Solutions to *Real and Complex Analysis*[*]

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November 13, 2009

Contents

1 Abstract Integration

1. **Exercise.** Does there exist an infinite σ-algebra which has only countably many members?

   Solution. The answer is no. Let X be a measurable set with an infinite σ-algebra $\mathfrak{M}$. Since $\mathfrak{M}$ is infinite, there exists nonempty $E \in \mathfrak{M}$ properly contained in X. Both E and E^c are measurable spaces by letting the measurable subsets of E (resp. E^c) be the intersections of measurable subsets of X with E (resp. E^c). Since $\mathfrak{M}$ is infinite, at least one of these two σ-algebras must be infinite.

   Now we define a rooted binary tree inductively as follows. The root is our set X. Given a vertex which is a measurable subset E of X, if it contains a proper measurable subset E', pick one such subset, and let its two successors be E' and $E \setminus E'$. The remarks above guarantee that this tree is infinite, and hence has infinite depth. So pick an infinite path consisting of subsets $E_0 \supsetneqq E_1 \supsetneqq E_2 \supsetneqq \dots$. Then the sets $F_i = E_i \setminus E_{i+1}$ form an infinite collection of disjoint nonempty measurable subsets of X by construction. At the very least, $\mathfrak{M}$ needs to contain every union of such sets, and this is in bijection with the set of subsets of $\mathbf{N}$, which is uncountable. Thus, $\mathfrak{M}$ must be uncountable. $\square$

2. **Exercise.** Prove an analogue of Theorem 1.8 for n functions.

   Solution. We need to prove the following: if $u_1, \dots, u_n$ are real measurable functions on a measurable space X, and Φ is a continuous map of $\mathbf{R}^n$ into a topological space Y, then $h(x) = \Phi(u_1(x), \dots, u_n(x))$ is a measurable function from X to Y.

   Define $f \colon X \to \mathbf{R}^n$ by $x \mapsto (u_1(x), \dots, u_n(x))$. By Theorem 1.7(b), to prove that h is measurable, it is enough to prove that f is measurable. If R is any open rectangle in $\mathbf{R}^n$ which is the Cartesian

[*]third edition, by Walter Rudin

product of n segments $I_1, \dots, I_n$, then $f^{-1}(R) = u_1^{-1}(I_1) \cap \cdots \cap u_n^{-1}(I_n)$, which is measurable since $u_1, \dots, u_n$ is measurable. Finally, every open set of $\mathbf{R}^n$ is the countable union of such rectangles, so we are done. $\square$

3. **Exercise.** Prove that if f is a real function on a measurable space X such that $\{x \mid f(x) \geq r\}$ is measurable for every rational r, then f is measurable.

   Solution. Let $U \subseteq \mathbf{R}^1$ be an open set. First, U can be written as a union of countably many open balls with rational radii that are centered at rational points. So to prove that $f^{-1}(U)$ is measurable, it is enough to prove this when U is an open ball of this form, say with radius r and center c. Since the set of measurable sets is closed under complements and finite intersections, every set of the form $\{x \mid r_1 > f(x) \geq r_2\}$ is measurable for rational r_1, r_2. Now note that $\{x \mid c + r > f(x) > c - r\}$ can be written as the countable union $\bigcup_{n \geq 1} \{x \mid c + r > f(x) \geq c - r + 1/n\}$, so $f^{-1}(U)$ is measurable. $\square$

4. **Exercise.** Let $\{a_n\}$ and $\{b_n\}$ be sequences in $[-\infty, \infty]$, and prove the following assertions:

   (a) $\limsup_{n\to\infty}(-a_n) = -\liminf_{n\to\infty} a_n$.

   (b) $\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$
   provided none of the sums is of the form $\infty - \infty$.

   (c) If $a_n \leq b_n$ for all n, then
   $$\liminf_{n\to\infty} a_n \leq \liminf_{n\to\infty} b_n.$$
   Show by an example that strict inequality can hold in (b).

   Solution. The supremum A_k of the set $\{-a_k, -a_{k+1}, \dots\}$ is the negative of the infimum A'_k of the set $\{a_k, a_{k+1}, \dots\}$. Hence $\inf_k\{A_k\} = -\sup_k\{A'_k\}$, which implies (a).

   The relation
   $$\sup\{a_k + b_k, a_{k+1} + b_{k+1}, \dots\} \leq \sup\{a_k, a_{k+1}, \dots\} + \sup\{b_k, b_{k+1}, \dots\}$$
   is clear, so this implies (b). To see that the inequality in (b) can be strict, consider $a_1 = 1$, $a_i = 0$ for $i > 1$, and $b_1 = -1$, $b_i = 0$ for $i > 1$. Then $\limsup(a_n + b_n) = 0$, but $\limsup a_n + \limsup b_n = 1$.

   Now suppose that $a_n \leq b_n$ for all n. Then $\inf\{a_k, a_{k+1}, \dots\} \leq \inf\{b_k, b_{k+1}, \dots\}$ for all k, so (c) follows. $\square$

5. **Exercise.**

   (a) Suppose $f\colon X \to [-\infty, \infty]$ and $g\colon X \to [-\infty, \infty]$ are measurable. Prove that the sets
   $$\{x \mid f(x) < g(x)\}, \quad \{x \mid f(x) = g(x)\}$$
   are measurable.

   (b) Prove the set of points at which a sequence of measurable real-valued functions converges (to a finite limit) is measurable.

   Solution. Let Y_+ and Y_- be the sets where $g(x) = \infty$ and $-\infty$, respectively, and define Z_+ and Z_- analogously for f. Then these subsets are measurable: for example, Y_+ is a countable intersection of the sets $\{x \in X \mid g(x) \geq n\}$ as n ranges over the positive integers. Let X' be the complement of these sets, i.e., the subset where both f and g take finite values.

So we can define the function $h = f - g$ on X', and it is a measurable function. The first set of (a) is

$$h^{-1}([-\infty, 0)) \cup (Y_+ \setminus Z_+) \cup (Z_- \setminus Y_-),$$

so is measurable. Also, the set where f and g agree is

$$(X' \setminus h^{-1}([-\infty, 0) \cup (0, \infty])) \cup (Y_+ \cap Z_+) \cup (Y_- \cap Z_-),$$

which is also measurable.

As for (b), let f_n be a sequence of measurable real functions, and let E be the set of x such that $f_n(x)$ converges as $n \to \infty$. Define $f = \limsup f_n$. Then f is measurable (Theorem 1.14), and f agrees with $\lim f_n$ on E. For each n, the function $f - f_n$ is measurable (1.22), so the set $E_{n,r}$ which is defined to be the preimage of $f - f_n$ of $(-r, r)$ is measurable. Then $E = \bigcap_{r=1}^{\infty} \bigcup_{n=1}^{\infty} E_{n,r}$, so is measurable. □

6. **Exercise.** Let X be an uncountable set, let $\mathfrak{M}$ be the collection of all sets $E \subset X$ such that either E or E^c is at most countable, and define $\mu(E) = 0$ in the first case, $\mu(E) = 1$ in the second. Prove that $\mathfrak{M}$ is a σ-algebra in X and that μ is a measure on $\mathfrak{M}$. Describe the corresponding measurable functions and their integrals.

   Solution. Since $X^c = \varnothing$ is at most countable, $X \in \mathfrak{M}$. Also, if $E \in \mathfrak{M}$, then either E or E^c is at most countable, so the same is true for E^c since $(E^c)^c = E$, and so $E^c \in \mathfrak{M}$. Now suppose $E_n \in \mathfrak{M}$ for all n, and put $E = \bigcup_{n \geq 1} E_n$. Let I be the set of n for which E_n is at most countable, and let J be the set of n for which E_n is uncountable, but E_n^c is at most countable, so that $E = \bigcup_{n \in I} E_n \cup \bigcup_{n \in J} E_n$. If $J = \varnothing$, then E is a countable union of countable sets, and hence is countable. Otherwise, $E^c = \bigcap_{n \in I} E_n^c \cap \bigcap_{n \in J} E_n^c$, so $E^c \subseteq \bigcap_{n \in J} E_n^c$, which is countable since $J \neq \varnothing$, so $E \in \mathfrak{M}$. Thus, $\mathfrak{M}$ is a σ-algebra.

   Now write a measurable set A as a disjoint union of measurable sets A_n. If A is at most countable, then so is each A_n, so $\mu(A) = \sum \mu(A_n) = 0$. In case A^c is at most countable, then A is uncountable, so at least one A_i is uncountable. Suppose that A_i and A_j are both uncountable for $i \neq j$. Then $A_i^c \cup A_j^c$ is countable and equal to X since A_i and A_j are disjoint. But this contradicts that X is uncountable, so exactly one A_i is uncountable, which means that $\mu(A) = \sum \mu(A_n) = 1$. Hence μ is a measure on $\mathfrak{M}$.

   The measurable functions on $\mathfrak{M}$ consist of those functions $f \colon X \to \mathbf{R}^1$ such that for each $r \in \mathbf{R}^1$, $f^{-1}(r)$ is either at most countable, or $f^{-1}(\mathbf{R}^1 \setminus \{r\})$ is at most countable. If we let $A \subset \mathbf{R}^1$ denote the set of points such that $f^{-1}(r)$ is not countable, then the integral of f is $\sum_{r \in A} r$. □

7. **Exercise.** Suppose $f_n \colon X \to [0, \infty]$ is measurable for $n = 1, 2, 3, \ldots$; $f_1 \geq f_2 \geq f_3 \geq \cdots \geq 0$, $f_n(x) \to f(x)$ as $n \to \infty$, for every $x \in X$, and $f_1 \in L^1(\mu)$. Prove that then

$$\lim_{n \to \infty} \int_X f_n \, d\mu = \int_X f \, d\mu$$

   and show that this conclusion does *not* follow if the condition "$f_1 \in L^1(\mu)$" is omitted.

