Radio-Wave Propagation and Antennas for Wireless Communications

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Final Exam
13 January 2022

Duration: 2 hours

Instructions: This paper consists of TWO pages comprising TWO parts each containing THREE questions with several sub-questions, altogether carrying a combined total of 100 marks. Part A contains short questions that require only brief answers. Hence, do not spend too much time on it. It is Part B that contains the bigger questions which carry more marks. So, allocate more time there.

Part A. (Total 20 marks)

1. Fundamentals (Total 10 marks)
   a. Write the relation between the electric field in the far-field region and the far-field function. (1 mark)

Answer:

$$\mathbf{E}(r,\theta,\varphi)=\frac{1}{r}e^{-jkr}\mathbf{G}(\theta,\varphi)$$

   b. Write or explain the relation between the magnetic field in the far-field region and the far-field function. (1 mark)

Answer:

$$\mathbf{H}(r,\theta,\varphi)=\frac{1}{\eta}\hat{\mathbf{r}}\times\mathbf{E}(r,\theta,\varphi)=\frac{1}{\eta}\hat{\mathbf{r}}\times\left[\frac{1}{r}e^{-jkr}\mathbf{G}(\theta,\varphi)\right]$$

   c. Explain how to transform a far-field function from one phase reference point to another. (2 marks)

Answer:

$$\mathbf{G}'(\theta,\varphi)=\mathbf{G}(\theta,\varphi)e^{-jk\mathbf{r}_0\cdot\hat{\mathbf{r}}}$$

where $\boldsymbol{r}_0$ is the position vector of the new phase reference point

   d. Explain how to transform a far-field function from one antenna location to another, when the antenna points in the same direction. (2 marks)

Answer:

$$\mathbf{G}_A(\theta,\varphi)=\mathbf{G}(\theta,\varphi)e^{jk\mathbf{r}_A\cdot\hat{\mathbf{r}}}$$

where $\boldsymbol{r}_A$ is the position vector of the new antenna location.

e. Write the E- and H- plane far-field functions of a Huygen's source. (2 marks)

Answer:

Slide 102 of CHAPTER THREE

$$G^{y}_{E_{Huygen}}(\theta) = \underbrace{2C_k \eta J_0 dS \cos^2\frac{\theta}{2} = \text{field function in } E\text{-plane } (yz \text{ plane}) \text{ when } \phi = 90°}_{\substack{EQUAL \\ \uparrow}}$$

$$G^{y}_{H_{Huygen}}(\theta) = \overbrace{2C_k \eta J_0 dS \cos^2\frac{\theta}{2} = \text{field function in } H\text{-plane } (xz \text{ plane}) \text{ when } \phi = 0}$$

f. State the condition for the far-field region. (1 mark)

Answer:

The far field region condition is

$$r \geq \frac{2D^2}{\lambda}$$

where D is the largest diameter of the antenna and λ is the wavelength.

g. If an antenna has an aperture of area A and works at frequency f (corresponding wavelength λ), what is the maximum available directivity? (1 mark)

Answer:

$$D_{\max} = \frac{4\pi}{\lambda^2} A$$

2. BOR$_1$ antenna (4 marks)

Consider a BOR$_1$ type antenna with an E-plane far-field function $G_E(\theta)$ and a H-plane far-field function $G_H(\theta)$. Answer all the following questions briefly.

a. Write the expression for the far-field function in the spherical coordinate system when the antenna is y-polarized. (1 mark)

Answer: from (2.83) Slide 185 of CHAPTER ONE

$$\mathbf{G}_y(\theta,\varphi) = G_E(\theta)\sin\varphi\,\hat{\boldsymbol{\theta}} + G_H(\theta)\cos\varphi\,\hat{\boldsymbol{\varphi}}$$

b. Write the expression for the far-field function when the antenna is x-polarized. (1 mark)

Answer: from (2.88a) Slide 200 of CHAPTER ONE

$$\mathbf{G}_x(\theta,\varphi) = G_E(\theta)\cos\varphi\,\hat{\boldsymbol{\theta}} - G_H(\theta)\sin\varphi\,\hat{\boldsymbol{\varphi}}$$

c. $G_E(\theta)$ and $G_H(\theta)$ must be identical for two directions. Which ones? (2 marks)

Answer:

$$\text{At } \theta = 0^\circ \text{ and } \theta = 180^\circ$$

3. Factorization of far-field function (6 marks)
   a. The far-field function of a straight wire antenna (e.g. an electric dipole) can be separated into three factors. State these three factors. (3 marks)

Answer

The three factors are i) the current amplitude at the terminal; ii) the radiation field function of an incremental electric current with unit amplitude, and iii) the Fourier transform of the current distribution along the dipole.

