ODE Mini-Report Assignment (By Patrice Harapeti)

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Damped Oscillator

A damped oscillator is an type of harmonic oscillator which possesses an identifiable characteristic known as dampening, which acts as a resistive or frictional force to oppose the direction of motion of the oscillator and aims to minimise the total energy of the system.

There are namely three types of damped oscillators. The type of a damped oscillator is determined by it's dampening ratio - which is often expressed as the determinant in the characteristic equation which describes the partial differential.

The first case is where the determinant is positive; also known as the Overdamped case. In the Overdamped case, the system exponentially decays to it's final steady state without oscillating. The larger the dampening ratio, the longer it takes for the oscillator to reach equilibrium.

The second case is where the determinant equals zero; this is known as the Critically damped case. In this case, the system reaches equilibrium as quickly possible without oscillating.

The last case is where the determinant is negative; also known as the Underamped case. This case describes an oscillator which osciallates at a given frequency and eventually reaches equilibrium. The rate at which the oscillator reaches equilibrium is further determined by the dampening ratio.

The angular frequency of an undamped oscillator can be determined by the following equation:

$$\omega = \sqrt{\frac{k}{m}}$$

The Q-Factor or Quality Factor is a scalar value which can be computed to describe how long it takes for a given oscillator to dampen it's motion to equilibrium. The following definition of the Q-Factor described how as the dampening ratio is minimised, the oscillators ability to resist energy loss increases:

$$Q = \frac{1}{2\xi}$$

Problem Specification

The following pertial differential equation describes a damped oscillator. This report will the derivation, implementation, analysis and valdiation of a numerical solution for this system.

$$-\frac{d^2x}{dt^2} - \gamma\frac{dx}{dt} - kx = 0$$

TODO INCLUDE DERIVIATINO OF ANALYTICAL HERE

Numerical Solutions to Systems

TODO make this section more original Briefly discuss why we numerically solve systems of equations A variety of physics problems and systems are not able to be dervied in terms of an analytical answer; due to complexity or non-linearity. In an effort to understand these complex systems, physicists have turned to numerical solutions which quantize and discretize parameters and/or data, such that it can be processed in a digital system such as a computer.

Since the inception of the techqinue to numerical solve problems, many numerical methods have gained popularity due to their ability to provide consistent and convergent solutions to a complex system. Numerical methods can also be classified in various types, but in an effort to keep this report brief select the Euler's method as our numerical method.

REFERENCE: https://en.wikipedia.org/wiki/Euler_method The Euler's method is a widely adopted numerical method which is able to numerically solving one or more ordinary differential equations. The main advantage of using this method is the method's simplicity and ease of implementation. The Euler's method is a first-order method, which means that the local error is proportional to the square of the step size, and the global error is proportional to the step size.

This means that the level of discretization used when determining a step size in the implementation of the Euler's method can be finely chosen to reach a sufficient minimum error criteria. The main limit to minimising the step size will lie in the computational power available during runtime.

Brief discuss on Validation and Convergence We must also consider the accuracy and validity of our numerical solution. A numerical solution is only valid if the accuracy of the solution is known, compared against other criteria such as other numerical solutions, an analytical solution, experimental results, intermediate results or convergence.

Validation of Numerical Solutions

REFERENCE: https://core.ac.uk/download/pdf/41765513.pdf Some types of numericals solutions are best suited for particular kinds of validation. For example, using an analytical solution to validate a numerical solution is only feasible if the analytical solution is simple to implement and derive. In reality, most systems are highly complex and require experimental data to sufficiently validate any of it's numerical solutions. Luckily in our case, we are have derived a simple analytical solution for our system - allowing us to compare the numerical solution against the analytical system by techniques such as converage.

Explore analytical solution with varying parameters and discuss the three cases (underdamped, critically damped, overdamped)

As discussed in the paragrahps above, the local error found in a numerical solution generated from the Euler's method grows proportinally with the step size used when discretizing the input dataset. Therefore, we would expect the local error to decrease linearly (TODO verify linearly) as the step size decreases.

By looking at the graph below, we can see that the Average Absolute Error and Relative Error Percentage both decrease as the step size is decreased. This proves that the Euler's method is an effective numerical method in solving the system; in comparison to the analytical solution. Further, as the both types of error decreases as the the dataset is more granularly discretized, it is clear that the analytical and numerical solution both converge.