   Solution. If we first assume that $f_1(x) < \infty$ for all x, then the conclusion is a consequence of Lebesgue's dominated convergence theorem (Theorem 1.34) using $g(x) = f_1(x)$ since $f_1(x) \geq f_n(x) \geq 0$ implies that $f_1(x) \geq |f_n(x)|$. Otherwise, let $E = \{x \in X \mid f_1(x) = \infty\}$. If $\mu(E) > 0$, then $\int_X |f_1| \, d\mu = \infty$, which contradicts $f_1 \in L^1(\mu)$. So we conclude that $\mu(E) = 0$, in which case, we can ignore E when integrating over X, and we are back to the above discussion.

Now suppose that $f_1 \in L^1(\mu)$ no longer holds. Take $X = \mathbf{R}^1$, and $\mu(E)$ is the length of E. Then define $f_n(x) = \infty$ for $x \in [0, 1/n]$, and 0 elsewhere, so that $f_n \to 0$. Then $\int_X f_n \, d\mu = \infty$ for all n, but $\int_X 0 \, d\mu = 0$. $\square$

8. **Exercise.** Put $f_n = \chi_E$ if n is odd, $f_n = 1 - \chi_E$ if n is even. What is the relevance of this example to Fatou's lemma?

   Solution. This is an example where

$$\int_X (\liminf_{n\to\infty} f_n) \, d\mu < \liminf_{n\to\infty} \int_X f_n \, d\mu,$$

   provided that $\mu(E) > 0$ and $\mu(X \setminus E) > 0$. To see this, first note that $\liminf f_n(x) = 0$ for all $x \in X$ because $f_n(E) = 1$ for n odd, $f_n(E) = 0$ for n even, and $f_n(X \setminus E) = 0$ for n odd, $f_n(X \setminus E) = 1$ for n even. So the integral on the left-hand side is 0. On the other hand, $\int_X f_n \, d\mu = \mu(E)$ if n is odd, and $\int_X f_n \, d\mu = \mu(X \setminus E)$ if n is even. Hence the right-hand side is $\min(\mu(E), \mu(X \setminus E)) > 0$. $\square$

9. **Exercise.** Suppose μ is a positive measure on X, $f\colon X \to [0, \infty]$ is measurable, $\int_X f \, d\mu = c$, where $0 < c < \infty$, and α is a constant. Prove that

$$\lim_{n\to\infty} \int_X n \log(1 + (f/n)^\alpha) \, d\mu = \begin{cases} \infty & \text{if } 0 < \alpha < 1, \\ c & \text{if } \alpha = 1, \\ 0 & \text{if } 1 < \alpha < \infty. \end{cases}$$

   Solution. First, we ignore the set where $f(x) = \infty$ since f is integrable and hence this set has measure 0. When $\alpha \geq 1$, we claim that the integrand is bounded from above by αf, so we may use Lebesgue's dominated convergence theorem. To see this, define a function $g\colon [0, \infty) \to \mathbf{R}$ by $g(x) = \alpha x - n \log(1 + (x/n)^\alpha)$. We need to show that $g(x) \geq 0$ for all $x \geq 0$. First, $g(0) = 0$. Then in general, we take the derivative of g to get

$$g'(x) = \alpha - n \frac{1}{1 + (x/n)^\alpha} \cdot \frac{\alpha x^{\alpha-1}}{n^\alpha} = \alpha - \frac{n\alpha x^{\alpha-1}}{n^\alpha + x^\alpha}.$$

   So it is enough to show that $nx^{\alpha-1} \leq n^\alpha + x^\alpha$. But this is clearly true: if $n \leq x$, then $nx^{\alpha-1} \leq x^\alpha$, and if $n \geq x$, then $nx^{\alpha-1} \leq n^\alpha$.

   For $\alpha = 1$, the integrand converges to $\log e^f = f$, so the integral is c. If $\alpha > 1$, then rewrite the integrand as

$$\frac{\log(1 + (f/n)^\alpha)^{n^\alpha}}{n^{\alpha-1}}.$$

   As $n \to \infty$, the numerator goes to f^α and the denominator goes to ∞, so the limit is 0, and hence the integral is also 0.

   If $0 < \alpha < 1$, then the integrand approaches ∞ since $n^{\alpha-1} \to 0$ as $n \to \infty$. Hence by Fatou's lemma, the limit of the integral is also infinite. $\square$

10. **Exercise.** Suppose $\mu(X) < \infty$, $\{f_n\}$ is a sequence of bounded complex measurable functions on X, and $f_n \to f$ uniformly on X. Prove that

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu,$$

and show that the hypothesis "$\mu(X) < \infty$" cannot be omitted.

Solution. Since $f_n \to f$ uniformly, there exists N such that $n \geq N$ implies that $|f_n(x) - f(x)| < 1$ for all $x \in X$. Then since $\{f_1, \ldots, f_{N-1}\}$ is finite and consists of bounded sets, we can take C to be the largest absolute value any of them obtains, and let C' be the maximum of the largest value of $|f(x) \pm 1|$ and C. Then $C' \geq |f_n(x)|$ for all n, and $C' \in L^1(\mu)$ because $\int_X C' \, d\mu = C'\mu(X) < \infty$. So by Theorem 1.34,

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu.$$

To see that $\mu(X) < \infty$ is necessary, let $X = \mathbf{R}^1$ with the usual measure. Define f_n to be the constant function $1/n$. Then $f_n \to 0$ uniformly, but the $\int_X f_n \, d\mu = \infty$ for all n, while $\int_X 0 \, d\mu = 0$, so the equality does not hold. □

11. **Exercise.** Show that

$$A = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k$$

in Theorem 1.41, and hence prove the theorem without any reference to integration.

Solution. Denote the right-hand side by B. Recall that A is the set of all x which lie in infinitely many E_k. Pick $x \in A$. Then $x \in \bigcup_{k=n}^{\infty} E_k$ for all k, so $x \in B$. If $x \notin A$, then x is contained in finitely many E_k, say $\{E_{i_1}, \ldots, E_{i_r}\}$ with $i_1 < \cdots < i_r$. So $x \notin \bigcup_{k=i_r+1}^{\infty} E_k$, which means $x \notin B$, and hence $A = B$.

Now set $B_n = \bigcup_{k=n}^{\infty} E_k$. Then $\mu(B_1) < \infty$ by assumption, and $B_1 \supset B_2 \supset \cdots$. By Theorem 1.19(e), $\mu(B_n) \to \mu(B)$ as $n \to \infty$. Since $\mu(B_n) \leq \sum_{k=n}^{\infty} \mu(E_k)$, and the bounding sum approaches 0 as $n \to \infty$, we get that $\mu(B) = 0$. □

13. **Exercise.** Show that Proposition 1.24(c) is also true for $c = \infty$.

Solution. We wish to prove that if $f \geq 0$, then

$$\int_E \infty f \, d\mu = \infty \int_E f \, d\mu.$$

Let F be the set where f is nonzero. Then F is measurable, being the preimage of an open set, and we can integrate over F instead of E and get the same result since $\infty \cdot 0$ is defined to be 0. Then the integral over F is ∞ on both sides of the above equation, so we are done. □

2 Positive Borel Measures

1. **Exercise.** Let $\{f_n\}$ be a sequence of real nonnegative functions on $\mathbf{R}^1$, and consider the following four statements:

   (a) If f_1 and f_2 are upper semicontinuous, then $f_1 + f_2$ is upper semicontinuous.

   (b) If f_1 and f_2 are lower semicontinuous, then $f_1 + f_2$ is lower semicontinuous.

   (c) If each f_n is upper semicontinuous, then $\sum_{n=1}^{\infty} f_n$ is upper semicontinuous.

   (d) If each f_n is lower semicontinuous, then $\sum_{n=1}^{\infty} f_n$ is lower semicontinuous.

   Show that three of these are true and that one is false. What happens if the word "nonnegative" is omitted? Is the truth of the statements affected if $\mathbf{R}^1$ is replaced by a general topological space?

Solution. First suppose that f_1 and f_2 are upper semicontinuous. The set $\{x \in \mathbf{R}^1 \mid f_1(x) + f_2(x) < \alpha\}$ is the union of the sets $\{x \in \mathbf{R}^1 \mid f_1(x) < \beta\} \cap \{x \in \mathbf{R}^1 \mid f_2(x) < \alpha - \beta\}$ where we range over all $\beta \leq \alpha$. Hence this set is open, so $f_1 + f_2$ is upper semicontinuous. If both f_1 and f_2 are instead lower semicontinuous, then an analogous argument shows that $f_1 + f_2$ is also lower semicontinuous. We have not used that the functions are nonnegative here, nor have we used that f_1 and f_2 are defined on $\mathbf{R}^1$.

Now suppose that we have a sequence $\{f_n\}$ of lower semicontinuous functions. Then the set $\{x \in \mathbf{R}^1 \mid \sum f_n(x) > \alpha\}$ is a union of the sets $\bigcup_{n\geq 1}\{x \in \mathbf{R}^1 \mid f_n(x) \geq \alpha_n\}$ where $\sum \alpha_n \geq \alpha$, and hence is open, so $\sum f_n$ is lower semicontinuous. Note that again have not used the fact that the f_n are nonnegative, nor have we used that they are defined on $\mathbf{R}^1$.