   b. The far-field function of a narrow rectangular slot in an infinite PEC (perfect electric conducting) ground plane can be separated into three factors. State these three factors. (3 marks)

Answer

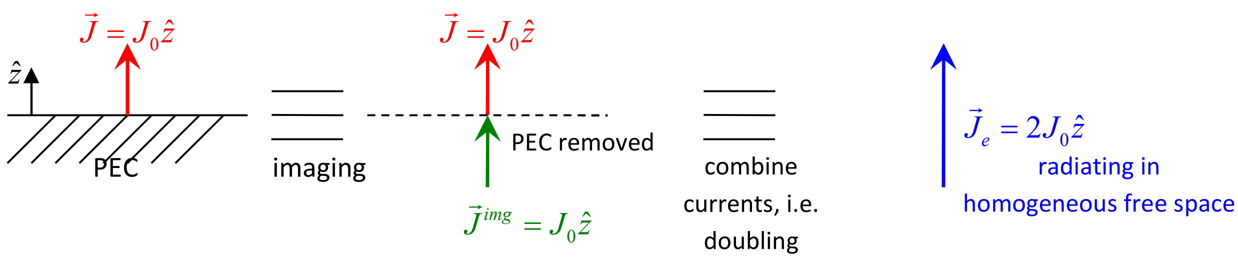
A rectangular slot in a ground plane radiates like a magnetic dipole. The radiation field function is proportional to i) the voltage over the center of the slot; ii) the radiation field function of an incremental magnetic current oriented along the slot; and iii) the Fourier transform of the field distribution along the slot.

Part B. (Total 80 marks)

1. Imaging of Vertical Electric Monopole (20 marks)

Consider a vertical short electric current source on an infinite PEC ground plane. This can, for example, be the center conductor of a coaxial line coming out of a hole in the ground plane. Use imaging to find the far-field function, and find thereafter the directivity when we assume that the vertical source is infinitesimal.

Solution:



Then, directly using (4.68) of Slide 71 of CHAPTER THREE:

$$\vec{G}^{z\text{-directed}}_{i_d(\hat{r})} = (-\eta C_k \underbrace{J_0^{line}\ell}_{2J_0 here} \sin\theta)\hat{\theta}$$

From (2.64) of slide 128 of CHAPTER ONE, radiated power is:

$$P_{rad} = \frac{1}{2\eta}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\left|\vec{G}(\theta,\phi)\right|^2 \underbrace{\sin\theta d\theta d\phi}_{d\Omega} \overset{for}{\underset{\substack{present\\case}}{=}} \frac{1}{2\eta}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\overbrace{\pi\to\pi/2}^{\substack{NOTE!\\ \text{JUST HALF}\\ \text{SPACE HERE}}}} \left|(-\eta C_k 2J_0 \sin\theta)\hat{\theta}\right|^2 \sin\theta d\theta d\phi$$

$$= 2\eta\left(J_0C_k\right)^2 \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2} \sin^3\theta d\theta d\phi = 4\pi\eta\left(J_0C_k\right)^2 \overbrace{\underbrace{\int_{\theta=0}^{\pi/2} \overbrace{\sin^3\theta}^{\sin\theta\left[\frac{1}{2}(1-\cos 2\theta)\right]} d\theta}}^{\frac{2}{3}} = \frac{8\pi\eta}{3}\left(J_0C_k\right)^2$$

$$\frac{1}{2}\left\{\left[-\cos\theta\right]_{\theta=0}^{\theta=\frac{\pi}{2}} - \int_{\theta=0}^{\pi/2} \sin\theta \overbrace{\cos 2\theta}^{2\cos^2\theta-1} d\theta\right\} =$$

$$= \frac{1}{2} - \frac{1}{2}\underbrace{\left\{2\underbrace{\overbrace{\int_{\theta=0}^{\pi/2} \cos^2\theta \sin\theta d\theta}^{\frac{2}{3}}}_{-\frac{1}{3}\left[\cos^3\theta\right]_{\theta=0}^{\theta=\frac{\pi}{2}}=\frac{1}{3}} - \underbrace{\overbrace{\int_{\theta=0}^{\pi/2} \sin\theta d\theta}^{1}}_{-\left[\cos\right]_{\theta=0}^{\theta=\frac{\pi}{2}}=1}\right\}}_{-\frac{1}{3}} = \frac{2}{3}$$

For the present $\hat{z}$ -directed vertical electric current (over PEC ground plane), the co-polar unit vector $\hat{co}$ is always fixed as the constant $\hat{\theta}$ for all ϕ because regardless of the ϕ-plane in which the line-of-sight (from source at origin to observation point) lies within, the projection of the $\hat{z}$ -directed vertical electric current onto the plane perpendicular to the line-of-sight is always $\hat{\theta}$ directed.

