Matematická analýza 1
Zkouškové a zápočtové písemky

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1 Zkoušková písemka z MA1 - varianta B

1.1 Najděte lokální extrémy funkce

$$\boldsymbol{f(x) \coloneqq (5 - 2x)x^3}.$$

To find local extrema we need to solve for $f'(x) = 0$.

$$\begin{aligned}
f'(x) &= \Big((5-2x)x^3\Big)' && (1)\\
&= (5-2x)'x^3 + (5-2x)(x^3)' && (2)\\
&= (-2)x^3 + (5-2x)(3x^2) && (3)\\
&= -2x^3 + 15x^2 - 6x^3 && (4)\\
&= 15x^2 - 8x^3 && (5)\\
&= x^2(15-8x) && (6)
\end{aligned}$$

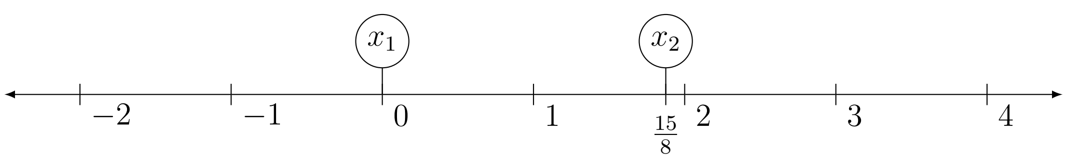
Now find $x_1 \dots x_n$ that satisfy this statement $x^2(15-4x) = 0$.

$$x^2(15-8x) = 0$$
$$(x^2 = 0) \vee (15 - 8x = 0) \Rightarrow (x^2(15-8x) = 0)$$

$$\begin{aligned}
x^2 &= 0 && (7)\\
x_1 &= 0 && (8)
\end{aligned}$$

$$\begin{aligned}
15 - 8x &= 0 && (9)\\
8x &= 15 && (10)\\
x_2 &= \frac{15}{8} && (11)
\end{aligned}$$

Now that we extracted the extrema we need to decide for each wheter it is a minimum or a maximum. To do that we can use a number line that is segmented by the points x_1 and x_2, and find out if the function $f'(x)$ is positive or negative at each interval. That will helps understand how the function behaves.



For the interval $(-\infty, 0)$ we will pick -1 as our input to the function $f(x)$.

$$\begin{aligned}
f'(x) &= (15-8x)x^2\\
f'(-1) &= (15 - 8(-1))(-1)^2\\
&= (15+8)1 = 24?
\end{aligned}$$

We got 24 which is positive so the function $f'(x)$ in the interval $(-\infty, 0)$ is negative.

For the interval $(0, \frac{15}{8})$ we will pick 1.

$$f'(x) = (15 - 8x)x^2$$
$$f'(1) = (15 - 2(1))1^2 = 7$$

The function f' in $(0, \frac{15}{8})$ is positive.

For the interval $(\frac{15}{8}, +\infty)$ we choose 2.

$$f(x) = (15 - 8x)x^2$$
$$f(2) = (15 - 8(2))2^2 = -2$$

The function f' is negative in $(\frac{15}{8}, +\infty)$.

