Matematická analýza 1 - cvičení 4
Limita, posloupnosti, faktoriál,
metoda dvou policajtů

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1 Limita funkce

Chování limita exponenciální funkce $f(q) \coloneqq q^n$.

$$\lim q^n \begin{cases} +\infty, & \text{if } q > 1 \\ 1, & \text{if } q = 1 \\ 0, & \text{if } q = 0 \\ 0, & \text{if } |q| < 1 \\ \text{DNF}, & \text{if } q \leq -1 \end{cases}$$

1.1 Vypočtěte limitu

$$\lim_{n \to +\infty} \left(n^4 + 5n^3 + 1\right)$$

$$\lim_{n \to +\infty} \left(n^4 + 5n^3 + 1\right)''' \tag{1}$$

$$= \lim_{n \to +\infty} 24n = 24\infty = \infty \tag{2}$$

1.2 Vypočtěte limitu

$$\lim_{n\to+\infty} \frac{-2n^3-4n+2}{5n^2+n-8}$$

$$\begin{aligned}
&\lim_{n\to+\infty} \frac{-2n^3-4n+2}{5n^2+n-8} && (3)\\
&= \lim_{n\to+\infty} \frac{(-2n^3-4n+2)'}{(5n^2+n-8)'} && (4)\\
&= \lim_{n\to+\infty} \frac{-6n^2-4}{10n+1} && (5)\\
&= \lim_{n\to+\infty} \frac{(-6n^2-4)'}{(10n+1)'} && (6)\\
&= \lim_{n\to+\infty} \frac{-10n}{10} && (7)\\
&= \lim_{n\to+\infty} -n = -\infty && (8)
\end{aligned}$$

1.3 Vypočtěte limitu

$$\lim_{n\to+\infty} \frac{5n^2+9n+6}{-6n^2+3n-1}$$

$$\lim_{n\to+\infty} \frac{(5n^2+9n+6)'}{(-6n^2+3n-1)'} \tag{9}$$

$$\lim_{n\to+\infty} \frac{10n+9}{-12n+3} \tag{10}$$

$$\lim_{n\to+\infty} \frac{(10n+9)'}{(-12n+3)'} \tag{11}$$

$$\lim_{n\to+\infty} \frac{10}{-12} = -\frac{5}{6} \tag{12}$$

$$\tag{13}$$

1.4 Vypočtěte limitu

$$\lim_{n\to+\infty} \frac{\sqrt{n^3} - n + 3}{2n - \sqrt{n}}$$

$$\lim_{n\to+\infty} \frac{n^{\frac{3}{2}} - n + 3}{2n - n^{\frac{1}{2}}} \tag{14}$$

$$\lim_{n\to+\infty} \frac{n^{\frac{3}{2}} - n + 3}{2n - n^{\frac{1}{2}}} \times \frac{2n + n^{\frac{1}{2}}}{2n + n^{\frac{1}{2}}} \tag{15}$$

$$\lim_{n\to+\infty} \frac{(n^{\frac{3}{2}} - n + 3)(2n + n^{\frac{1}{2}})}{(2n - n^{\frac{1}{2}})(2n + n^{\frac{1}{2}})} \tag{16}$$

$$\lim_{n\to+\infty} \frac{2n^{\frac{5}{2}} - 2n^2 + 6n + n^2 - n^{\frac{3}{2}} + 3n^{\frac{1}{2}}}{4n^2 - 2n^{\frac{3}{2}} + 2n^{\frac{3}{2}} - n} \tag{17}$$

$$\lim_{n\to+\infty} \frac{2n^{\frac{5}{2}} - n^2 - n^{\frac{3}{2}} + 6n + 3n^{\frac{1}{2}}}{4n^2 - n} \tag{18}$$

$$\lim_{n\to+\infty} \frac{2n^{\frac{5}{2}}}{4n^2} = \frac{n^{\frac{5}{2}}}{2n^2} = \frac{n^{\frac{5}{2}}}{2n^2} \div \frac{n^2}{n^2} = \frac{n^{\frac{1}{2}}}{2} = \infty \tag{19}$$

$$\tag{20}$$

1.5 Vypočtěte limitu

$$\lim_{n\to+\infty}\left(\sqrt{n+2}-\sqrt{n-2}\right)$$

$$\lim_{n\to+\infty}\left(\sqrt{n+2}-\sqrt{n-2}\right) \tag{21}$$

$$\lim_{n\to+\infty}\left(\sqrt{n+2}-\sqrt{n-2}\right)\times\frac{\sqrt{n+2}+\sqrt{n-2}}{\sqrt{n+2}+\sqrt{n-2}} \tag{22}$$

$$\lim_{n\to+\infty}\left(\frac{(\sqrt{n+2})^2-(\sqrt{n-2})^2}{\sqrt{n+2}+\sqrt{n-2}}\right) \tag{23}$$