Hence, the co-polar field function is simply:

$$\vec{G}^{\substack{z\text{-}direc\\ \text{elec curr}}}_{co(\theta,\phi)} = \vec{G}^{z\text{-directed}}_{i_d(\hat{r})} \cdot \overbrace{\hat{co}^*}^{\hat{\theta}} = (-\eta C_k \underbrace{J_0^{line}\ell}_{2J_0 here} \sin\theta)\hat{\theta}\cdot\hat{\theta} = -\eta C_k 2J_0 \sin\theta$$

Then finally, from (2.68a) of Slide 134 of CHAPTER ONE: the co-polar directive gain is

$$G^D_{co}(\theta,\phi)\Big|_{dBi} = 10\log_{10}\left[\frac{2\pi\left|G_{co}(\theta,\phi)\right|^2}{\eta P_{rad}}\right] dBi$$

$$= 10\log_{10}\left[\frac{2\pi\left|-\eta C_k 2J_0 \sin\theta\right|^2}{\eta\frac{8\pi\eta}{3}\left(J_0 C_k\right)^2}\right] dBi = 10\log_{10}\left(3\left|\sin\theta\right|^2\right) dBi$$

It is known that maximum directive gain for this electric dipole antenna placed vertically above a PEC ground is towards the broadside θ = 90° direction. Hence,

$$\therefore G^D_{co,\max}\Big|_{dBi} = G^D_{co}(\theta = 90°,\phi)\Big|_{dBi} = 10\log_{10}\left(3\right) dBi = 4.77 dBi$$

2. Imaging of Horizontal Monopole (20 marks)

Consider a horizontal incremental electric dipole located at a height *h* above an infinite PEC ground plane. Derive the expression for the radiation field by using imaging when *h* = *λ*/4.

Solution:

Assume, arbitrarily, a *y*-directed horizontal incremental electric dipole at height *z* = *d* above PEC ground plane (vertical direction is *z*).

Then from (5.64) of Slide 98 of CHAPTER FOUR:

$$G^{\substack{y\text{-}direc \text{ elec dip}\\ \text{height } h \text{ above}\\ \text{PEC } xy \text{ ground}}}_{J_{actual}\&J_{image}}(\theta,\phi) = \eta C_k I_0 \tilde{j}(\overbrace{k\sin\theta\sin\phi}^{k_y})\Big[(\cos\theta\sin\phi)\hat{\theta}+(\cos\phi)\hat{\phi}\Big]\overbrace{j2\sin(kh\cos\theta)}^{\text{ground factor}}$$

$$\text{where } \tilde{j}(\overbrace{k\sin\theta\sin\phi}^{k_y}) = \int_{y=-\frac{\ell}{2}}^{y=\frac{\ell}{2}} j(y)e^{jk_y y}\,dy =$$

$$= \text{Fourier transform of normalized } y\text{-directed electric line current distribution}$$

Since here we consider incremental electric dipole without any spatial distribution, so *j*(*y*) becomes just a *y*-directed point source and $\tilde{j}(\overbrace{k\sin\theta\sin\phi}^{k_y})$ is simply just 1.

So,

$$G^{\substack{y\text{-}direc \text{ increm}\\ \text{elec dipole}\\ \text{height } h \text{ above}\\ \text{PEC } xy \text{ ground}}}_{J_{actual}\&J_{image}}(\theta,\phi) = \eta C_k I_0 \underbrace{\tilde{j}(\overbrace{k\sin\theta\sin\phi}^{k_y})}_{\substack{UNITY\\ \text{for increm point-source type}\\ \text{electric dipole of zero length}\\ \text{(zero spatial distribution)}}} \Big[(\cos\theta\sin\phi)\hat{\theta}+(\cos\phi)\hat{\phi}\Big]\overbrace{j2\sin(kh\cos\theta)}^{\text{ground factor}}$$

$$= j2\eta C_k I_0 \sin(kh\cos\theta)\Big[(\cos\theta\sin\phi)\hat{\theta}+(\cos\phi)\hat{\phi}\Big]$$

For *h* = λ/4,

$$G^{\substack{y\text{-}direc \text{ increm}\\ \text{elec dipole}\\ \text{height } h=\lambda/4 \text{ above}\\ \text{PEC } xy \text{ ground}}}_{J_{actual}\&J_{image}}(\theta,\phi) = j2\eta C_k I_0 \sin\left(\frac{2\pi}{\lambda}\frac{\lambda}{4}\cos\theta\right)\Big[(\cos\theta\sin\phi)\hat{\theta}+(\cos\phi)\hat{\phi}\Big]$$

$$= j2\eta C_k I_0 \sin\left(\frac{\pi}{2}\cos\theta\right)\Big[(\cos\theta\sin\phi)\hat{\theta}+(\cos\phi)\hat{\phi}\Big]$$

