Computational Electromagnetics

Finite Differences in One Dimension

1. Introduction

Consider an electromagnetic plane wave incident on a large window of glass as shown in Fig. 1. We want to compute the reflected and transmitted wave. The thickness of the glass window is $2a$.

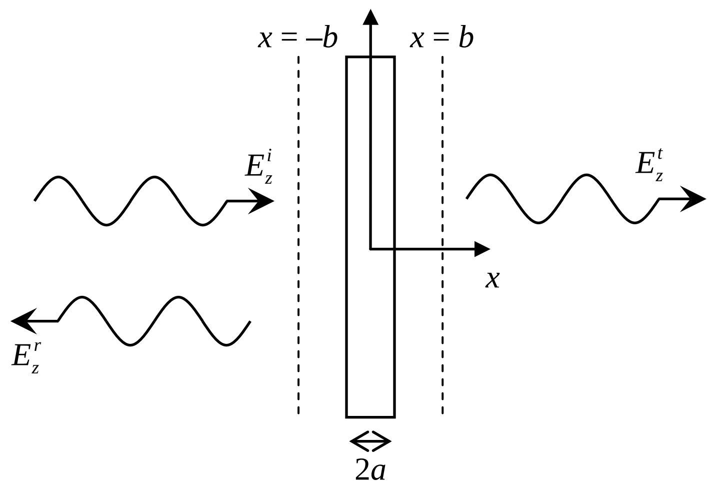

Fig. 1: Incident z-polarized field E_z^i on a glass window, reflected field E_z^r, and transmitted field E_z^t

In the glass, $-a \le x \le a$, the material parameters are $\varepsilon(x)$, $\mu(x) = \mu_0$ and $\sigma(x)$. The medium outside the window $|x| > a$ is air, with $\varepsilon(|x| > a) = \varepsilon_0$, $\mu(|x| > a) = \mu_0$, and $\sigma = 0$. The total field satisfies the wave equation:

$$-\frac{d^2E_z(x)}{dx^2} + \mu_0\left[j\omega\sigma(x) - \omega^2\varepsilon(x)\right]E_z(x) = 0 \tag{1}$$

2. Preparation

This problem introduces some of the techniques used in electromagnetic scattering. First of all, the incoming wave is injected by matching to the numerical solution in the vacuum region, where we know how both the incoming and outgoing waves behave. For this 1D example, this can be done by deriving boundary conditions in front of and behind the window, e.g. at $x = -b < -a$ and $x = b > a$ respectively.

2.1 Formulate Boundary Conditions

The first task is to derive analytical boundary conditions at $x = \pm b$. Here is some guidance.

Call the total field $E_z(x)$. At $x = -b$, the incoming field is $E_z^i(x) = E_0 e^{-jk_0 x}$ and we introduce the reflected field as $E_z^r(x) = E_r e^{+jk_0 x}$. At $x = b$, the transmitted field is $E_z^t(x) = E_t e^{-jk_0 x}$. The coefficients E_r and E_t are what we really want to compute. From Eq. (1), one can see that k0 = ω/c_0 where $c_0 = 1\big/\sqrt{\varepsilon_0 \mu_0}$ is the speed of light in vacuum.

We start by asking how a boundary condition can be formulated at $x = b$. At $x = b$, the total field is $E_z(x) = E_t e^{-jk_0 x}$. Then the derivative of this field must be:

$$\frac{dE_z(x)}{dx} = -jk_0 E_t e^{-jk_0 x} = -jk_0 E_z(x) \tag{2}$$

This gives the boundary condition $E_z'(b) + jk_0 E_z(b) = 0$ for $x = b$. This is a standard (homogeneous) boundary condition for an outgoing wave.

Now, do the same thing at $x = -b$ where the total field is $E_z(x) = E_0 e^{-jk_0 x} + E_r e^{+jk_0 x}$. Take the derivative and relate it to E_z and other *known* quantities (what is known here?). Now you should get an *inhomogeneous* boundary condition.

2.2 Generation of Grid and System Matrix

We choose a grid with N intervals, where N is even. Although it is wasting some unnecessary space, we can choose $b = 2a$ and choose the grid as

$$x_n = 4a\frac{n-1}{N} - 2a\,, \qquad n = 1, 2, \ldots, N+1 \tag{3}$$

Call the unknowns on this grid $E_z(x_n) = \zeta_n$. Discretize the differential equation (1) and your boundary conditions using finite differences with an error $O(h^2)$. The boundary conditions involving the function and its first derivative are best centered on the half grid. Alternatively you can center it also on the integer grid, using higher-order expressions for the first derivative. From the differential equation and the boundary conditions, one gets a system of linear equations $\mathbf{Az} = \mathbf{b}$ to solve, where $\mathbf{z} = [\zeta_1\ \zeta_1\ \ldots, \zeta_N]^{\mathrm{T}}$.

Preliminary Tasks:

1. Write down the matrix $\mathbf{A}$ and the right-hand side vector $\mathbf{b}$ in the special case when $N = 8$.

2. Find a way of computing the reflection and transmission coefficients given the numerical solution.

3. Numerical Experiment

Implement your numerical algorithm in a computer programming language (e.g. MATLAB) for an arbitrary number of points N and test it on the case when the glass window has constant relative permittivity ε_r and conductivity σ. The reflection and transmission coefficient can be calculated analytically in this case and they are given by:

$$R = \frac{k_0^2 - k_1^2}{\Delta} e^{j2ak_0} \left(e^{j4ak_1} - 1 \right) \tag{4a}$$

$$T = \frac{k_0 k_1}{\Delta} 4e^{j2a(k_0 + k_1)} \tag{4b}$$

where

$$\Delta = \left(k_0 + k_1\right)^2 e^{j4ak_1} - \left(k_0 - k_1\right)^2 \tag{5a}$$

$$k_0 = \omega / c_0 \tag{5b}$$

$$k_1 = \sqrt{\varepsilon_r k_0^2 - j\omega\mu_0\sigma} \tag{5c}$$

Choose the material parameters such that there is a couple of wavelengths inside the glass window.

Main Tasks

- Compute R and T numerically for $N = 2^p$ for some $p \geq 3$ when $b = 2a$. Do the numerically computed values of R and T converge towards the analytical values? Which order of convergence do you find? How are the interfaces located on the grid? How do you choose $\varepsilon(\pm a)$>
- Now change to $N = 2^p + 2$, compute R and T, and find out the order of convergence. What changed in the computation?
- Choose a = 2 cm, ε_r = 2.5 and σ = 0.02 ohm^{-1}m^{-1} and compute R and T as functions of frequency between 0 and 10 GHz.

Write a report that includes your formulation, computer program code, numerical results, and answers to the questions.