Matematická analýza 1

Zkoušková písemka - varianta B

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1

1.1

Najděte intervaly ryzí monotonie funkce

$$f(x) := x^2 - 3x + \ln x.$$

Get the first derivative of f and solve for zero to find the points where the function is changing direction.

$$\begin{aligned}
f'(x) &= \left(x^2 - 3x + \ln x\right)' && (1)\\
f'(x) &= 2x - 3 + \frac{1}{x} && (2)\\
0 &= 2x - 3 + \frac{1}{x} && (3)\\
3 &= 2x + x^{-1} && (4)\\
3 &= 2x + x^{-1} / \ \cdot x && (5)\\
3 &= 2x^2 + x^0 && (6)\\
3 &= 2x^2 + 1 && (7)\\
2 &= 2x^2 && (8)\\
x &= 1 && (9)
\end{aligned}$$

Now check if the point $x = 1$ is an actual extremum.

$$\begin{aligned}
f'(-1) &= 2(-1) - 3 + \frac{1}{-1} && (10)\\
&= -2 - 3 - 1 = -6 && (11)\\
f'(2) &= 2(2) - 3 + \frac{1}{2} && (12)\\
&= 4 - 3 + \frac{1}{2} = 1\frac{1}{2} && (13)\\
& && (14)
\end{aligned}$$

$x \in$	$(-\infty, 1)\backslash\{0\}$	$(1, +\infty)$
$f'(x)$	$-$	$+$

The intervals of pure monotony are $(-\infty, 1)\backslash\{0\}$, decreasing, and $(1, +\infty)$, increasing.

1.2

Najděte inflexní body funkce

$$f(x) \coloneqq 3x^5 - 10x^4 + 10x^3 + 7x - 9.$$

Get the seond derivative of f and solve for zero, $f''(x) = 0$.

$$\begin{aligned}
f''(x) &= \left(3x^5 - 10x^4 + 10x^3 + 7x - 9\right)'' && (15)\\
&= \left(15x^4 - 40x^3 + 30x^2 + 7\right)' && (16)\\
&= 1x^3 - 2x^2 + 1x && (17)\\
0 &= 1x^3 - 2x^2 + 1x && (18)\\
0 &= x(1x^2 - 2x + 1) && (19)\\
x_0 &= 0 && (20)\\
0 &= 1x^2 - 2x + 1 && (21)\\
0 &= x^2 - 2x + 1 && (22)\\
0 &= (x-1)^2 && (23)\\
x_1 &= \frac{2 \pm \sqrt{4-4}}{2} = 1 && (24)
\end{aligned}$$

Check if they are extrema.

$x \in$	$(-\infty, 0)$	$(0, 1)$	$(1, +\infty)$
$f''(x)$	$-$	$+$	$+$

The inflection point of function f is $x_0 = 0$.

1.3

Vypočtěte limitu
$$\lim_{x\to 0} \frac{\ln(1+x) - x}{x \sin x}.$$

Try direct placement.

$$\lim_{x\to 0} \frac{\ln(1+x) - x}{x \sin x} \tag{25}$$
$$= \frac{\ln(1+0) - 0}{0 \sin 0} \tag{26}$$
$$= \frac{\ln 1}{0} \tag{27}$$

This will result in division by zero. $L'H\hat{o}pital's$ method.

$$\lim_{x\to 0} \frac{\ln(1+x) - x}{x \sin x} \tag{28}$$
$$\to \lim_{x\to 0} \frac{(\ln(1+x) - x)'}{(x \sin x)'} \tag{29}$$
$$= \frac{\frac{1}{(1+x)} - 1}{\sin x + x \cos x} \tag{30}$$
$$= \frac{\frac{1}{(1+x)} - 1}{\sin x + x \cos x} \tag{31}$$
$$= \frac{(\frac{1}{(1+x)} - 1)'}{(\sin x + x \cos x)'} \tag{32}$$
$$= \frac{(\frac{1}{(1+x)})'}{\cos x + \sin x} \tag{33}$$
$$= \frac{\frac{1}{(1+x)^2}}{\cos x + \sin x} \tag{34}$$
$$= \frac{-\frac{1}{(1+0)^2}}{\cos 0 + \sin 0} \tag{35}$$
$$\tag{36}$$

$$\lim_{x\to 0} \frac{\ln(1+x) - x}{x \sin x} = -1$$

1.4

Určete Taylorův polynom 2. řádu funkce f se středem v bodě x_0, je-li

$$f(x) \coloneqq \sin x + \cos x + \tan x,\ \ x_0 = 0.$$

Get the first and second derivative of function f.
The chain rule:

$$\text{Let } f : U \subseteq \mathbb{R} \mapsto \mathbb{R} \text{ and } g : U \subseteq \mathbb{R} \mapsto \mathbb{R}, \text{ then}$$

$$\begin{aligned} (\ f(g(x))\ )' &= f'(g(x)) \cdot g'(x) \\ (f \circ g)'(x) &= f'(g(x)) \cdot g'(x) \end{aligned}$$