Exploring solutions

Systems are often analysed through various techniques relevant to the type of system and behaviour present in the system. In our case, the damped oscillator can be analysed through the lense of Fourier Analysis. In short, Fourier Analysis is the study of representing a function as a sum of trigonometric functions; other through the Fourier Series.

Performing Fourier Analysis via MATLAB can be greatly simplified by using the Fast Fourier Transform method; which produces a Fast Fourier Transformation matrix, which can be applied to a set of input domain to trasform the domain into one of frequency and space. Fourier Analysis can be used on our system to identify key features such as the power, center frequency and decay time of our damped oscillator.

Discuss FWHM and frequency (compare this with calculated oscillation frequency from above)

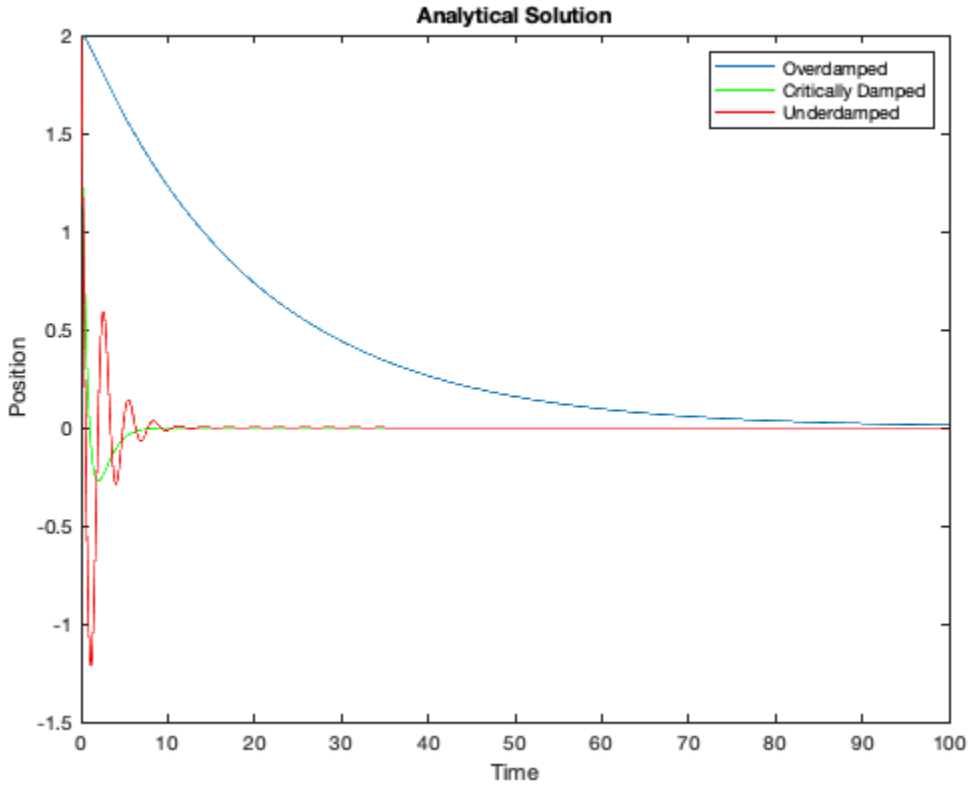
Part Zero : Setup

```
% Clear existing workspace
clear; clc; close all

% Setup parameters
timestep = 0.01; % timestep (seconds)
totalTime = 100; % total time of simulation (seconds)
timeSeries = 0:timestep:totalTime;

% Define initial conditions of system
x_initial = 2; % initial position (metres)
v_initial = 0; % intial velocity (metres / second)
```

Part One : Analytical Solution

```
% Plot analytical solution
figure('NumberTitle', 'off', 'Name', 'Analytical Solution of each
 case');
yline(0, '--');
grid on;

% Overdamped case
plot(timeSeries, generateAnalyticalSolution(timeSeries, 2, 0.1,
 x_initial));
hold on;

% Critically Damped case
plot(timeSeries, generateAnalyticalSolution(timeSeries, 2, 1,
 x_initial), 'g');
hold on;
```

```
% Underdamped case
plot(timeSeries, generateAnalyticalSolution(timeSeries, 1, 5,
 x_initial), 'r');
title('Analytical Solution');
xlabel('Time');
ylabel('Position');
legend('Overdamped', 'Critically Damped', 'Underdamped');
hold off;
```