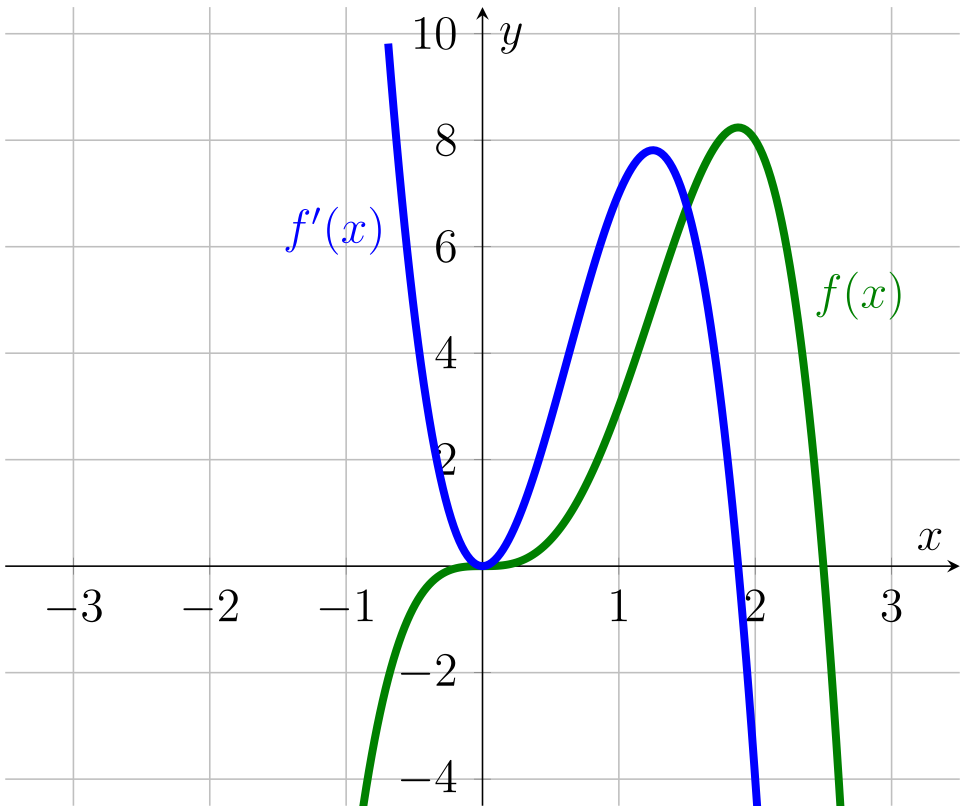
Now that we evaluated all the intervals let's put all of our findings into a separate table.

	$(-\infty, 0)$	$(0, \frac{15}{8})$	$(\frac{15}{8}, +\infty)$
$f'(x)$	$+$	$+$	$-$

This tells us that our x_1 isn't a local extremum because the intervals at either side are positive, theyx don't switch signs. This implies that f' is decreasing, touching zero and then increasing, but never switches signs. The x_2 on the other hand is a local extermum, a maximum, becuase the interval on left is positive and negative on the right implying f' is dropping down, touching zero and descreasing further. The sign switch from positive to negative as they cross $x = \frac{15}{8}$.

The function f and its fisrt derivative f'

$$f(x) \coloneqq (5-2x)x^3$$
$$f'(x) \coloneqq (15-8x)x^2$$



1.2 Najděte inflexní body funkce

$$\boldsymbol{f(x) \coloneqq 2x^3 + \frac{3}{2}\ln x.}$$

To find the inflection points we need the second derivative

$$f'' \coloneqq \frac{d^2 f}{dx^2}$$

and solve for

$$f''(x) = 0$$

Take the second derivative of function f.

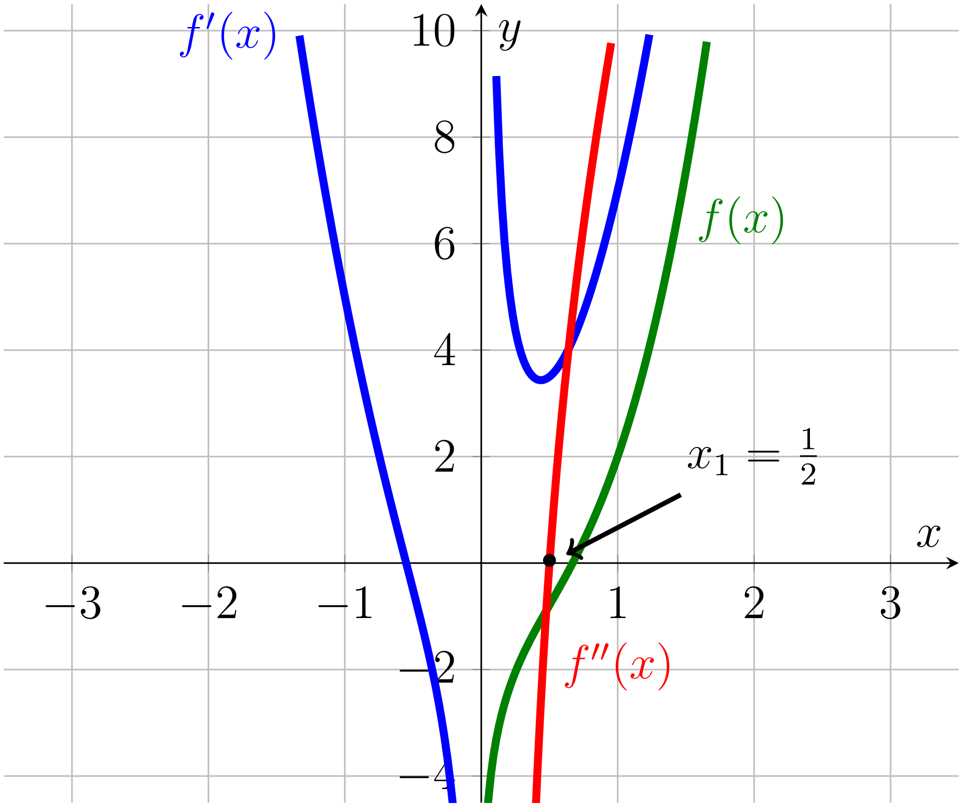
$$\begin{aligned}
f'(x) &= \left(2x^3 + \frac{3}{2}\ln x\right)' && (12)\\
&= 6x^2 + \frac{3}{2}\frac{1}{x} && (13)\\
&= 6x^2 + \frac{3}{2x} && (14)\\
f''(x) &= \left(6x^2 + \frac{3}{2x}\right)' && (15)\\
&= 12x + \frac{2x(3)' - 3(2x)'}{(2x)^2} && (16)\\
&= 12x - \frac{6}{4x^2} && (17)\\
&= 12x - \frac{3}{2x^2} && (18)
\end{aligned}$$