However, (c) is a false statement. Define $f_1(x) = 0$ on $(-1, 1)$ and $f_1(x) = 1$ on the rest of $\mathbf{R}^1$. For $n > 1$, define $f_n(x) = 1$ on $\left[\frac{1}{n}, \frac{1}{n-1}\right] \cup \left[-\frac{1}{n-1}, -\frac{1}{n}\right]$ and 0 elsewhere. Then each f_n is upper semicontinuous since the set of x such that $f_n(x) = 0$ is open. However, $\sum f_n$ is 0 at 0 and greater than 0 elsewhere, so is not upper semicontinuous. □

2. **Exercise.** Let f be an arbitrary complex function on $\mathbf{R}^1$, and define

$$\begin{aligned}\varphi(x, \delta) &= \sup\{|f(s) - f(t)| \mid s, t \in (x - \delta, x + \delta)\},\\ \varphi(x) &= \inf\{\varphi(x, \delta) \mid \delta > 0\}.\end{aligned}$$

Prove that φ is upper semicontinuous, that f is continuous at a point x if and only if $\varphi(x) = 0$, and hence that the set of points of continuity of an arbitrary complex function is a G_δ.

Formulate and prove an analogous statement for general topological spaces in place of $\mathbf{R}^1$.

Solution. We formulate the general statement and prove that. Let X be a topological space, let $f\colon X \to \mathbf{C}$ be an arbitrary function, and define

$$\varphi(x) = \inf_{U \ni x} \sup\{|f(s) - f(t)| \mid s, t \in U\},$$

where U ranges over open sets containing x. Then φ is upper semicontinuous and f is continuous at x if and only if $\varphi(x) = 0$.

Pick a real number α, and consider the set $E = \{x \in X \mid \varphi(x) < \alpha\}$. Pick $x \in E$ and $\varepsilon > 0$ such that $\varphi(x)+\varepsilon < \alpha$. Then there exists an open set $U \ni x$ such that $\sup\{|f(s)-f(t)| \mid s, t \in U\} < \alpha$. In particular, this means that for every $t \in U$, $\varphi(t) < \alpha$. So E is open, and hence φ is upper semicontinuous.

Now suppose that f is continuous at x. Then for every $\varepsilon > 0$, there is a neighborhood $U_\varepsilon \ni x$ such that $f(U_\varepsilon) \subset B_\varepsilon(f(x))$, where $B_\varepsilon(f(x))$ denotes the ball of radius ε around $f(x)$. In particular, this means that $\varphi(x) < \varepsilon$, so we conclude that $\varphi(x) = 0$. Conversely, suppose that $\varphi(x) = 0$. To show that f is continuous at x, it is enough to show that for every $\varepsilon > 0$, there is an open set $U_\varepsilon \ni x$ such that $f(U_\varepsilon) \subset B_\varepsilon(f(x))$, but this is clear from the definition.

We conclude that the set of points for which f is continuous is a G_δ since it is in the countable intersection of open sets $\bigcap_{n\geq 0} E_n$ where $E_n = \{x \in X \mid \varphi(x) < 1/n\}$. □

3. **Exercise.** Let X be a metric space, with metric ρ. For any nonempty $E \subset X$, define

$$\rho_E(x) = \inf\{\rho(x, y) \mid y \in E\}.$$

Show that ρ_E is uniformly continuous function on X. If A and B are disjoint nonempty closed subsets of X, examine the relevance of the function

$$f(x) = \frac{\rho_A(x)}{\rho_A(x) + \rho_B(x)}$$

to Urysohn's lemma.

Solution. Pick $\varepsilon > 0$, and put $\delta = \varepsilon/2$. We claim that if $\rho(x, y) < \delta$, then $|\rho_E(x) - \rho_E(y)| < \varepsilon$ for all $x, y \in X$. We can find $z \in E$ such that $\rho(y, z) < \rho_E(y) + \delta$ by definition of ρ_E. Then

$$\rho(x, y) + \rho_E(y) + \delta > \rho(x, y), +\rho(y, z) \geq \rho(x, z) \geq \rho_E(x),$$

so

$$\rho(x, y) + \delta > \rho_E(x) - \rho_E(y).$$

By symmetry, we conclude that

$$\rho(x, y) + \delta > |\rho_E(x) - \rho_E(y)|.$$

But the left-hand side is less than ε, so we have established that ρ_E is a uniformly continuous function on X.

Now let A and B be disjoint nonempty closed subsets of X, and consider f as defined above. Then $f(x) = 1$ for $x \in B$, $f(x) = 0$ for $x \in A$, and $f(x) \leq 1$ on $X \backslash (A \cup B)$. Then $\chi_B \leq f \leq \chi_{X \backslash A}$, so that this is an analogous result to Urysohn's lemma. □

4. **Exercise.** Examine the proof of the Riesz theorem and prove the following two statements:

   (a) If $E_1 \subset V_1$ and $E_2 \subset V_2$, where V_1 and V_2 are disjoint open sets, then $\mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2)$, even if E_1 and E_2 are not in $\mathfrak{M}$.

   (b) If $E \in \mathfrak{M}_F$, then $E = N \cup K_1 \cup K_2 \cup \cdots$, where $\{K_i\}$ is a disjoint countable collection of compact sets and $\mu(N) = 0$.

   Solution. Recall that the definition is $\mu(E_i) = \inf\{\mu(V) \mid E_i \subset V,\ V \text{ open}\}$. Note that Step I of the proof of the Riesz theorem does not use the fact that $E_i \cap K \in \mathfrak{M}_F$ for every compact set K. Since this is the only difference between sets in $\mathfrak{M}$ and sets not in $\mathfrak{M}$, the proof follows just as before, so $\mu(E_1 \cup E_2) \leq \mu(E_1) + \mu(E_2)$. Conversely, let U be a subset containing $E_1 \cup E_2$. Then $\mu(U) = \mu(U \cap V_1) + \mu(U \cap V_2) \geq \mu(E_1) + \mu(E_2)$, where the first equality follows since V_1 and V_2 are disjoint. Hence $\mu(E_1 \cup E_2) \geq \mu(E_1) + \mu(E_2)$ by definition, and we have established (a).

   Now pick $E \in \mathfrak{M}_F$, and set $E_0 = E$. By Step V of the proof of the Riesz theorem, there is a compact set K_1 and an open set V_1 such that $K_1 \subset E_0 \subset V_1$ and $\mu(V_1 \setminus K_1) < 1$. Then $E_0 \setminus K_1 \in \mathfrak{M}_F$ by Step VI, so set $E_1 = E_0 \setminus K_1$. Inductively, we can find a compact set K_n and open set V_n such that $K_n \subset E_{n-1} \subset V_n$ and $\mu(V_n \setminus K_n) < 1/n$, and define $E_n = E_{n-1} \setminus K_n$. Then set $N = E \setminus \bigcup_{n \geq 1} K_n$. Then $\mu(N) < 1/n$ for all n, so $\mu(N) = 0$, and we have (b). □

5. **Exercise.** Let E be Cantor's familiar "middle thirds" set. Show that $m(E) = 0$, even though E and $\mathbf{R}^1$ have the same cardinality.

   Solution. First define $E_1 = [0, 1]$, and inductively define E_n to be the the result of removing the open middle third of each connected component of E_{n-1}. Then $m(E_n) = (2/3)^{n-1}$. Letting $E = \bigcap_{n \geq 0} E_n$, we then get $m(E) = 0$. However, E contains uncountably many points because each decimal in base 3 with either no 1's or exactly one 1 at the end is an element of E. □

6. **Exercise.** Construct a totally disconnected compact set $K \subset \mathbf{R}^1$ such that $m(K) > 0$.

   If v is lower semicontinuous and $v \le \chi_K$, show that actually $v \le 0$. Hence χ_K cannot be approximated from below by lower semicontinuous functions, in the sense of the Vitali–Carathéodory theorem.

   Solution. Define $K_0 = [0,1]$, and inductively define K_n to be K_{n-1} with an open interval of length 2^{-2n} removed from the middle of each connected component, then take $K = \bigcap_{n\ge 0} K_n$. Since K_n has 2^n connected components, we see that

$$m(K_n) = 1 - \sum_{i=1}^{n} \frac{1}{2^{2i}} \cdot 2^{i-1} = 1 - \sum_{i=1}^{n} \frac{1}{2^{i+1}} = 1 - \left(\frac{1 - \frac{1}{2^{n+2}}}{1 - \frac{1}{2}} - 1 - \frac{1}{2} \right),$$

   so $m(K) = \lim_{n\to\infty} m(K_n) = 1/2$. Furthermore, K is bounded and closed, since it is the intersection of closed sets, so K is compact. Finally, K is totally disconnected: if there were a connected component of K consisting of more than a point, then K contains an interval (a,b) for $a < b$. But for n sufficiently large, $2^{-2n} < b - a$, so we have a contradiction. Hence K is also totally disconnected.

   Now let v be a lower semicontinuous function with $v \le \chi_K$. The set of x where $v(x) > 0$ lies inside of K and is open, so must be empty, because K has no interior. So $v \le 0$. □

7. **Exercise.** If $0 < \varepsilon < 1$, construct an open set $E \subset [0,1]$ which is dense in $[0,1]$, such that $m(E) = \varepsilon$.

   Solution. Note that in (Ex. 2.6), we could have replaced $\frac{1}{2}$ with an arbitrary number in $(0,1)$ (start with a smaller or larger set for K_1). Then we just need to take the complement in $[0,1]$ to get the desired example. □

9. **Exercise.** Construct a sequence of continuous functions f_n on $[0,1]$ such that $0 \le f_n \le 1$, such that

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = 0,$$

   but such that the sequence $\{f_n(x)\}$ converges for no $x \in [0,1]$.