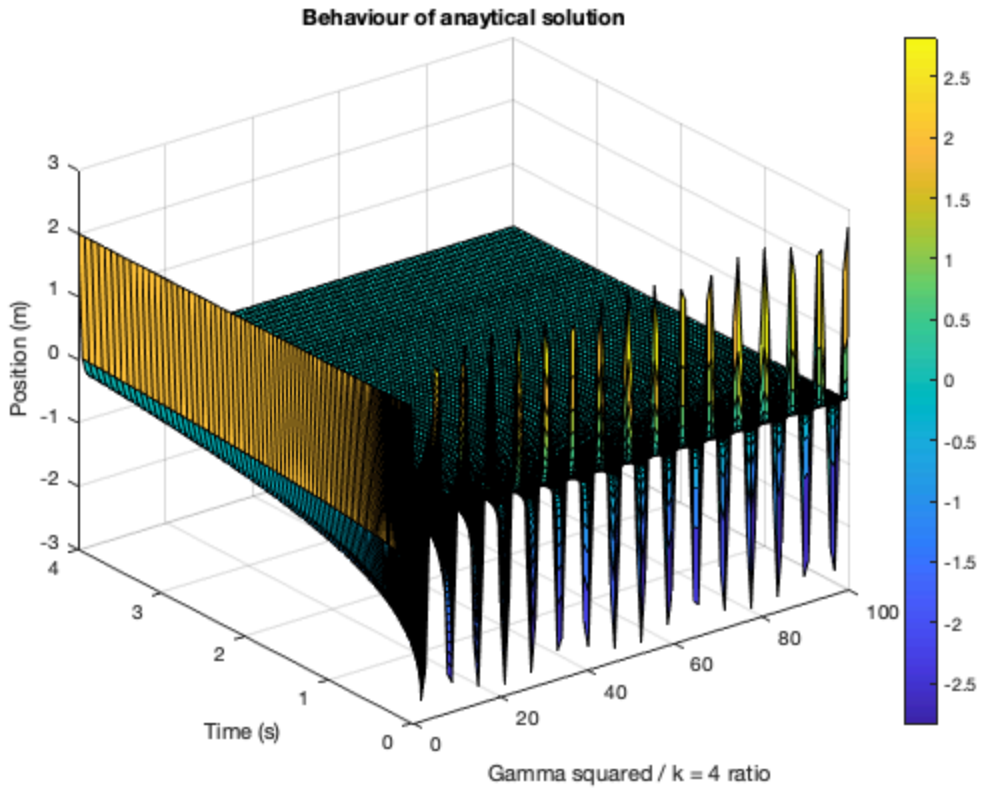


Part Two : Exploration of Analytical Solution

Plot surface plot of gamma/k vs time to visualise how the ratio (dampening) of the parameters affect the function