Solve for $f''(x) = 0$.

$$\begin{aligned}
12x - \frac{3}{2x^2} &= 0 \ / \ \cdot x^2 && (19)\\
12x^3 - \frac{3}{2} &= 0 && (20)\\
24x^3 &= 3 && (21)\\
x^3 &= \frac{1}{8} && (22)\\
x_1 &= \frac{1}{2} && (23)
\end{aligned}$$

The inflection point of the function f is x_1, where $x_1 = \frac{1}{2}$.

The funciton f and its first f' and second derivative f''



1.3 Najděte asymptotu v $-\infty$ funkce

$$\boldsymbol{f(x) \coloneqq x + 5 + \frac{3}{2x+1}}.$$

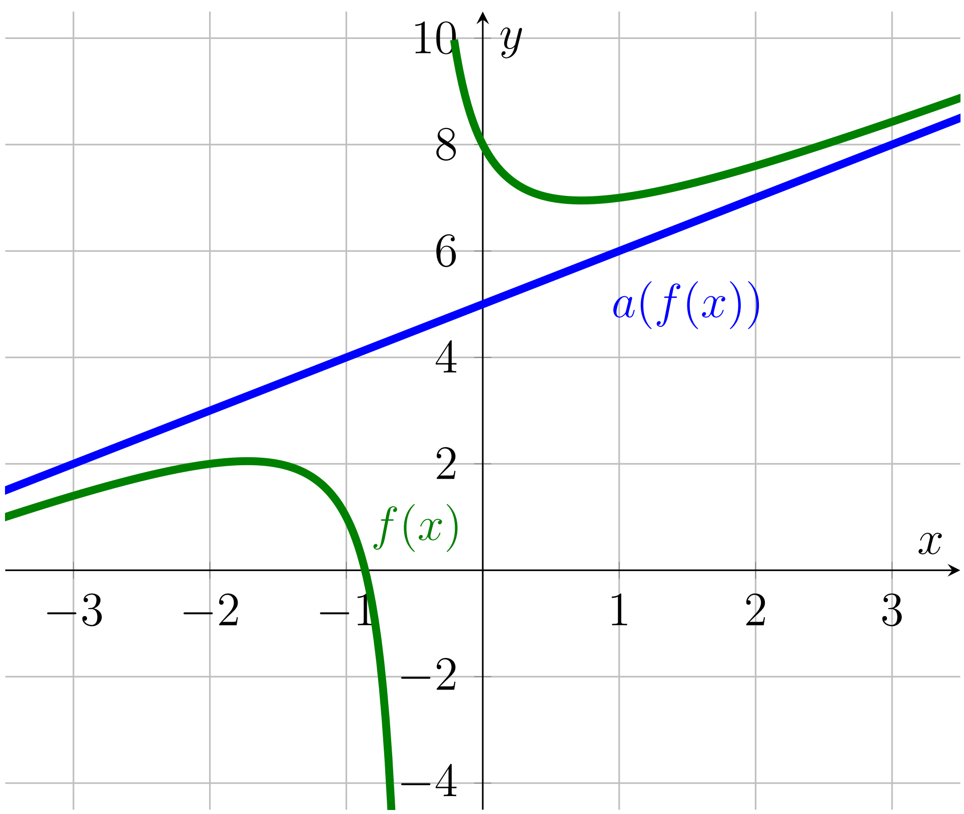
To know where the asymptote for $-\infty$ lies on the graph get the limit of te function f.