   Solution. For a given n, define n functions $g_{n,i}$ for $i = 0, \dots, n-1$ by $g_{n,i}(x) = 1$ on $\left[\frac{i}{n}, \frac{i+1}{n}\right]$ and define $g_{n,i}(x) = nx - (i-1)$ on $\left[\frac{i-1}{n}, \frac{i}{n}\right]$ and $g_{n,i}(x) = -nx + i + 2$ on $\left[\frac{i+1}{n}, \frac{i+2}{n}\right]$, and 0 elsewhere. Then think of $g_{n,i}$ as functions on $[0,1]$, note that they are continuous. Let $\{f_1, f_2, \dots,\}$ be the sequence $\{g_{1,0}, g_{2,0}, g_{2,1}, \dots, g_{n,0}, \dots, g_{n,n-1}, \dots\}$. Then $\int_0^1 g_{n,i}\,dx = 2/n$ if $0 < i < n-1$ and otherwise the integral is equal to $3/2n$. Hence $\int_0^1 f_n(x)\,dx \to 0$ as $n \to \infty$. However, $\{f_n(x)\}$ does not converge for any $x \in [0,1]$ because there are infinitely many values of n for which $f_n(x) = 1$ and infinitely many values of n for which $f_n(x) = 0$ for each $x \in [0,1]$. □

11. **Exercise.** Let μ be a regular Borel measure on a compact Hausdorff space X; assume $\mu(X) = 1$. Prove that there is a compact set $K \subset X$ (the *carrier* or *support* of μ) such that $\mu(K) = 1$ but $\mu(H) < 1$ for every proper compact subset H of K.

    Solution. Let K be the intersection of all compact K_α such that $\mu(K_\alpha) = 1$. Each K_α is compact, and hence closed since X is Hausdorff, so K is closed, and hence compact since X is compact.

    Let V be an open set which contains K. Then V^c is closed and hence compact. Since the K_α are compact, they are closed, so $K_\alpha^c \cap V^c$ forms an open cover of V^c, and by compactness, we can write $V^c = (K_1^c \cup \cdots \cup K_n^c) \cap V^c$. Since $\mu(K_1^c) = 0$, this shows that $\mu(V^c) = 0$, and hence $\mu(V) = 1$. So all open sets containing K have measure 1, which implies $\mu(K) = 1$ since μ is regular. Finally, if H is a compact set properly contained in V, then $\mu(H) < 1$. If not, then $\mu(H) = 1$, which contradicts the definition of K. $\square$

13. **Exercise.** Is it true that every compact subset of $\mathbf{R}^1$ is the support of a continuous function? If not, can you describe the class of all compact sets in $\mathbf{R}^1$ which are supports of continuous functions? Is your description valid in other topological spaces?

Solution. A point is a compact subset of $\mathbf{R}^1$, but cannot be the support of any continuous function. Any nonzero continuous function f has some real number α in its image, and hence its support is either all of $\mathbf{R}^1$. Otherwise, f contains 0 in its image, and so its support must contain an open interval $f^{-1}(0, \alpha)$. Restricting to each connected component of $\operatorname{Supp} f$, we see that each component needs to contain an open interval. Conversely, if K is a compact set with nonempty interior, it is the support of some continuous function. To construct such a function, we need only construct it on each connected component, so assume that K is connected. Let φ be a homeomorphism of the interior of K to $\mathbf{R}^1$. Then we can define $f\colon \mathbf{R}^1 \to \mathbf{R}^1$ by $x \mapsto e^{-x^2}$, and we can extend the function $f \circ \varphi$ to K by defining it to be 0 at the end points of K. It is clear that the support of this function is K.

However, this description will not carry to general topological spaces, namely because compact sets need not be closed. For an example, take the indiscrete topology on a set $X = \{a, b, c\}$ with open sets $\{\varnothing, \{a\}, X\}$. Then $\{a, b\}$ is a compact set with nonempty interior, but it is not closed. $\square$

14. **Exercise.** Let f be a real-valued Lebesgue measurable function on $\mathbf{R}^k$. Prove that there exist Borel functions g and h such that $g(x) = h(x)$ a.e. $[m]$, and $g(x) \le f(x) \le h(x)$ for every $x \in \mathbf{R}^k$.

Solution. Pick $I = (i_1, \ldots, i_k) \in \mathbf{Z}^k$, we will define F on the box $B_I = (i_1, i_1{+}1] \times \cdots \times (i_k, i_k{+}1]$. Let E_n be the set $f^{-1}((n, n+1]) \cap B_I$ for all $n \in \mathbf{Z}$. Then $|f|$ is bounded on E_n, so there exist compactly supported continuous functions $F_{n,r}(x)$ such that if we define $F_n = \lim\limits_{r \to \infty} F_{n,r}$, then $F_n = f$ a.e. on E_n by the corollary to Theorem 2.24. Since continuous functions are Borel measurable, F_n is Borel measurable (Corollary to Theorem 1.14). Finally, define F on I to be F_n on the set E_n. Then F is also Borel measurable. We repeat for every $I \in \mathbf{Z}^n$ and define a global function in this fashion. The set where F and f differ is then a countable union of sets of measure zero, so $f = F$ a.e.

Let $X = \{x \mid f(x) \neq F(x)\}$. We can partition X into measurable sets $X_n = (f - F)^{-1}((n, n + 1]) \cap X$. Now define a function $\varphi\colon \mathbf{R}^k \to \mathbf{R}^1$ by $\varphi(x) = n$ if $x \in X_n$, and $\varphi(x) = 0$ if $x \notin X$. Then φ is a measurable function. Now define $g = F + \varphi$ and $h = F + \varphi + \chi_X$. Then we see that g and h are Borel measurable, that $g = h$ a.e., and that $g(x) \le f(x) \le h(x)$ for all $x \in \mathbf{R}^k$. $\square$

15. **Exercise.** It is easy to guess the limits of

$$\int_0^n \left(1 - \frac{x}{n}\right)^n e^{x/2}\, dx \quad \text{and} \quad \int_0^n \left(1 + \frac{x}{n}\right)^n e^{-2x}\, dx,$$

as $n \to \infty$. Prove that your guesses are correct.

Solution. Define a function f_n to be $\left(1 - \frac{x}{n}\right)^n e^{x/2}$ on $[0, n]$ and 0 for $x > n$. Then $f_n \to e^{-x/2}$ as $n \to \infty$, and furthermore, $e^{-x/2} \ge |f_n(x)|$ for all n, and $e^{-x/2} \in L^1(\mathbf{R}^1)$. So by Lebesgue's dominated convergence theorem,

$$\lim_{n\to\infty}\int_0^\infty f_n\,dx = \int_0^\infty e^{-x/2}\,dx = 2,$$

and the left-hand side is the first limit to compute. By similar considerations, the second integral is

$$\int_0^\infty e^{-x}\,dx = 1. \quad \square$$

16. **Exercise.** Why is $m(Y) = 0$ in the proof of Theorem 2.20(e)?

Solution. In this case, Y lies in a proper linear subspace of $\mathbf{R}^k$. It is easy to see that the measure of a proper linear subspace must be 0 because we can take arbtrarily thin open sets that contain the subspace. $\square$

17. **Exercise.** Define the distance between two points (x_1, y_1) and (x_2, y_2) in the plane to be

$$|y_1 - y_2| \quad \text{if } x_1 = x_2, \qquad 1 + |y_1 - y_2| \quad \text{if } x_1 \neq x_2.$$

Show that this is indeed a metric, and that the resulting metric space X is locally compact.

If $f \in C_c(X)$, let $x_1, \ldots, x_n$ be those values of x for which $f(x, y) \neq 0$ for at least one y, and define

$$\Lambda f = \sum_{j=1}^n \int_{-\infty}^\infty f(x_j, y)\,dy.$$

Let μ be the measure associated with this Λ by Theorem 2.14. If E is the x-axis, show that $\mu(E) = \infty$ although $\mu(K) = 0$ for every compact $K \subset E$.

Solution. Let ρ be the metric defined. It is obvious that $\rho(x, y) \geq 0$ for all x, y, and that $\rho(x, y) = 0$ if and only if $x = y$. It is also obvious that $\rho(x, y) = \rho(y, x)$. We just need to verify the triangle inequality. Let $\alpha = (x_1, y_1)$, $\beta = (x_2, y_2)$, and $\gamma = (x_3, y_3)$. Then

$$\rho(\alpha,\gamma) \leq 1 + |y_1 - y_3| \leq |y_1 - y_2| + 1 + |y_2 - y_3| \leq \rho(\alpha,\beta) + \rho(\beta,\gamma),$$

so ρ is a metric. Note that a set under ρ is open if and only if its intersection with each vertical line is open when considered as a copy of $\mathbf{R}^1$. Every point in the plane has an open neighborhood in its vertical line whose closure is compact when thought of as a set in $\mathbf{R}^1$. But each vertical line is closed since it is the complement of the other vertical lines, so the closure of such a neighborhood is the same as the closure when considering it as a subset of $\mathbf{R}^1$. So X is locally compact.

Note that if $f \in C_c(X)$, then the values of x for which $f(x, y) \neq 0$ for at least one y must be finite because any collection of vertical lines is open, and hence only a finite union of vertical lines can be compact. The fact that Λ is a linear functional follows from the fact that one could sum over all $x \in \mathbf{R}^1$ and not change the value of Λ, and the fact that the integral is a linear functional. Also, $\Lambda f < \infty$ because f is compactly supported, and such a compact set in X is a union of closed sets in finitely many vertical lines.

Now let E be the x-axis. Any open set V containing E must contain a segment of the form $((x, -\delta(x)), (x, \delta(x)))$ for each $x \in \mathbf{R}^1$ where $\delta(x) > 0$. To show that $\mu(E) = \infty$, it is enough to show that the open set $U_\delta = \{(x, y) \mid -\delta(x) < y < \delta(x)\}$ has infinite measure for all arbitrary functions $\delta\colon \mathbf{R}^1 \to \mathbf{R}^1_{>0}$. To see this, consider the sets $X_1 = [1, \infty)$, and $X_n = \left[\frac{1}{n}, \frac{1}{n-1}\right)$ for

$n = 2, 3, \ldots$. Then for all $x \in \mathbf{R}^1$, $\delta(x)$ lies in some set. Since we have countably many sets, and $\mathbf{R}^1$ is uncountable, there is some set X_i such that $\delta^{-1}(X_i)$ is infinite. In particular, let $x_1, x_2, \ldots$ be distinct values inside of X_i. Then for each n, we can find a compactly supported continuous function f that is 1 for points of the form (x, y) where $x \in \{x_1, \ldots, x_n\}$ and $-\delta(x)/2 < y < \delta(x)/2$. Then $\Lambda f \geq \frac{n}{2i}$. In particular, as $n \to \infty$, this shows that $\mu(U_\delta) = \infty$, so $\mu(E) = \infty$.