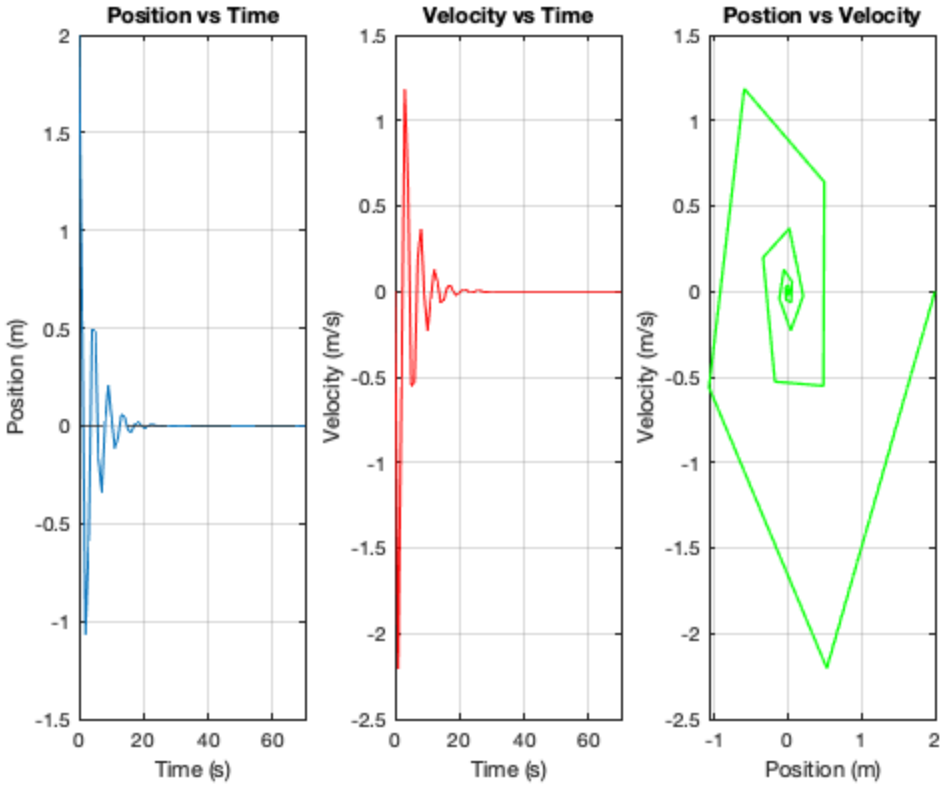
```
% Fix value of parameter k while gamma changes
kExplore = 1;

% Determine number of points required in discretization
noPoints = 100;

timeSeries = linspace(0, totalTime, noPoints);
gammaSeries = linspace(0, 2, noPoints);
ratioSeries = nan(size(gammaSeries));
positionSeries = nan(size(gammaSeries));
```

```
% Build up vectors
for i = 1:length(gammaSeries)
    % Calculate gamma using kExplore
    gamma = gammaSeries(i);

    % Calculate ratio using gamma and kExplore
    ratioSeries(i) = gamma.^2 ./ kExplore;

    % Calculate position based on time and gamma ratio
    positionSeries(i, :) = generateAnalyticalSolution(timeSeries,
 gamma, kExplore, x_initial);
end

% Plot ratio of parameters vs position and time
figure('NumberTitle', 'off', 'Name', 'Function Behavioural Analysis');
surf(timeSeries, ratioSeries, positionSeries);

% Decorate surface plot
colorbar
title('Behaviour of anaytical solution')
xlabel('Gamma squared / k = 4 ratio');
ylabel('Time (s)');
zlabel('Position (m)');

% Adjust camera viewport
%view([-15 3 4]);
```



Part Three : Numerical Solution (TODO USE EULERS OR ANOTHER METHOD)

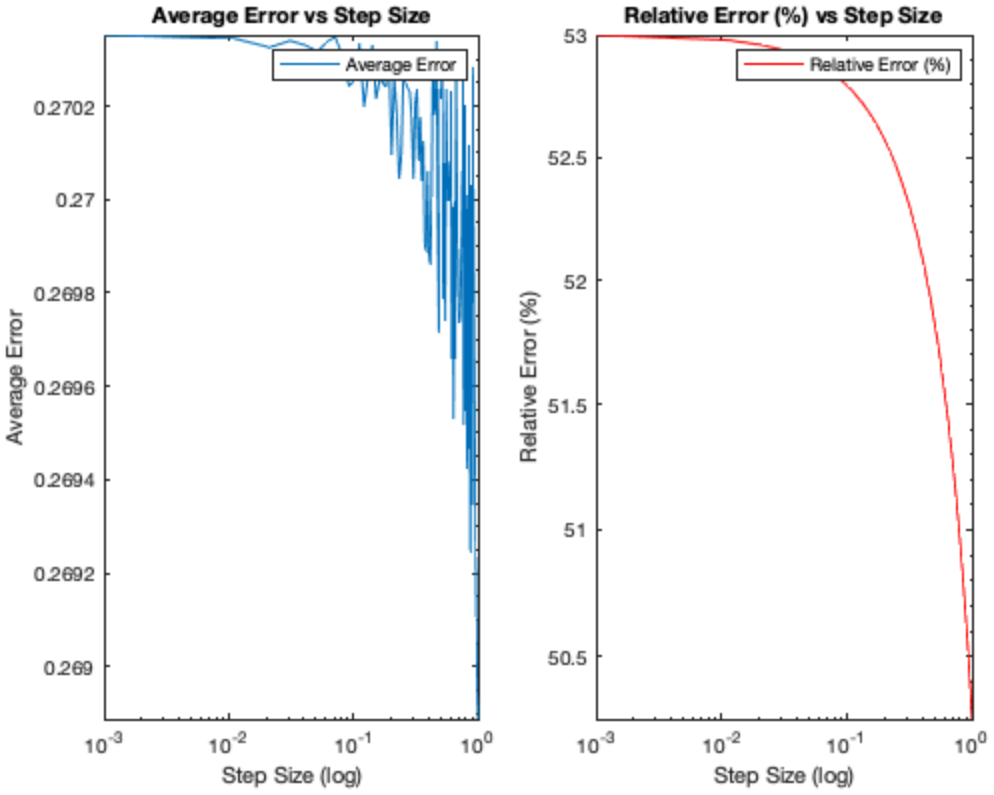
```
% Generate numerical solution for the underdamped case
[numerical_position, numerical_velocity] =
 generateNumericalSolution(timeSeries, 0.5, 2, x_initial, v_initial);

% Plot Position vs Time
figure('NumberTitle', 'off', 'Name', 'Numerical Solution of
 underdamped case');
subplot(1, 3, 1);
plot(timeSeries, numerical_position);

% Draw horizontal line at y = 0 to represent convergence value
yline(0, '--');
grid on;
title('Position vs Time');
xlim([0, 70]);
xlabel('Time (s)');
ylabel('Position (m)');

% Plot Velocity vs Time
subplot(1, 3, 2);
plot(timeSeries, numerical_velocity, 'r');
grid on;
title('Velocity vs Time');
xlim([0, 70]);
xlabel('Time (s)');
ylabel('Velocity (m/s)');

% Plot Position vs Velocity
subplot(1, 3, 3);
plot(numerical_position, numerical_velocity, 'g', 'LineWidth', 1.2);
grid on;
title('Postion vs Velocity');
xlabel('Position (m)');
ylabel('Velocity (m/s)');
```