$$\lim_{x\to-\infty}\left(x + 5 + \frac{3}{2x+1}\right) \tag{24}$$

$$\lim_{x\to-\infty}(x+5) + \lim_{x\to-\infty}\left(\frac{3}{2x+1}\right) \tag{25}$$

The first term $\lim_{x\to-\infty}(x+5)$ converges to $-\infty$ and the second $\lim_{x\to-\infty}\left(\frac{3}{2x+1}\right)$ converges to 0, meaning the asymptote for the function f is $x+5$.

The graph of function $f(x)$ and its asymptote $a(f(x))$



1.4 Vypočtěte limitu

$$\lim_{x\to 0} \frac{x^{2019} + \sin(2019x)}{e^{2019x} - x - 1}.$$

Let's try direct substituting $x = 0$.

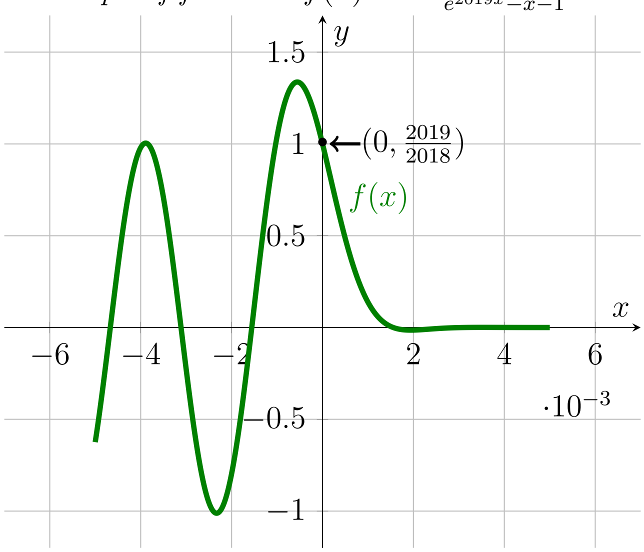
$$\begin{aligned}
&\lim_{x\to 0} \frac{x^{2019} + \sin(2019x)}{e^{2019x} - x - 1} && (26)\\
&= \frac{0^{2019} + \sin(2019(0))}{e^{2019(0)} - 0 - 1} && (27)\\
&= \frac{\sin 0}{e^0 - 1} = \frac{0}{0} && (28)
\end{aligned}$$

We get division by zero which is undefined, so we must go a different route. Let's try $L'H\hat{o}pital's$ rule.

$$\begin{aligned}
&\lim_{x\to 0} \frac{x^{2019} + \sin(2019x)}{e^{2019x} - x - 1} && (29)\\
&= \frac{(x^{2019} + \sin(2019x))'}{(e^{2019x} - x - 1)'} && (30)\\
&= \frac{2019x^{2018} + 2019\cos(2019x)}{2019e^{2019x} - 1} && (31)\\
&= \frac{2019(0)^{2018} + 2019\cos(2019(0))}{2019e^{2019(0)} - 1} && (32)\\
&= \frac{2019\cos(0)}{2019e^0 - 1} = \frac{2019}{2018} && (33)
\end{aligned}$$

The result of limit $\lim_{x\to 0} \frac{x^{2019}+\sin(2019x)}{e^{2019x}-x-1}$ equals to $\frac{2019}{2018}$.

Graph of function $f(x) := \frac{x^{2019}+\sin(2019x)}{e^{2019x}-x-1}$



1.5 Určete Taylorův polynom 2. řádu funkce f se středem v bodě x_0, je-li

$$\boldsymbol{f(x) = \sin x - \cos x, \quad x_0 = 0}.$$

To determine Taylor's polynom of 2nd degree, P^2, we need the 1st, $f' = \frac{d}{dx}f$, and 2nd derivative, $f'' = \frac{d^2}{dx^2}f$, of function $f(x)$. Then get the result of $f(0), f'(0), f''(0)$ and match with the modifiable values of our polynom P^2.

