Rule-based Control

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1 Control barrier functions

This section gives an overview of control Lyapunov functions and control barrier functions and provides an example for an adaptive cruise control application. The material here is found in [XB19] unless otherwise noted.

1.1 Theory

Assume our system has control affine dynamics, with state space of dimension n and input dimension m.

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t)) + g(\mathbf{x}(t))\mathbf{u}(t) \tag{1}$$

Define the following operators on a test function $V : \mathbb{R}^n \to \mathbb{R}$:

$$\mathcal{L}_f V(\mathbf{x}) = \frac{\partial V}{\partial x} f(\mathbf{x}) \tag{2}$$

$$\mathcal{L}_g V(\mathbf{x}) = \frac{\partial V}{\partial x} g(\mathbf{x}) \tag{3}$$

1.1.1 Control Lyapunov Functions

A continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$ that is 0 at an equilibrium point of (or a subset of the states in) (1) , and strictly positive elsewhere is a globally and exponentially stabilizing control Lyapunov function (CLF) if there exists positive constants such that

$$c_1 ||\mathbf{x}||^2 \leq V(\mathbf{x}) \leq c_2 ||\mathbf{x}||^2 \tag{4}$$

$$\inf_{u \in U} \mathcal{L}_f V(\mathbf{x}) + \mathcal{L}_g V(\mathbf{x})\mathbf{u} + c_3 V(\mathbf{x}) \leq 0 \tag{5}$$

A controller $K_{\text{clf}} : \mathbb{R}^n \to \mathbb{R}^m$ stabilizes (1) to its zero dynamics if

$$K_{\text{clf}}(\mathbf{x}) := \{\mathbf{u} \in U \ : \ \mathcal{L}_f V(\mathbf{x}) + \mathcal{L}_g V(\mathbf{x})\mathbf{u} + c_3 V(\mathbf{x}) \leq 0\} \tag{6}$$

1.1.2 Control Barrier Functions

A function $\alpha : [0, a) \rightarrow [0, \inf),\ a > 0$ is said to be class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. The set

$$C := \{\mathbf{x} \in \mathbb{R}^n : b(\mathbf{x}) \geq 0\} \tag{7}$$

is forward invariant w.r.t. (1) if b is a barrier function satisfying

$$\sup_{\mathbf{u} \in U} \mathcal{L}_f b(\mathbf{x}) \mathcal{L}_g b(\mathbf{x}) \mathbf{u} + \alpha(b(\mathbf{x})) \geq 0 \tag{8}$$

for all $\mathbf{x} \in C$. A controller $K_{\text{cbf}} : \mathbb{R}^n \rightarrow \mathbb{R}^m$ renders C forward invariant if

$$K_{\text{cbf}}(\mathbf{x}) := \{\mathbf{u} \in U\ :\ \mathcal{L}_f b(\mathbf{x}) + \mathcal{L}_g b(\mathbf{x}) \mathbf{u} + \alpha(b(\mathbf{x})) \geq 0\} \tag{9}$$

1.1.3 High-order Control Barrier Functions

If the control input $\mathbf{u}$ appears after differentiating $b(\mathbf{x})$ once with respect to time, (9) can be incorporated directly as a constraint in an optimization program to render C forward invariant. However, if, for example b is a function of position, and the control input is an acceleration, we will need to formulate a Higher-order Control Barrier Function (HOCBF). Define a series of functions $\psi_i : \mathbb{R}^n \times [t_0, \infty) \rightarrow \mathbb{R}$ in the form

$$\psi_0(\mathbf{x}, t) := b(\mathbf{x}, t) \tag{10}$$
$$\psi_1(\mathbf{x}, t) := \dot{\psi}_0(\mathbf{x}, t) + \alpha_1(\psi_0(\mathbf{x}, t)) \tag{11}$$
$$\vdots \tag{12}$$
$$\psi_m(\mathbf{x}, t) := \dot{\psi}_{m-1}(\mathbf{x}, t) + \alpha_m(\psi_{m-1}(\mathbf{x}, t)) \tag{13}$$

where α_i are class $\mathcal{K}$ functions. Further define a series of sets in the form

$$C_1(t) := \{\mathbf{x} \in \mathbb{R}^n\ :\ \psi_0(\mathbf{x}, t) \geq 0\} \tag{14}$$
$$C_2(t) := \{\mathbf{x} \in \mathbb{R}^n\ :\ \psi_1(\mathbf{x}, t) \geq 0\} \tag{15}$$
$$\vdots \tag{16}$$
$$C_m(t) := \{\mathbf{x} \in \mathbb{R}^n\ :\ \psi_{m-1}(\mathbf{x}, t) \geq 0\} \tag{17}$$