However, every compact set K contained in E must be a finite set of points $\{(x_1, 0), \ldots, (x_r, 0)\}$, so $\mu(K) = 0$ necessarily because for all n, the set $U_n = \{(x, y) \mid x \in \{x_1, \ldots, x_r\},\ -\frac{1}{n} < y < \frac{1}{n}\}$ contains K and has measure $\mu(U_n) = r/n$. □

20. **Exercise.** Find continuous functions $f_n\colon [0,1] \to [0, \infty)$ such that $f_n(x) \to 0$ for all $x \in [0,1]$ as $n \to \infty$, $\int_0^1 f_n(x)\,dx \to 0$, but $\sup_n f_n$ is not in L^1.

    Solution. For each n, define n^2 functions $g_{n,r}\colon \mathbf{R}^1 \to [0, \infty)$ for $r = 0, \ldots, n^2 - 1$ by $g_{n,r}(x) = n$ on $\left[\frac{i}{n^2}, \frac{i+1}{n^2}\right]$, $g_{n,r}(x) = n(n^2x - i + 1)$ on $\left[\frac{i-1}{n^2}, \frac{i}{n^2}\right]$, and $g_{n,r}(x) = n(i + 2 - n^2x)$ on $\left[\frac{i+1}{n^2}, \frac{i+2}{n^2}\right]$. Then $\int_0^1 g_{n,r}(x)\,dx \leq \frac{2}{n}$. Letting the sequence $\{f_1, f_2, \ldots, \}$ be $\{g_{1,0}, g_{2,0}, \ldots, g_{2,3}, \ldots, g_{n,0}, \ldots, g_{n,n^2-1}, \ldots\}$, we get $\int_0^1 f_n(x)\,dx \to 0$, and $\sup_n f_n = \infty$, so is not L^1. □

21. **Exercise.** If X is compact and $f\colon X \to (-\infty, \infty)$ is upper semicontinuous, prove that f attains its maximum at some point of X.

    Solution. The sets $f^{-1}((-\infty, \alpha))$ are open for all $\alpha \in \mathbf{R}^1$, so by compactness, there are finitely many that cover X, and hence f is bounded. In particular, we only need to take one such set. Now let $\alpha = \sup\{f(x) \mid x \in X\}$. We claim that $f(x) = \alpha$ for some $x \in X$. If not, then we can find some sequence α_n such that $0 < \alpha - \alpha_n < \frac{1}{n}$. In particular, the sets $f^{-1}((-\infty, \alpha - \frac{1}{n}))$ cover X, and there is no finite subcover, which is a contradiction. Hence f attains its maximum at some point of X. □

22. **Exercise.** Suppose that X is a metric space, with metric d, and that $f\colon X \to [0, \infty]$ is lower semicontinuous, $f(p) < \infty$ for at least one $p \in X$. For $n = 1, 2, 3, \ldots$; $x \in X$, define

$$g_n(x) = \inf\{f(p) + nd(x, p) \mid p \in X\}$$

and prove that

(i) $|g_n(x) - g_n(y)| \leq nd(x, y)$,

(ii) $0 \leq g_1 \leq g_2 \leq \cdots \leq f$,

(iii) $g_n(x) \to f(x)$ as $n \to \infty$, for all $x \in X$.

Thus f is the pointwise limit of an increasing sequence of continuous functions.

Solution. Pick $x, y \in X$ and $p \in X$ with $f(p) < \infty$. Without loss of generality, suppose that $g_n(y) \geq g_n(x)$. We have

$$nd(x, y) \geq nd(y, p) - nd(x, p) = nd(y, p) + f(p) - nd(x, p) - f(p) \geq g_n(y) - g_n(x),$$

so this shows (i).

It is clear that $g_1 \geq 0$. Also, $f(p) + nd(x, p) \geq f(p) + (n-1)d(x, p) \geq g_{n-1}(x)$, so $g_{n-1}(x) \leq g_n(x)$ for all x. Furthermore, if $f(x) = \infty$, then $f \geq g_n$ for all n. Otherwise, $g_n(x) = f(x)$ by taking $p = x$, so $f \geq g_n$ for all n in this case, too. So (ii) is established.

Again, if $f(x) = \infty$, then $d(x, p) > 0$ for all p with $f(p) < \infty$, so $g_n(x) \to \infty$ as $n \to \infty$. Otherwise, $g_n(x) = f(x)$ for all n, so in both cases we have $g_n \to f$ as $n \to \infty$. □

24. **Exercise.** A *step function* is, by definition, a finite linear combination of characteristic functions of bounded intervals in $\mathbf{R}^1$. Assume $f \in L^1(\mathbf{R}^1)$, and prove that there is a sequence $\{g_n\}$ of step functions so that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} |f(x) - g_n(x)|\, dx = 0.$$

Solution. Let $g_n = \sum_{i=-n^2}^{n^2} f(i)\chi_{[\frac{i}{n},\frac{i+1}{n}]}$. One can partition $\mathbf{R}^1$ with sets E_r for $r \in \mathbf{Z}$ where $r \le f(x) < r+1$ for $x \in E_r$. For a given r, for n sufficiently large, on each the subintervals of length we can bound $|f(x)-g_n(x)|$ on their intersection E_r by an error directly proportional to r and inversely proportional to n. We omit the precise details. Since $\int |f|\, dx < \infty$, the expression

$$S_n = \int_{-\infty}^{-n} |f|\, dx + \int_n^{\infty} |f|\, dx \to 0$$

as $n \to \infty$. Hence for a given ε, we can choose n large enough so that $S_n < \varepsilon$, and also so that $\int_{-n}^{n} |f-g|\, dx < \varepsilon$. This is enough to guarantee that $\int_{-\infty}^{\infty} |f-g|\, dx < 2\varepsilon$ by noting that

$$S_n \le \int_{-\infty}^{-n} |f-g|\, dx + \int_n^{\infty} |f-g|\, dx.$$

From this it is clear that the limit of the integral above is 0. $\square$

25. **Exercise.**

 (i) Find the smallest constant c such that

$$\log(1+e^t) < c + t \quad (0 < t < \infty).$$

 (ii) Does

$$\lim_{n\to\infty} \frac{1}{n} \int_0^1 \log(1+e^{nf(x)})\, dx$$

 exist for every real $f \in L^1$? If it exists, what is it?

Solution. Since exp is an increasing function, $\log(1+e^t) < c+t$ becomes $1+e^t < e^{c+t}$, from which we divide by e^t to get $e^{-t}+1 < e^c$, and finally, we apply log (which is an increasing function) to get $\log(e^{-t}+1) < c$. The left-hand side is decreasing with t, so the smallest c satisfying this inequality is $\lim_{t\to 0} \log(e^{-t}+1) = \log 2$.

Let $X \subset [0,1]$ be the set where $f(x) \ge 0$. This implies

$$\int_X \log(1+e^{nf(x)})\, dx \le \int_X (\log 2 + nf(x))\, dx = \log 2 + \int_X nf(x)\, dx,$$

so as $n \to \infty$, the integral becomes $\int_X f(x)\, dx$ since the left-hand side increases toward the second term as $n \to \infty$, and the integral approaches 0 on $[0,1] \setminus X$ as $n \to \infty$. $\square$

3 L^p-Spaces

1. **Exercise.** Prove that the supremum of any collection of convex functions on (a,b) is convex on (a,b) (if it is finite) and that pointwise limits of sequences of convex functions are convex. What can you say about upper and lower limits of sequences of convex functions?

Solution. Let $\{f_\alpha\}$ be a collection of convex functions on (a,b), let $f = \sup_\alpha f_\alpha$, and assume that f is finite. Pick $\lambda \in [0,1]$ and $x, y \in (a,b)$. Then for all α, we have

$$(1-\lambda)f(x) + \lambda f(y) \geq (1-\lambda)f_\alpha(x) + \lambda f_\alpha(y) \geq f_\alpha((1-\lambda)x + \lambda y).$$

So by definition of supremum, we conclude that $(1-\lambda)f(x) + \lambda f(y) \geq f((1-\lambda)x + \lambda y)$, and hence f is convex.

Now let $\{f_n\}$ be a sequence of convex functions that converges pointwise to f. Pick $\lambda \in [0,1]$ and $x, y \in (a,b)$. Since

$$(1-\lambda)f_n(x) + \lambda f_n(y) \to (1-\lambda)f(x) + \lambda f(y)$$

and

$$f_n((1-\lambda)x + \lambda y) \to f((1-\lambda)x + \lambda y)$$

as $n \to \infty$, and we have

$$(1-\lambda)f_n(x) + \lambda f_n(y) \geq f_n((1-\lambda)x + \lambda y)$$

for all n, we conclude that

$$(1-\lambda)f_n(x) + \lambda f_n(y) \geq f_n((1-\lambda)x + \lambda y),$$

so that f is convex.

Now consider the sequence of functions on $(0,2)$ defined by $f_n(x) = x$ if n is even and $f_n(x) = 2 - x$ if n is odd. Then the lower limit f is defined by $f(x) = x$ if $x \in [0,1]$ and $f(x) = 2 - x$ if $x \in [1,2]$, and this is not convex: pick $x = 1/2$, $y = 3/2$, $\lambda = 1/2$. Then $f(x)/2 + f(y)/2 = 1/2$, but $f(1) = 1$. So we conclude that the lower limit of a sequence of convex functions need not be convex.