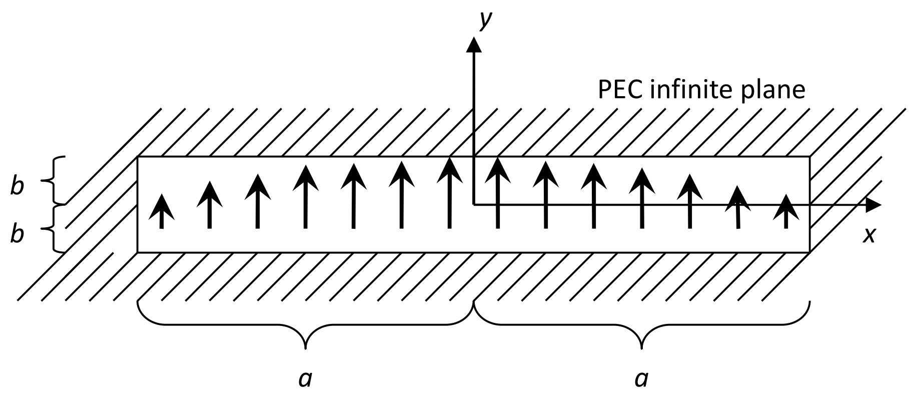
3. Magnetic Currents and Radiating Slot (40 marks)

Consider a small rectangular radiating slot in an infinite PEC ground plane. Assume that the *E*-field in the slot is:

$$\vec{E} = \hat{y}\cos\left(\frac{\pi x}{2a}\right) \text{ for } -a < x < a \text{ and } -b < y < b, \text{ where } a \ll \lambda \;\&\; b \ll \lambda .$$

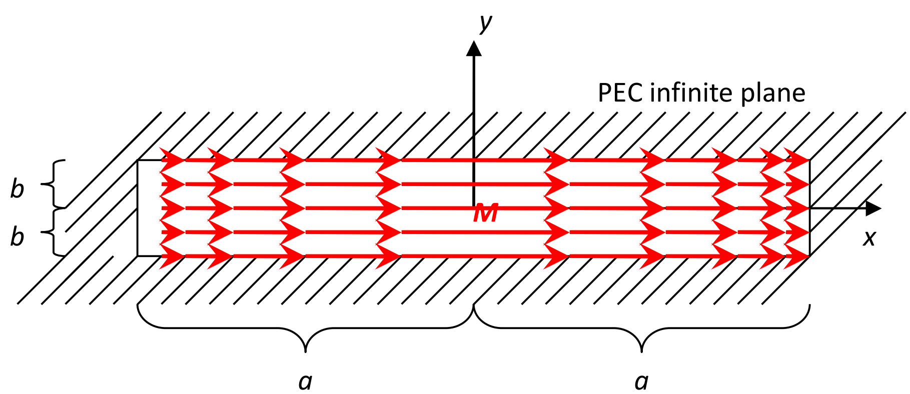
a) By using PEC equivalence, imaging and the radiation field of a magnetic current distribution in free space, write the expression for the radiation field of the slot.

Solution a)



PEC equivalence: Fill up slot with PEC and place magnetic current over slot:

$$\vec{M} = \vec{E} \times \underbrace{\hat{n}}_{\hat{z}} = \begin{pmatrix} 0 \\ \cos(\pi x/2a) \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \hat{x}\cos\left(\frac{\pi x}{2a}\right)$$



Imaging: Remove PEC plane and double magnetic current: the resultant magnetic current radiates in unbounded homogeneous free space.

$$\vec{M}_{eq} = 2\vec{M} = 2\cos\left(\frac{\pi x}{2a}\right)\hat{x}$$

From (4.50) and (4.52) of Slide#53,

$$\vec{E}_M(\hat{r}) = \frac{e^{-jkr}}{r}\vec{G}_M(\hat{r});\ \vec{G}_M(\hat{r}) = \vec{I}_M \times r;\ \vec{I}_M = C_k \iint_{S'} \vec{M}(\vec{r}')e^{jk(\vec{r}'\cdot\hat{r})}dS';\ C_k = -jk/(4\pi)$$