A function $b : \mathbb{R}^n \times [t_0, \infty)$ is a high order control barrier function (HOCBF) of relative degree m for system (1) if there exist class $\mathcal{K}$ functions α_i such that

$$\sup_{\mathbf{u} \in U} \psi_m(\mathbf{x}(t), t) \geq 0 \tag{18}$$
$$\sup_{\mathbf{u} \in U} \mathcal{L}_f^m b(\mathbf{x}, t) + \mathcal{L}_g \mathcal{L}_f^{m-1} b(\mathbf{x}, t) \mathbf{u} + \frac{\partial^m b(\mathbf{x}, t)}{\partial t^m} + O(b(\mathbf{x}, t)) + \alpha_m(\psi_{m-1}(\mathbf{x}, t)) \geq 0 \tag{19}$$

For all $\mathbf{x}(t_0) \in C_1(t_0) \cap C_2(t_0) \cap ... \cap C_m(t_0)$. $O(\cdot)$ denotes the remaining Lie derivatives along f and partial derivatives with respect to t with degree less than or equal to $m-1$. Given a HOCBF a controller

$$K_{\text{HOCBF}} = \left\{ \mathbf{u} \in U \ : \ \mathcal{L}_f^m b(\mathbf{x},t) + \mathcal{L}_g \mathcal{L}_f^{m-1} b(\mathbf{x},t)\mathbf{u} + \frac{\partial^m b(\mathbf{x},t)}{\partial t^m} + O(b(\mathbf{x},t)) + \alpha_m(\psi_{m-1}(\mathbf{x},t)) \geq 0 \right\} \tag{20}$$

renders $C_1(t_0) \cap C_2(t_0) \cap ... \cap C_m(t_0)$ forward invariant. Note that rearranging (20) introduces a constraint on the maximum (or minimum) of the input

$$\mathbf{u} \leq \frac{\mathcal{L}_f^m b(\mathbf{x},t) + O(b(\mathbf{x},t) + \alpha_m(\psi_{m-1}(\mathbf{x},t)}{\mathcal{L}_g \mathcal{L}_f^{m-1} b(\mathbf{x},t)} \tag{21}$$

Hence if the right hand side of (21) is less than $\mathbf{u}_{\max}$ the system performance can be reduced. Additionally if $\mathbf{u}_{\max}$ is small one must take care choose α_i such that (18) is indeed satisfied. See Figure 1.

1.2 Example

The dynamics for vehicle i are given by

$$\begin{bmatrix} \dot{x}_i(t) \\ \dot{v}_i(t) \end{bmatrix} = \begin{bmatrix} v_i(t) \\ -\frac{1}{m_i} F_r(v_i(t)) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m_i} \end{bmatrix} u_i(t) \tag{22}$$

$$F_r(v) = c_0 \text{sgn}(v) + c_1 v + c_2 v^2 \tag{23}$$

where F_r is a combination of the rolling resistance and drag force. The control limits are given by

$$-c_d m_i g \leq u_i \leq c_a m_i g \tag{24}$$

A control Lyapunov function $V(\mathbf{x}_i(t)) = (v_i(t) - v_d)^2$ is used to track a set speed v_d. Ensuring our controller is stabilizing, as in (6), introduces the following constraint on our input

$$-\frac{2(v_i(t)-v_d)}{m_i} F_r(v_i(t)) + \frac{2(v_i(t)-v_d)}{m_i} u_i(t) + \epsilon (v_i(t)-v_d)^2 \leq \delta_{\text{acc}}(t) \tag{25}$$

$$\delta_{\text{acc}}(t) \geq 0 \tag{26}$$

where $\delta_{\text{acc}}(t)$ softens the constraint if it is positive. Speed limits $v_{\max}$ and $v_{\min}$ are enforced with barrier functions $b_1(\mathbf{x}_i(t)) := v_{\max} - v_i(t)$ and $b_1(\mathbf{x}_i(t)) := v_i(t) - v_{\min}$. Ensuring our controller renders these sets invariant (9) is accomplished by

$$\frac{F_r(v_i(t))}{m_i} + \frac{-1}{m_i} u_i(t) + v_{\max} - v_i(t) \geq 0 \tag{27}$$

$$\frac{-F_r(v_i(t))}{m_i} + \frac{1}{m_i} u_i(t) - v_{\max} + v_i(t) \geq 0 \tag{28}$$