However, the upper limit of convex functions will be convex. The proof is similar to the proof for pointwise convergent functions, except that we use that for any $\varepsilon > 0$, there exist infinitely many values of n (rather than all sufficiently large n) for which $(1-\lambda)f_n(x) + \lambda f_n(y)$ is within ε of $(1-\lambda)f(x) + \lambda f(y)$, and similarly for $f_n((1-\lambda)x + \lambda y)$ and $f((1-\lambda)x + \lambda y)$. $\square$

2. **Exercise.** If φ is convex on (a,b) and if ψ is convex and nondecreasing on the range of φ, prove that $\psi \circ \varphi$ is convex on (a,b). For $\varphi > 0$, show that the convexity of $\log \varphi$ implies the convexity of φ, but not vice versa.

   Solution. Pick $\lambda \in [0,1]$ and $x, y \in (a,b)$. Then $(1-\lambda)\varphi(x) + \lambda\varphi(y) \geq \varphi((1-\lambda)x + \lambda y)$, and since ψ is convex and nondecreasing,

$$(1-\lambda)\psi(\varphi(x)) + \lambda\psi(\varphi(x)) \geq \psi((1-\lambda)\varphi(x) + \lambda\varphi(y)) \geq \psi(\varphi((1-\lambda)x + \lambda y)),$$

   so $\psi \circ \varphi$ is convex on (a,b). For $\varphi > 0$, the convexity of $\log \varphi$ implies the convexity of $\exp \circ \log \varphi = \varphi$ since exp is a nondecreasing and convex function. However, the converse is not true: the identity function x is convex, but $\log x$ is not convex. $\square$

3. **Exercise.** Assume that φ is a continuous real function on (a,b) such that

$$\varphi\left(\frac{x+y}{2}\right) \leq \frac{1}{2}\varphi(x) + \frac{1}{2}\varphi(y)$$

for all $x, y \in (a, b)$. Prove that φ is convex.

Solution. Pick $\lambda \in [0, 1]$ and $x, y \in (a, b)$. Without loss of generality, assume that $\varphi(y) \geq \varphi(x)$. By repeated iterations of the above inequality, if λ is a rational number whose denominator is a power of 2, then we can conclude that $\varphi((1 - \lambda)x + \lambda y) \leq (1 - \lambda)\varphi(x) + \lambda y$. The general case follows by continuity of φ: we can arbitrarily approximate $\varphi((1 - \lambda)x + \lambda y)$ by $\varphi((1 - r)x + ry)$ where r is some rational number whose denominator is a power of 2, and we can choose r such that $(1 - r)\varphi(x) + r\varphi(y) \leq (1 - \lambda)\varphi(x) + \lambda\varphi(y)$. □

10. **Exercise.** Suppose $f_n \in L^p(\mu)$, for $n = 1, 2, 3, \ldots$, and $\|f_n - f\|_p \to 0$ and $f_n \to g$ a.e., as $n \to \infty$. What relation exists between f and g?

Solution. Since $\|f_n - f\|_p \to 0$, we know that $f_n \to f$ a.e. Let E be the set where $\lim f_n \neq g$ and let F be the set where $\lim f_n \neq f$. Then $f = g$ except possibly on $E \cup F$, which has measure 0. Hence $f = g$ a.e. □

11. **Exercise.** Suppose $\mu(\Omega) = 1$, and suppose f and g are positive measurable functions on Ω such that $fg \geq 1$. Prove that

$$\int_\Omega f \, d\mu \cdot \int_\Omega g \, d\mu \geq 1.$$

Solution. First note that $fg \geq 1$ implies that $\sqrt{fg} \geq 1$. Using the Cauchy–Schwarz inequality on $\sqrt{f}$ and $\sqrt{g}$, we get

$$\int_\Omega f \, d\mu \cdot \int_\Omega g \, d\mu \geq \left(\int_\Omega \sqrt{fg} \, d\mu \right)^2 \geq \left(\int_\Omega 1 \, d\mu \right)^2 = \mu(\Omega)^2 = 1. \quad \square$$

12. **Exercise.** Suppose $\mu(\Omega) = 1$ and $h \colon \Omega \to [0, \infty]$ is measurable. If

$$A = \int_\Omega h \, d\mu,$$

prove that

$$\sqrt{1 + A^2} \leq \int_\Omega \sqrt{1 + h^2} \, d\mu \leq 1 + A.$$

If μ is Lebesgue measure on $[0, 1]$ and h is continuous, $h = f'$, the above inequalities have a simple geometric interpretation. From this, conjecture (for general Ω) under what conditions on h equality can hold in either of the above inequalities, and prove your conjecture.

Solution. The function $\varphi(x) = \sqrt{1 + x^2}$ is a convex function because its second derivative $\frac{\sqrt{x^2+1}}{x^4+2x^2+1}$ is always positive. Hence the first inequality follows from Jensen's inequality. The second inequality is equivalent to $\int_\Omega (\sqrt{1 + h^2} - 1) \, d\mu \leq \int_\Omega h \, d\mu$ since $\mu(\Omega) = 1$. This new inequality follows from the fact that $\sqrt{1 + x^2} \leq x + 1$ for all nonnegative x. To see this, square both sides to get $1 + x^2 \leq x^2 + 2x + 1$.

In the case that $\Omega = [0, 1]$ and μ is the Lebesgue measure, and $h = f'$ is continuous, then $\int_0^1 \sqrt{1 + (f')^2} \, d\mu$ is the formula for the arc length of the graph of f. Then $A = f(1) - f(0)$, and the second inequality says that the longest path from $(0, f(0))$ to $(1, f(1))$ is following along the line $y = f(0)$ from $x = 0$ to $x = 1$, and then going up the line $x = 1$ until $y = f(1)$. And $\sqrt{1 + A^2}$ is the length of the hypotenuse of the right triangle whose legs are the path just described, so the first inequality says that the straight path is the shortest path.

The intuition from this suggests that the second inequality is equality if and only if $h = 0$ a.e., and the first inequality is equality if and only if $h = A$ a.e. The first claim is easy to establish, we go back to the above discussion and note that $\sqrt{1+x^2} < x + 1$ if $x > 0$. If h is constant a.e., then trivially the first inequality holds. Conversely, if $\sqrt{1+A^2} = \int_\Omega \sqrt{1+h^2}\,d\mu$, then an examination of the proof of Jensen's inequality, namely equation (3), shows that $\varphi(A) = \varphi(h(x))$ a.e., so $h = A$ a.e. since φ is injective on $[0, \infty)$. □

13. **Exercise.** Under what conditions on f and g does equality hold in the conclusions of Theorems 3.8 and 3.9? You may have to treat the cases $p = 1$ and $p = \infty$ separately.

    Solution. The inequality in question for Theorem 3.8 is

$$\|fg\|_1 \le \|f\|_p \|g\|_q$$

for p and q conjugate exponents. If $1 < p < \infty$, this is Hölder's inequality, so assuming that both quantities are finite, we know that equality holds if and only if there are constants α and β, not both 0, such that $\alpha f^p = \beta g^q$ a.e. If $p = \infty$, then $|f(x)g(x)| = \|f\|_\infty |g(x)|$ holds if and only if for all x, either $g(x) = 0$ or $f(x) = \|f\|_\infty$. The case for $p = 1$ is analogous.

For Theorem 3.9, we are interested in the inequality

$$\|f + g\|_p \le \|f\|_p + \|g\|_p.$$

For $1 < p < \infty$, this follows from Minkowski's inequality. Examining the proof of Minkowski's inequality, this has equality if and only if equality is obtained for f and $f + g$ in Hölder's inequality and is obtained for g and $f + g$. In the case $p = \infty$ or $p = 1$, this is the inequality $|f + g| \le |f| + |g|$, and this obtains equality if and only if f and g are nonnegative functions. □

15. **Exercise.** Suppose $\{a_n\}$ is a sequence of positive numbers. Prove that

$$\sum_{N=1}^{\infty} \left(\frac{1}{N} \sum_{n=1}^{N} a_n \right)^p \le \left(\frac{p}{p-1} \right)^p \sum_{n=1}^{\infty} a_n^p$$

if $1 < p < \infty$.

Solution. Set $f = \sum_{n \ge 1} a_n \chi_{[n,n+1]}$. Then $f \in L^p$ if and only if $\sum_{n\ge1} a_n^p < \infty$. If $f \notin L^p$, then the above inequality trivially holds. Otherwise, we can use (Ex. 3.14(a)) to get $\|F\|_p \le \frac{p}{p-1}\|f\|_p$, where

$$F(x) = \frac{1}{x}\left(\sum_{n=1}^{\lfloor x \rfloor} a_n + (x - \lfloor x \rfloor) a_{n+1} \right).$$

If we assume that $a_n \ge a_{n+1}$ for all n, then this inequality implies the desired inequality by noting that $F(\lceil x \rceil) \le F(x)$.

In the general case, note that the right-hand side stays the same if we rearrange the a_n to be nondecreasing since the a_n are positive and hence the sum is absolutely convergent. Among all permutations of the sequence $\{a_n\}$, the sum on the left-hand side is biggest when they are nondecreasing because the earlier terms appear more often. Hence we deduce the general case from the nondecreasing case. □

16. **Exercise.** Prove Egoroff's theorem: If $\mu(X) < \infty$, if $\{f_n\}$ is a sequence of complex measurable functions which converges pointwise at every point of X, and if $\varepsilon > 0$, there is a measurable set $E \subset X$, with $\mu(X \setminus E) < \varepsilon$, such that $\{f_n\}$ converges uniformly on E.

Show that the theorem does not extend to σ-finite spaces.

Show that the theorem does extend, with essentially the same proof, to the situation in which the sequence $\{f_n\}$ is replaced by a family $\{f_t\}$, where t ranges over the positive reals; the assumptions are now that, for all $x \in X$,

(i) $\lim_{t\to\infty} f_t(x) = f(x)$ and

(ii) $t \to f_t(x)$ is continuous.