$$\lim_{n\to+\infty}\left(\frac{(n+2)-(n-2)}{\sqrt{n+2}+\sqrt{n-2}}\right) \tag{24}$$

$$\lim_{n\to+\infty}\left(\frac{4}{\sqrt{n+2}+\sqrt{n-2}}\right) \tag{25}$$

$$\lim_{n\to+\infty}\left(\frac{4}{\sqrt{\infty+2}+\sqrt{\infty-2}}\right) \tag{26}$$

$$\lim_{n\to+\infty}\left(\frac{4}{\sqrt{\infty+2}+\sqrt{\infty-2}}\right)=\frac{4}{\infty}=0 \tag{27}$$

$$\tag{28}$$

1.6 Vypočtěte limitu

$$\lim_{n \to +\infty} \cos(2\pi n)$$

$$\lim_{n \to +\infty} \cos(2\pi n) = 1 \tag{29}$$

1.7

$$\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n$$

Euler number

$$e \coloneqq \lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n$$

$$e \coloneqq \sum_{n=0}^{\infty}\frac{1}{n!}$$

$$e \coloneqq 2.71828182845904523536028747135266249775 72\ldots$$

$$\begin{aligned}
e \approx \left(1+\frac{1}{3}\right)^3 &= \left(1+\frac{1}{3}\right)^2\left(1+\frac{1}{3}\right)\\
&= \left(1+\frac{2}{3}+\frac{1}{3^2}\right)\left(1+\frac{1}{3}\right)\\
&= \left(1+\frac{2}{3}+\frac{1}{3^2}+\frac{1}{3}+\frac{2}{3^2}+\frac{1}{3^3}\right)\\
&= \left(1+\frac{1}{3}+\frac{2}{3}+\frac{1}{9}+\frac{2}{9}+\frac{1}{27}\right)\\
&= 2\frac{4}{27}
\end{aligned}$$

$$\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n = 1+\frac{1}{n}+\frac{2}{n}+\frac{1}{n^2}+\frac{2}{n^2}+\frac{1}{n^3}+\ldots+\frac{1}{n^n}+\frac{2}{n^n}$$

$$\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n = e \tag{30}$$

(31)

1.8 Vypočtěte limitu

$$\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^{2n}$$

$$\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^{2n} \tag{32}$$

$$\lim_{n\to+\infty}\left(\left(1+\frac{1}{n}\right)^{n}\right)^{2} \tag{33}$$

$$\lim_{n\to+\infty}(e)^{2}=e^{2} \tag{34}$$

$$\tag{35}$$

1.9

$$\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^{3n+6}$$

$$\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^{3n}\cdot\left(1+\frac{1}{n}\right)^{6} \tag{36}$$

$$\lim_{n\to+\infty}e^{3}\cdot\left(1+\frac{1}{n}\right)^{6} \tag{37}$$

$$\lim_{n\to+\infty}e^{3}\cdot 1 \tag{38}$$

$$\tag{39}$$

1.10

$$\lim_{n\to+\infty}\left(1+\frac{1}{5n}\right)^{9n+8}$$

$$\lim_{n\to+\infty}\left(\left(1+\frac{1}{5n}\right)^{5n}\right)^{\frac{9}{5}}\left(1+\frac{1}{5n}\right)^{8} \tag{40}$$

$$\lim_{n\to+\infty}\left(\left(1+\frac{1}{5n}\right)^{5n}\right)^{\frac{9}{5}}\left(1+\frac{1}{\infty}\right)^{8} \tag{41}$$

$$\lim_{n\to+\infty}\left(\left(1+\frac{1}{n}\right)^{n}\right)^{\frac{9}{5}}(1)^{8} \tag{42}$$

$$\lim_{n\to+\infty}(e)^{\frac{9}{5}}(1)^{8}=e^{\frac{9}{5}} \tag{43}$$

$$\tag{44}$$

Chování e v limitě

$$\left(1+\frac{1}{n}\right)^{n}=e$$

$$\left(1+\frac{1}{an}\right)^{an}=e$$

1.11

$$\lim_{n\to+\infty}\left(3^n+(-3)^n\right)$$

Funkce není kontinuální - růst se rozchází při přechodu definičního oboru ze sudých čísel na lichá a naopak.

Pro sudá x definičního oboru platí

$$\lim_{n\to+\infty} = 3^n + 3^n \to \infty$$

a pro lichá platí

$$\lim_{n\to+\infty} = 3^n - 3^n \to 0.$$

Proto nemá limitu - do nekonečna osciluje mezi nekonečnem a nulou.