Safety constraints are given by the barrier function

$$b(\mathbf{x}_i(t)) = x_{\text{lead}}(t) - x_i(t) - \delta \tag{29}$$

where x_{lead} is the position of the lead vehicle. (20) is enforced with $p > 0$ as follows (this is Form 2 in [XB19])

$$\psi_1(\mathbf{x}_i(t)) := \dot{b}(\mathbf{x}_i(t)) + pb(\mathbf{x}_i(t)) \tag{30}$$
$$\psi_2(\mathbf{x}_i(t)) := \dot{\psi}_1(\mathbf{x}_i(t)) + p\psi_1(\mathbf{x}_i(t)) \tag{31}$$

any control $u_i(t)$ should satisfy

$$\dot{\psi}_1(\mathbf{x}_i(t)) + p\psi_1(\mathbf{x}_i(t)) \geq 0 \tag{32}$$
$$\ddot{b}(\mathbf{x}_i(t)) + p\dot{b}(\mathbf{x}_i(t)) + p\dot{b}(\mathbf{x}_i(t)) + p^2 b(\mathbf{x}_i(t)) \geq 0 \tag{33}$$
$$\frac{F_r(v_i(t))}{m_i} + \frac{-1}{m_i}u_i(t) + 2p\dot{b}(\mathbf{x}_i(t)) + p^2 b(\mathbf{x}_i(t)) \geq 0, \tag{34}$$

where the last inequality leverages the assumption that the lead vehicle has constant acceleration (i.e. $\dot{v}_{\text{lead}}(t) \equiv 0$). To ensure that the controller respects the braking limits as the barrier function approaches 0, one can plot (21) over different values of p, and select $p = 1$.

At time t the optimal control input is computed by solving the program

$$\min_{u_i(t),\delta_{\text{acc}}} \begin{bmatrix} u_i(t) \\ \delta_{\text{acc}} \end{bmatrix}^\top \begin{bmatrix} 2/m_i^2 & 0 \\ 0 & 2p_{\text{acc}} \end{bmatrix} \begin{bmatrix} u_i(t) \\ \delta_{\text{acc}} \end{bmatrix} + \begin{bmatrix} -2F_r(v_i(t))/m_i^2 \\ 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \delta_{\text{acc}} \end{bmatrix} \tag{35}$$
$$\text{s.t. control limits (24)} \tag{36}$$
$$\text{Lyapunov stability (25),(26)} \tag{37}$$
$$\text{Speed limits (27),(28)} \tag{38}$$
$$\text{Barrier function (32)} \tag{39}$$

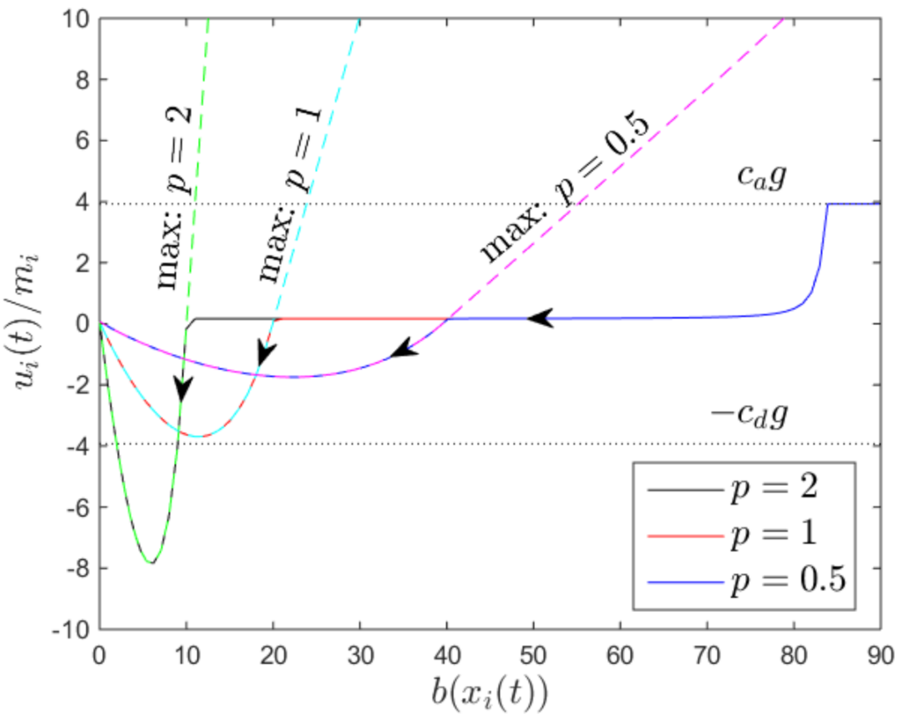

Figure 1: Control limitations for different values of p, computed using the HOBCF constraint (32), rearranged as in (21). A value of $p = 1$ should be selected to maximize performance, but still respect the braking limitations of the ego vehicle.

References

[XB19] Wei Xiao and Calin Belta. "Control barrier functions for systems with high relative degree". In: *2019 IEEE 58th conference on decision and control (CDC)*. IEEE. 2019, pp. 474–479.