q_r(\cdot, B)$ is measurable for all $B \in \mathcal{B}(Z) \times \mathcal{B}(E)$.

  Easy to show since $(z, e) \mapsto A(z, e)$ is continuous function on a compact state space.
- Q_r is an increasing map: need to show that $(z, e) \mapsto A(z, e)$ is jointly increasing, so that "higher" (z, e) makes $Q_r((z, e), \cdot)$ place bigger probability mass on "higher" (z', e').
- Monotone mixing holds.

Weak convergence of probability measures. Then we have

$$\lim_{t\to\infty}\int A(z,e;r)d\psi_t = \int A(z,e;r)d\psi,$$

for every continuous and bounded real-valued function $A : Z \times E \to \mathbb{R}$.

$\mathbb{E}a(r)$ is continuous w.r.t. r:

- Continuity of $A(\cdot; r)$: Can show that $A(\cdot; r)$ converges to $A(\cdot; r^*)$ uniformly since $Z \times E$ is compact (e.g. Thm 3.8 SLP).
- Continuity of $\psi(r)$ in the sense of weak convergence (Thm 12.13 SLP). Assume
  1. $Z \times E$ compact,
  2. If $(z_n, e_n; r_n)_n$ a sequence in $Z \times E \times [-1, \rho)$ converging to $(z, e; r^*)$, then $Q_{r_n}(z_n, e_n, \cdot)$ converges weakly to $Q_{r^*}(z, e, \cdot)$ on $M(Z \times E, \mathcal{B}(Z) \times \mathcal{B}(E))$;
  3. For each r, $\lim_{t\to\infty} \int A(z, e; r) d\psi_t = \int A(z, e; r) d\psi$
- If (r_n) is a sequence converging to r^*, then (ψ_{r_n}) converges weakly to ψ_{r^*}.

Stationary Equilibrium: Existence

Recap:

- The above show that $\mathbb{E}a(r)$ is well defined for $r \in [-1, \rho)$.
  Figure SE
- $\mathbb{E}a(r)$ is continuous w.r.t. r.
- Given "nice" $K(r)$, exists stationary equilibrium.

Stationary Equilibrium: Uniqueness?

- Need to show monotonicity of $\mathbb{E}a(r)$ w.r.t. r. No analytical results.
- Solution: Plot $\mathbb{E}a(r)$ for many r.
- Example with multiple equilibria: model with seigniorage (see Ljungqvist and Sargent 17.11).

Assume E is a finite set, $|E| = m$, and (e_t) is Markov-(P, μ_0). Since leisure does not enter u, labor supply is inelastic. Calculate aggregate labor supply as

$$N = \sum_{i=1}^{m} e_i \mu_i,$$

where $\mu = (\mu_1, ..., \mu_m)$ is ergodic distribution of the e.

N is fixed number – can calculate outside of loops.

Pseudocode

Step 1 Pick initial guess $r_0 \in (-\delta, \rho)$. Set $r = r_0$.

Step 2 Given r, find $K(r)$ satisfying

$$r = F_K(K, N) - \delta.$$

Then find $w(r) = F_N(K(r), N)$.

Step 3 Given relative prices $(r, w(r))$, solve agent's DP problem. Get solutions $A(a, e; r)$ and $C(a, e; r)$ (latter via budget constraint).

Step 4 Given policy $A(a, e; r)$ and $C(a, e; r)$, and P, initial ψ_0, find

$$\psi_r = \lim_{t \to \infty} W_r^t(\psi_0).$$

Step 5 Given aggregate demand for capital $K(r)$, compute

$$\mathbb{E}a(r) = \int_{A \times E} A(a, e; r) d\psi_r.$$

Step 6 Check market clearing condition. Excess demand: $D(r) = K(r) - \mathbb{E}a(r)$. If $D(r) > 0$, then next guess $r' > r$. Else if $D(r) < 0$, then next guess $r' < r$. Can exploit concavity of F here:

$$r' = F_K(\mathbb{E}a(r), N) - \delta.$$

Step 7 Set new r using smoothing, $\lambda \in [0, 1]$:

$$r = \lambda r + (1 - \lambda) r'.$$

Repeat Step 2-7, $n \geq 1$, until $|r_{n+1} - r_n| < \epsilon$, for ϵ small.

Numerical Example

Some results:

ρ	σ	K	r	$Welfare$	$Gini$
0.0000	0.2000	5.6014	0.0395	-3.1781	0.2701
0.0000	0.4000	5.9243	0.0353	-2.9093	0.2568
0.3000	0.2000	5.6429	0.0389	-3.2309	0.3001
0.3000	0.4000	6.1567	0.0325	-3.0001	0.2896
0.6000	0.2000	5.7818	0.0371	-3.4444	0.3671
0.6000	0.4000	6.8281	0.0253	-3.4142	0.3661
0.8000	0.2000	5.8211	0.0366	-4.7170	0.5000
0.8000	0.4000	6.9735	0.0239	-6.2047	0.5160

Reading List

1. Aiyagari (1994, QJE)
2. Aiyagari (1992, FRB Minneapolis Working Paper)
3. Ljungqvist and Sargent (2004) 17.3 - 17.7