For this case,

$$\vec{I}_M = \hat{x}2C_k \int_{y'=-b}^{y'=-b}\int_{x'=-a}^{x'=a} \cos\left(\frac{\pi x'}{2a}\right) e^{jk(\vec{r}'\cdot\hat{r})} dx'dy' \underset{\substack{\text{such that}\\ \vec{r}'\cdot\hat{r}=x'\sin\theta\cos\phi+\\ +y'\sin\theta\sin\phi+zero}}{\overset{\substack{\vec{r}'=\hat{x}x'+yy'\\ \hat{r}=x\sin\theta\cos\phi+\\ +\hat{y}\sin\theta\sin\phi+\\ +\hat{z}\cos\theta}}{=}} \hat{x}2C_k \int_{y'=-b}^{y'=-b}\int_{x'=-a}^{x'=a} \cos\left(\frac{\pi x'}{2a}\right) e^{jk\sin\theta(x'\cos\phi+y'\sin\phi)} dx'dy'$$

$$= \underbrace{\hat{x}}_{\begin{pmatrix}\hat{r}\sin\theta\cos\phi\\ \hat{\theta}\cos\theta\cos\phi\\ -\hat{\phi}\sin\phi\end{pmatrix}} 2\underbrace{C_k}_{-\frac{jk}{4\pi}} \underbrace{\int_{x'=-a}^{x'=a} \cos\left(\frac{\pi x'}{2a}\right) e^{j\overbrace{(k\sin\theta\cos\phi)}^{k_x}x'} dx' \int_{y'=-b}^{y'=-b} e^{j\overbrace{(k\sin\theta\sin\phi)}^{k_y}y'} dy'}_{\substack{\updownarrow\\ \overbrace{\tilde{\tilde{E}}_{a_y}(k_x,k_y)=-\frac{8\pi a\cos\overbrace{(ka\sin\theta\cos\phi)}^{k_x a}\sin\overbrace{(kb\sin\theta\sin\phi)}^{k_y b}}{k\sin\theta\sin\phi(2ka\sin\theta\cos\phi+\pi)(2ka\sin\theta\cos\phi-\pi)}}\\ \text{of below later for alternative method}}}$$

$$= \overbrace{\begin{pmatrix}\hat{r}\sin\theta\cos\phi\\ \hat{\theta}\cos\theta\cos\phi\\ -\hat{\phi}\sin\phi\end{pmatrix}}^{\hat{x}} \frac{j4a\cos\overbrace{(ka\sin\theta\cos\phi)}^{k_x a}\sin\overbrace{(kb\sin\theta\sin\phi)}^{k_y b}}{\sin\theta\sin\phi(2ka\sin\theta\cos\phi+\pi)(2ka\sin\theta\cos\phi-\pi)} = \begin{pmatrix}I_{M_r}\\ I_{M_\theta}\\ I_{M_\phi}\end{pmatrix}$$

$$\therefore \vec{G}_M(\hat{r}) = \begin{pmatrix}I_{M_r}\\ I_{M_\theta}\\ I_{M_\phi}\end{pmatrix} \times \underbrace{\begin{pmatrix}1\\ 0\\ 0\end{pmatrix}}_{\hat{r}} = \begin{pmatrix}0\\ I_{M_\phi}\\ -I_{M_\theta}\end{pmatrix}$$

$$= \frac{j4a\cos(ka\sin\theta\cos\phi)\sin(kb\sin\theta\sin\phi)}{\sin\theta\sin\phi(2ka\sin\theta\cos\phi+\pi)(2ka\sin\theta\cos\phi-\pi)}\left[\hat{\theta}(-\sin\phi)+\hat{\phi}(-\cos\theta\cos\phi)\right]$$

$$= \frac{4a\cos(ka\sin\theta\cos\phi)\sin(kb\sin\theta\sin\phi)}{j\sin\theta\sin\phi(2ka\sin\theta\cos\phi+\pi)(2ka\sin\theta\cos\phi-\pi)}\left[\hat{\theta}(\sin\phi)+\hat{\phi}(\cos\theta\cos\phi)\right]$$

being valid only for half space: $0 < \theta < \pi/2$. This is identical to the alternative method below.

The final radiation field expression of the slot is simply just this $\boldsymbol{G}_M$ multiplied by e^{-jkr}/r.

ALTERNATIVE METHOD:

From (7.19) of Slides being the general expression, repeated below:

$$\boldsymbol{G}(\theta,\varphi) = -2C_k(\cos\varphi\hat{\theta} - \cos\theta\sin\varphi\hat{\varphi})\tilde{\tilde{E}}_{ax}(k\sin\theta\cos\varphi, k\sin\theta\sin\varphi)$$
$$-2C_k(\sin\varphi\hat{\theta} + \cos\theta\cos\varphi\hat{\varphi})\tilde{\tilde{E}}_{ay}(k\sin\theta\cos\varphi, k\sin\theta\sin\varphi)$$