Taylor's polynom is defined as such

$$P_T^n(x) \coloneqq \sum_{i=0}^{n} a_i x^i = a_0x^0 + a_1x^1 + \ldots + a_nx^n$$

$$f(x) = \sin x - \cos x \tag{34}$$
$$f'(x) = (\sin x - \cos x)' = \cos x + \sin x \tag{35}$$
$$f''(x) = (\cos x + \sin x)' = -\sin x + \cos x \tag{36}$$
$$f(0) = \sin 0 - \cos 0 = 0 - 1 = -1 \tag{37}$$
$$f'(0) = \cos 0 + \sin 0 = 1 + 0 = 1 \tag{38}$$
$$f''(0) = -\sin 0 + \cos 0 = 0 + 1 = 1 \tag{39}$$
$$P_T^2(x) = \sum_{i=0}^{2} a_i x^i = a_0x^0 + a_1x^1 + a_2x^2 \tag{40}$$
$$\frac{d}{dx}P_T^2(x) = a_1 + 2a_2x \tag{41}$$
$$\frac{d^2}{dx^2}P_T^2(x) = 2a_2 \tag{42}$$

We can get the correct values for $a_0 \ldots a_n$ by solving for $\frac{d^n}{dx^n}P_T^n = \frac{d^n}{dx^n}f$, meaning that

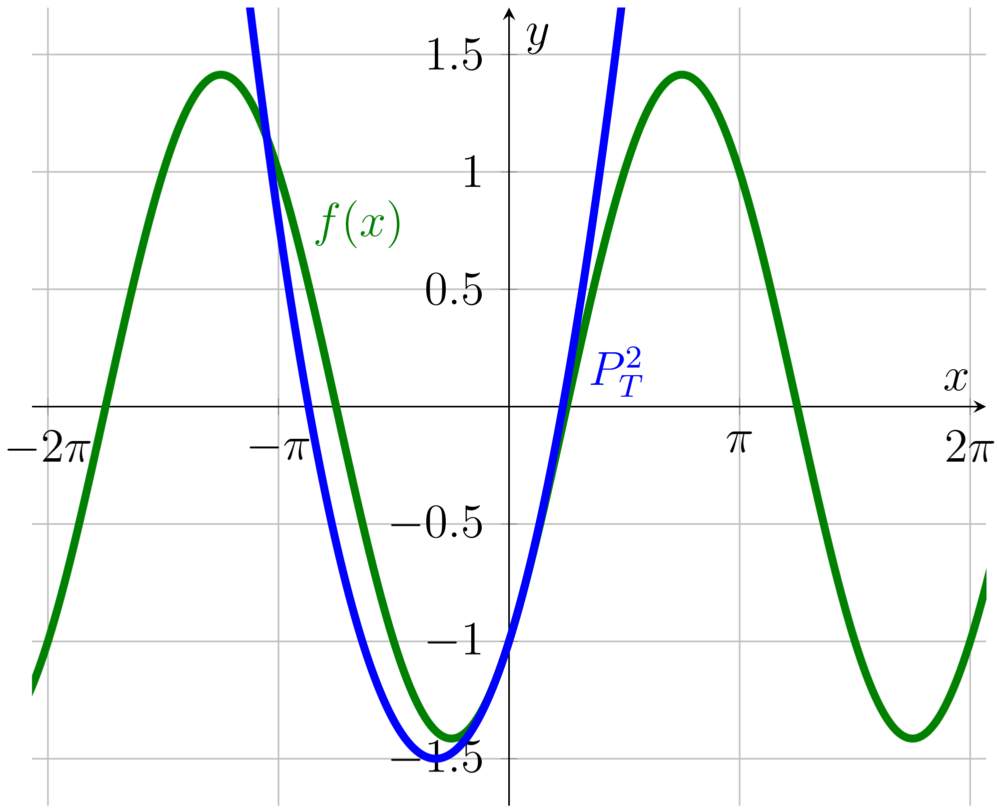
$$P(x) = f(x),\ P'(x) = f'(x),\ P''(x) = f''(x).$$

$$P(x) = a_0x^0 + a_1x^1 + a_2x^2 \tag{43}$$
$$f(0) = -1 \rightarrow\ a_0 = \frac{-1}{0!} = -1 \tag{44}$$
$$f'(0) = 1 \rightarrow\ a_1 = \frac{1}{1!} = 1 \tag{45}$$
$$f''(0) = 1 \rightarrow\ a_2 = \frac{1}{2!} = \frac{1}{2} \tag{46}$$

