Non-Life Insurance — Assignment 3

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Question 1

We have that:

$$S(x) = \exp\left(-\int_0^x \mu(t)\,dt\right)$$

and

$$\mu(t) = a + bc^t.$$

Therefore:

$$
\begin{aligned}
S(x) &= \exp\left(-\int_0^x (a + bc^t)\,dt\right) \\
&= \exp\left(-\left(a\int_0^x dt + b\int_0^x c^t\,dt\right)\right) \\
&= \exp\left(-a\,[t]_0^x - b\left[\frac{c^t}{\log c}\right]_0^x\right) \\
&= \exp\left(-ax - b\left(\frac{c^x}{\log c} - \frac{1}{\log c}\right)\right) \\
&= \exp\left(-ax - \frac{b}{\log c}\,(c^x - 1)\right), \quad x \geq 0.
\end{aligned}
$$

Question 2

We will use the following:

$$\mu(x) = \frac{f(x)}{1 - F(x)}, \quad x \geq 0.$$

We have that $Z \sim \mathrm{Exp}(a)$.

Therefore,

$$
\begin{aligned}
\mu_z(x) &= \frac{ae^{-ax}}{1 - (1 - e^{-ax})} \\
&= \frac{ae^{-ax}}{e^{-ax}} \\
&= a.
\end{aligned}
$$

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Question 3

Let

$$Y = \frac{\log\left(1 + \frac{V \log c}{b}\right)}{\log c}, \quad b \geq 0, \ c \geq 1.$$

where $V \sim \text{Exp}(1)$. Hence:

$$P[V \leq x] = 1 - e^{-x}$$
$$P[V > x] = e^{-x}$$

We will compute:

$$S_Y(x) = P[Y > x].$$

$$
\begin{aligned}
P[Y > x] &= P\left[\frac{\log\left(1 + \frac{V \log c}{b}\right)}{\log c} > x\right] \\
&= P\left[\log\left(1 + \frac{V \log c}{b}\right) > x \cdot \log c\right] \\
&= P\left[\log\left(1 + \frac{V \log c}{b}\right) > \log\left(c^x\right)\right] \\
&= P\left[1 + \frac{V \log c}{b} > c^x\right] \\
&= P\left[\frac{V \log c}{b} > c^x - 1\right] \\
&= P\left[V > \frac{b\left(c^x - 1\right)}{\log c}\right] \\
&= \exp\left(-\frac{b}{\log c} \cdot \left(c^x - 1\right)\right), \quad x \geq 0.
\end{aligned}
$$

Therefore, $Y \sim \text{Gompertz}(b, c)$.

Question 4

(a) For values of $a > 0$, `gen.Sample` considers the pairwise minimum between values generated by a Gompertz(b, c) distribution and the contribution of the term `rexp(n,a)`, so this quantity can only get smaller when a positive constant a is present as an argument.

(b) The probability of dying from old age is *much higher* (0.959 in our sample), *i.e.* the inclusion of $a > 0$ only altered $1 - \texttt{mean(M==G)} = 0.041$ of our sample.

Question 5

We note that the Makeham density takes values less than e^{-3} everywhere. Now, when a computer encounters values of the order e^{-900} as our likelihood function does, it simply returns zero, despite the fact that these terms are non-zero. Conversely, by considering the logarithm of the likelihood function, the values that this function takes become representable on a computer *e.g* $\log e^{-900} = -900$.