Solution. Define
$$S(n,k) = \bigcap_{i,j>n} \{x \in X \mid |f_i(x) - f_j(x)| < 1/k\}.$$
For each k, note that $S(n,k) \subset S(n+1,k)$, and that $\bigcup_{n\geq 1} S(n,k) = X$ by our assumptions that the f_n converge pointwise. Hence $\mu(S(n,k)) \to \mu(X)$ as $n \to \infty$. So for $\varepsilon > 0$, we can find n_k for each k such that $\mu(S(n_k,k)) > \mu(X) - \varepsilon/2$. Then take $E = \bigcap_{k\geq 1} S(n_k,k)$. Note that $\mu(E) \geq \mu(X) - \varepsilon/2$, so $\mu(X \setminus E) < \varepsilon$. Also, by definition, $\{f_n\}$ will converge uniformly on E because for every $\varepsilon > 0$, there is a k such that $1/k \leq \varepsilon$, and then every $x \in E$ satisfies $|f_i(x) - f_j(x)| < 1/k$ for all $i, j > n_k$.

If we drop the condition that $\mu(X) < \infty$ and replace it with X is σ-finite, the conclusion does not necessarily hold. For an example, take $X = \mathbf{R}^1$ with the Lebesgue measure. Then the functions $f_n(x) = x/n$ converge pointwise to 0, but cannot converge uniformly on any unbounded set.

For the extension to functions $\{f_t\}$ as t ranges over positive real numbers, we can make the same definitions, and now we just used that $t \mapsto f_t(x)$ is continuous for all x to find the n_k used in the above proof. □

21. **Exercise.** Call a metric space Y a *completion* of a metric space X if X is dense in Y and Y is complete. In Sec. 3.15 reference was made to "the" completion of a metric space. State and prove a uniqueness theorem which justifies this terminology.

Solution. We claim that if (Y,d) and (Y',d') are both completions of a metric space X, then Y and Y' are isomorphic metric spaces. More precisely, there exists bijective maps $f\colon Y \to Y'$ and $g\colon Y' \to Y$ which preserve the metric, i.e., $d(x,y) = d'(f(x),f(y))$ for all $x,y \in Y$, and such that f and g are inverses of one another.

So suppose that X can be embedded in both Y and Y', and identify these two images. We define f as follows. If $y \in X$, then $f(y) = y$. Otherwise, choose a sequence $\{x_n\}$ where $x_n \in X$ that converges to y. Then define $f(y)$ to be the limit of this sequence in Y'. This makes sense since X is dense in Y and since Y' is complete. That it is well-defined is a consequence of the uniqueness of limits. We can define $g\colon Y' \to Y$ in an analogous manner. From our definition it is obvious that f and g are inverses of one another. Also, note that both f and g are continuous since they preserve limits.

We just need to check that f and g preserve the metrics. By symmetry, it is enough to do so for f. Pick $x,y \in Y$. If $x,y \in X$, then $d(x,y) = d'(f(x),f(y))$ by our identification of the image of X in both Y and Y'. If $x \in X$ and $y \notin X$, then let $\{x_n\}$ be a sequence converging to y. We then have $d(x,x_n) = d'(f(x),f(x_n))$ for all n, so by continuity of the function $z \mapsto d'(f(x),f(z))$, and taking $n \to \infty$, we see that $d(x,y) = d'(f(x),f(y))$. The case $x \notin X$ and $y \in X$ is handled by the symmetric property of metrics. Finally, suppose that $x \notin X$ and $y \notin X$. We can show that the metric is preserved in this case by taking a double limit instead of a single limit in the previous argument. Hence we have shown that Y and Y' are isomorphic. □

22. **Exercise.** Suppose X is a metric space in which every Cauchy sequence has a convergent subsequence. Does it follow that X is complete?

    Solution. Let $\{x_n\}$ be a Cauchy sequence in X with a convergent subsequence $\{x_{n_i}\}$ with limit x. We claim that x is the limit of $\{x_n\}$. Pick $\varepsilon > 0$. Then there exists N such that $n, m \geq N$ implies that $d(x_n, x_m) < \varepsilon/2$, and there exists I such that $i \geq I$ implies that $d(x, x_{n_i}) < \varepsilon$. Then for n sufficiently large, there exists i such that $d(x, x_n) \leq d(x, x_{n_i}) + d(x_{n_i}, x_n) < \varepsilon$, so we are done. Hence X is complete. $\square$

4 Elementary Hilbert Space Theory

Notation. In this section, H denotes a Hilbert space, and T denotes the unit circle.

1. **Exercise.** If M is a closed subspace of H, prove that $M = (M^\perp)^\perp$. Is there a similar true statement for subspaces M which are not necessarily closed?

   Solution. The inclusion $M \subseteq (M^\perp)^\perp$ is obvious. Conversely, pick $x \in (M^\perp)^\perp$. We can write $x = y + z$ where $y \in M$ and $z \in M^\perp$ (Theorem 4.11(a)). Since $0 = (x, z) = (y, z) + (z, z) = (z, z)$, we conclude that $z = 0$, so $x \in M$. Hence $M = (M^\perp)^\perp$.

   Now suppose M is not necessarily closed. Then one can conclude that $M^\perp = ((M^\perp)^\perp)^\perp$ because $M^\perp$ is a closed subspace of H. Also, one can say that $\overline{M} = (M^\perp)^\perp$. Indeed, $M \subseteq (M^\perp)^\perp$, and $(M^\perp)^\perp$ is a closed set, so $\overline{M} \subseteq (M^\perp)^\perp$. On the other hand, if $x \in M^\perp$, then $x \in \overline{M}^\perp$ because $y \mapsto (y, x)$ is a continuous map. $\square$

2. **Exercise.** Let $\{x_n \mid n = 1, 2, 3, \dots\}$ be a linearly independent set of vectors in H. Show that the following construction yields an orthonormal set $\{u_n\}$ such that $\{x_1, \dots, x_N\}$ and $\{u_1, \dots, u_N\}$ have the same span for all N.

   Put $u_1 = x_1/\|x_1\|$. Having $u_1, \dots, u_{n-1}$ define

   $$v_n = x_n - \sum_{i=1}^{n-1} (x_n, u_i) u_i, \quad u_n = v_n / \|v_n\|.$$

   Note that this leads to a proof of the existence of a maximal orthonormal set in separable Hilbert spaces which makes no appeal to the Hausdorff maximality principle.

   Solution. It is obvious from the definition, that $\|u_n\| = 1$ for all n. Now we need to show orthogonality, and we do showing that $(u_n, u_m) = 0$ for $m < n$ by induction on n. If $n = 1$, there is nothing to show, so pick $n > 1$. Then we will show that $(u_n, u_m) = 0$ for all $m < n$ by induction on m. Since they u_n and v_n differ only by scalars, it is enough to show that $(v_n, u_m) = 0$. We get

   $$(v_n, u_m) = \left(x_n - \sum_{i=1}^{n-1} (x_n, u_i) u_i, u_m \right) = (x_n, u_m) - (x_n, u_m) = 0$$

   since $(u_i, u_m) = 0$ for $i < n$.

   It is clear that $\{x_1, \dots, x_N\}$ lies inside of the span of $\{u_1, \dots, u_N\}$ for each N by definition of the u_i. Since $u_1 = x_1/\|x_1\|$, we can inductively build up the set $\{u_1, \dots, u_N\}$ from $\{x_1, \dots, x_N\}$ also by the definition of the u_i.

If H is separable, then there is a countable dense subset for H, from which we can extract an at most countable basis for it. Doing the above transformation, we may assume that this basis is orthonormal. Then by Theorem 4.18, we know that it is a maximal orthonormal basis for H. $\square$

4. **Exercise.** Show that H is separable if and only if H contains a maximal orthonormal system which is at most countable.

   Solution. If H is separable, then H contains a maximal orthonormal system which is at most countable by (Ex. 4.2). Conversely, suppose that H contains a maximal orthonormal system which is at most countable. Then the subspace spanned by this basis is dense by Theorem 4.18, and must be countable, since it consists of finite linear combinations of an at most countable set, so H is separable. $\square$

5. **Exercise.** If $M = \{x \mid Lx = 0\}$, where L is a continuous linear functional on H, prove that $M^\perp$ is a vector space of dimension 1 (unless $M = H$).

   Solution. By Theorem 4.12, there is a unique $y \in H$ such that $Lx = (x, y)$ for all $x \in H$. We claim that $M^\perp = N$, where N is the subspace spanned by y. Indeed, the N is closed because any sequence in this subspace consists of multiples of y, and hence can only converge to some multiple of y. Then $N = (N^\perp)^\perp$ by (Ex. 4.1), and $N^\perp = M$. In the case that $M \neq H$, $y \neq 0$, so $\dim N = 1$, and we are done. $\square$

7. **Exercise.** Suppose $\{a_n\}$ is a sequence of positive numbers such that $\sum a_n b_n < \infty$ whenever $b_n \geq 0$ and $\sum b_n^2 < \infty$. Prove that $\sum a_n^2 < \infty$.

   Solution. Suppose that $\sum a_n^2 = \infty$. Then one can find an infinite number of disjoint sets $E_1, E_2, \ldots$ such that $S_k = \sum_{n \in E_k} a_n^2 > 1$. Now put $c_k = \frac{1}{k\sqrt{S_k}}$, and define $b_n = c_k a_n$ when $n \in E_k$. Then
$$\sum_{n\geq 1} a_n b_n = \sum_{k\geq 1} c_k \sum_{n\in E_k} a_n^2 = \sum_{k\geq 1} \frac{S_k}{k\sqrt{S_k}} \geq \sum_{k\geq 1} \frac{1}{k} = \infty,$$
   but
$$\sum_{n\geq 1} b_n^2 = \sum_{k\geq 1}\sum_{n\in E_k} c_k^2 a_n^2 = \sum_{k\geq 1} \frac{1}{k^2} = \frac{\pi^2}{6},$$
   which contradicts our hypothesis on $\{a_n\}$, and hence $\sum a_n^2 < \infty$. $\square$

8. **Exercise.** If H_1 and H_2 are two Hilbert spaces, prove that one of them is isomorphic to a subspace of the other.