What we get is

$$P_T^2 = -1x^0 + 1x^1 + \frac{1}{2}x^2.$$

The graph of the function $f(x)$ and the Taylor's polynom P_T^2



1.6 Vypočtěte integrál

$$\int \left(\mathbf{2x^6 - \frac{3x^2}{\sqrt[3]{x}} - 7\sin x + \frac{2}{x^2+1} + \frac{11}{x}} \right) \mathbf{dx}.$$

Rules for the integral of arctan and the summation.

$$\arctan x \coloneqq \tan^{-1} x$$
$$(\arctan x)' = \frac{1}{x^2+1}$$
$$\int \frac{1}{x^2+1}\,\mathrm{d}x = \arctan x + C$$
$$\int \sum_{i=1}^{n} t_i\,\mathrm{d}x = \int t_1\,\mathrm{d}x + \int t_2\,\mathrm{d}x + \ldots + \int t_n\,\mathrm{d}x$$

$$\int \left(2x^6 - \frac{3x^2}{\sqrt[3]{x}} - 7\sin x + \frac{2}{x^2+1} + \frac{11}{x}\right)\mathrm{d}x \tag{47}$$
$$= \int 2x^6\,\mathrm{d}x - \int \frac{3x^2}{\sqrt[3]{x}}\,\mathrm{d}x - \int 7\sin(x)\,\mathrm{d}x + \int \frac{2}{x^2+1}\,\mathrm{d}x + \int \frac{11}{x}\,\mathrm{d}x \tag{48}$$
$$= 2\int x^6\,\mathrm{d}x - 3\int \frac{x^2}{x^{\frac{1}{3}}}\,\mathrm{d}x - 7\int \sin(x)\,\mathrm{d}x + 2\int \frac{1}{x^2+1}\,\mathrm{d}x + 11\int \frac{1}{x}\,\mathrm{d}x \tag{49}$$
$$= 2\frac{x^7}{7} - 3\int x^{\frac{5}{3}}\,\mathrm{d}x + 7\cos(x) + 2\arctan x + 11\ln|x| \tag{50}$$
$$= \frac{2x^7}{7} - 3\frac{x^{\frac{8}{3}}}{\frac{8}{3}} + 7\cos(x) + 2\arctan x + 11\ln|x| \tag{51}$$
$$= \frac{2x^7}{7} - 3\frac{3x^{\frac{8}{3}}}{8} + 7\cos(x) + 2\arctan x + 11\ln|x| \tag{52}$$
$$= \frac{2x^7}{7} - \frac{9x^{\frac{8}{3}}}{8} + 7\cos(x) + 2\arctan x + 11\ln|x| \tag{53}$$