Question 6

```
## i. Makeham Sample G
neg.loglik <- function(par) {
  -sum(log.fx(G, par[1], par[2], par[3]))
}

o3 <- optim(c(a,b,c), neg.loglik)
o3$par     ##  a = 1.59e-04, b = 5.9e-05, c = 1.084
o3$value   ##  8245.877

## ii. Gompertz Sample G
neg.loglik <- function(par) {
  -sum(log.fx(G, 0, par[1], par[2]))
}

o4 <- optim(c(b,c), neg.loglik)
o4$par     ##  b = 7.09e-05, c = 1.082
o4$value   ##  8248.117
```

Question 7

```
## calculations

myAIC <- function(l, k) -2*l + 2*k

myBIC <- function(l, k, n) -2*l + k*log(n)

## sample M
n <- length(M)

##  Makeham
l <- -o1$value
k <- length(o1$par)
aic.Makeham.M <- myAIC(l, k)
bic.Makeham.M <- myBIC(l, k, n)

##  Gompertz
l <- -o2$value
k <- length(o2$par)
aic.Gompertz.M <- myAIC(l, k)
bic.Gompertz.M <- myBIC(l, k, n)
```

```
## sample G
n <- length(G)

## Makeham
l <- -o3$value
k <- length(o3$par)
aic.Makeham.G <- myAIC(l, k)
bic.Makeham.G <- myBIC(l, k, n)

## Gompertz
l <- -o4$value
k <- length(o4$par)
aic.Gompertz.G <- myAIC(l, k)
bic.Gompertz.G <- myBIC(l, k, n)
```

(a) The model denoted by * is preferred.

```
## Sample M, AIC
aic.Makeham.M     # [1] 16862.98 *
aic.Gompertz.M    # [1] 16916.52

## Sample G, AIC
aic.Makeham.G     # [1] 16497.75 *
aic.Gompertz.G    # [1] 16500.23
```
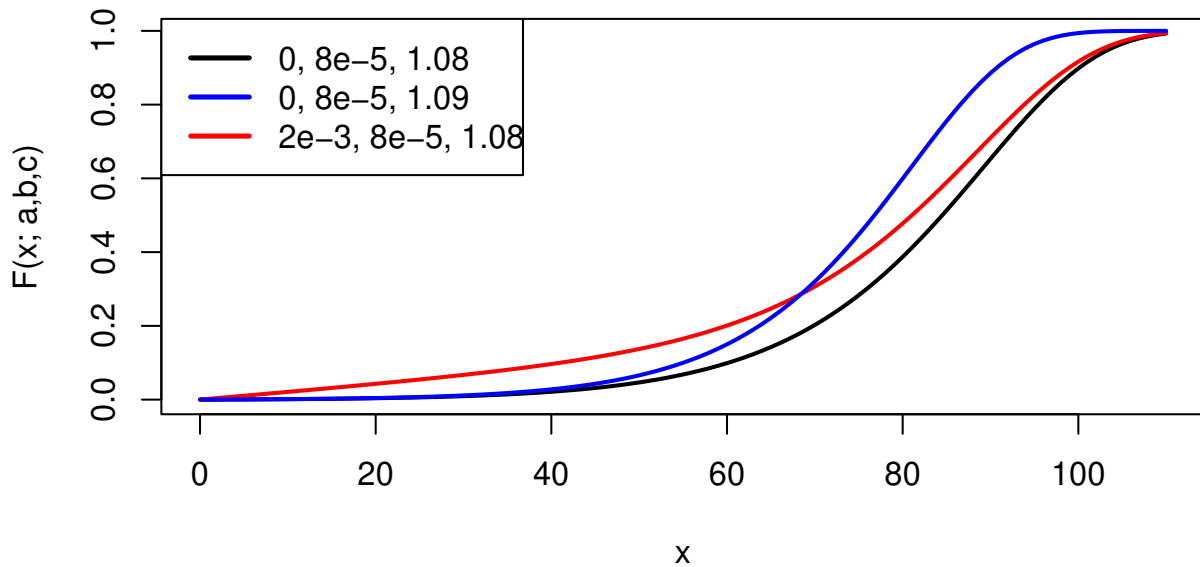
(b)

```
## Sample M, BIC
bic.Makeham.M     # [1] 16879.78 *
bic.Gompertz.M    # [1] 16927.72

## Sample G, BIC
bic.Makeham.G     # [1] 16514.56
bic.Gompertz.G    # [1] 16511.44 *
```