Part Four : Analyis of error with varying step size

```
% Pick particular case, underdamped in this case
gammaErrorAnalysis = 0.1;
kErrorAnalysis = 3;

% Generate range of step sizes
stepSizes = linspace(0.001, 1);
averageErrorAtStepSize = nan(size(stepSizes));
relativePercentErrorAtStepSize = nan(size(stepSizes));

% Loop over each discretized step size
for i = 1:length(stepSizes)
    % Generate timeseries for discretized total time based on step
 size
    varyingTimeSeries = 0:stepSizes(i):totalTime;

    % Generate analytical solution with timeseries
    analyticalPos = generateAnalyticalSolution(varyingTimeSeries,
 gammaErrorAnalysis, kErrorAnalysis, x_initial);

    % Generate numerical solution with same timeseries
```

```
    [numericalPos, ~] = generateNumericalSolution(varyingTimeSeries,
 gammaErrorAnalysis, kErrorAnalysis, x_initial, v_initial);

    % Calculate error at the current step size
    [averageError, relativeError] = calculateError(analyticalPos,
 numericalPos.');
    averageErrorAtStepSize(i) = averageError;
    relativePercentErrorAtStepSize(i) = relativeError .* 100;
end

% Plot each type of error vs step size
% We expect the error to be reduced as the step size is minimised
figure('NumberTitle', 'off', 'Name', 'Error Analysis');
subplot(1, 2, 1);
loglog(stepSizes, averageErrorAtStepSize);
title('Average Error vs Step Size');
xlabel('Step Size (log)');
ylabel('Average Error');
legend('Average Error');

subplot(1, 2, 2);
loglog(stepSizes, relativePercentErrorAtStepSize, 'r');
title('Relative Error (%) vs Step Size');
xlabel('Step Size (log)');
ylabel('Relative Error (%)');
legend('Relative Error (%)');
```



Part Five : Fourier Analysis of numerical solution

```
% Determine equation parameters for the underdamped case
gammaFourier = 0.5;
kFourier = 2;

% Generate positions of analytical solution
analytical_position = generateAnalyticalSolution(timeSeries,
 gammaFourier, kFourier, x_initial);

% Define new domain to transformed into frequency space
x = linspace(-1,1,length(timeSeries)).' * 10;
power_real = abs(analytical_position).^2;

% Plot real power of analytical solution vs x
figure('NumberTitle', 'off', 'Name', 'Fourier Analysis of critically
 damped case');
subplot(1, 3, 1);
plot(x, power_real, 'Color','#008000');
title('Power vs x');
xlim([-3, 3]);
xlabel('x');
ylabel('Power');
legend('Power');

% Perform Fast Fourier Transform on Analytical Solution
N = length(x); % Number of samples
Y = fft(analytical_position); % Compute Fast Fourier Transformation
dx = mean(diff(x)); % Determine sample spacing
df = 1/(N*dx); % Determine frequency spacing
fi = (0:(N-1)) - floor(N/2); % Generate unfolded index
frequency = df * fi; % Generate frequency vector
power_freq = abs(Y) .^ 2; % Calculate absolute power in frequency
 space

% Plot Power vs Frequency
subplot(1, 3, 2);
plot(frequency, power_freq);
title('Power vs Frequency');
xlabel('Hz');
ylabel('Power');
legend('Power');

% Limit power (in frequency domain) to be positive for frequency
 analysis
frequencyPositive = frequency .* (frequency > 0);
powerPositive = power_freq .* (power_freq > 0);

% Plot power (frequency domain) vs frequency for positive frequencies
subplot(1, 3, 3);
plot(frequencyPositive, powerPositive, '-r');
```

```
title('Power vs Frequency');
subtitle('For positive frequencies');
xlabel('Hz');
ylabel('Power');
legend('Power');
hold on;

% Estimate center frequency in frequency domain and include in plot
powerSum_freq = sum(powerPositive);
weightedPowerSum_freq = sum(powerPositive .* frequencyPositive.');
expectedCenterFrequency_freq = weightedPowerSum_freq ./ powerSum_freq;

% Determine FWHM and include in plot
```