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f :: R -> R                              -- R could also be R -> R
g :: R -> R                              -- ' equals derivative
f $ g x = ( f' $ g x ) * g' x
-- or
f (g x) = f'(g x) * g' x
```

The behaviour of secant function:

$$\begin{aligned} \sec^2 x &= (\sec x)^2 \\ \left(\sec^2 x\right)' &= \left((\sec x)^2\right)' \\ 2\sec^{2-1} x \cdot (\sec x)' &= 2\sec x \left(\frac{1}{\cos x}\right)' \\ 2\sec x \frac{\sin x}{\cos^2 x} &= 2\sec x \frac{\sin x}{\cos x}\frac{1}{\cos x} \\ 2\sec x \tan x \sec x &= 2\sec^2 x \tan x \end{aligned}$$

Taylor's polynom of second degree.

$$P_T^2 \coloneqq c_0 + c_1 x + c_2 x^2$$

$$\begin{aligned} P_T^2 &= c_0 + c_1 x + c_2 x^2 \\ \frac{d}{dx} P_T^2 &= c_1 + 2c_2 x \\ \frac{d^2}{dx^2} P_T^2 &= 2c_2 \end{aligned}$$

$$
\begin{aligned}
f(x) &\coloneqq \sin x + \cos x + \tan x && (37)\\
f'(x) &= (\sin x + \cos x + \tan x)' && (38)\\
&= \cos x - \sin x + \left(\frac{\sin x}{\cos x}\right)' && (39)\\
&= \cos x - \sin x + \frac{\sin x(\cos x)' - (\sin x)'\cos x}{\cos^2 x} && (40)\\
&= \cos x - \sin x + \frac{\cos x(\sin x)' - \sin x(\cos x)'}{\cos^2 x} && (41)\\
&= \cos x - \sin x + \frac{\cos x\cos x - \sin x(-\sin x)}{\cos^2 x} && (42)\\
&= \cos x - \sin x + \frac{\cos^2 x + sin^2 x}{\cos^2 x} && (43)\\
&= \cos x - \sin x + \frac{1}{\cos^2 x} && (44)\\
&= \cos x - \sin x + \sec^2 x && (45)\\
f''(x) &= \left(\cos x - \sin x + \sec^2 x\right)' && (46)\\
&= -\sin x - \cos x + 2\sec^2 x\tan x && (47)\\
f'''(x) &= \left(-\sin x - \cos x + 2\sec^2 x\tan x\right)' && (48)\\
&= -\cos x + \sin x + \left(2\sec^2 x\tan x\right)' && (49)\\
&= -\cos x + \sin x + 2\left((\sec^2 x)'\tan x + \sec^2 x(\tan x)'\right) && (50)\\
&= -\cos x + \sin x + 4\sec^2 x\tan^2 x + 2\sec^4 x && (51)\\
& && (52)
\end{aligned}
$$

$$
\begin{aligned}
f(x_0) &= \sin x_0 + \cos x_0 + \tan x_0 && (53)\\
&= \sin 0 + \cos 0 + \tan 0 = 1 && (54)\\
f'(x_0) &= \cos x_0 - \sin x_0 + \sec^2 x_0 && (55)\\
&= \cos 0 - \sin 0 + \sec^2 0 && (56)\\
&= 1 - 0 + 1 = 2 && (57)\\
f''(x_0) &= -\sin x_0 - \cos x_0 + 2\sec^2 x_0\tan x_0 && (58)\\
&= -0 - 12\cdot 1\cdot 0 = 0 && (59)\\
f'''(x_0) &= -\cos x_0 + \sin x_0 + 4\sec^2 x_0\tan^2 x_0 + 2\sec^4 x_0 && (60)\\
&= -\cos 0 + \sin 0 + 4\sec^2 0\tan^2 0 + 2\sec^4 0 && (61)\\
&= -1 + 0 + 0 + 2 = 1 && (62)
\end{aligned}
$$

Taylor's polynom of third degree for our function f is

$$P_T^3 = 1 + 2x + 0x^2 + \frac{1}{6}x^3$$

1.5

Vypočtěte

$$\int (2x+3)e^x \, \mathrm{d}x.$$

I will use the per partes becuase I don't see how the sub-method will help me here.

Per partes method

$$\int uv' = uv - \int vu'$$

The *per partes* method is derived from the product rule in derivation.

$$(uv)' = u'v + uv'$$

1.6

Vypočtěte
$$\int_0^{\frac{\pi}{2}} \sin^3 x \cos^2 x \,\mathrm{d}x.$$

1.7

finish