Question 8

The code provided generates the following plot of the Makeham *cdf* $F(x)$ for various values of the parameters a, b, c:


```
## include legends
legend("topleft", c("0, 8e-5, 1.08", "0, 8e-5, 1.09", "2e-3, 8e-5, 1.08"),
       lty=c(1), lwd=c(2.5), col=c("black" ,"blue", "red"))
```

(a) Moving from the black to the blue curve, increasing the parameter c has the effect of increasing the value of the lifetime distribution $F(x)$ for any given value of x, which means that mortality has been higher.

Similarly, moving from the black curve to the red curve illustrates the effect of adding the parameter a to the model. For any given value of x, the *cdf* $F(x)$ of the model with $a > 0$ is larger, indicating higher levels of mortality for a given age.

Continued on next page.

(b) Now, for mortality rates μ:

```
## Plot the corresponding Makeham mortality rates
mu <- function(x, a, b, c) a + b*(c^x)
x <- 0:110
plot(x, mu(x, 0, 8e-5, 1.08), type="l", ylab="mu(x; a,b,c)", lwd=2,
     xlim = c(0, 80), ylim = c(0, 0.02))
lines(x, mu(x, 0, 8e-5, 1.09), col="royalblue", lwd=2)
lines(x, mu(x, 2e-3, 8e-5, 1.08), col="forestgreen", lwd=2)


## include legends
legend("topleft", c("0, 8e-5, 1.08", "0, 8e-5, 1.09", "2e-3, 8e-5, 1.08"),
       lty=c(1), lwd=c(2.5), col=c("black" ,"royalblue", "forestgreen"))
```
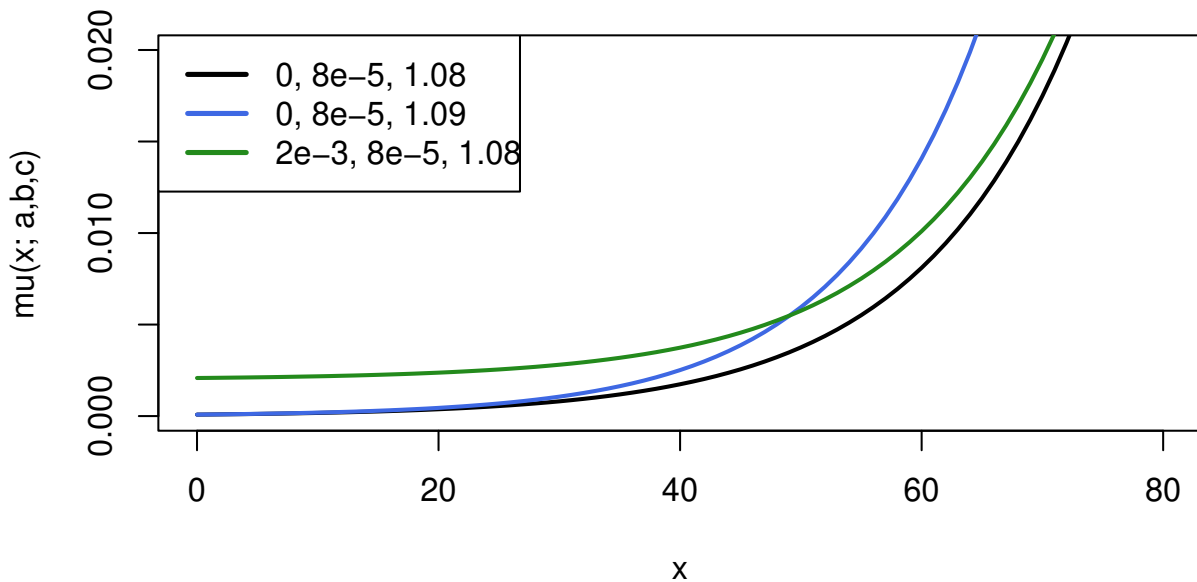

Plotting the mortality rates $\mu(x)$, we are able to explain the mechanism driving the effects seen in part (a). The changes in parameters correspond to higher rates of mortality for a given age x, which in turn leads to higher levels of mortality (indicated by $F(x)$) for a given age x.