(7.19)

in which

$$\tilde{\tilde{E}}_{a_{x \text{ or } y}}(k_x,k_y) = \int_{x=-\infty}^{x=\infty}\int_{y=-\infty}^{y=\infty} E_{a_{x \text{ or } y}}(x,y)e^{jk_x x}e^{jk_y y}dydx$$

in which $\vec{E}_a(x,y) = \hat{x}E_{a_x}(x,y) + \hat{y}E_{a_y}(x,y)$

For present case, aperture E-field is:

$$\vec{E}_a(x) = \hat{y}\underbrace{E_{a_y}(x)}_{\cos\left(\frac{\pi x}{2a}\right)} \text{ with } E_{a_y} = 0 \text{ and thus } \tilde{\tilde{E}}_{a_x} = 0$$

$$\therefore \tilde{\tilde{E}}_{a_y}(k_x,k_y)=\int_{x=-\infty}^{x=\infty}\int_{y=-\infty}^{y=\infty}\overbrace{\cos\left(\frac{\pi x}{2a}\right)}^{\substack{\frac{e^{j\pi x/2a}+e^{-j\pi x/2a}}{2}\\ \updownarrow}} e^{jk_x x}e^{jk_y y}\,dy\,dx=\frac{1}{2}\underbrace{\int_{x=-a}^{x=a}\left[e^{j\left(k_x+\frac{\pi}{2a}\right)x}+e^{j\left(k_x-\frac{\pi}{2a}\right)x}\right]dx}_{\left[\frac{e^{j\left(k_x+\frac{\pi}{2a}\right)x}}{j\left(k_x+\frac{\pi}{2a}\right)}+\frac{e^{j\left(k_x-\frac{\pi}{2a}\right)x}}{j\left(k_x-\frac{\pi}{2a}\right)}\right]_{x=-a}^{x=a}}\underbrace{\left[\frac{e^{jk_y y}}{jk_y}\right]_{y=-b}^{y=b}}_{\frac{2\sin(k_y b)}{k_y}}$$

$$=\left\{\left[\frac{e^{j\left(k_x a+\frac{\pi}{2}\right)}}{j\left(k_x+\frac{\pi}{2a}\right)}+\frac{e^{j\left(k_x a-\frac{\pi}{2}\right)}}{j\left(k_x-\frac{\pi}{2a}\right)}\right]-\left[\frac{e^{-j\left(k_x a+\frac{\pi}{2}\right)}}{j\left(k_x+\frac{\pi}{2a}\right)}+\frac{e^{-j\left(k_x a-\frac{\pi}{2}\right)}}{j\left(k_x-\frac{\pi}{2a}\right)}\right]\right\}\frac{\sin(k_y b)}{k_y}$$

$$=\left[\frac{2\overbrace{\sin\left(k_x a+\frac{\pi}{2}\right)}^{\cos(k_x a)}}{\frac{2k_x a+\pi}{2a}}+\frac{2\overbrace{\sin\left(k_x a-\frac{\pi}{2}\right)}^{-\cos(k_x a)}}{\frac{2k_x a-\pi}{2a}}\right]\frac{\sin(\overbrace{k_y}^{k\sin\theta\sin\phi}b)}{\underbrace{k\sin\theta\sin\phi}_{k_y}}=$$

$$=4a\cos(\overbrace{k_x}^{k\sin\theta\cos\phi}a)\frac{\sin\overbrace{(kb\sin\theta\sin\phi)}^{k_y b}}{\underbrace{k\sin\theta\sin\phi}_{k_y}}\overbrace{\left(\frac{1}{2k_x a+\pi}-\frac{1}{2k_x a-\pi}\right)}^{\frac{\overbrace{2k_x a-\pi-2k_x a-\pi}^{-2\pi}}{(2k_x a+\pi)(2k_x a-\pi)}}$$

$$=-\frac{8\pi a\cos\overbrace{(ka\sin\theta\cos\phi)}^{k_x a}\sin\overbrace{(kb\sin\theta\sin\phi)}^{k_y b}}{k\sin\theta\sin\phi(2ka\sin\theta\cos\phi+\pi)(2ka\sin\theta\cos\phi-\pi)}$$

Then, with this latter equation and (7.19), and $\tilde{\tilde{E}}_{a_x}=0$,

$$\vec{G}^{\substack{\text{apert in}\\ \text{inf PEC}\\ xy\text{ plane}}}(\theta,\phi)=2\underbrace{C_k}_{-\frac{jk}{4\pi}}\left[(\sin\phi)\hat{\vec{\theta}}+(\cos\theta\cos\phi)\hat{\vec{\phi}}\right]\frac{8\pi a\cos(ka\sin\theta\cos\phi)\sin(kb\sin\theta\sin\phi)}{k\sin\theta\sin\phi(2ka\sin\theta\cos\phi+\pi)(2ka\sin\theta\cos\phi-\pi)}$$

$$=\frac{4a\cos(ka\sin\theta\cos\phi)\sin(kb\sin\theta\sin\phi)\left[(\sin\phi)\hat{\vec{\theta}}+(\cos\theta\cos\phi)\hat{\vec{\phi}}\right]}{j\sin\theta\sin\phi(2ka\sin\theta\cos\phi+\pi)(2ka\sin\theta\cos\phi-\pi)}$$

being valid only for half space: 0 < θ < $\pi/2$.