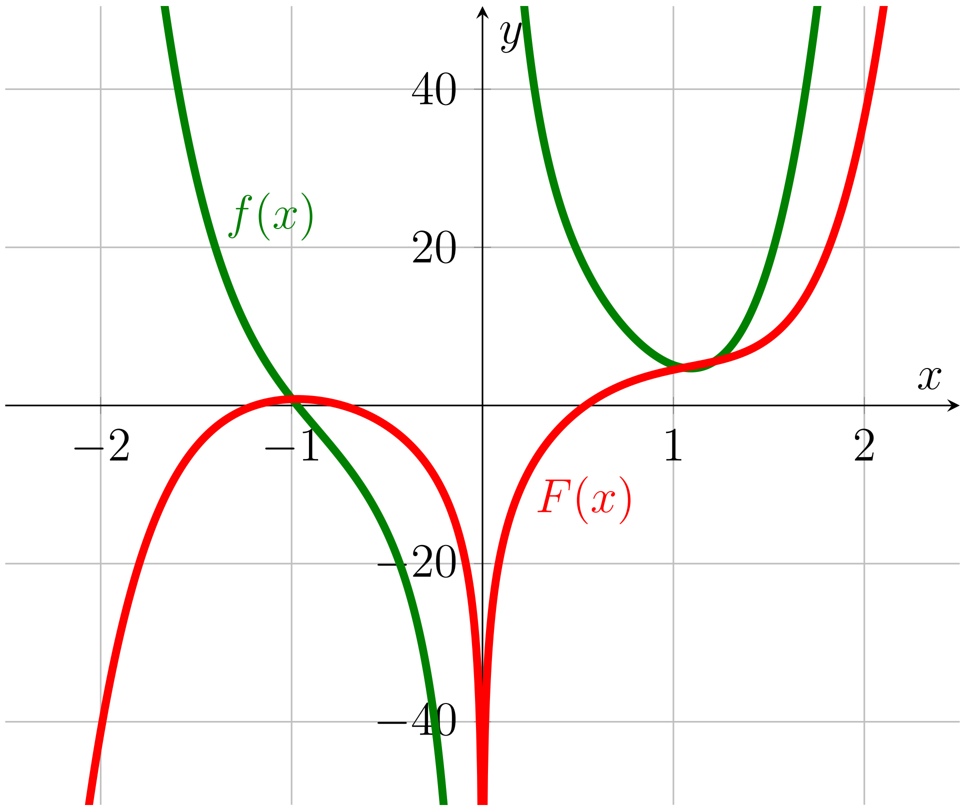
The result is

$$\int \left(2x^6 - \frac{3x^2}{\sqrt[3]{x}} - 7\sin x + \frac{2}{x^2+1} + \frac{11}{x}\right)\mathrm{d}x = \frac{2x^7}{7} - \frac{9x^{\frac{8}{3}}}{8} + 7\cos(x) + 2\arctan x + 11\ln|x| + C$$

The graph of the function $f(x)$ and its integral $F(x)$

$$f(x) = 2x^6 - \frac{3x^2}{\sqrt[3]{x}} - 7\sin x + \frac{2}{x^2+1} + \frac{11}{x}$$

$$F(x) = \frac{2x^7}{7} - \frac{9x^{\frac{8}{3}}}{8} + 7\cos(x) + 2\arctan x + 11\ln|x| + C$$



1.7 Rozhodněte, která z následujících tvrzení jsou pravdivá či nepravdivá.

(a) Je-li funkce f spojitá v intervalu $I = \langle 0, 1 \rangle$, pak f musí být na I omezená.
(b) Je-li $f'(5) = 0$, má funkce f v bodě 5 lokální extrém.
(c) Funkce mající nekonečnou derivaci v x_0 může být v bodě x_0 spojitá.
(d) Na intervalu $(0, +\infty)$ platí $\int \frac{1}{2x} dx = ln(2x)$.
(e) Existuje omezená funkce, která má asymptotu.

1. otázka

Ano, vychází z toho, že její krajní body jsou pevně určené a funkce je nutně spojitá, tedy její hodnoty nemohou nabývat $+\infty$ nebo $-\infty$. To znamená, že funkce musí být v tomto intervalu I omezená.

2. otázka

Není jednoznačné, že se v bodě $x = 5$ nachází lokální minimum. Tvrzení je příliš obecné. Může se tam nacházet inflexní bod.

3. otázka

Žádná hodnota v jakémkoli bodě derivace se nemůže *rovant* $\pm\infty$, pouze se k přibližovat (hodnota v jinak nedefinovaném bodě x diverguje[a] k nekonečnu). Pokud se otázka bere ve smyslu, že se hodnota derivace k nekonečnu diverguje, tak potom je odpověd ano. Platí to např. u funkce $f(x) = \sqrt{x}$. Její derivace $f'(x) = (x^{\frac{1}{2}})' = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$. V bodě $x = 0$ je funkce f definována a spojitá. Ve funkci $f'(0)$ bod $x = 0$ není definován, $\lim_{x\to 0} f'(x)$ konverguje k nekonečnu.

[a] Divergence generally means two things are moving apart while convergence implies that two forces are moving together. zdroj

4. otázka

Neplatí.

$$\int \frac{1}{2x}\,\mathrm{d}x = \left| t = 2x\,,\ \mathrm{d}t = 2\,\mathrm{d}x \right| = \tag{54}$$

$$= \int \frac{1}{t}\frac{1}{2}\,\mathrm{d}t = \frac{1}{2}\int \frac{1}{t}\,\mathrm{d}t = \frac{1}{2}\ln|t| = \frac{1}{2}\ln|2x| \tag{55}$$

V intervalu $(0, +\infty)$ platí $\frac{1}{2}\ln(2x)$.

5. otázka

Ano, jsou jimi funkce, které mají horizontální asymptotu. Např. funkce $f(x) = \arctan x$. Ten má dvě horizontální asymptoty $a_0 = 0x + \frac{\pi}{2}$ a $a_1 = 0x - \frac{\pi}{2}$.