Question 9

```
## Makeham
## (a)
S <- function(x, a, b, c) exp(-a*x - b/log(c)*(c^x-1))

neg.loglik <- function(par) {
  sur.x <- S(0:nages, par[1], par[2], par[3])
  q.x <- -diff(sur.x) / sur.x[-length(sur.x)]
  -sum(dbinom(D.x, e.x, q.x, log=TRUE))
}

## (b)
o9b <- optim(c(5e-4, 8e-5, 1.08), neg.loglik)

> o9b$par
[1] 0.0009208569 0.0000184291 1.1135530947

## (c)
## recall this beauty
myAIC <- function(l, k) -2*l + 2*k

l <- -o9b$value
k <- length(o9b$par)
AIC.Makeham <- myAIC(l, k)

> AIC.Makeham
[1] 382753
```
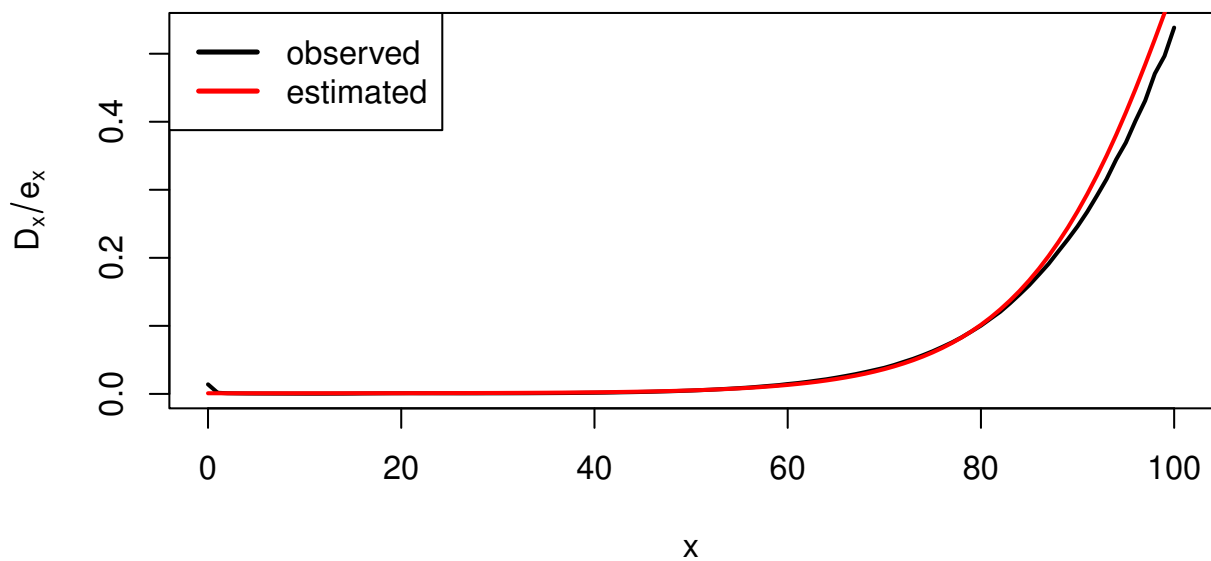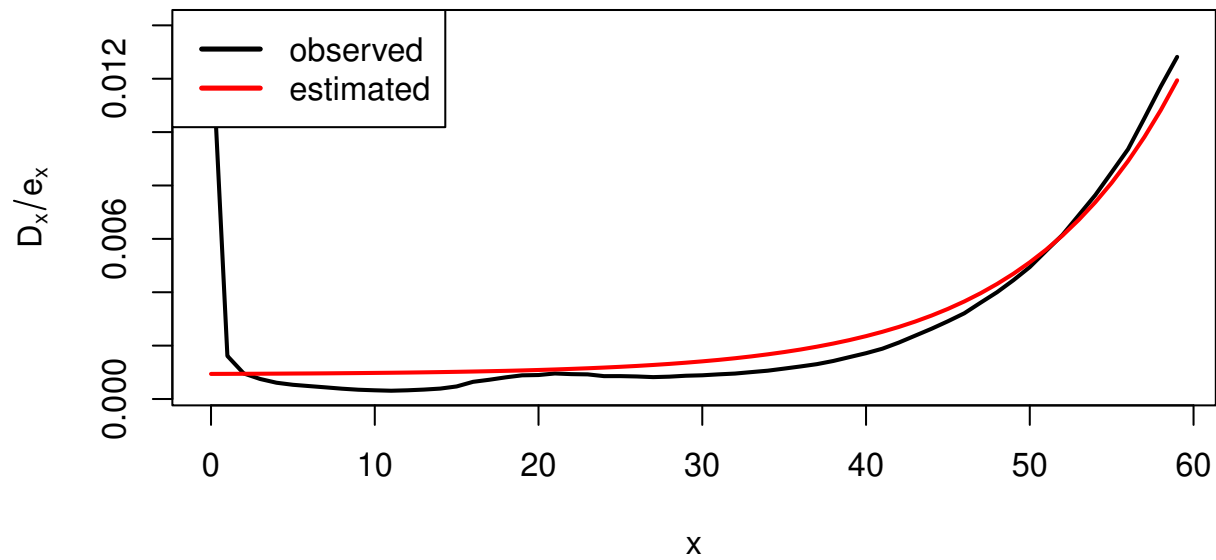
Question 10

```
## (a)
## compute optimal estimates
mleS.x <- S(0:nages, o9b$par[1], o9b$par[2], o9b$par[3])
mleq.x <- -diff(mleS.x) / mleS.x[-length(mleS.x)]
observed <- D.x / e.x

## plot
x <- 0:(length(observed)-1)
plot(x, observed, type="l", ylab=expression('D'[x] / 'e'[x]), lwd=2)
lines(x, mleq.x, col="red", lwd=2)

legend("topleft", c("observed","estimated"),
       lty=c(1), lwd=c(2.5), col=c("black" ,"red"))
```

```
## (b) zoom in
x <- 0:(length(1:60)-1)
plot(x, observed[1:60], type="l", ylab=expression('D'[x]/'e'[x]), lwd=2)
lines(x, mleq.x[1:60], col="red", lwd=2)
legend("topleft", c("observed","estimated"),
       lty=c(1), lwd=c(2.5), col=c("black" ,"red"))
```