This is identical to the earlier method above.

b) Find the directivity.

Solution b)

From (2.64) of Slides: total radiated power is

$$P_{rad} = \frac{1}{2\eta}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\overbrace{\pi\to\pi/2}^{NOTE!}} \underbrace{\left(\left|G_{co}(\theta,\phi)\right|^2+\left|G_{xp}(\theta,\phi)\right|^2\right)}_{\left|\vec{G}(\theta,\phi)\right|^2}\underbrace{\sin\theta d\theta d\phi}_{d\Omega}$$

$$= \frac{1}{2\eta}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\overbrace{\pi/2}^{\substack{HALF\\SPACE\\ONLY!}}} \left|\frac{4a\cos\left(ka\sin\theta\cos\phi\right)\sin\left(kb\sin\theta\sin\phi\right)\left[\left(\sin\phi\right)\hat{\theta}+\left(\cos\theta\cos\phi\right)\hat{\phi}\right]}{j\sin\theta\sin\phi\left(2ka\sin\theta\cos\phi+\pi\right)\left(2ka\sin\theta\cos\phi-\pi\right)}\right|^2\sin\theta d\theta d\phi$$

noting the change of former π to $\pi/2$ for the upper limit of integration with respect to θ since now power radiates only into the upper half space.

Since given in question that $\{a,b\} \ll \lambda \Rightarrow \frac{\{a,b\}}{\lambda} \ll 1 \Rightarrow k\{a,b\} = \frac{2\pi\{a,b\}}{\lambda} \ll 2\pi$

and it is well-known that $\cos\alpha \underset{\alpha\to 0}{\approx} 1$ and $\sin\alpha \underset{\alpha\to 0}{\approx} \alpha$, hence this above P_{rad} may be simplified to

$$P_{rad} \approx \frac{1}{2\eta}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2}\left|\frac{4akb\,\cancel{\sin\theta\sin\phi}\left[\left(\sin\phi\right)\hat{\theta}+\left(\cos\theta\cos\phi\right)\hat{\phi}\right]}{j\,\cancel{\sin\theta\sin\phi}\,(\pi)(-\pi)}\right|^2\sin\theta d\theta d\phi$$

$$= \frac{1}{2\eta}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2}\left|\frac{j4\left(2\pi/\lambda\right)ab\left[\left(\sin\phi\right)\hat{\theta}+\left(\cos\theta\cos\phi\right)\hat{\phi}\right]}{\pi^2}\right|^2\sin\theta d\theta d\phi$$

$$= \frac{1}{2\eta}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2}\left|\frac{j8ab\left[\left(\sin\phi\right)\hat{\theta}+\left(\cos\theta\cos\phi\right)\hat{\phi}\right]}{\lambda\pi}\right|^2\sin\theta d\theta d\phi = \frac{32}{\eta}\left(\frac{ab}{\lambda\pi}\right)^2\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2}\left\|\left[\left(\sin\phi\right)\hat{\theta}+\left(\cos\theta\cos\phi\right)\hat{\phi}\right]\right\|^2\sin\theta d\theta d\phi$$

$$= \frac{32}{\eta}\left(\frac{ab}{\lambda\pi}\right)^2\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2}\left[\underbrace{\sin^2\phi}_{(1-\cos 2\phi)/2}+\left(\cos\theta\cos\phi\right)^2\right]\sin\theta d\theta d\phi$$

$$= \frac{32}{\eta}\left(\frac{ab}{\lambda\pi}\right)^2\left\{\frac{1}{2}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2}\left(1-\cos 2\phi\right)\sin\theta d\theta d\phi+\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2}\cos^2\theta\sin\theta\underbrace{\cos^2\phi}_{(1+\cos 2\phi)/2}d\theta d\phi\right\}$$