   Solution. Let A_1 and A_2 be the cardinalities of maximal orthonormal bases β_1 and β_2 of both H_1 and H_2. Then either $A_1 \leq A_2$ or $A_2 \leq A_1$. Without loss of generality, suppose that $A_1 \leq A_2$. Then we can find a subset of β_2 with cardinality A_1, and let H be the closure of the subspace generated by this subset. Then H is a Hilbert space that has a maximal orthonormal basis of cardinality A_1, so we can find an isomorphism from H_1 to H. $\square$

10. **Exercise.** Let $n_1 < n_2 < n_3 < \cdots$ be positive integers, and let E be the set of all $x \in [0, 2\pi]$ at which $\{\sin n_k x\}$ converges. Prove that $m(E) = 0$.

    Solution. Let $f(x) = \lim_{k\to\infty} \sin n_k x$ on E. From the relation $2\sin^2\alpha = 1 - \cos 2\alpha$, we see that $2f(x)^2 = 1 - \lim_{k\to\infty} \cos 2n_k x$. The integral of the right-hand side is 0 by (Ex. 4.9), so $2f(x)^2 = 1$

a.e. on E, and hence $f(x) = \pm\frac{1}{\sqrt{2}}$ a.e. on E. Let E_1 be the set where $f(x) = \frac{1}{\sqrt{2}}$ and let E_2 be the set where $f(x) = -\frac{1}{\sqrt{2}}$. Using (Ex. 4.9) again, we have

$$0 = \int_{E_1} f(x) = \frac{m(E_1)}{\sqrt{2}},$$

so we conclude that $m(E_1) = 0$. Similarly, $m(E_2) = 0$, and so $m(E) = m(E_1) + m(E_2) = 0$. □

11. **Exercise.** Find a nonempty closed set E in $L^2(T)$ that contains no element of smallest norm.

Solution. Let $u_n(t) = e^{int}$, and define $x_n = \frac{n+1}{n}u_n$ for all positive integers n. Then the set $X = \{x_1, x_2, \dots\}$ contains no element of smallest norm. For $n \neq m$,

$$\|x_n - x_m\| = \|x_n\| + \|x_m\| = \frac{n+1}{n} + \frac{m+1}{m} \geq 2$$

because the x_n form an orthogonal set. If x is a limit point of the sequence $\{x_{n_i}\}$, then for $\varepsilon > 0$, and for sufficiently large i, $\|x - x_{n_i}\| < \varepsilon$. But $\|x_{n_j} - x_{n_i}\| = \|x - x_{n_i} - (x - x_{n_j})\| \leq \|x - x_{n_i}\| + \|x - x_{n_j}\| < 2\varepsilon$, which is a contradiction for $\varepsilon < 1/2$ if $i \neq j$. Hence $\{x_{n_i}\}$ is a constant sequence, which means that X is closed. □

16. **Exercise.** If $x_0 \in H$ and M is a closed linear subspace of H, prove that

$$\min\{\|x - x_0\| \mid x \in M\} = \max\{|(x_0, y)| \mid y \in M^\perp, \|y\| = 1\}.$$

Solution. By Theorem 4.11, we can write $x_0 = Px_0 + Qx_0$ where $Px_0 \in M$ and $Qx_0 \in M^\perp$, and furthermore, that $\|Px_0 - x_0\| = \min\{\|x - x_0\| \mid x \in M\}$. We claim that $y = Qx_0/\|Qx_0\|$ maximizes the quantity $|(x_0, y)|$ where $y \in M^\perp$ and $\|y\| = 1$. Pick $y' \in M^\perp$ such that $\|y'\| = 1$. Then

$$|(x_0, y')| = |(Qx_0, y')| = \|Qx_0\||(y, y')| \leq \|Qx_0\|,$$

where the last inequality is the Schwarz inequality. But $|(x_0, y)| = \|Qx_0\|$, so we have proved our claim. Finally, $\|Qx_0\| = \|Px_0 - x_0\|$, so we have established the desired equality. □

17. **Exercise.** Show that there is a continuous one-to-one mapping γ of $[0, 1]$ into H such that $\gamma(b) - \gamma(a)$ is orthogonal to $\gamma(d) - \gamma(c)$ whenever $0 \leq a \leq b \leq c \leq d \leq 1$. ($\gamma$ may be called a "curve with orthogonal increments.")

[I think this is to be interpreted as there exists some Hilbert space H such that this is true, because this is impossible for $H = \mathbf{R}^1$.]

Solution. Take $H = L^2([0, 1])$, and define $\gamma(a) = \chi_{[0,a]}$. This is clearly continuous and injective, and furthermore, $\gamma(b) - \gamma(a) = \chi_{(a,b]}$ whenever $a < b$. So in the case that $a < b \leq c < d$, we have $(\chi_{(a,b]}, \chi_{(c,d]}) = \int_0^1 \chi_{(a,b]}\chi_{(c,d]}\, dt = 0$ since the integrand is 0. If $a = b$, then $\gamma(b) - \gamma(a) = 0$, so the orthogonality relation is obvious (similarly if $c = d$). □

18. **Exercise.** Define $u_s(t) = e^{ist}$ for all $s \in \mathbf{R}^1$, $t \in \mathbf{R}^1$. Let X be the complex vector space consisting of all finite linear combinations of these functions u_s. If $f \in X$ and $g \in X$, show that

$$(f, g) = \lim_{A\to\infty} \frac{1}{2A} \int_{-A}^{A} f(t)\overline{g(t)}\, dt$$

exists. Show that this inner product makes X into a unitary space whose completion is a nonseparable Hilbert space H. Show also that $\{u_s \mid s \in \mathbf{R}^1\}$ is a maximal orthonormal set in H.

Solution. To show that (f, g) is well-defined, it is enough to do so when $f = u_r$ and $g = u_s$ for $r, s \in \mathbf{R}^1$. In this case, for $r \neq s$,

$$(u_r, u_s) = \lim_{A\to\infty} \frac{1}{2A} \int_{-A}^{A} \cos((s-r)t) + i\sin((s-r)t)\,dt = \lim_{A\to\infty} \frac{2i\sin((s-r)A)}{2A(s-r)} = 0.$$

For $r = s$, we get

$$(u_r, u_r) = \lim_{A\to\infty} \frac{1}{2A} \int_{-A}^{A} dt = 1,$$

so this shows that (f, g) is well-defined, as well as showing that $\{u_s \mid s \in \mathbf{R}^1\}$ is an orthonormal set.

Now we verify that this is an inner product structure on X. By definition, it is clear that $(f, g) = \overline{g, f}$, and by linearity of integrals, $(f_1 + f_2, g) = (f_1, g) + (f_2, g)$, and $(\alpha f, g) = \alpha(f, g)$ for $\alpha \in \mathbf{C}$. Since $f(x)\overline{f(x)} = |f(x)|^2$, it follows that $(f, f) \geq 0$, and that $(f, f) = 0$ implies that $f = 0$ a.e. on $\mathbf{R}^1$, but since f is a finite linear combination of exponential functions, this implies $f = 0$. Hence X is a unitary space.

Now let H be the completion of X. Since X is dense in H, it follows that $\{u_s \mid s \in \mathbf{R}^1\}$ is a maximal orthonormal set in H by Theorem 4.18. If H were to have a countable dense subset, then one could find a countable basis for H, which contradicts the fact that we have an uncountable orthonormal set. Hence H is a nonseparable Hilbert space. □

19. **Exercise.** Fix a positive integer N, put $\omega = e^{2\pi i/N}$, prove the orthogonality relations

$$\frac{1}{N}\sum_{n=1}^{N} \omega^{nk} = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } 1 \leq k \leq N-1 \end{cases}$$

and use them to derive the identities

$$(x, y) = \frac{1}{N}\sum_{n=1}^{N} \|x + \omega^n y\|^2 \omega^n$$

that hold in every inner product space if $N > 3$. Show also that

$$(x, y) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \|x + e^{i\theta} y\|^2 e^{i\theta}\,d\theta.$$

Solution. The identity $\frac{1}{N}\sum_{n=1}^{N} \omega^0 = 1$ is obvious. If k is relatively prime to N, then the sum $\sum_{n=1}^{N} \omega^{nk}$ is the sum of all primitive Nth roots of unity, which is 0 for $N > 1$, and otherwise, if k is not relatively prime, then the sum is k times the sum over all primitive $\frac{N}{k}$th roots of unity, which is also 0.

To show the second identity, expand the right-hand side:

$$\begin{aligned}
\frac{1}{N}\sum_{n=1}^{N} \|x + \omega^n y\|^2 \omega^n &= \frac{1}{N}\sum_{n=1}^{N} (x + \omega^n y, x + \omega^n y)\omega^n \\
&= \frac{1}{N}\left(\sum_{n=1}^{N}(x, x)\omega^n + \sum_{n=1}^{N}(x, y)\omega^{-n} + \sum_{n=1}^{N}(y, x)\omega^{2n} + \sum_{n=1}^{N}(y, y)\omega^{3n}\right) \\
&= (x, y)
\end{aligned}$$

using the first identity.

The last identity is obtained by taking the limit as $N \to \infty$ of the sum on the right-hand side of the second identity. The result is the Riemann integral $\frac{1}{2\pi}\int_{-\pi}^{\pi} \|x + e^{i\theta}y\|^2 e^{i\theta}\, d\theta$. Of course, since the left-hand side of the second identity is independent of N, we get the desired result. □

Acknowledgements.

I thank Jorun Bomert and Brian Streit for pointing out corrections to some of the solutions.