Part Six : Function Definitions

```
function position = generateAnalyticalSolution(timeSeries, gamma, k,
 x_init)
    %   Derivation
    %   Let x = e^bt
    %   therefore... xdot = b * e^bt
    %   therefore... xddot = b^2 * e^bt
    %
    %   Plugging into the original PDE give us...
    %   -b^2 * e^bt - (gamma * b * e^bt) - ke^bt = 0
    %
    %   Pull e^bt out as common factor, this leaves us...
    %   e^bt (-b^2 - gamma*b - k) = 0
    %
    %   Therefore b^2 + gamma*b + k must equal 0
    %   Solving for b
    %   b = (gamma ± sqrt((gamma)^2 - (4k)) / -2
    %
    %   Overdamped when...         gamma^2 - 4k > 0
    %   Critically damped when...  gamma^2 - 4k = 0
    %   Underdamped when...        gamma^2 - 4k < 0

    % Calculate roots of characteristics equations
    b_1 = (-gamma + sqrt(gamma.^2 - (4 .* k))) ./ 2;
    b_2 = (-gamma - sqrt(gamma.^2 - (4 .* k))) ./ 2;

    % Define discriminant of characteristic equation
    discriminant = gamma.^2 - (4 .* k);

    % Define function in three cases based on the determinant of the
 roots
    % Reference: https://nrich.maths.org/11054
    if discriminant == 0 % critically damped
        % Solve for A and B constants
        A = x_init;
        B = x_init .* b_1;

        position = (A + B.*timeSeries) .* exp(b_1 .* timeSeries);
```

```
    elseif discriminant > 0 % overdamped
        % Solve for A and B constants
        A = (x_init .* b_2) ./ (b_2 - b_1);
        B = (x_init .* b_1) ./ (b_1 - b_2);

        position = (A .* exp(b_1 .* timeSeries)) + ...
            (B .* exp(b_2 .* timeSeries));

    else % underdamped if discriminant is less than 0
        % Separate real and imaginary parts of roots
        alpha = real(b_1);
        beta = imag(b_2);

        % Solve for A and B constants
        A = x_init;
        B = -x_init ./ beta;

        position = exp(alpha .* timeSeries) .* ...
            (A.*cos(beta.*timeSeries) + B.*sin(beta.*timeSeries));
    end
end

function [position, velocity] = generateNumericalSolution(timeSeries,
 gamma, k, x_initial, v_initial)
    % Negatively dampened (will converge at y = 0)
    % Also known as an underdamped system
    A = [0 1; -k -gamma];

    % Use ode45 to numerically solve system of equations
    [~, x] = ode45(@(t, x) A * x, timeSeries, [x_initial, v_initial]);
    position = x(:, 1);
    velocity = x(:, 2);
end

function [averageError, relativeError] =
 calculateError(analyticalSolution, numericalSolution)
    % Absolute error between analytical and numerical solution
    absoluteError = abs(analyticalSolution - numericalSolution);

    % Calculate Average Absolute Error
    % Reference: https://sutherland.che.utah.edu/wiki/index.php/
Iteration_and_Convergence
    averageError = norm(absoluteError) ./
 sqrt(length(analyticalSolution));

    % Calculate Relative Error
    relativeError = norm(absoluteError) ./
 norm(abs(analyticalSolution));
end
```

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