Question 11

We have:

$$\tilde{\mu}_{xt} = \exp\left(\tilde{\alpha}_x + \tilde{\beta}_x\, \tilde{\kappa}_t\right)$$

$$= \exp\left(\alpha_x + \beta_x\, \bar{\kappa} + \frac{\beta_x}{\beta_\Sigma} \cdot \beta_\Sigma(\kappa_t - \bar{\kappa})\right)$$

$$= \exp\left(\alpha_x + \beta_x\, \bar{\kappa} + \beta_x\, \kappa_t - \beta_x\, \bar{\kappa}\right)$$

$$= \exp\left(\alpha_x + \beta_x\, \kappa_t\right) = \mu_{xt}$$

with

$$\sum_x \tilde{\beta}_x = \sum_x \left(\frac{\beta_x}{\beta_\Sigma}\right)$$

$$= \sum_x \frac{\beta_x}{\sum_x \beta_x}$$

$$= \sum_x \beta_x$$

$$= 1$$

and

$$\sum_t \tilde{\kappa}_t = \sum_t \beta_\Sigma\, (\kappa_t - \bar{\kappa})$$

$$= \beta_\Sigma \sum_t \kappa_t - \beta_\Sigma \sum_t \bar{\kappa}$$

$$= \beta_\Sigma \left(0 - \sum_t (\frac{1}{T} \cdot \sum_t \kappa_t)\right)$$

$$= 0.$$

Question 12

```
## (a)
## extract parameters
coef.gg <- coef(gg)
alpha.gg <- coef.gg[1:nages]
beta.gg <- coef.gg[(nages+1):(2*nages)]
kappa.gg <- coef.gg[(2*nages+1):length(coef.gg)]

## transform parameters
alpha.gnm <- alpha.gg + mean(kappa.gg)*beta.gg
beta.gnm <- beta.gg/sum(beta.gg)
kappa.gnm <- sum(beta.gg)*(kappa.gg-mean(kappa.gg))

## check
sum(beta.gnm)     #  [1] 1
sum(kappa.gnm)    #  [1] -8.826273e-15 (approx. 0)

## (b)
plot(alpha.gnm[1:65]+beta.gnm[1:65]*kappa.gnm[1], type="l",
     lwd=2, col="gray80", ylim=c(-9.3,-3.6), xlab="Age", ylab="Log-mortality")
lines(alpha.gnm[1:65]+beta.gnm[1:65]*kappa.gnm[25], lwd=2, col="gray60")
lines(alpha.gnm[1:65]+beta.gnm[1:65]*kappa.gnm[40], lwd=2, col="gray40")
lines(alpha.gnm[1:65]+beta.gnm[1:65]*kappa.gnm[55], lwd=2, col="gray20")
```
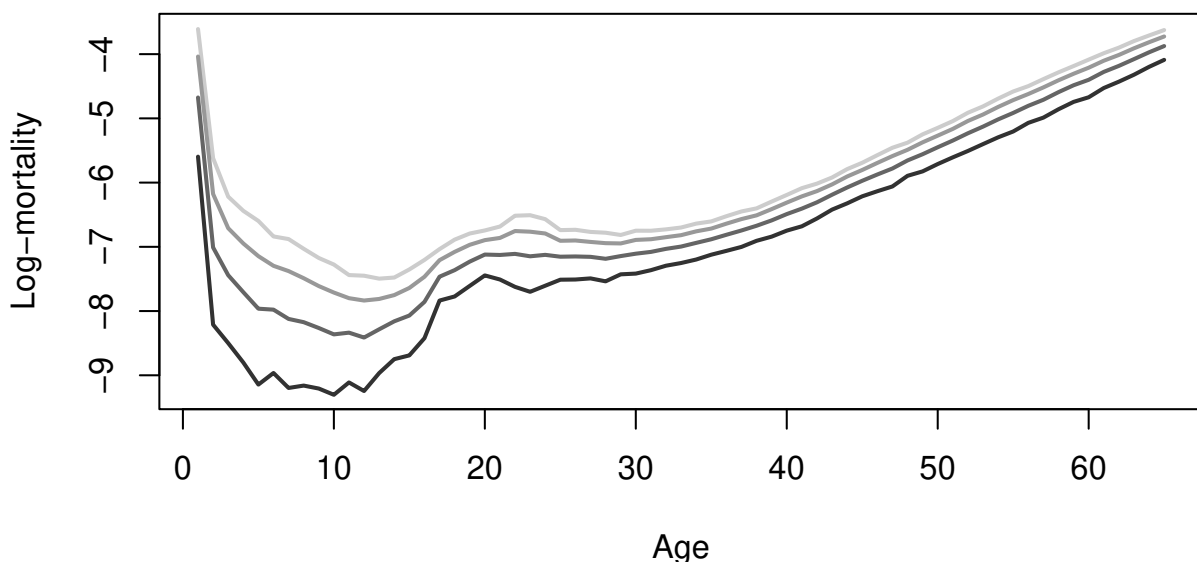

We observe a consistent downward shift of the log-mortality curve over time t, which corresponds to lower mortality for a given age *i.e.* people are living longer.