$$= \frac{32}{\eta}\left(\frac{ab}{\lambda\pi}\right)^2\left\{\overbrace{\frac{1}{2}\underbrace{\int_{\phi=0}^{2\pi}\left(1-\cos 2\phi\right)d\phi}_{\left[\phi-\frac{\sin 2\phi}{2}\right]_{\phi=0}^{\phi=2\pi}=2\pi}}^{\pi}\overbrace{\underbrace{\int_{\theta=0}^{\pi/2}\sin\theta d\theta}_{\left[-\cos\theta\right]_{\theta=0}^{\theta=\pi/2}=1}}^{1}+\overbrace{\frac{1}{2}\underbrace{\int_{\phi=0}^{2\pi}\left(1+\cos 2\phi\right)d\phi}_{\left[\phi+\frac{\sin 2\phi}{2}\right]_{\phi=0}^{\phi=2\pi}=2\pi}}^{\pi}\overbrace{\underbrace{\int_{\theta=0}^{\pi/2}\cos^2\theta\sin\theta d\theta}_{-\left[\frac{\cos^3\theta}{3}\right]_{\theta=0}^{\theta=\pi/2}=\frac{1}{3}}}^{\frac{1}{3}}\right\} = \frac{32}{\eta\pi}\left(\frac{ab}{\lambda}\right)^2\frac{4}{3}$$

Then from (2.68a) of Slides: the co-polar directive gain is

$$G_{co}^D(\theta,\phi)\Big|_{dBi} = 10\log_{10}\left[\frac{2\pi\left|G_{co}(\theta,\phi)\right|^2}{\eta P_{rad}}\right] dBi$$

Considering just the directive gain, $G^D(\theta,\phi)\Big|_{dBi} = 10\log_{10}\left[\frac{2\pi\left|\vec{G}\ (\theta,\phi)\right|^2}{\eta P_{rad}}\right] dBi$

whereby for the present case,

$\vec{G}(\theta,\phi) = \dfrac{4a\cos\left(ka\sin\theta\cos\phi\right)\sin\left(kb\sin\theta\sin\phi\right)\left[\left(\sin\phi\right)\hat{\theta}+\left(\cos\theta\cos\hat{\phi}\right)\phi\right]}{j\sin\theta\sin\phi\left(2ka\sin\theta\cos\phi+\pi\right)\left(2ka\sin\theta\cos\phi-\pi\right)}$ found earlier, which upon invoking the approximations due to $\left\{a,b\right\} \ll \lambda \Rightarrow \dfrac{\left\{a,b\right\}}{\lambda} \ll 1 \Rightarrow k\left\{a,b\right\} = \dfrac{2\pi\left\{a,b\right\}}{\lambda} \ll 2\pi$ and $\cos\alpha \underset{\alpha\to 0}{\approx} 1$ and $\sin\alpha \underset{\alpha\to 0}{\approx} \alpha$ as were done above, becomes

$$\vec{G}(\theta,\phi) \approx \frac{4a\left(2\pi/\lambda\right)b\,\cancel{\sin\theta\sin\phi}\left[\left(\sin\phi\right)\hat{\theta}+\left(\cos\theta\cos\phi\right)\hat{\phi}\right]}{j\,\cancel{\sin\theta\sin\phi}\,\left(\pi\right)\left(-\pi\right)} = \frac{j8ab\left[\left(\sin\phi\right)\hat{\theta}+\left(\cos\theta\cos\phi\right)\hat{\phi}\right]}{\lambda\pi}$$

and also with $P_{rad} = \dfrac{32}{\eta\pi}\left(\dfrac{ab}{\lambda}\right)^2\dfrac{4}{3}$

$$\therefore G^D(\theta,\phi)\Big|_{dBi} = 10\log_{10}\left[\frac{2\pi\left|\dfrac{j8ab\left[\left(\sin\phi\right)\hat{\theta}+\left(\cos\theta\cos\phi\right)\hat{\phi}\right]}{\lambda\pi}\right|^2}{\cancel{\eta}\,\dfrac{32}{\cancel{\eta}\pi}\left(\dfrac{ab}{\lambda}\right)^2\dfrac{4}{3}}\right] dBi =$$

$$= 10\log_{10}\left[\frac{\dfrac{\cancel{128}}{\cancel{\pi}}\cancel{\left(\dfrac{ab}{\lambda}\right)^2}\left[\sin^2\phi+\left(\cos\theta\cos\phi\right)^2\right]}{\dfrac{\cancel{32}}{\cancel{\pi}}\cancel{\left(\dfrac{ab}{\lambda}\right)^2}\dfrac{\cancel{4}}{3}}\right] dBi = 10\log_{10}\left\{3\left[\sin^2\phi+\left(\cos\theta\cos\phi\right)^2\right]\right\} dBi$$

It is known that maximum directive gain for this slot antenna is towards the broadside $\theta = 0$ direction. Hence,

$$\therefore G_{\max}^D\Big|_{dBi} = G^D\left(\theta=0,\phi\right)\Big|_{dBi} = 10\log_{10}\left[3\overbrace{\left(\sin^2\phi+\cos^2\phi\right)}^{1}\right] dBi = 10\log_{10}\left(3\right) dBi = 4.77 dBi$$