1.12

$$\lim_{n\to+\infty}\frac{3^n+(-3)^n}{3\cdot 6^n}$$

$$\lim_{n\to+\infty}\frac{3^n+(-3)^n}{3\cdot 6^n} \tag{45}$$

$$\lim_{n\to+\infty}\frac{6^n\left(\frac{1}{2}-\frac{1}{2}\right)}{6^n\cdot 3} \tag{46}$$

$$\lim_{n\to+\infty}\frac{6^n\cdot 0}{6^n\cdot 3} \tag{47}$$

$$\lim_{n\to+\infty}\frac{0}{3}=0 \tag{48}$$

1.13

$$\lim_{n\to+\infty} \frac{5^{n+2}+3^n}{5^n-2^n}$$

$$\lim \frac{5^n\left(5^2+\left(\frac{3}{5}\right)^n\right)}{5^n\left(1-\left(\frac{2}{5}\right)^n\right)} \tag{49}$$

$$\lim \frac{25+\frac{3^n}{5^n}}{1-\frac{2^n}{5^n}} \tag{50}$$

$$\lim \frac{25}{1} = 25 \tag{51}$$

Zákon adice termů stejného řádu

$$a^n+b^n = a^n\left(1+\left(\frac{b}{a}\right)^n\right)$$

1.14

$$\lim_{n\to+\infty} 5^n - 2^n$$

$$\lim_{n\to+\infty} 5^n \left(1 - \left(\frac{2}{5}\right)^n\right) \to \infty \tag{52}$$

1.15

$$\lim_{n\to+\infty} \frac{1}{n^2} + \frac{2}{n^2} + \ldots + \frac{n-1}{n^2}$$

$$\lim_{n\to+\infty} \frac{1+2+\ldots+(n-1)}{n^2} \tag{53}$$

$$\lim_{n\to+\infty} \frac{n(n-1)}{2n^2} \tag{54}$$

$$\lim_{n\to+\infty} \frac{n^2-n^1}{2n^2} \tag{55}$$

$$\lim_{n\to+\infty} \frac{n^2(1-n^{-1})}{2n^2} \tag{56}$$

$$\lim_{n\to+\infty} \frac{1-\frac{1}{n}}{2} = \frac{1}{2} \tag{57}$$

Nekonečná řada sumy inkrementujicích se integrálů

$$1+2+\ldots+n = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\begin{aligned} S &= 1+2+\ldots+(n-1)+n \\ 2S &= ((n)+(1))+((n-1)+(2))+\ldots+((2)+(n-1))+((1)+(n)) \\ &= (n+1)+(n+1)+\ldots+(n+1)+(n+1) \\ &= n(n+1)S = \frac{n(n+1)}{2} \end{aligned}$$

1.16

$$\lim_{n\to+\infty} \frac{(n+1)! - 2(n!)}{3(n+1)! + 1}$$

$$\lim_{n\to+\infty} \frac{(n+1)!(1-\frac{2}{n+1})}{(n+1)!(3+\frac{1}{(n+1)!})} \tag{58}$$

$$\lim_{n\to+\infty} \frac{(1-\frac{2}{n+1})}{(3+\frac{1}{(n+1)!})} \tag{59}$$

$$\lim_{n\to+\infty} \frac{1+0}{3+0} = \frac{1}{3} \tag{60}$$

Vytýkání faktoriálu

$$a(n!) = \frac{a}{n+1}(n+1)!$$

$$a(n!) = \frac{a}{(n+2)(n+1)}(n+2)!$$

$$a(n!) = \frac{a}{(n+3)(n+2)(n+1)}(n+3)!$$

1.17

$$\lim_{n\to+\infty} \frac{3(n+2)! - (n+1)!}{(n+3)!}$$

$$\lim_{n\to+\infty} \frac{(n+3)!\left(\frac{3}{n+3} - \frac{1}{(n+3)(n+2)}\right)}{(n+3)!} \tag{61}$$

$$\lim_{n\to+\infty} \left(\frac{3}{n+3} - \frac{1}{(n+3)(n+2)}\right) \tag{62}$$

$$\lim_{n\to+\infty} \left(\frac{3}{\infty} - \frac{1}{\infty}\right) = 0 \tag{63}$$

1.18

$$\lim_{n\to+\infty} \frac{(-1)^n}{n^3\sqrt{18}}$$

$$\frac{(-1)}{n^3\sqrt{18}} \leq \frac{(-1)^n}{n^3\sqrt{18}} \leq \frac{(1)}{n^3\sqrt{18}}$$

$$\lim_{n\to+\infty} \frac{(-1)}{n^3\sqrt{18}} \leq \lim_{n\to+\infty} \frac{(-1)^n}{n^3\sqrt{18}} \leq \lim_{n\to+\infty} \frac{(1)}{n^3\sqrt{18}}$$

$$0 \leq \lim_{n\to+\infty} \frac{(-1)}{n^3\sqrt{18}} \leq 0$$

$$\lim_{n\to+\infty} \frac{(-1)^n}{n^3\sqrt{18}} = 0$$

Metoda dvou policajtů

$$a_n \leq b_n \leq c_n \ , \forall n \in \mathbb{N}$$
$$\lim a_n \leq \lim b_n \leq \lim c_n$$