Question 13

```
## Bailey-Simon (successive substitution)
kappa.glm <- kappa.LC; kappas <- kappa.glm[t]
g1 <- glm(Dxt.vec ~ 0 + x + x:kappas + offset(lnExt.vec), poisson)
g1$deviance; g1$iter   ## 27605.14  4
c1 <- coef(g1)


##
alpha.glm <- c1[1:nages]
beta.glm <- c1[(nages+1):(2*nages)]
```

Question 14

```
betas <- beta.glm[x]
g2 <- glm(Dxt.vec ~ 0 + x + t:betas + offset(lnExt.vec), poisson,
          mustart=fitted(g1))
g2$deviance; g2$iter          ## 23594.36 ;  4
c2 <- coef(g2)
round(tail(c2),2)
## t53:betas t54:betas t55:betas t56:betas t57:betas t58:betas
## 24.96     21.34     18.48     15.25     5.62         NA

alpha.glm <- c2[1:nages]
kappa.glm <- c2[(nages+1):length(c2)]
kappa.glm["t58:betas"] <- 0
```

Question 15

```
## reconstruct
fitted(g2)[532]    ## 57.28315

myfitg2 <- exp(alpha.glm[x[532]] +
               beta.glm[x[532]]*kappa.glm[t[532]] + log(Ext.vec[532]))

> myfitg2
[1] 57.28315
```

Question 16

```
## Iteration method
kappa.glm <- kappa.LC
oldDeviance <- 0; TotnIter <- 0; start <- NULL
repeat {
    kappas <- kappa.glm[t]
    g1 <- glm(Dxt.vec~ 0 + x + x:kappas + offset(lnExt.vec), poisson,
              mustart=start)
    c1 <- coef(g1)
    alpha.glm <- c1[1:nages]
    beta.glm <- c1[(nages+1):(2*nages)]

    betas <- beta.glm[x]
    g2 <- glm(Dxt.vec ~ 0 + x + t:betas + offset(lnExt.vec), poisson,
              mustart=fitted(g1))
    c2 <- coef(g2)
    alpha.glm <- c2[1:nages]
    kappa.glm <- c2[(nages+1):length(c2)]
    kappa.glm["t58:betas"] <- 0                             ## sneaky

    alpha.glm <- alpha.glm + mean(kappa.glm)*beta.glm      ## identification
    kappa.glm <- sum(beta.glm)*(kappa.glm-mean(kappa.glm)) ## identification
    beta.glm <- beta.glm/sum(beta.glm)                     ## identification

    TotnIter <- TotnIter + g1$iter + g2$iter
    newDeviance <- g2$deviance;
    done <- isTRUE(all.equal(oldDeviance, newDeviance, tol=1e-6))
    cat(g1$deviance, "\t", g2$deviance, "\n")
    oldDeviance <- newDeviance; start <- fitted(g2)
    if (done) break
}

## output

## 27605.14    23594.36
## 23416.02    23406.77
## 23406.28    23406.25
## 23406.25    23406.25

> TotnIter
[1] 20
```

```
## results

> range(abs(alpha.glm - alpha.gnm))
[1] 2.465098e-09 6.967275e-06


> range(abs(beta.glm - beta.gnm))
[1] 6.812729e-10 6.608703e-07


> range(abs(kappa.glm - kappa.gnm))
[1] 3.845795e-07 2.071147e-04
```

Question 17

```
AIC(gg)          ## [1] 67209.76 *
AIC.Makeham      ## [1] 382753
```

Despite the large number of extra parameters, the *gnm* (Model **gg**) is preferred according to the AIC.